RESIDUES FOR HOLOMORPHIC FOLIATIONS OF SINGULAR PAIRS

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Abstract. Let X be a (possibly singular) subvariety of a complex manifold M and Y a subvariety of X. We assume that Y is the intersection locus of X with a submanifold $P \subset M$ and this intersection is generically transverse. For such a pair (X, Y), we prove a generalization of the classical Camacho-Sad residue theorem, in case there exists a holomorphic foliation $\mathcal{F}$ of X leaving Y invariant. Also, we compute explicitly the residues at isolated singular points.

Introduction

The classical Camacho-Sad residue (or index) theorem [CS] states that if X is a two dimensional complex manifold, $Y \subset X$ a non-singular compact complex curve invariant by a holomorphic foliation $\mathcal{F}$ of X, then the first Chern class $c_1(N_{Y,X})$ of the normal bundle $N_{Y,X}$ of Y in X localizes at the singularities of $\mathcal{F}$ in Y. That is to say, to each point p in $Y \cap \mathrm{Sing}(\mathcal{F})$ one can associate a complex number $\mathrm{Res}(\mathcal{F}, Y; p)$, called the *residue* of $\mathcal{F}$ at p relative to Y, depending only on the behavior of $\mathcal{F}$ near p, such that $\sum \mathrm{Res}(\mathcal{F}, Y; p) = \int_Y c_1(N_{Y,X})$. C. Camacho and P. Sad used their theorem to settle a problem raised by Poincaré on the existence of separatrices for germs of holomorphic foliations in $\mathbb{C}^2$. On the other hand the Camacho-Sad theorem can be seen as an obstruction to the existence of foliations having a given curve as invariant. Especially in this optic, the Camacho-Sad theorem has been generalized by several authors. Just to name a few, A. Lins Neto [Li] for the case of a singular curve Y in $X = \mathbb{CP}^2$, the second named author [Su2] for the general case Y singular, and D. Lehmann and the second named author [LS] for the case X is a complex manifold of arbitrary dimension and Y is an arbitrary (co)dimensional strongly locally complete intersection in X (we refer to [Su3] for a

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detailed history). Very recently, a paper by V. Cavalier, D. Lehmann and M. Soares [CLS] generalizes the Camacho-Sad theorem to the case Y is an arbitrary singular subvariety of a complex manifold X.

The basic idea in all these papers (which is however rather hidden in the first three) is that a holomorphic foliation $\mathcal{F}$ provides a vanishing of certain characteristic classes away from the singular locus of Y and $\mathcal{F}$. Provided that there exists a "good extension" of $N_{Y,X}$ on such a singular locus, the Čech-de Rham theory produces localization at some cohomological level and Poincaré and Alexander duality give then the residue theorems. Thus, taking this machinery for granted, to have a Camacho-Sad type theorem one has to face two problems: find an extension ("natural" to some extent) of the normal bundle of Y in X to all the singularities of Y, and then find some "good action" of $\mathcal{F}$ on such extension in order to get some vanishing theorems. According to this picture, in order to solve problems in discrete dynamics, the first named author together with M. Abate and F. Tovena (see [Ab], [Br], [BT], [ABT]) developed a way to obtain generalizations of the Camacho-Sad theorem in case the foliation $\mathcal{F}$ is replaced by a holomorphic map $f : X \to X$ pointwise fixing Y.

In all the previous works the ambient manifold X is supposed to be non-singular, which allows to have natural extensions of $N_{Y,X}$. The next step would be then to allow some singularities for X. This is not, however, just a merely technical game. Indeed, in some dynamical problems one has to face a singular ambient space. In the case of foliations (see [Ca]), this does not cause really a serious problem, for one can always resolve the singularities and pull back the foliation to a non-singular ambient, however, in the case of holomorphic maps this is no longer possible. Thus in [BS], for answering a question of discrete dynamics, the authors were forced to define residues for a singular X of dimension two and proved a Camacho-Sad type theorem in that case. The aim of this paper is to give a general version of the residue theorem introduced in [BS] with no restrictions on the dimension of X and (co)dimension of Y.

The setting is as follows. We let X be an analytic variety in a complex manifold M and Y a subvariety of X. We assume that Y is presented as the intersection of X with a submanifold $P \subset M$, in such a way that the intersection is generically transverse. We call such a pair (X, Y) *adequate* (see Definition 1.2 below). Many examples of algebraic varieties come up this way. The bundle $N_{Y,X}$, defined only on the non-singular part of Y in X, has then a natural extension given by $N_{P,M}|_Y$. If $\mathcal{F}$ is a holomorphic foliation of X (actually, we do not need it to be defined outside X as required in [BS]) leaving Y invariant, we prove a localization theorem for $\varphi(N_{P,M}|_Y)$ near the singularities of Y and of $\mathcal{F}$ in Y, where φ is a homogeneous symmetric polynomial of an appropriate degree (see Theorem 2.1). This can be seen as a natural generalization of the Camacho-Sad theorem for the case X is singular. We also compute explicitly the residues at isolated singular points in the top degree case (see Theorem 3.1).

The first section of the paper recalls some basic facts about holomorphic foliations and introduces adequate singular pairs, especially discussing their singularities. In the second section we derive our main residues theorem. In the third section we provide a computation of residues at isolated points.

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1. Preliminaries

Let M be a connected complex manifold. The symbol $\mathcal{C}^\infty_M$ will denote the sheaf of C^∞-functions on M, while $\mathcal{O}_M$ is the sheaf of holomorphic functions. If E is a (complex) vector bundle on M we denote by $\mathcal{C}^\infty(E)$ the sheaf of C^∞-sections of E while we reserve the italic symbol $\mathcal{E} = \mathcal{O}_M(E)$ to denote the sheaf of *holomorphic sections* of E. Moreover we denote by TM the holomorphic tangent bundle of M and $\Theta_M = \mathcal{O}_M(TM)$ the sheaf of germs of holomorphic vectors fields on M.

