SOLVING THE SCHRÖDER EQUATION AT THE BOUNDARY IN SEVERAL VARIABLES.

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ABSTRACT. Let $\mathbb{H} := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \mathsf{Im}\, w > \|z\|^2\}$ be the unbounded Siegel realization of the Euclidean unit ball of $\mathbb{C}^{n+1}$. Let $F \in \mathsf{Hol}\,(\mathbb{H},\mathbb{H})$ be a holomorphic self-map of $\mathbb{H}$ with no fixed points in $\mathbb{H}$ and let $P \in \partial\mathbb{H}$ be its Wolff point. Assume the boundary dilatation coefficient of F at P is strictly smaller than 1. Under natural hypothesis on the expandability of F at P, we prove that there exists a nonconstant holomorphic map $\sigma : \mathbb{H} \to \mathbb{C}^{n+1}$ such that $\sigma \circ F = A_P \sigma^t$, where A_P is a $(n+1) \times (n+1)$ matrix representing the differential of F at P. The Schröder map σ turns out to be unique in an appropriate class and under a suitable normalization. We apply these results to characterize families of commuting holomorphic maps.

PROLOGUE

Let D be a domain of $\mathbb{C}$ containing the origin 0 and let $f \in \mathsf{Hol}(D,\mathbb{C})$ be a given holomorphic function defined on D with $f(0) = 0$ and $f'(0) \neq 0$. The classical Schröder equation [21] is the functional equation

$$\sigma \circ f = \lambda \sigma, \tag{0.1}$$

where σ is an unknown function, called a *Schröder map* for f, and λ is an unknown complex number.

This equation has a solution if, for instance, $|f'(0)| < 1$ (see Königs [15]). In particular if $f \in \mathsf{Hol}\,(\Delta,\Delta)$ is a self-map (not an automorphism) of the unit disc $\Delta \subset \mathbb{C}$, then there exists a solution σ of (0.1) which is defined and holomorphic on Δ, and this solution is unique if the value of $\sigma'(0)$ is chosen. This last fact can be stated as the possibility of "representing" the given $f \in \mathsf{Hol}\,(\Delta,\Delta)$ by means of the linear part of its expansion at the fixed point 0.

If $f \in \mathsf{Hol}\,(\Delta,\Delta)$ has no fixed points in Δ, then the situation becomes more complicated and the existence of a solution σ of (0.1) depends on the "dynamics" of f. It is well known (see, *e.g.*, [1]) that if $f \in \mathsf{Hol}\,(\Delta,\Delta)$ has no fixed points in Δ, then there exists a unique point $\tau \in \partial\Delta$ such that f has non-tangential limit τ at τ—*i.e.* τ is a boundary fixed point for f—and f' has non-tangential limit $f'(\tau) = \alpha_\tau \in (0,1]$ at τ. The real number α_τ is called the *boundary*

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dilatation coefficient of f at τ. If $\alpha_\tau < 1$ the function f is called *hyperbolic*, otherwise it is called *parabolic*. If f is hyperbolic then (0.1) has a solution. The proof of this fact dates back to the times of Valiron [22]. If f is parabolic there are no (nonconstant) solutions of (0.1), and one is led to solve the so-called Abel's equation ([11], [18], [3]).

The Schröder (and Abel) equation can be written in a more general framework as follows. Let $f \in \mathsf{Hol}\,(\Delta, \Delta)$ and consider the equation

$$\sigma \circ f = \Phi_f \circ \sigma, \tag{0.2}$$

where Ω is a complex manifold, $\sigma : \Delta \to \Omega$ is an unknown holomorphic function (called an *intertwining map*) and Φ_f is a biholomorphism of Ω. A solution of (0.2) gives rise to a "representation model" for f, described by the following commuting diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{f} & \Delta \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Omega & \xrightarrow{\Phi_f} & \Omega \end{array}$$

The more information one has on σ, Ω, Φ_f appearing in the above diagram, the more useful the model turns out to be. Among many interesting papers appeared on the subject, we recall here, Pommerenke [18], Baker-Pommerenke [3], Cowen [11], Bourdon-Shapiro [10] and Poggi Corradini [17]. In this last paper a "reversed" Schröder equation is solved.

