RESIDUES FOR SINGULAR PAIRS AND DYNAMICS OF BIHOLOMORPHIC MAPS OF SINGULAR SURFACES

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0. Introduction

Let X be a two dimensional complex manifold and $Y \subset X$ a (possibly singular) compact complex curve. Also let S be a finite set of points of Y including the singularities of Y and set $Y_0 = Y \setminus S$. If there is some *holomorphic action* of the tangent bundle of Y_0 on the normal bundle of Y_0 in X, then, by the well-known Bott vanishing theorem (see [Bo], [Su2]), the first Chern form for an appropriate connection of the normal bundle vanishes on Y_0. As a consequence, the first Chern class of the line bundle associated to the divisor Y localizes at S and gives rise to a *residue* at each point in S. Thus we have a residue (index) formula saying that the self-intersection number of Y is given as the sum of these residues.

Applications of this principle are given in the paper [Su1] (see also [LS]) by the second named author (where the action comes from Y being a leaf of a foliation on X) and in the joint work [BT] of the first named author with F. Tovena (where the action comes from Y being the fixed points set of a *non-degenerate* holomorphic maps of X into itself). These works were motivated by the previous paper by C. Camacho and P. Sad [CS], in the first, and the one by M. Abate [Ab2], in the second. In these papers, Y was assumed to be non-singular and came out from a blowing-up and the index theorems were used to solve problems about dynamics of germs of vector fields or maps. After these, some more generalizations of the previous situations (both for foliations and maps) came up; here we refer the interested reader to [Su2], [ABT] and [Bra] and the bibliography therein.

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In the case where X has singularities at some points of Y, there have been no index theorems available. Usually in applications one is lead to first solve the singularities of X and then apply the above mentioned index theorems. In fact, this is the way Camacho [Cam] proves the existence of separatrices for a germ of a vector field at an isolated normal singular point of a two dimensional variety, under the hypothesis that the dual graph of the resolution is a tree, generalizing the result in [CS].

One would be tempted to mimick this strategy even for proving the existence of *petals* for germs of biholomorphisms tangent to the identity at an isolated normal singular point of a two dimensional variety (with some hypothesis on the type of singularity allowed), in order to generalize the result of [Ab2], but she/he would miserably fail. Indeed, first, the usual method of desingularization (see, *e.g.*, [La]) cannot be applied (for it is possible in general to lift the biholomorphim—see [Ab1]—only after a one point blow-up and not after the blow-up along a submanifold as in the approach followed in [Cam]). Secondly (and mainly) because of this intrinsic difference between maps and vector fields: a vector field always has a singularity at a singular point of X whereas a biholomorphism may not (for the definition of *singularity* of a biholomorphism see section 2). Thus, in order to solve the problem of the existence of petals for biholomorphisms tangent to the identity at a singular point, one has to find a new way.

With the aim of giving an answer to such a problem, in this paper we present an index theorem for the case X is a singular two dimensional variety nicely embedded in some complex manifold W. This case will be naturally the one we get after a blow-up of a singular point of X (under some hypothesis on such a singularity).

The setting is as follows. Let W be a complex manifold of dimension r, $P \subset W$ a non-singular hypersurface and X a two dimensional subvariety of W such that, if $Y := P \cap X$, then the singular set $\mathrm{Sing}(X)$ of X is a finite set in Y. Assume P intersects $X^r := X \setminus \mathrm{Sing}(X)$ generically transversally. This allows a natural extension of the normal bundle of the regular part of Y in X, namely the restriction of the normal bundle $N_{P,W}$ of P in W to Y. Moreover assume there exists a biholomorphic map f of W into itself so that $f(X) \subset X$ and $f|_P = Id_P$ (one might substitute the map f by a one-dimensional foliation leaving X and P invariant obtaining a similar result; however we are not going to discuss further of it in here). Suppose f is *non-degenerate* on the non-singular part of Y. Roughly speaking f is non-degenerate on Y if Y is "well-fixed" in X (see section 2), and this condition is fulfilled whenever it is so at only one point. Then we obtain an index theorem (see Theorem 2.2) stating that the evaluation of the first Chern class of $N_{P,W}$ on Y equals the sum of the *residues* of f on Y. As often happens, an index theorem is useful only when the residues are explicitly calculated, and we thus perform an explicit calculation of such residues (see eq. (2.7)). Also in section 3 we determine the behavior of residues under one point blow-ups (see eq. (3.3)). With such tools at hands we can solve the question about the existence of petals for germs of biholomorphisms tangent to the identity at *t-absolutely isolated points* of X whose resolution graph is a tree (see Theorem 4.2 and section 1 for the terminology). We point out that the proof of such a result is constructive and it has been inspired by [Bra] which in turn was partially a generalization of a paper of J. Cano [Can].

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1. Preliminaries

Let X be a complex variety embedded in a complex manifold W and suppose that X has only one isolated singularity p. The point p is said to be an *absolutely isolated singularity* if it can be resolved by a finite number of quadratic blow-ups. That is to say, if there exist
(1) complex manifolds $W_0 = W, W_1, \dots, W_m$,
(2) varieties $X_0 = X, X_1, \dots, X_m$ such that each $X_j \subset W_j$ has only isolated singularities and X_m is non-singular, and
(3) proper holomorphic maps $\pi_j : W_j \to W_{j-1}$ such that each $\pi_j : W_j \to W_{j-1}$ is a quadratic blow-up with center a singularity of $X_{j-1} \subset W_{j-1}$ and the variety X_j is the strict transform of X_{j-1} under π_j.
Thus $\pi_1 \circ \dots \circ \pi_m : X_m \to X$ is a *resolution* of p.

If X is a (abstract) variety, an isolated singularity $p \in X$ is said to be an *absolutely isolated singularity* if p can be resolved by finitely many quadratic blow-ups of a local embedding of X near p.

In this paper, we consider absolutely isolated singularities satisfying also the following:
(4) at each step of the above process, X_j intersects generically transversally with the exceptional divisor.

In the sequel, an absolutely isolated singularity satisfying (4) is simply called a *t-absolutely isolated singularity.*

Example 1.1. *The variety* $\{(x, y, z) \in \mathbb{C}^3 : x^2 - y^2 + z^3 = 0\}$ *has a t-absolutely isolated singularity at the origin* 0, *whereas the Du Val singularity* $\{x^2 + y^3 + z^3 = 0\}$ *at* 0 *is an absolutely isolated singularity which does not satisfy* (4).