1.1 Foliations.

Definition 1.1. A *(singular) foliation* of M is a coherent subsheaf $\mathcal{F} \subset \Theta_M$ which is involutive, i.e., for any $p \in M$,

$$[\mathcal{F}_p, \mathcal{F}_p] \subseteq \mathcal{F}_p.$$

Let $\mathcal{Q} := \Theta_M/\mathcal{F}$ be the quotient sheaf. The singular set $\mathrm{Sing}(\mathcal{F})$ of a foliation is defined as

$$\mathrm{Sing}(\mathcal{F}) = \{\, p \in M : \mathcal{Q}_p \ \text{ is not } \ \mathcal{O}_{M,p}\text{-free} \,\}.$$

Note that $\mathrm{Sing}(\mathcal{F})$ is a closed subvariety of M. The *dimension* of $\mathcal{F}$ is the rank of $\mathcal{F}_p$ at some (and hence any) point $p \in M \setminus \mathrm{Sing}(\mathcal{F})$.

Remarks 1. If $\mathcal{Q}_p$ is $\mathcal{O}_{M,p}$-free, then so is $\mathcal{F}_p$. Therefore on $M^0 = M \setminus \mathrm{Sing}(\mathcal{F})$ there exists a holomorphic vector bundle $F \subset TM^0$ whose germs of holomorphic sections form $\mathcal{F}|_{M^0}$.

2. Sometimes the definition of foliation requires that $\mathcal{F}$ be also *reduced* (or *full*). This is a technical condition meaning that, for any open set $U \subset M$ and any section $s \in \Gamma(U, \Theta_M)$, if $s_p \in \mathcal{F}_p$ for any $p \in U \setminus \mathrm{Sing}(\mathcal{F})$, then actually s is in $\Gamma(U, \mathcal{F})$. However there is a canonical way to obtain a reduced foliation from a non-reduced one (see [BB], [Su1]).

Let X be a possibly singular subvariety in M. We denote by $\mathrm{Sing}(X)$ the singular set of X and by $X' = X \setminus \mathrm{Sing}(X)$ the non-singular part. In the following we need holomorphic foliations on X. First we need to define the sheaf of holomorphic “tangent vectors” Θ_X to X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. This is defined as $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}_X$, where $\mathcal{I}_X \subset \mathcal{O}_M$ is the ideal sheaf of germs of holomorphic functions identically vanishing on X. Then Θ_X is defined as $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ with Ω_X being given by the following exact sequence of $\mathcal{O}_X$-modules:

$$\mathcal{I}_X/\mathcal{I}_X^2 \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow 0,$$

where $\Omega_M = \mathcal{O}_M(T^*M)$ and the first morphism is given by $[f] \mapsto df \otimes 1$. Note that on X', $\Theta_X = \mathcal{O}_X(TX')$. Note also that Θ_X acts on $\mathcal{O}_X$ as derivations, as in the case of non-singular base spaces. A *holomorphic foliation* on X is a coherent subsheaf $\mathcal{F} \subset \Theta_X$ such that $\mathcal{F}|_{X'}$ is a holomorphic foliation. We set $\mathrm{Sing}(\mathcal{F}) = \mathrm{Sing}(\mathcal{F}|_{X'}) \cup \mathrm{Sing}(X)$.

First examples of holomorphic foliations on X come from restriction of foliations of M. Namely, let $\mathcal{F}$ be a holomorphic foliation of M. We say that X is $\mathcal{F}$-invariant if every vector v in $\mathcal{F}$ leaves the ideal $\mathcal{I}_X$ invariant, which is equivalent to saying that v is tangent to X'. We denote by $\mathcal{F}|_X$ the image of $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_X$ in $\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_X$. If X is $\mathcal{F}$-invariant, then actually $\mathcal{F}|_X \subset \Theta_X$ and if, moreover, $\dim(\mathrm{Sing}(\mathcal{F}) \cap X) < \dim X$, then $\mathcal{F}|_X$ is a foliation of X of the same dimension as $\mathcal{F}$ (cf. [Su3, Ch.VI, 6]).

Other natural examples of foliations on X come from the case X is pointwise fixed by a (nontrivial) holomorphic self-map of M (see [ABT]).

1.2 Adequate singular pairs.

Definition 1.2. Let M be a complex manifold of dimension m. Let $X \subset M$ be an analytic variety of pure dimension n, $Y \subset X$ a subvariety of pure dimension $l > 0$. We say that (X, Y) is an *adequate singular pair* in M if there exists a submanifold $P \subset M$ of dimension $r = m + l - n$ such that
(1) $Y = X \cap P$ (set-theoretically),
(2) $\dim(\mathrm{Sing}(X) \cap P) < l$,
(3) X' intersects P generically transversally.

Example 1.1. Let X be an n-dimensional algebraic subvariety of $\mathbb{CP}^m$ and $P \subset \mathbb{CP}^m$ an r-dimensional general linear subspace (with $r - m + n > 0$). Then $(X, X \cap P)$ is an adequate singular pair in $\mathbb{CP}^m$.

Example 1.2. Let $X_1 \subset \mathbb{C}^m$ be a germ of a singular variety of dimension n at the origin O. Assume that the singularity of X at O is isolated and can be resolved with only a finite number of blow-ups. Blow up the point O and let X be the strict transform of X_1 and P the exceptional divisor. Let $\tilde{Y} = P \cap X$. Let Y be a connected component of $\tilde{Y}$. Then either X intersects nontransversally P everywhere on Y or (X, Y) is an adequate singular pair in the blow up of $\mathbb{C}^m$ at O.