The reasons of all such efforts for solving the Schröder (and Abel) equation in its form (0.1), or in a more general setting as in (0.2), aside a natural beauty contained in the problem itself, have to be found in the number of questions that can be affirmatively solved by means of a solution of (0.1) (and 0.2). By means of a Schröder map, for instance, one can study the dynamics of holomorphic self-maps of Δ (see, *e.g.*, [1]), or questions concerning the composition operators associated to a given map ([10], [13]). Other interesting problems that can be attacked by means of "representation models" of type (0.2) concern the study of families of commuting mappings ([1], [12], [8], [19]) and their boundary properties ([9], [4]). These last results state that, roughly speaking, one can characterize the families of holomorphic maps of Δ which commute under composition as those families of mappings which "share" a common Schröder map.

In the case of several complex variables, one can in principle follow the one dimensional approach of Cowen [11] in order to find out an "abstract" representation model. Namely, arguing as in [11] (with some additional hypotheses), one can effectively solve (0.2) for a given holomorphic self-map f of the open unit $(n+1)-$ball. In this approach the domain Ω turns out to be an abstract, non-compact, simply connected, complex manifold and therefore the lack of an uniformization theorem in several complex variables makes such a model not particularly interesting.

Recently, there have been several attempts to construct useful representation models in the unit ball of $\mathbb{C}^{n+1}$. Up to now, only the case of an "isolated inner fixed point" has been solved. In fact a paper by Cowen and MacCluer [14] deals with, and solves, the Schröder equation for holomorphic self-maps of the open unit $(n+1)-$ball having an isolated inner fixed point (see also [5] where a different approach to this problem is presented). Cowen and McCluer study the equation

$$\sigma \circ F = M\sigma^t, \tag{0.3}$$

where F is a holomorphic self-map of the open unit ball in $\mathbb{C}^{n+1}$ fixing O and M is a $(n+1) \times (n+1)$-matrix. Under the condition that dF_O is semisimple, invertible and has no eigenvalues of modulus one, Cowen and MacCluer characterize the existence of a Schröder map σ, holomorphic in the ball, by means of the semisimplicity of the matrix associated to the composition operator induced by F. In [14] (and [5]) the problem of uniqueness for the Schröder map is investigated. It is shown how (contrarily to the one-dimensional case) the Schröder map is no longer unique in general—even when $d\sigma_O$ is given—but it depends on the presence of *resonances* among the eigenvalues of dF_O. Later, MacCluer [16] used the above mentioned results to characterize families of commuting holomorphic self-mappings of the open unit ball fixing an isolated point.

In this paper we solve the Schröder equation in several variables for holomorphic mappings with "no inner fixed points".

We actually deal with (0.3) for a given holomorphic self-map F of $\mathbb{H} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathsf{Im}\, w > ||z||^2\}$ (the unbounded Siegel realization of the Euclidean unit ball of $\mathbb{C}^{n+1}$) with no fixed points in $\mathbb{H}$. As it happens in the one-dimensional case, if $F \in \mathsf{Hol}\,(\mathbb{H}, \mathbb{H})$ has no inner fixed points, then there exists a unique point $P \in \partial\mathbb{H}$, the *Wolff point* of F, such that F has non-tangential limit P at P and the normal (with respect to $\partial\mathbb{H}$ at P) component of dF has non-tangential limit $\alpha_P \in (0, 1]$ at P (see, *e.g.*, [1], [20]). If $\alpha_P < 1$ we call F a *hyperbolic* map. Otherwise F is called *parabolic*.

For a hyperbolic map having a "good" expansion at its Wolff point, we prove that the Schröder map exists and is *unique* in an appropriate class of maps.