Let X be a normal two dimensional variety and let $p \in X$ be a t-absolutely isolated singularity. Also, let f be a (germ of) biholomorphic self-map of X such that the point p is an isolated fixed point of f and that $df_p = Id$; such a germ is sometimes called a *biholomorphism tangent to the identity* at p. Let T_pX be the Zariski tangent space of X at p and assume $T_pX = \mathbb{C}^r$ for some $r \geq 2$. Then there exists an embedding of an open neighborhood $U \subseteq X$ of p into $\mathbb{C}^r$. Let $j : U \to \mathbb{C}^r$ be such an embedding and suppose $j(p) = 0$.

Lemma 1.1. *There exists a germ of holomorphic self-map $F : \mathbb{C}^r \to \mathbb{C}^r$ at 0 such that $F \circ j = j \circ f$ and $dF_0 = Id$.*

Proof. Let $V \subseteq X$ be an open set such that $f(V) \subseteq U$. Let $g : j(V) \to j(U)$ be the map defined by $j \circ f \circ j^{-1}|_{j(V)}$. Then g has holomorphic components defined on the locally closed set $j(V)$ and therefore they extend to some open neighborhood of $j(V)$. Let F be the germ defined by such extensions. By the very definition it follows that $F \circ j = j \circ f$. Thus $dF_0 \circ dj_p = dj_p$. Since dj_p is an isomorphism (for $T_pX = \mathbb{C}^r$), we have $dF_0 = Id$. $\square$

By Lemma 1.1, we may assume that $X \subset \mathbb{C}^r$ is a germ of a normal two dimensional variety with a t-absolutely isolated singularity at 0, the map f is a germ of a biholomorphic self-map of $\mathbb{C}^r$ at 0 such that $f(X) = X$, 0 is an isolated fixed point of $f|_X$ and $df_0 = Id$.

Let $W_0 = \mathbb{C}^r$, $X_0 = X$ and $\pi_1 : W_1 \to W_0$ the quadratic blow-up centered at 0. Let $X_1 := \overline{\pi_1^{-1}(X_0 \setminus \{0\})}$ be the strict transform of X_0 under π_1, $P_1 := \pi^{-1}(0)$ the exceptional divisor and $Y_1 := X_1 \cap P_1$. Note that Y_1 is a (possibly singular) curve.

Let $\mathrm{Sing}(X_1)$ be the set of singular points of X_1 and $X_1^r := X_1 \setminus \mathrm{Sing}(X_1)$ the regular (non-singular) part. Also let $X_1^{nt} := \{p \in X_1^r \cap P_1 : T_pX_1^r \subset T_pP_1\}$ be the set of points where X_1^r and P_1 do not intersect transversally. Note that by hypothesis X_1^{nt} is a discrete set in X_1^r (actually a finite set as a consequence of the following Lemma 1.2).

Lemma 1.2. *We have $X_1^r \cap \mathrm{Sing}(Y_1) = X_1^{nt}$.*

Proof. Clearly $X_1^r \cap \mathrm{Sing}(Y_1) \subset X_1^{nt}$, for if $X_1 \pitchfork P_1$ at p, Y_1 is non-singular at p. On the other hand suppose that $p \in X_1^r \cap (Y_1 \setminus \mathrm{Sing}(Y_1))$. Then one can choose local coordinates $(z_1, \ldots, z_r)$ on an open set $U \subset W$ such that $p \in U$, $p = (0, \ldots, 0)$, $X_1 \cap U = \{z_2 = \ldots = z_{r-1} = 0\}$ and $Y_1 \cap U = \{z_1 = \ldots, z_{r-1} = 0\}$. In this coordinate system, let $P_1 \cap U = \{h(z_1, \ldots, z_r) = 0\}$ for some holomorphic function h defined on U. The aim is to show that $\frac{\partial h}{\partial z_1}(0, \ldots, 0) \neq 0$ meaning that $X_1 \pitchfork P_1$ at p. Since $Y_1 \cap U = X_1 \cap P_1 \cap U$, $h(z_1, 0 \ldots, 0, z_r) = z_1^t b(z_1, z_r)$ for some $t \geq 1$ and holomorphic function b such that $b(0, z_r) \not\equiv 0$. Moreover $b(0, 0) \neq 0$ for otherwise $p \in \mathrm{Sing}(Y_1)$. If $t > 1$ then $T_qX_1 \subset T_qP_1$ for any $q \in U \cap Y_1$ (and indeed this would hold for all the connected component of Y_1 containing p) against our assumption on the kind of singularity of X_1. Thus $t = 1$ and $\frac{\partial h}{\partial z_1}(0, \ldots, 0) \neq 0$ as wanted. $\square$

We have $\mathrm{Sing}(X_1) \subset \mathrm{Sing}(Y_1)$, for P_1 is generically transversal to X_1^r. Thus $\mathrm{Sing}(Y_1) = \mathrm{Sing}(X_1) \cup X_1^{nt}$, by Lemma 1.2. We set $X_1' := X_1 \setminus (\mathrm{Sing}(X_1) \cup X_1^{nt})$ and $Y_1' := Y_1 \cap X_1' = Y_1 \setminus \mathrm{Sing}(Y_1)$.

It is possible to define a holomorphic self-map $f^1 : W_1 \to W_1$ near P_1 in such a way that $\pi_1 \circ f^1 = f \circ \pi_1$ and $f^1(v) = df_0(v) = v$ for any $v \in P_1$ (see [Ab1]). Thus $f^1(X_1) = X_1$ and f^1 pointwise fixes P_1.

Let N_{P_1,W_1} be the normal bundle of P_1 in W_1 so that we have the following exact sequence:

$$0 \to TP_1 \to TW_1|_{P_1} \xrightarrow{\rho} N_{P_1,W_1} \to 0.$$

Also, let $N_{Y_1',X_1'}$ be the normal bundle of Y_1' in X_1'. Then it follows from the transversality $X_1' \pitchfork P_1$ that $N_{Y_1',X_1'} = N_{P_1,W_1}|_{Y_1'}$. Thus $N_{Y_1',X_1'}$ has a natural extension over Y_1.