We examine closer the singularities of adequate singular pairs. We set $Y' = (Y \setminus \mathrm{Sing}(Y)) \cap X'$ and

$$Y^{nt} = \{\, q \in X' \cap P : T_qX' + T_qP \neq T_qM \,\}.$$

Thus Y^{nt} is the set where X' and P are not transverse. In [BS] the following lemma is proved for $n = 2$ and $r = m - 1$.

Lemma 1.1. *If (X, Y) is an adequate pair, then $Y^{nt} = X' \cap Sing(Y)$.*

Proof. Clearly $X' \cap \mathrm{Sing}(Y) \subset Y^{nt}$, for if X' is transverse to P at a point q, then Y is non-singular at q. Conversely, suppose $q \in Y'$. We wish to show that X' intersects transversally P at q. Let $(z_1, \dots, z_m)$ be local coordinates on an open set $U \subset M$ such that $q \in U$, $X \cap U = \{z_{n+1} = \ldots = z_m = 0\}$ and $Y \cap U = \{z_{l+1} = \ldots = z_m = 0\}$, where we recall that $l = n + r - m \geq 1$. There exist holomorphic functions $h_1, \dots, h_{m-k}$ on U such that $P \cap U = \{h_1(z) = \ldots = h_{m-r}(z) = 0\}$. Then the target is to show that

$$\det \left(\frac{\partial h_j}{\partial z_k} \right) (q) \neq 0. \tag{1.1}$$

In the above, $1 \leq j \leq m-r$ and $l+1 \leq k \leq n$. Let $\overline{h}_j := h_j|_X$, for $j = 1, \dots, m-r$. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf of holomorphic functions on X identically vanishing on Y. Note that $\mathcal{I}_Y$ is generated in $\mathcal{O}_X$ by $z_{l+1}, \dots, z_n$. Since $\overline{h}_j \in \mathcal{I}_{Y,q}$, there exist holomorphic functions $a_{jk} \in \mathcal{O}_X$ on X, for $j = 1, \dots, m - r$ and $k = l + 1, \dots, n$, such that $\overline{h}_j = \sum_{k=l+1}^{n} a_{jk} z_k$. Since

$$\det \left(\frac{\partial h_j}{\partial z_k} \right) (q) = \det \left(\frac{\partial \overline{h}_j}{\partial z_k} \right) (q)$$

and the latter term is non-zero if and only if

$$\det(a_{jk})(q) \neq 0, \tag{1.2}$$

we are then left to prove (1.2). For any point $p \in U \cap Y$ let $\mathcal{G}_p \subset \mathcal{O}_{X,p}$ be the ideal generated by $\overline{h}_1, \dots, \overline{h}_{m-r}$. Note that

$$\mathcal{I}_{Y,p} = \mathcal{G}_p \tag{1.3}$$

if and only if $\det(a_{jk})(p) \neq 0$ and thus (1.3) is equivalent to X being transverse to P at p. Therefore we are left to prove that (1.3) holds for $p = q$. Suppose not. Since Y is smooth at q, $\{\overline{h}_1, \dots, \overline{h}_{m-r}\}$ are non-reduced defining equations of Y at q. Since Y is locally complete intersection at q, $\{\overline{h}_1, \dots, \overline{h}_{m-r}\}$ are non-reduced defining equations for Y at nearby points (see, *e.g.*, [Lo, p.50]). But then for any point $p \in Y$ in a neighborhood of q it follows that $\mathcal{G}_p \neq \mathcal{I}_{Y,p}$ and thus X is non-transversal to P on an open set in Y, against our hypothesis (3). $\square$

As a corollary of Lemma 1.1 we have

Corollary 1.2. *The subvariety $Y^{nt} \subset Y$ is made of connected components isolated in Y.*

2. Residue theorems for foliations of adequate singular pairs

Let (X, Y) be an adequate singular pair in M. Let $N_{P,M}$ denote the normal bundle of P in M defined by the following exact sequence of holomorphic vector bundles:

$$0 \longrightarrow TP \longrightarrow TM|_P \longrightarrow N_{P,M} \longrightarrow 0.$$

Similarly define $N_{Y',X'}$ the normal bundle of Y' in X'. It is easy to see that

$$N_{P,M}|_{Y'} = N_{Y',X'}$$

for P and X' are transverse at each point of Y' by Lemma 1.1. This apparently harmless observation will allow us to obtain localization of $N_{P,M}|_Y$ (which can be seen as a natural extension of $N_{Y',X'}$) regardless the singularities of Y and X.

Let $\mathcal{F}$ be a holomorphic foliation of X of dimension $d \leq l$ and assume that Y is $\mathcal{F}$-invariant and $\dim(\mathrm{Sing}(\mathcal{F}) \cap Y) < l$. Let φ be a symmetric homogeneous polynomial of degree $t > l - d$. Let $X^0 := X' \setminus \mathrm{Sing}(\mathcal{F})$ and $Y^0 := Y' \cap X^0$. Let $F \subset TX^0$ denote the associated holomorphic bundle to $\mathcal{F}$. Then there is a natural holomorphic action Ξ of $F|_Y \subset TY^0$ on N_{Y^0,X^0} (see [Le]). That is to say, one can define an operator $\Xi : F|_Y \times N_{Y^0,X^0} \to N_{Y^0,X^0}$ such that
(i) $\Xi([u, v], s) = \Xi(u, \Xi(v, s)) - \Xi(v, \Xi(u, s))$ for $u, v \in \mathcal{C}^\infty(F|_Y)$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(ii) $\Xi(hu, s) = h\Xi(u, s)$ for $h \in \mathcal{C}^\infty_Y$, $u \in \mathcal{C}^\infty(F|_Y)$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(iii) $\Xi(u, hs) = h\Xi(u, s) + u(h)s$ for $h \in \mathcal{C}^\infty_Y$, $u \in \mathcal{C}^\infty(F|_Y)$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(iv) $\Xi(u, s) \in \mathcal{O}_Y(N_{Y^0,X^0})$ for $u \in \Theta_Y$, $s \in \mathcal{O}_Y(N_{Y^0,X^0})$.