Lemma 1.3. *The map f^1 acts as the identity on $N_{Y_1',X_1'}$.*

Proof. By the previous remark, $N_{Y_1',X_1'} = N_{P_1,W_1}|_{Y_1'}$ and therefore we only need to check that f^1 acts as the identity on N_{P_1,W_1}. To see this, let us introduce local coordinates $(x, y) \in \mathbb{C}\times\mathbb{C}^{r-1}$ around $0 \in W_0$ in such a way that $f = (f_1(x, y), f_2(x, y)) \in \mathbb{C} \times \mathbb{C}^{r-1}$ is given by

$$f_1(x, y) = x + A_h(x, y) + \dots ,$$
$$f_2(x, y) = y + B_k(x, y) + \dots ,$$

where $A_h : \mathbb{C}\times\mathbb{C}^{r-1} \to \mathbb{C}$ is a non-zero homogeneous polynomial of degree $h \geq 2$ and $B_k : \mathbb{C}\times\mathbb{C}^{r-1} \to \mathbb{C}^{r-1}$ is a vector whose coordinates are (not all zero) homogeneous polynomial of degree $k \geq 2$. Let $(u, v) \in \mathbb{C} \times \mathbb{C}^{r-1}$ be local coordinates on W_1 so that $\pi_1(u, v) = (u, uv)$. In such coordinates, $P_1 = \{u = 0\}$. If $f^1(u, v) = (f^1_1(u, v), f^1_2(u, v)) \in \mathbb{C} \times \mathbb{C}^{r-1}$, from $\pi_1 \circ f^1 = f \circ \pi_1$ we obtain

$$f^1_1(u, v) = u + u^h A_h(1, v) + o(u^h),$$
$$f^1_2(u, v) = v + u^{k-1}B_k(1, v) - vu^{h-1}A_h(1, v) + o(u^{\min(k-1,h-1)}).$$

Thus the differential df^1 on P_1 is given by

$$df^1{}_{(0,v)} = \begin{pmatrix} 1 & 0 \\ * & Id \end{pmatrix},$$

where $* = u^{k-2}B_k(1, v) - u^{h-2}vA_h(1, v)$ evaluated at $u = 0$ (it is 0 if, e.g., $k > 2, h > 2$). In these coordinates, $\rho(\frac{\partial}{\partial u})$ is a frame for N_{P_1,W_1} and the action of f^1 on N_{P_1,W_1} is given by

$$\rho(\frac{\partial}{\partial u}) \mapsto \rho(df^1(\frac{\partial}{\partial u})) = \rho(\frac{\partial}{\partial u}),$$

as wanted. □

2. Residues for singular pairs

As noted in the previous section, the normal bundle $N_{Y_1',X_1'}$ has an extension N_{P_1,W_1} to P_1. We set $N_{Y_1}=N_{P_1,W_1}|_{Y_1}$. In this section we describe a method to localize the first Chern class of the bundle N_{Y_1} and to define residues around either the "singularities" of f^1 on Y_1' or those of Y_1.

The general situation is as follows. Let W be a complex manifold of dimension r, $P\subset W$ a complex manifold of dimension $r-1$, X a two dimensional variety in W with isolated singularities. Let $X^r=X\setminus\mathrm{Sing}(X)$ be the non-singular part of X and X^{nt} the set of non-transversal points in $P\cap X^r$ as before. Let Y be a curve in $P\cap X$ such that $Y\cap X^{nt}$ is a discrete set. We set $X':=X\setminus(\mathrm{Sing}(X)\cup X^{nt})$ and $Y':=Y\cap X'$ (note that Y' is a non-singular curve, for $P\pitchfork X'$). We assume that Y is connected, for otherwise one can work on each connected component separately. Suppose $f:W\to W$ is holomorphic and such that $f|_P=Id_P$, $f(X)\subset X$. Moreover assume that f acts as the identity on the normal bundle $N_{P,W}$ of P in W.

In the above situation, we may construct a one-dimensional distribution (with singularities) in $TX|_{Y'}$ as follows. Let V be a neighborhood of a point $p\in Y'$ in W. Shrinking V if necessary, we endow V with local coordinates $(z_1,\dots,z_r)$ in such a way that $P\cap V=\{z_1=0\}$, $X'\cap V=\{z_3=\ldots=z_r=0\}$ and hence $Y'\cap V=\{z_1=z_3=\ldots=z_r=0\}$. Let $g:=f|_{X'}$. By hypothesis $g(X')\subseteq X'$ and $g|_{Y'}=Id$. Moreover the transversality $P\pitchfork X'$ implies that $N_{Y',X'}=N_{P,W}|_{Y'}$, and thus g acts as the identity on $N_{Y',X'}$. In the local coordinates (z_1,z_2) on X', setting $g=(g_1,g_2)$, we may write

$$g_1(z_1,z_2)=z_1+z_1^{\nu}a_1(z_1,z_2),\quad g_2(z_1,z_2)=z_2+z_1^{\mu}a_2(z_1,z_2) \tag{2.1}$$

with $a_1(0,z_2)\not\equiv 0$, $a_2(0,z_2)\not\equiv 0$. Using the coherence of the sheaf of ideals of Y one can show that $\nu_g:=\min(\nu,\mu)\geq 1$ is constant on each connected component of Y, and, since Y is assumed to be connected, it is actually constant on Y. We call ν_g the *order of g on Y* (see [Ab2] or [BT]). Let us define the (local) holomorphic vector field

$$X_g:=z_1^{\nu-\nu_g}a_1\frac{\partial}{\partial z_1}+z_1^{\mu-\nu_g}a_2\frac{\partial}{\partial z_2}=\frac{z_1\circ g-z_1}{z_1^{\nu_g}}\frac{\partial}{\partial z_1}+\frac{z_2\circ g-z_2}{z_1^{\nu_g}}\frac{\partial}{\partial z_2}. \tag{2.2}$$

To see what happens under a coordinate change, let $(\hat z_1,\dots,\hat z_r)$ be another local coordinate system as above. Then, for $j=1,2$, we have

$$\hat z_j\circ g-\hat z_j=\sum_{k=1}^{2}\frac{\partial\hat z_j}{\partial z_k}(z_k\circ g-z_k)+R_{2\nu_g}, \tag{2.3}$$

where $R_{2\nu_g}$ denotes a term divisible by $z_1^{2\nu_g}$. Now, there exists a nowhere vanishing holomorphic function C such that $\hat z_1=C(z_1,z_2)z_1$. Therefore by (2.3), if $\hat X_g$

denotes the vector field defined as (2.2) in the $\hat{z}$ coordinate system, we have

$$\hat{X}_g = \sum_{j=1}^{2} \frac{\hat{z}_j \circ g - \hat{z}_j}{\hat{z}_1^{\nu_g}} \frac{\partial}{\partial \hat{z}_j} = \frac{1}{C^{\nu_g}} \sum_{j=1}^{2} \frac{z_j \circ g - z_j}{z_1^{\nu_g}} \frac{\partial}{\partial z_j} + R_{\nu_g} = C^{-\nu_g} X_g + R_{\nu_g}. \tag{2.4}$$

Thus if we let $v_g := X_g|_{Y'}$, it is determined uniquely up to multiplications by non-vanishing holomorphic functions and the set

$$\mathrm{Sing}(g) := \{p \in Y' : v_g(p) = 0\}$$

is well-defined. Also the v_g's define a one-dimensional distribution (without singularities) Ξ_g in $TX|_{Y' \setminus \mathrm{Sing}(g)}$. We say that g is *non-degenerate* or *tangential* to Y' if $\Xi_g = TY'$ on $Y' \setminus \mathrm{Sing}(g)$.

Note that, if g is given by (2.1), then g is non-degenerate at p if and only if $\nu > \nu_g$ and $\mu = \nu_g$. Thus we recover the definition of *non-degenerate map* along a curve of fixed points given in [Ab2] and [BT]. From the coherence of the sheaf of ideals of Y' it follows that g is non-degenerate at one point if and only if it is non-degenerate everywhere on the connected component of Y' containing such a point (see [BT]).

¿From the point of view of dynamics the degenerate situation is trivial. First we recall that a *parabolic curve* (sometimes called a *petal*) for a biholomorphism $f : W \to W$ at a point $p \in W$ is a holomorphic map $\varphi : \Delta \to W$, where Δ denotes the disk $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$, such that $\varphi \in C^0(\overline{\Delta})$, $p \in \varphi(\partial\Delta)$, $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$ and for any $\zeta \in \Delta$ it holds $\lim_{k\to\infty} f^{\circ k}(\varphi(\zeta)) = p$. Then we have (see [Bra]):

Proposition 2.1. *Let M be a two dimensional complex manifold, $C \subset M$ a non-singular (possibly non-compact) curve and let $f : M \to M$ be holomorphic ($f \neq Id_M$). Suppose $f|_C = Id_C$, f acts as the identity on the normal bundle $N_{C,M}$ and f is degenerate on C. Then for every but a discrete set of points in C there exist parabolic curves for f.*

Thus for our aim we assume that g is non-degenerate on Y'. In this case we may write g locally as $g = (g_1, g_2)$ with

$$g_1(z_1, z_2) = z_1 + z_1^{\nu_g+1} b_1(z_1, z_2), \quad g_2(z_1, z_2) = z_2 + z_1^{\nu_g} b_2(z_1, z_2), \tag{2.5}$$

where $b_1 = z_1^{\nu-\nu_g-1} a_1$ and $b_2 = a_2$ (cf. (2.1)), and the vector field X_g is given by

$$X_g = z_1 b_1 \frac{\partial}{\partial z_1} + b_2 \frac{\partial}{\partial z_2}.$$

Let $Y_0 := Y' \setminus \mathrm{Sing}(g)$. We may define a holomorphic action (see, *e.g.*, [Su2, Ch.II, 9])

$$\alpha : \Xi_g \times N_{Y_0,X'} \to N_{Y_0,X'}$$

as follows. Let ρ also denote the canonical projection $TX'|_{Y_0} \to N_{Y_0,X'}$. An element $v \in \Xi_g$ is given by $v = \xi X_g|_{Y_0}$ for some C^∞ function ξ; an element $w \in N_{Y_0,X'}$ is given by $w = \rho(\tilde{w}|_{Y_0})$ for some $\tilde{w} \in TX'$. Then we define

$$\alpha(\xi X_g|_{Y_0}, \rho(\tilde{w}|_{Y_0})) := \rho([\xi X_g, \tilde{w}]|_{Y_0}). \tag{2.6}$$

One easily sees that α is independent of the extension $\tilde{w}$ chosen to define it. Also, whereas X_g depends on the local coordinates chosen, α does not. This is clear if $\nu_g > 1$ because of (2.4) and the very definition of α. To see that this is always the case we need a refinement of (2.4) in the non-degenerate case. In such a case, using the notations as in (2.4) and taking into account that $(z_1 \circ g - z_1)$ is divisible by $z_1^{\nu_g+1}$ and $\hat{z}_1 = C(z_1, z_2)z_1$, we have

$$\begin{aligned} \hat{z}_1 \circ g - \hat{z}_1 &= \sum_{j=1}^{2} \frac{\partial \hat{z}_1}{\partial z_j}(z_j \circ g - z_j) + \sum_{j,k=1}^{2} \frac{\partial^2 \hat{z}_1}{\partial z_j \partial z_k}(z_j \circ g - z_j)(z_k \circ g - z_k) + R_{3\nu_g} \\ &= \sum_{j=1}^{2} \frac{\partial \hat{z}_1}{\partial z_j}(z_j \circ g - z_j) + R_{2\nu_g+1}. \end{aligned}$$

¿From this equation and from (2.3) we obtain

$$\hat{X}_g = \sum_{j=1}^{2} \frac{\hat{z}_j \circ g - \hat{z}_j}{\hat{z}_1^{\nu_g}} \frac{\partial}{\partial \hat{z}_j} = C^{-\nu_g} X_g + T_1 + R_2,$$

where T_1 is a vector of the form $\kappa(z_1, z_2)v$ with $\kappa(0, z_2) \equiv 0$ and $\rho(v|_{Y_0}) = 0$ and, as usual, R_2 is a vector whose coefficients are divisible by z_1^2. Using this expression in the very definition of α it is easy to see that it is well-defined.