In fact, for $u \in \mathcal{F}|_Y$ and $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$ the operator Ξ is defined as

$$\Xi(u, s) = \chi([\tilde{u}, \tilde{s}]|_Y), \tag{2.1}$$

where $\tilde{u} \in \mathcal{F}$ is such that $\tilde{u}|_Y = u$, $\tilde{s} \in \mathcal{C}^\infty(TX^0)$ and $\chi(\tilde{s}|_Y) = s$, where $\chi : TX^0|_Y \to N_{Y^0,X^0}$ is the canonical projection. One can prove that Ξ is well-defined and verifies the properties (i)–(iv). Starting from Ξ one can define a Ξ-connection ∇_0 for N_{Y^0,X^0}. Namely ∇_0 is a connection for N_{Y^0,X^0} such that

$$(\nabla_0)_u \cdot = \Xi(u, \cdot)$$

for any $u \in \mathcal{F}|_Y$ (see [BB], [Le], [Su3]).

In particular, for any Ξ-connection ∇_0 for N_{Y^0,X^0}, we have the "Bott vanishing" (see, e.g., [Su3, Ch. II, 9])

$$\varphi(\nabla_0) = 0. \tag{2.2}$$

Now, using the Čech-de Rham cohomology we can localize $\varphi(N_{P,M}|_Y)$ as follows. Let U_0 be a tubular neighborhood of Y^0 in P, and let $\rho : U_0 \to Y^0$ be the C^∞-retraction. Since $N_{P,M}|_{Y^0} = N_{Y^0,X^0}$, we can take ∇_0 a Ξ-connection for $N_{P,M}|_{Y^0}$ on $U_0 \cap Y^0$ and consider the connection $\rho^*(\nabla_0)$ for $\rho^*(N_{P,M})$ on U_0. Note that $\rho^*(N_{P,M})$ is C^∞-equivalent to $N_{P,M}$ on U_0. With some abuse of notation we simply denote $\rho^*(N_{P,M})$ by $N_{P,M}$ and also we denote by ∇_0 its connection $\rho^*(\nabla_0)$.

We set $\Sigma := (\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Sing}(Y) \cup \mathrm{Sing}(X)) \cap Y)$, which is a subvariety of dimension strictly less than l in Y. Let U_1 be an open neighborhood of Σ in P such that U_1 is union of disjoint open sets $U_{1,\gamma}$ each of them containing exactly one connected component—say Σ_γ—of Σ and such that $U_{1,\gamma}$ is a regular neighborhood of Σ_γ for every γ (this is possibly by Corollary 1.2). Moreover we may assume that $U_0 \cup U_1$ is a regular neighborhood of Y in P. On U_1 we choose an arbitrary connection ∇_1 for $N_{P,M}$. Let $H^*_D(\mathcal{U})$ be the Čech-de Rham cohomology associated to the covering $\mathcal{U} = \{U_0, U_1\}$ (see, *e.g.*, [Su3, Ch.II,3, Ch.IV,2, Ch.VI,4]). Recall that $H^*_D(\mathcal{U}) \simeq H^*(Y;\mathbb{C})$. Then, from (2.1), it follows that $\varphi(N_{P,M})$ in $H^{2t}_D(\mathcal{U})$ is represented by the cocycle

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0,\nabla_1)) = (0, \varphi(\nabla_1), \varphi(\nabla_0,\nabla_1)), \tag{2.2}$$

where $\varphi(\nabla_0,\nabla_1)$ is the so-called *Bott difference form* (it is a $(2t-1)$-form (of P) on $U_0 \cap U_1$ such that $d\varphi(\nabla_0,\nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0)$). Therefore $[\varphi(\nabla_*)] \in H^{2t}_D(\mathcal{U}, U_0)$, the relative Čech-de Rham cohomology. Recall that $H^*_D(\mathcal{U}, U_0) \simeq H^*(Y, Y \setminus \Sigma; \mathbb{C})$. Then we have a first localization of $\varphi(N_{P,M}|_Y) \in H^{2t}(Y;\mathbb{C})$ at cohomology level, which we denote by $\varphi_\Sigma(N_{P,M}) \in H^{2t}(Y, Y\setminus\Sigma;\mathbb{C})$. If Σ is compact we can consider the Alexander homomorphism $A : H^*((Y, Y\setminus\Sigma;\mathbb{C}) \to H_{2l-*}(\Sigma;\mathbb{C})$. The image of $A(\varphi_\Sigma(N_{P,M})) \in H_{2l-2t}(\Sigma;\mathbb{C})$ is denoted by $\overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y; \Sigma)$. Note that, since $H_{2l-2t}(\Sigma;\mathbb{C}) = \oplus_\gamma H_{2l-2t}(\Sigma_\gamma;\mathbb{C})$, $\overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y;\Sigma) = \sum_\lambda \overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y;\Sigma_\gamma)$ with $\overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y;\Sigma_\gamma) \in H_{2l-2t}(\Sigma_\gamma;\mathbb{C})$. If $\iota_\gamma : \Sigma_\gamma \hookrightarrow Y$ is the inclusion and $(\iota_\gamma)_*$ is the morphism induced in homology by ι_γ, we denote by $\mathrm{Res}_\varphi(\mathcal{F}, Y;\Sigma_\gamma) = (\iota_\gamma)_*\overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y;\Sigma_\gamma)$. Thus if Y is compact we have

$$\sum_\gamma \mathrm{Res}_\varphi(\mathcal{F}, Y;\Sigma_\gamma) = \mathrm{Poi}(\varphi(N_{P,M}|_Y)),$$

where $\mathrm{Poi} : H^*(Y;\mathbb{C}) \to H_{2l-*}(Y;\mathbb{C})$ denotes the Poincaré homomorphism, which is given by the cap product with the fundamental cycle $[Y]$ of Y. Summing up, we have proved

Theorem 2.1. *Let (X, Y) be an adequate singular pair in M and let $\mathcal{F}$ be a foliation of X of dimension d which leaves Y invariant. Let $\Sigma = (Sing(\mathcal{F}) \cup Sing(Y) \cup Sing(X)) \cap Y)$ and assume that Σ is compact. Let $\Sigma = \cup_\gamma \Sigma_\gamma$ be the decomposition*

into connected components. Let φ be a symmetric homogeneous polynomial of degree $t > l - d$. Then
(i) *For each component Σ_γ there exists a residue* $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) \in H_{2l-2t}(Y; \mathbb{C})$ *which is determined only by the local behavior of $\mathcal{F}$ near Σ_γ.*
(ii) *If Y is compact we have*

$$\sum_\gamma \mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) = \varphi(N_{P,M}) \frown [Y] \qquad \text{in} \;\; H_{2l-2t}(Y; \mathbb{C}).$$

3. Computation of residues.

In this section we compute the residues given by Theorem (2.1) in some special cases and compare them with the ones obtained by [Le] and [LS] for the case X is non-singular. We retain the notation introduced in the previous sections.

First we find a generic expression for $\mathrm{Res}(\mathcal{F}, Y; \Sigma_\lambda)$. Let $\mathcal{U} = \{U_0, U_1\}$ be as in section 2. Let $\tilde{R}_{1,\gamma}$ be a compact real $2r$-dimensional manifold with C^∞ boundary in $U_{1,\gamma}$ such that Σ_λ is contained in the interior of $\tilde{R}_{1,\gamma}$ and that the boundary $\partial\tilde{R}_{1,\gamma}$ is transverse to Y. We set $R_{1,\gamma} = \tilde{R}_{1,\gamma} \cap Y$. Thus (see, *e.g.*, [Su3, p. 189]) it follows that $\overline{\mathrm{Res}}_\varphi(\mathcal{F}, Y; \Sigma_\gamma)$ is represented by the $(2l-2t)$-cycle C in Σ_γ such that for any closed $(2l-2t)$-form τ_1 on $U_{1,\gamma}$ it follows

$$\int_C \tau_1 = \int_{R_{1,\gamma}} \varphi(\nabla_1) \wedge \tau_1 - \int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1) \wedge \tau_1, \tag{3.1}$$

where ∇_0, ∇_1 and $\varphi(\nabla_0, \nabla_1)$ are defined in (2.2). In particular if $t = l$, then the residue is a complex number.

Our aim is to compute explicitly the residue in case $t = l$, $d = 1$ and $\Sigma_\gamma = \{q_\gamma\}$. Moreover we assume $\mathcal{F}_{q_\gamma}$ is generated on $\mathcal{O}_{X,q_\gamma}$ by a single element of Θ_{X,q_γ}.

Since Σ_γ is reduced to one point, we can take $\tilde{U}_{1,\gamma}$ an open neighborhood of q_γ with local coordinates $\{z_1, \ldots, z_m\}$ centered at q_γ and such that $P \cap \tilde{U}_{1,\gamma} = \{z_{r+1} = \ldots = z_m = 0\}$. Moreover we may assume that on $U_{1,\gamma} := \tilde{U}_{1,\gamma} \cap P$ the bundle $N_{P,M}|_{U_{1,\gamma}}$ is holomorphically trivial and both X and Y are non-singular on $\tilde{U}_{1,\gamma} \setminus \{q_\gamma\}$. Let $\sigma : TM|_P \longrightarrow N_{P,M}$ be the canonical projection. Let $\delta_j := \sigma(\frac{\partial}{\partial z_j})$ for $j = r+1, \ldots, m$. Then $\{\delta_{r+1}, \ldots, \delta_m\}$ is a holomorphic base frame of $N_{P,M}$ on $U_{1,\gamma}$. We can take ∇_1 to be the trivial connection for $N_{P,M}$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$. Thus $\varphi(\nabla_1) = 0$ and

$$\mathrm{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = -\int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1). \tag{3.2}$$

Now we find "good coordinates" to express the Bott difference form. First, by the local parameterization theorem, we can find an open set $\tilde{U}_0$ in M containing $Y^0 \cap \tilde{U}_{1,\gamma}$ and holomorphic functions $h_1, \dots, h_l \in \mathcal{O}_M(\tilde{U}_{1,\gamma})$ such that, if $j : X \hookrightarrow M$, then

$$j^*(dh_1 \wedge \ldots \wedge dh_l \wedge dz_{r+1} \wedge \ldots \wedge dz_m|_{\tilde{U}_0 \cap X}) \neq 0. \tag{3.3}$$

This means that $\{h_1, \dots, h_l, z_{r+1}, \dots, z_m\}$ can be thought of as local coordinates of X^0 and $\{h_1, \dots, h_l\}$ as local coordinates of Y^0 on $\tilde{U}_0$.

¿From the coherence of $\mathcal{F}$ and since $\mathcal{F}_{q_\gamma}$ is generated by only one element of Θ_{X,q_γ}, up to shrink $\tilde{U}_{1,\gamma}$ we can assume that on $\tilde{U}_{1,\gamma} \cap X$ the foliation $\mathcal{F}$ is generated by the holomorphic vector field $\xi \in \Theta_X$. With a slight abuse of notation, by (3.3), we can write ξ on $\tilde{U}_0 \cap X$ as

$$\xi = \sum_{j=r+1}^{m} \xi(z_j) \frac{\partial}{\partial z_j} + \sum_{i=1}^{l} \xi(h_j) \frac{\partial}{\partial h_i}, \tag{3.4}$$

where, for $\xi(z_j)$ and $\xi(h_i)$ to make sense, one should think of z_j and h_i as elements of $\mathcal{O}_X$ via the surjection $\mathcal{O}_M \to \mathcal{O}_X$.