¿From the holomorphic action α we can define an α-connection for $N_{Y_0,X'}$ on Y_0 (see [Su2, Ch. II, 9]), that is a connection ∇_0 such that $(\nabla_0)_v(w) = \alpha(v, w)$ for $v \in TY_0 = \Xi_g$ and $w \in N_{Y_0,X'}$. If ∇_0 is an α-connection for $N_{Y_0,X'}$ on Y_0 we have the "Bott vanishing" $c^1(\nabla_0) = 0$, where c^1 is the first Chern polynomial ([Bo], [Su2]). Now using the Čech-de Rham cohomology we can localize $c^1(N_{P,W}|_Y)$ as follows.

Let U_0 be a tubular neighborhood of Y_0 in P. If ∇_0 is an α-connection for $N_{Y_0,X'}$ on Y_0, we may endow $N_{P,W}$ on U_0 with the connection given by the pull-back of ∇_0 by the retraction, which we also denote by ∇_0. Let U_1 be a neighborhood of $\Sigma := \mathrm{Sing}(X) \cup X^{nt} \cup \mathrm{Sing}(g)$ in P such that U_1 is the union of disjoint open sets $U_{1,\gamma}$ each of them containing exactly one point—say p_γ—of Σ and that $N_{P,W}$ is trivial on each $U_{1,\gamma}$. Let $\tilde{R}_{1,\gamma} \subset U_{1,\gamma}$ be a small real $2(r-1)$-dimensional closed disk containing p_γ in its interior and such that $\partial \tilde{R}_{1,\gamma}$ intersects transversally Y_0. Let

$R_{1,\gamma} := \tilde{R}_{1,\gamma} \cap Y$ and $L_\gamma := \partial R_{1,\gamma}$, which is the link of the singularity p_γ of Y. Let ∇_1 be a connection for $N_{P,W}$ on U_1. Since U_1 is a trivializing set for $N_{P,W}$, we may choose ∇_1 to be trivial on each $U_{1,\gamma}$ with respect to some frame. We have the Bott difference form $c^1(\nabla_0, \nabla_1)$ of the two connections, which is a 1-form on $U_0 \cap U_1$ with $dc^1(\nabla_0, \nabla_1) = c^1(\nabla_1) - c^1(\nabla_0)$. Let $H^*_D(\mathcal{U})$ denote the Čech-de Rham cohomology associated to the covering $\mathcal{U} = \{U_0, U_1\}$ (see, *e.g.*, [Su2, Ch.II, 3, Ch.IV, 2], [Su3]). Then $c^1(N_{P,W})$ in $H^2_D(\mathcal{U})$ is represented by the cocycle

$$(c^1(\nabla_1), c^1(\nabla_0), c^1(\nabla_0, \nabla_1)) = (0, 0, c^1(\nabla_0, \nabla_1)),$$

which defines a "localization" of $c^1(N_{P,W})$ in the relative Čech-de Rham cohomology $H^2_D(\mathcal{U}, U_0)$. Thus we have the following "residue theorem" (cf. [Su2, Ch.IV, Theorem 2.4]).

Theorem 2.2. *In the above situation,*
(1) *For each point p_γ in Σ, we have a residue* $\mathrm{Res}(f, Y; p_\gamma)$*, which is determined only by the local behavior of f near p_γ and is given by*

$$\mathrm{Res}(f, Y; p_\gamma) = -\int_{L_\gamma} c^1(\nabla_0, \nabla_1).$$

(2) *If Y is compact, Σ is a finite set and we have*

$$\sum_\gamma \mathrm{Res}(f, Y; p_\gamma) = \int_Y c^1(N_{P,W}).$$

Remark 2.3. *Suppose P is a projective space (as in the blow-up situation) and H is the hyperplane bundle on P. Then $N_{P,W} = -H$ and*

$$\int_Y c^1(N_{P,W}) = \int_Y c^1(-H) = -H \cdot Y.$$

Now we wish to find an explicit expression for $\mathrm{Res}(f, Y; p_\gamma)$. For this purpose, let us recall briefly how the form $c^1(\nabla_0, \nabla_1)$ is defined. Consider the vector bundle $E := N_{P,W} \times \mathbb{R}$ over $(U_1 \cap U_0) \times \mathbb{R}$ and let $\overline{\nabla}$ be the connection for E given by $\overline{\nabla} := (1-t)\nabla_0 + t\nabla_1$. Then $c^1(\nabla_0, \nabla_1) := \beta_*(c^1(\overline{\nabla}))$, where β_* is the integration along the fibers of the projection $\beta : (U_1 \cap U_0) \times [0, 1] \to U_1 \cap U_0$.

We may assume that there exists an open set $V_{1,\gamma}$ in W with local coordinates $(z_1, \ldots, z_r)$ such that $V_{1,\gamma} \cap P = U_{1,\gamma} = \{z_1 = 0\}$. We may take $\rho(\frac{\partial}{\partial z_1})$ as a frame for $N_{P,W}$ and assume that ∇_1 is $\rho(\frac{\partial}{\partial z_1})$-trivial, *i.e.*, the connection form θ_1 of ∇_1 with respect to $\rho(\frac{\partial}{\partial z_1})$ is zero; $\theta_1 = 0$. Therefore, if $\overline{\theta}$ is the connection form of $\overline{\nabla}$, it follows $\overline{\theta} = (1-t)\theta_0$, where θ_0 is the connection form of ∇_0. Then we compute

$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi}\beta_* d\overline{\theta} = \frac{1}{2\pi\sqrt{-1}}\theta_0.$$

By the parameterization theorem we may find a holomorphic function h on $V_{1,\gamma}$ such that the restriction of $dz_1 \wedge dh$ to X' does not vanish on a neighborhood of $Y_0 \cap U_{1,\gamma}$. Thus (z_1, h) are local coordinates on X' and h is a local coordinate on Y', near each point of $Y_0 \cap U_{1,\gamma}$. Once the pair (z_1, h) is chosen as above, the form $\theta_0|_{Y_0}$ can be calculated from the very definition of the action α. The restriction $g = f|_{X'}$ is given by (2.5) with z_2 replaced by h. Recall that $\rho(\frac{\partial}{\partial z_1})$ is the frame chosen for $N_{P,W}$ on $U_{1,\gamma}$ and θ_0 is defined by $\nabla_0(\rho(\frac{\partial}{\partial z_1})) = \theta_0 \otimes \rho(\frac{\partial}{\partial z_1})$. From (2.6), setting $\xi := 1/b_2(z_1, h)$, we compute

$$\begin{aligned}(\nabla_0)_{\frac{\partial}{\partial h}}\rho(\frac{\partial}{\partial z_1}) :&= \alpha(\xi X_g|_{Y_0}, \rho(\frac{\partial}{\partial z_1})) = \rho([\frac{b_1(z_1,h)}{b_2(z_1,h)}z_1\frac{\partial}{\partial z_1} + \frac{\partial}{\partial h}, \frac{\partial}{\partial z_1}]|_{Y_0})\\ &= -\frac{b_1(0,h)}{b_2(0,h)}\rho(\frac{\partial}{\partial z_1}).\end{aligned}$$