Remark 3.1. *In case $\mathcal{F}$ comes from the restriction of a foliation of M given in $\tilde{U}_{1,\gamma}$ by a vector field $\hat{\xi}$, then $\mathcal{F}$ on $X^0 \cap \tilde{U}_{1,\gamma}$ is given by*

$$\xi = \hat{\xi}|_{X^0} = \sum_{j=r+1}^{m} \hat{\xi}(z_j)|_X \frac{\partial}{\partial z_j} + \sum_{i=1}^{l} \hat{\xi}(h_j)|_X \frac{\partial}{\partial h_i}. \tag{3.5}$$

Note that, since q_γ is an isolated singularity, $\xi(p) \neq 0$ for $p \in X^0 \cap \tilde{U}_{1,\gamma}$, and thus

$$\bigcap_{j=r+1}^{m} \{p \in \tilde{U}_0 | \xi(z_j)(p) = 0\} \bigcap_{i=1}^{l} \{p \in \tilde{U}_0 | \xi(h_i)(p) = 0\} = \emptyset.$$

Let $\mathfrak{M}$ be the $(m-r)$-square matrix with entries $\frac{\partial \xi(z_j)}{\partial z_k}$ for $j, k = r+1, \dots, m$. Recall that for a symmetric homogeneous polynomial $\varphi = \sum c_k \sigma_k^{j_k}$ with σ_k elementary symmetric functions and $c_k \in \mathbb{C}$, one can define $\varphi(\mathfrak{M}) := \sum c_k \sigma_k^{j_k}(\mathfrak{M})$ where $\sigma_k(\mathfrak{M})$ are defined by means of the following relation

$$\det(I + t\mathfrak{M}) = 1 + t\sigma_1(\mathfrak{M}) + \ldots + t^{m-r}\sigma_{m-r}(\mathfrak{M}).$$

With the above notation we have

Theorem 3.1. *Let $d = 1$, $l = t$ and $\Sigma_\gamma = \{q_\gamma\}$. Assume that $\mathcal{F}$ is generated by $\xi \in \Theta_X$ near q_γ. Then there exists a small $\epsilon > 0$ such that*

$$\mathrm{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^l \int_\Gamma \frac{\varphi(\mathfrak{M}) dh_1 \wedge \ldots \wedge dh_l}{\xi(h_1)\cdots\xi(h_l)},$$

where $\Gamma = \{\, p \in U_0 \cap Y : |\xi(h_i)(p)| = \epsilon \,\}$ is a real l-cycle oriented so that $d(\arg \xi(h_1) \wedge \ldots \wedge d(\arg \xi(h_l)) \geq 0$.

The proof of Theorem 3.1 for $l = 1$, $n = 2$ can be inferred from that of [BT, eq. (2.7)]. We also explicitly note that once appropriate coordinates as before are introduced the argument is similar to the one in [LS, Thm. 1']. However for sake of clearness we give here a proof of Theorem 3.1 for $l = 2$. For $l > 2$ the argument is the same and is left to the reader.

Proof of Theorem 3.1 for $l = 2$. Let $U = \tilde{U}_0 \cap Y$. Let $W_j = \{\, p \in U : \xi(h_j)(p) \neq 0 \,\}$, $j = 1, 2$. Note that $W_1 \cup W_2 = U$. Let $\mathcal{W} = \{W_1, W_2\}$. We recall briefly what the Čech-de Rham complex $CDR^*(\mathcal{W})$ associated to $\mathcal{W}$ is (for details see, *e.g.*, [Su3,Ch.II]). An element $\alpha \in CDR^k(\mathcal{W})$ is made of a triple $(\alpha_1, \alpha_2, \alpha_{12})$ with α_j a differential form of Y of degree k on W_j ($j = 1, 2$) and α_{12} a differential form of Y of degree $k - 1$ on $W_1 \cap W_2$. The differential D of $CDR^*(\mathcal{W})$ is given by

$$D(\alpha_1, \alpha_2, \alpha_{12}) = (d\alpha_1, d\alpha_2, \alpha_2 - \alpha_1 - d\alpha_{12}).$$

One can define a linear operator $\int_{\partial R_{1,\gamma}} : CDR^3(\mathcal{W}) \to \mathbb{C}$ as follows. Let $T_1 = \{\, p \in \partial R_{1,\gamma} : |\xi(h_1)(p)| \geq |\xi(h_2)(p)| \,\}$ with positive orientation, $T_2 = \{\, p \in \partial R_{1,\gamma} : |\xi(h_2)(p)| \geq |\xi(h_1)(p)| \,\}$ with positive orientation and Γ as defined in Theorem 3.1. Then

$$\int_{\partial R_{1,\gamma}} (\alpha_1, \alpha_2, \alpha_{12}) := \int_{T_1} \alpha_1 + \int_{T_2} \alpha_2 + \int_\Gamma \alpha_{12}.$$

It is easy to show that $\int_{\partial R_{1,\gamma}} \circ D = 0$.