Therefore, noting that $b_1(z_1, h) = \frac{(z_1\circ f - z_1)|_X}{z_1^{\nu_g+1}}$ and $b_2(z_1, h) = \frac{(h\circ f - h)|_X}{z_1^{\nu_g}}$, we get an expression for θ_0 on $Y_0 \cap U_{1,\gamma}$:

$$\theta_0 = -\frac{(z_1 \circ f - z_1)|_X}{z_1(h\circ f - h)|_X}dh.$$

Hence we have

$$\text{Res}(f, Y; p_\gamma) = \frac{1}{2\pi\sqrt{-1}}\int_{L_\gamma}\frac{(z_1\circ f - z_1)|_X}{z_1(h\circ f - h)|_X}dh. \tag{2.7}$$

Note that the residue as defined above is "additive" with respect to decompositions of Y. Namely, let $Y = \cup Y_\lambda$ be a local decomposition of Y at p_γ, with the Y_λ's curves with no common irreducible components, then

$$\text{Res}(f, Y; p_\gamma) = \sum_\lambda \text{Res}(f, Y_\lambda; p_\gamma), \tag{2.8}$$

since the link L_γ is a disjoint union of those of the Y_λ's.

Remarks 2.4. *1. Since* $f|_P = Id$, *we may write* $h\circ f - h = Hz_1^T$ *for some holomorphic function* H *not divisible by* z_1 *and integer* $1 \leq T \leq \nu_g$. *Thus if* $H(p_\gamma) \neq 0$*—and necessarily* $T = \nu_g$*—the integrand in* (2.7) *is holomorphic and then* $\text{Res}(f, Y; p_\gamma) = 0$. *In particular if* $df_{p_\gamma} \neq Id$ *then* $H(p_\gamma) \neq 0$.

2. Suppose that p_γ *is a non-singular point of* X*. If* z_1 *is a defining equation of* Y *in* X*, i.e., if* $Y = P \cap X$ *near* p_γ*, then the residue in* (2.7) *coincides with the index* $\mathrm{Ind}(f, Y; p_\gamma)$ *defined in* [Ab2] *in case* Y *is non-singular at* p_γ*, and with the one defined in* [BT, Def. 6] *in case* Y *is singular at* p_γ *(see also* [Bra]*).*

3. To be precise, when $Y \subsetneq P \cap X$ *near* p_γ*, the residue* $\mathrm{Res}(f, Y; p_\gamma)$ *defined as above should be considered as the residue of* f *on* Y *relative to* $P \cap X$ *and, to be consistent with the notation in* [BT] *or* [Su1]*, it should be denoted as* $\mathrm{Res}(f, Y, P \cap X; p_\gamma)$*. This is reflected on the additivity* (2.8) *above. If* p_γ *is a non-singular point of* X*, the residue coincides with* $\mathrm{Ind}(f, Y, P \cap X; p_\gamma)$ *in* [BT, Def. 5]*. It would be interesting but seems to be difficult to define an "absolute" residue on* Y *when* p_γ *is a singular point of* X*.*

3. Behavior of residues under blow-ups

In this section we examine how the residues introduced in the previous section behave under blow-ups. We use the same notations as in section 2. Let $p \in \Sigma$ be such that $\mathrm{Res}(f, Y; p) \neq 0$.

Let $\pi : \tilde{W} \to W$ be the blowing-up at p. Let $D := \pi^{-1}(p)$. Thus $\pi : \tilde{W} \setminus D \to W \setminus \{p\}$ is biholomorphic. Let $\tilde{X} := \overline{\pi^{-1}(X \setminus \{p\})}$ be the strict transform of X, $\tilde{Y} := \overline{\pi^{-1}(Y \setminus \{p\})}$ the strict transform of Y and $\tilde{P} := \overline{\pi^{-1}(P \setminus \{p\})}$ the strict transform of P. By construction, $\tilde{Y}$ is a curve in $\tilde{P} \cap \tilde{X}$ satisfying the conditions as Y in the previous section. In view of Remark 2.4.1, we have $df_p = Id$. Thus there exists a holomorphic map $\tilde{f} : \tilde{W} \to \tilde{W}$ such that $\tilde{f}|_D = Id$ and $\pi \circ \tilde{f} = f \circ \pi$. Note that $\tilde{f}|_{\tilde{X}'}$ is non-degenerate on $\tilde{Y}'$. First suppose that Y is irreducible at p. Thus $\tilde{Y} \cap D := \{q\}$. We wish to calculate $\mathrm{Res}(\tilde{f}, \tilde{Y}; q)$. To do that, let us introduce coordinates $(z_1, \ldots, z_r)$ in a neighborhood V of p in W so that $P \cap V = \{z_1 = 0\}$ and that the restriction of $dz_1 \wedge dz_r$ to X' does not vanish near $Y \cap V$. Then z_r is a local coordinate on Y'. Let $(w_1, \ldots, w_{r-1}, u)$ be local coordinates on $\tilde{W}$ in such a way that $\pi(w_1, \ldots, w_{r-1}, u) = (uw_1, \ldots, uw_{r-1}, u)$. Thus $\tilde{P} = \{w_1 = 0\}$ and $D = \{u = 0\}$. Moreover the restriction of $dw_1 \wedge du$ to $\tilde{X}' \setminus \{q\}$ does not vanish near $\tilde{Y}$ for, since the restriction $\pi|_{\tilde{X}'}$ of π is biholomorphic outside D,

$$0 \neq (\pi|_{\tilde{X}'})^*(dz_1 \wedge dz_r) = d(uw_1) \wedge du = udw_1 \wedge du.$$