Recall that $\varphi(\nabla_0, \nabla_1)$ is a $(2t - 1)$-form of P on $U_0 \cap U_1$ and we need to integrate it on $\partial R_{1,\gamma} \subset Y$. Thus we can consider the restriction $\varphi(\nabla_0, \nabla_1)|_Y$, denoted by $\kappa(\nabla_0, \nabla_1)$. Thus $\kappa(\nabla_0, \nabla_1)$ is a $(2l-1)$-form (that is a 3-form) on $Y \cap U$. Thus

$$(\kappa(\nabla_0, \nabla_1)|_{W_1}, \kappa(\nabla_0, \nabla_1)|_{W_2}, 0)$$

defines an element $\varpi \in CDR^3(\mathcal{W})$. Note that, since

$$d\kappa(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0) = 0 - 0 = 0,$$

$D\varpi = 0$. On the other hand on $W_1 \cap W_2$ we have the l-form (that is a 2-form) $\varphi(\mathfrak{M})\frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}$. We can consider the element $\phi \in CDR^3(\mathcal{W})$ given by

$$\phi = (0, 0, (2\pi\sqrt{-1})^{-2}\varphi(\mathfrak{M})\frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}).$$

Since the form $\varphi(\mathfrak{M})\frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}$ is holomorphic on $W_1 \cap W_2$ it follows that $D\phi = 0$. The aim is now to show that there exists $\tau \in CDR^2(\mathcal{W})$ such that

$$\phi - \varpi = D\tau. \tag{3.6}$$

Suppose we proved (3.6). Then $\int_{\partial R_{1,\gamma}} \varpi = \int_{\partial R_{1,\gamma}} \phi$ and the result follows.

Thus we are left to prove (3.6). On W_1, since $\xi(h_1) \neq 0$, a basis of TY is given by $\{\xi|_Y, \frac{\partial}{\partial h_2}\}$. Similarly, on W_2, $\{\xi|_Y, \frac{\partial}{\partial h_1}\}$ is a basis of TY. On W_i, $i = 1, 2$, we define the following Ξ-connection ∇^i of type $(1, 0)$:

$$\begin{aligned} \nabla^i_{\xi|_Y}\delta_j &= \Xi(\xi|_Y, \delta_j), \\ \nabla^i_{\frac{\partial}{\partial h_k}}\delta_j &= 0, \end{aligned}$$

for $k = i+1 \mod (2)$, $j = r+1, \ldots, m$ (here we recall that Ξ is defined by (2.1) and $\{\delta_{r+1}, \ldots, \delta_m\}$ is the holomorphic frame for $N_{P,M}$ on $\tilde{U}_{1,\gamma}$ fixed at the beginning of this section). Recall that for $k \leq 2l = 4$ connections $\tilde{\nabla}_j$'s for a vector bundle E over some open set of Y, one can define a $2l - k + 1$-form $\varphi(\tilde{\nabla}_1, \ldots, \tilde{\nabla}_k)$ (see, *e.g.*, [Su3,p.69]). In particular

$$\sum_{a=1}^{k}(-1)^{a-1}\varphi(\tilde{\nabla}_1, \ldots, \widehat{\tilde{\nabla}_a}, \ldots, \tilde{\nabla}_k) + (-1)^{k-1}d\varphi(\tilde{\nabla}_1, \ldots, \tilde{\nabla}_k) = 0. \tag{3.7}$$

Moreover, if all $\tilde{\nabla}_1, \ldots, \tilde{\nabla}_k$ are α-connection with respect to some holomorphic action α on E, then $\varphi(\varphi(\tilde{\nabla}_1, \ldots, \tilde{\nabla}_k) = 0$ (see, *e.g.*, [Su3, Ch.II,Thm. 9.11]). Going back to our situation, we define

$$\tau = (\varphi(\nabla_0, \nabla_1, \nabla^1), \varphi(\nabla_0, \nabla_1, \nabla^2), \varphi(\nabla_0, \nabla_1, \nabla^1, \nabla^2)),$$

where we recall that ∇_0 is the original Ξ-connection for $N_{Y^0,X^0} = N_{P,M}|_{Y^0}$ on $U_0 \cap Y$ and ∇_1 is the trivial connection for $N_{P,M}$ on $U_{1,\gamma}$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$ (and here we consider its restriction to Y). Now we calculate $D\tau$. By (3.7) we get

$$d\varphi(\nabla_0, \nabla_1, \nabla^1) = -\varphi(\nabla_1, \nabla^1) + \varphi(\nabla_0, \nabla^1) - \varphi(\nabla_0, \nabla_1) = -\varphi(\nabla_0, \nabla_1),$$

for ∇_1, ∇^1 are both α_1-connections with respect to the trivial action given by $\alpha_1(\frac{\partial}{\partial z_1}, \delta_j) = 0$, and ∇_0, ∇^1 are both Ξ-connections. Similarly we get

$$d\varphi(\nabla_0, \nabla_1, \nabla^2) = -\varphi(\nabla_0, \nabla_1),$$

and finally by (3.7) and since $\nabla_0, \nabla^1, \nabla^2$ are all Ξ-connections

$$\begin{aligned} &\varphi(\nabla_0, \nabla_1, \nabla^2) - \varphi(\nabla_0, \nabla_1, \nabla^1) - d\varphi(\nabla_0, \nabla_1, \nabla^1, \nabla^2) \\ &\qquad = -\varphi(\nabla_1, \nabla^1, \nabla^2) + \varphi(\nabla_0, \nabla^1, \nabla^2) = -\varphi(\nabla_1, \nabla^1, \nabla^2). \end{aligned}$$

Thus

$$D\tau = (-\varphi(\nabla_0, \nabla_1), -\varphi(\nabla_0, \nabla_1), -\varphi(\nabla_1, \nabla^1, \nabla^2)).$$

Therefore (3.6) will follow as soon as we show that

$$-\varphi(\nabla_1, \nabla^1, \nabla^2) = (2\pi\sqrt{-1})^{-2}\varphi(\mathfrak{M})\frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}. \tag{3.8}$$

To prove (3.6) we first calculate the connection matrices $\theta_1, \theta^1, \theta^2$ of $\nabla_1, \nabla^1, \nabla^2$ with respect to the frame $\{\delta_{r+1}, \dots, \delta_m\}$. Clearly $\theta_1 \equiv 0$. As for θ^1 (and similarly for θ^2) we have

$$\Xi(\xi|_Y, \delta_j) = \nabla^1_{\xi|_Y}\delta_j = \nabla^1_{\xi(h_1)\frac{\partial}{\partial h_1} + \xi(h_2)\frac{\partial}{\partial h_2}}\delta_j = \xi(h_1)\nabla^1_{\frac{\partial}{\partial h_1}}\delta_j. \tag{3.9}$$