Thus by (2.7) we have

$$\mathrm{Res}(\tilde{f}, \tilde{Y}; q) = \frac{1}{2\pi\sqrt{-1}} \int_{\tilde{L}} \frac{(w_1 \circ \tilde{f} - w_1)|_{\tilde{X}}}{w_1(u \circ \tilde{f} - u)|_{\tilde{X}}} du,$$

where $\tilde{L}$ is the link of the singularity q of $\tilde{Y}$. Note that $\tilde{L} = \pi^* L$, with L the link of the singularity p of Y. Since $w_1 = u^{-1}(z_1 \circ \pi)$ and $\tilde{f} = \pi^{-1} \circ f \circ \pi$ outside D we have (omitting to write $|_{\tilde{X}}$ in the integrands)

$$
\begin{aligned}
(3.1)\quad & \int_{\tilde{L}} \frac{w_1 \circ \tilde{f} - w_1}{w_1(u \circ \tilde{f} - u)} du \\
&= \int_{\pi^* L} \frac{z_1 \circ f \circ \pi}{z_1 \circ \pi} \cdot \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du + \int_{\pi^* L} \pi^* \left(\frac{z_1 \circ f - z_1}{z_1(z_r \circ f - z_r)} dz_r \right) \\
&= \int_{\pi^* L} \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du + 2\pi\sqrt{-1} \cdot \mathrm{Res}(f, Y; p).
\end{aligned}
$$

Now $\tilde{f}|_{\tilde{X}} = Id + w_1 H$ for some holomorphic map $H = (H_1, H_2)$ near $\tilde{Y}$. Thus, indicating by $< .,. >$ the scalar product between vectors, we have

$$
\frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} = \frac{v < \partial u^{-1}, H > + o(w_1)}{u^{-1} w_1 < \partial u, H > + o(w_1)} = -\frac{1 + o(1)}{u + o(1)}.
$$

Therefore

$$
(3.2)\qquad \int_{\pi^* L} \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du = -\int_{\pi^* L} \frac{du}{u} = -\int_L \frac{dz_r}{z_r} = -2\pi\sqrt{-1} \cdot m_p,
$$

where $m_p \geq 1$ is the multiplicity of the singularity p of Y (note that $m_p = 1$ if and only if p is a non-singular point). From (3.1) and (3.2) we obtain

$$
(3.3)\qquad \mathrm{Res}(\tilde{f}, \tilde{Y}; q) = \mathrm{Res}(f, Y; p) - m_p.
$$

Remark 3.1. *In case $p \in X^r$ then $(z_1 \circ \pi)|_{\tilde{X}} = u^{m_p} v$, where v is a defining function of $\tilde{Y}$ in $\tilde{X}$ and m_p is the multiplicity of the singularity p of Y. Thus $w_1|_{\tilde{X}} = u^{m_p - 1} v$. Therefore, arguing as before,*

$$
\begin{aligned}
\mathrm{Res}(\tilde{f}, \tilde{Y}; q) &= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\tilde{L}} \frac{v \circ \tilde{f} - v}{v(u \circ \tilde{f} - u)} du + (m_p - 1) \int_{\tilde{L}} \frac{du}{u} \right) \\
&= \mathrm{Ind}(\tilde{f}, \tilde{Y}; q) + m_p(m_p - 1),
\end{aligned}
$$

where $\mathrm{Ind}(\tilde{f}, \tilde{Y}; q)$ *is the index defined in* [BT]. *Therefore in such a case, from* Remark 2.4.2 *and* (3.3) *we recover* [BT, (5)], *that is*

$$
\mathrm{Ind}(\tilde{f}, \tilde{Y}; q) = \mathrm{Ind}(f, Y; p) - m_p^2.
$$

Now suppose Y may not be not irreducible at p and let $Y = \cup_{\lambda=1}^N Y_\lambda$ be the irreducible decomposition at p. Then the link L is a disjoint union $L = \cup_{\lambda=1}^N L_\lambda$, with L_λ the link of Y_λ, and we have the identity (2.8). Let $m_{p,\lambda}$ be the multiplicity of Y_λ at p. We set $\tilde{Y} \cap D = \{q_1, \dots, q_s\}$ (note that $N \geq s$). Then from the relation obtained by replacing Y (resp. $\tilde{Y}$) with Y_λ (resp. $\tilde{Y}_\lambda$) in (3.3) and the identity (2.8), we have

$$\sum_{j=1}^{s} \operatorname{Res}(\tilde{f}, \tilde{Y}; q_j) = \operatorname{Res}(f, Y; p) - \sum_{\lambda=1}^{N} m_{p,\lambda}. \tag{3.4}$$

4. Existence of parabolic curves for t-absolutely isolated singularities whose resolution graph is a tree

Now we go back to the situation of section 1. Let X be a two dimensional variety with a normal t-absolutely isolated singularity at $p \in X$ whose resolution graph is a tree. Let f be a germ of biholomorphic map of X at p_0 such that p_0 is an isolated fixed point of f and $df_{p_0} = Id$.

By the consideration of section 1, we may assume that $X \subset \mathbb{C}^r = W_0$. Blow up the point p_0. We find an r-dimensional complex manifold W_1, a holomorphic map $f^1 : W_1 \to W_1$ and an $(r-1)$-dimensional projective space $P_1 \subset W_1$ such that $f^1|_{P_1} = Id$ and f^1 acts as the identity on N_{P_1,W_1}. Moreover there is a 2-dimensional variety $X_1 \subset W_1$ so that $f^1(X_1) \subset X_1$, $\operatorname{Sing}(X_1) \cup X_1^{nt}$ is a finite set contained in $Y_1 = X_1 \cap P_1$ and $X_1' := X_1 \setminus (\operatorname{Sing}(X_1) \cup X_1^{nt}$ intersects P_1 transversally.

If $f^1|_{Y_1}$ is degenerate on $Y_1' := Y \cap X_1'$ there exist infinitely many parabolic curves not contained in P_1 by Proposition 2.1 which project down to parabolic curves in X for f at p_0.