Now, by the very definition of Ξ (see (2.1) and (3.4)) we have

$$\begin{aligned} \Xi(\xi|_Y, \delta_j) &= \chi([\xi, \frac{\partial}{\partial z_j}]|Y) = \chi([\sum_{k=r+1}^m \xi(z_k)\frac{\partial}{\partial z_j} + \sum_{i=1}^2 \xi(h_i)\frac{\partial}{\partial h_i}, \frac{\partial}{\partial z_j}]|_Y) \\ &= -\sum_{k=r+1}^m \frac{\partial \xi(z_k)}{\partial z_j}|_Y\delta_k. \end{aligned} \tag{3.10}$$

¿From (3.9) and (3.10) we find

$$\theta^i = -\frac{dh_i}{\xi(h_i)}\mathfrak{M}, \quad i = 1, 2.$$

Now let $\tilde\nabla$ denote the connection for the bundle $TY \times \mathbb{R}^2$ over $W_1 \cap W_2 \times \mathbb{R}^2$ defined by $\tilde\nabla = (1 - \sum_{k=1}^2 t_k)\nabla_1 + \sum_{j=1}^2 t_j\nabla^j$. The connection matrix of $\tilde\nabla$ is then given by $\tilde\theta = -\sum_{i=1}^2 t_i/\xi(h_i)\mathfrak{M}\, dh_i$ and the curvature matrix $\tilde K$ is

$$\tilde K = -\sum_{i=1}^2 dt_i \wedge \frac{dh_i}{\xi(h_i)}\mathfrak{M} + \text{terms not containing } dt_i.$$

Let Δ^2 be the standard 2-simplex in $\mathbb{R}^2$ and denote by $\beta : Y \times \Delta^2 \to Y$ the projection. By the very definition $\varphi(\nabla_1, \nabla^1, \nabla^2)$ is defined as $(2\pi\sqrt{-1})^{-2}\beta_*(\varphi(\tilde{K}))$, where $\beta_* : \Omega^*(Y \times \Delta^2) \to \Omega^{*-2}(Y)$ denotes the integration along the fibers. From the expression of $\tilde{K}$ formula (3.8) follows and we are done. $\square$

Final Remarks. 1. In case $\{q_\gamma\}$ is such that $q_\gamma \in X'$, then one can take $h_1, \dots, h_l$ to be part of local coordinates in an open (in M) neighborhood of q_γ. Thus in such a case Theorem (3.1) reduced to [LS, Thm.2], which indeed is the classical Camacho-Sad formula for $l = 1$, $n = 2$.

2. In case $l = n-1$, instead of a foliation $\mathcal{F}$ of X one can consider a holomorphic self $f : X \to X$ which pointwise fixes Y. Generically (see [ABT]) this allows to define a one-dimensional foliation $\mathcal{F}_f$ of Y and a holomorphic action Ξ_f of $\mathcal{F}_f$ on $N_{Y',X'}$ outside some "singularities" of f on Y. Arguing as in section 2 one has a residue theorem for this case as well, which generalizes [BS, Thm. 2.2] where this result was achieved for $l = 1, n = 2$ and under the assumption that f were a holomorphic self-map of all the ambient M pointwise fixing P as well.

References

[Ab] M. Abate, *The residual index and the dynamics of holomorphic maps tangent to the identity*, Duke Math. J. **107** (2001), 173-207.

[ABT] M. Abate, F. Bracci and F. Tovena, *Index theorems for holomorphic self-maps* (Preprint 2002).

[BB] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Diff. Geom. **7**, 279-342.

[Bo] R. Bott, *Lectures on characteristic classes and foliations*, Lectures on Algebraic and Differential Topology, Lecture Notes in Mathematics 279, Springer-Verlag, New York, Heidelberg, Berlin, 1972, pp. 1-94.

[Br] F. Bracci, *The dynamics of holomorphic maps near curves of fixed points* (preprint 2001).

[BS] F. Bracci and T. Suwa, *Residues for singular pairs and dynamics of biholomorphic maps of singular surfaces* (preprint 2002).

[BT] F. Bracci and F. Tovena, *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*, Math. Z. **242** (2002), 481-490.

[Ca] C. Camacho, *Quadratic forms and holomorphic foliations on singular surfaces*, Math. Ann. **282** (1988), 177–184.

[CS] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. **115** (1982), 579-595.

[CLG] V. Cavalier, D. Lehmann and M. Soares, *Classes de Chern virtuelles des ensembles analytiques et applications* (preprint 2002).

[Le] D. Lehmann, *Résidus des sous-variétés invariantes d'un feuilletage singulier*, Ann. Inst. Fourier **41** (1991), 211-258.

[LS] D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*, J. Differential Geom. **42** (1995), 165-192.

[Li] A. Lins Neto, *Algebraic solutions of polynomial differential equations and foliations in dimension two* (1988), Holomorphic Dynamics, Mexico 1986, Lecture Notes in Mathematics 1345, Springer-Verlaf, New York, Heidelberg, Berlin, 192-232.

[Lo] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note 77, Cambridge University Press, Cambridge, London, 1984.

[Su1] T. Suwa, *Unfoldings of complex analytic foliations with singularities*, Japanese J. of Math. **9** (1983), 181-206.

[Su2] T. Suwa, *Indices of holomorphic vector fields relative to invariant curves on surfaces*, Proc. Amer. Math. Soc. **123** (1995), 2989-2997.

[Su3] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Hermann, Paris, 1998.

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