Thus assume f^1 is non-degenerate on Y_1'. We first recall the following result from [Bra], adapted to our needs (we can apply such a result to the residue as defined in this paper because by Remark 2.4.2 it coincides with the residue defined in [Ab2, BT, Bra] in the case of non-singular surfaces):

Proposition 4.1. *Assume f is a germ of holomorphic self map of $\mathbb{C}^2$ at 0. Let E be the set of fixed points of f at 0. Suppose that E has dimension one and f is non-degenerate on the non-singular part of E.*
1. *If E is non-singular at 0 and*

$$\operatorname{Res}(f, E; 0) \notin \mathbb{Q}^+ \cup \{0\},$$

then there exists at least one parabolic curve for f at 0.
2. *If E has two irreducible non-singular branches E_1, E_2 at 0 intersecting transversally and such that*

$$\operatorname{Res}(f, E_1; 0) \notin \{a \in \mathbb{Q} : a \geq -1\},$$
$$\operatorname{Res}(f, E_2; 0) \in \{a \in \mathbb{Q} : a \leq -1\},$$

then there exists at least one parabolic curve for f at 0.

Let $R(f^1) := \{p \in Y^1 : \mathrm{Res}(f^1, Y_1; p) \notin \mathbb{Q}^+ \cup \{0\}\}$. By Theorem 2.2 and Remark 2.3 it follows that $R(f^1) \neq \emptyset$. If $Y_1' \cap R(f^1) \neq \emptyset$ then we apply Proposition 4.1.1 and we are done. Thus assume $R(f^1) \subset \mathrm{Sing}(X) \cup X_1^{nt}$. Let $p \in R(f^1)$. Blow up the point p in W_1 and let $\pi_2 : W_2 \to W_1$ be the quadratic blow-up. By Remark 2.4.1, $df_p^1 = Id$ and hence there exists a holomorphic map $f^2 : W_2 \to W_2$ which is the identity on the exceptional divisor P_2 and acts as the identity on N_{P_2,W_2} by Lemma 1.3. Let X_2 be the strict transform of X_1 and $Y_2 := P_2 \cap X_2$. We denote by the same symbol Y_1 the strict transform of Y_1 in W_2. By hypothesis $\mathrm{Sing}(X_2) \cup X_2^{nt}$ is a finite set contained in $Y_1 \cup Y_2$. Again, if $f^2|_{X_2}$ is degenerate on $Y_2' := Y_2 \cap X_2'$ then we are done. Suppose f^2 is non-degenerate on Y_2'. Let

$$R(f^2) := \{p \in Y_1 \cup Y_2 : \mathrm{Res}(f^2, Y_j; p) \notin \mathbb{Q}^+ \cup \{0\} \text{ for } j = 1 \text{ or } 2\}.$$

By Theorem 2.2 and Remark 2.3, $R(f^2) \neq \emptyset$. Again by Proposition 4.1.1, we may suppose $R(f^2) \subset \mathrm{Sing}(X_2) \cup X_2^{nt}$ for otherwise we are done.

We continue arguing this way: we define inductively the blow-up $\pi_{s+1} : W_{s+1} \to W_s$ at points of the set

$$R(f^s) := \{p \in \bigcup_{j=1}^{s} Y_j : \mathrm{Res}(f^s, Y_j; p) \notin \mathbb{Q}^+ \cup \{0\} \text{ for some } j = 1, \dots, s\},$$

where the curve Y_j is defined inductively as the intersection of the exceptional divisor P_j with the strict transform X_j of X_{j-1} and f^j is defined inductively to be the lift of f^{j-1}. If for some s the map f^s is degenerate on Y_s' then we are done. Therefore we suppose that at each step f^s is non-degenerate on Y_s'. Note that at each step $R(f^s) \neq \emptyset$ by Theorem 2.2 and Remark 2.3 and that Y_s is a locally and globally irreducible curve (by hypothesis on p_0).

By hypothesis on (X, p_0) we end up with a two dimensional variety X_l such that $\mathrm{Sing}(X_l) \cap R(f^l) = \emptyset$. Moreover, up to blowing up some more, we may assume that if a point $p \in R(f^l)$ then p is a non-singular point of a curve Y_j and either there is no other Y_k $(k \neq j)$ passing through p or there exists only one $k \neq j$ such that p is a non-singular point of Y_k and Y_j intersects transversally Y_k at p. In the first case we may apply Proposition 4.1.1 and we are done. Thus we assume that all the points of $R(f^l)$ are normal crossings. Since the resolution graph of X is a tree, the dual graph of the union of the Y_j's is a tree as well. Let Y_{j_0} be an end point of such a graph, $j_0 \neq 1$. Theorem 2.2 implies that $Y_{j_0} \cap R(f^l) \neq \emptyset$, and since each point of this set is a normal crossing then there exists a unique point q_{j_0} such that $\{q_{j_0}\} = R(f^l) \cap Y_{j_0}$; again by Theorem 2.2, $\mathrm{Res}(f^l, Y_{j_0}; q_{j_0}) \in \{a \in \mathbb{Q} : a \leq -1\}$. Also there exists

$j_1 \neq j_0$ such that $\{q_{j_0}\} = Y_{j_0} \pitchfork Y_{j_1}$. We claim that $\mathrm{Res}(f^l, Y_{j_1}; q_{j_0}) \notin \{a \in \mathbb{Q} : a \geq -1\}$ (and thus we may we apply Proposition 4.1.2 and we are done). Seeking for a contradiction we assume $\mathrm{Res}(f^l, Y_{j_1}; q_{j_0}) \in \{a \in \mathbb{Q} : a \geq -1\}$. Let $\pi_{j_0} : X_l \to X_{l-1}$ be the blow up such that Y_{j_0} is in the exceptional divisor and let f^{l-1} be such that $\pi_{j_0} \circ f^l = f^{l-1} \circ \pi_{j_0}$. By (3.3)

$$\mathrm{Res}(f^{l-1}, \pi_{j_0}(Y_{j_1}); \pi_{j_0}(q_{j_0})) = \mathrm{Res}(f^l, Y_{j_1}; q_{j_0}) + m_{q_{j_0}} \in \mathbb{Q}^+ \cup \{0\}.$$

This means that Y_{j_0} comes from the blow-up of the point $\pi_{j_0}(q_{j_0})$ which is not in $R(f^{l-1})$, against our construction.

Therefore we proved:

Theorem 4.2. *Let X be a two dimensional variety with a normal t-absolutely isolated singularity $p \in X$ whose resolution graph is a tree. Let $f : X \to X$ be a holomorphic map such that p is an isolated fixed point of f and $df_p = Id$. Then there exists at least one parabolic curve for f at p.*

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