In order to give a better explanation of our hypotheses on the "good" expandability of the map, we need a few definitions. For $\epsilon > 0$, we say that H is *expandable* of order $1 + \epsilon$ at $P \in \partial\mathbb{H}$—and we write $H \in \mathcal{E}^{1+\epsilon}(P)$—provided H, holomorphic on $\mathbb{H}$ with values in $\mathbb{C}^{n+1}$, has a first order expansion at P whose remainder satisfies certain "directional bounds" with respect to the normal direction to $\partial\mathbb{H}$ at P . Roughly speaking, a map H belongs to $\mathcal{E}^{1+\epsilon}(P)$ if it "follows up to the order $1 + \epsilon$" the behavior "prescribed" by the Julia-Wolff-Carathéodory Theorem for a holomorphic self-map of $\mathbb{H}$ at its Wolff point (see the first section for precise definitions and comments). We also define the *linear part* A_P of $H \in \mathcal{E}^{1+\epsilon}(P)$ as the $(n + 1) \times (n + 1)$ matrix given by the linear part of the expansion of H at P. This linear part A_P can be thought of as the differential of H at P, and indeed this is exactly what it is in most cases. However we warn the reader that, for a map, neither being expandable of order $1 + \epsilon$ implies being differentiable, nor the converse statement is true. The reason for considering the class of expandable maps instead of the class of differentiable maps becomes clear when one tries to follow the approach of Valiron or Bourdon-Shapiro to define the Schröder map in the one-dimensional case (see section one).

If $\varphi \in \mathcal{E}^{1+\epsilon}(P) \cap \mathsf{Hol}(\mathbb{H}, \mathbb{H})$, then its linear part A_P defines a natural action, which we indicate by τ_φ, on the complex tangent space to $\partial\mathbb{H}$ at P. For instance, if φ is differentiable at P, one can think of τ_φ has the action of $d\varphi_P$ on $T_P^{\mathbb{C}}\partial\mathbb{H}$. We say that the action τ_φ is *normal* provided it is so with respect to the hermitian structure of $T_P^{\mathbb{C}}\partial\mathbb{H}$ induced by that of $\mathbb{C}^{n+1}$.

With these notations our main result is the following.

Theorem 0.1. *For some $P \in \partial\mathbb{H}$, let $\varphi \in \mathcal{E}^{1+\epsilon}(P) \cap \mathsf{Hol}(\mathbb{H}, \mathbb{H})$. Let $\alpha_P > 0$ be the boundary dilatation coefficient of φ at P and let A_P be the linear part of φ at P. Suppose that $\alpha_P < 1$, the action $\tau_\varphi : T_P^{\mathbb{C}}\partial\mathbb{H} \to T_P^{\mathbb{C}}\partial\mathbb{H}$ is normal and its eigenvalues are all of modulus strictly greater than $\alpha_P^{(1+\epsilon)/2}$. Then there exists $\sigma \in \mathcal{E}^{1+\epsilon}(P)$ with invertible linear part at P such that*

$$\sigma \circ \varphi = A_P \sigma^t. \tag{0.4}$$

Moreover $\tilde{\sigma} \in \mathcal{E}^{1+\epsilon}(P)$, with invertible linear part, is another solution of (3.1) *if and only if there exists a $(n+1) \times (n+1)$ invertible matrix E such that $[A_P, E] = 0$ and $\tilde{\sigma} = E\sigma^t$.*

Thus, contrarily to what happens in the inner fixed points case, for expandable hyperbolic maps having a boundary fixed point, one can select a "special", essentially unique, solution of the Schröder equation.

At this point we apply our construction to the study of commuting holomorphic maps with no fixed points in $\mathbb{H}$. We prove that, even in the case of a boundary fixed point, families of maps which commute under composition can be characterized as those families of maps which "share" a common Schröder map (see Theorems 5.1 and 5.2).

The plan of the paper goes as follows. In the first section we discuss of differentiable maps versus expandable maps and set up the basic environment to prove our results. Section two is devoted to the proof of the main theorem for hyperbolic maps with Wolff point at infinity. In section three the proof is then extended to the case of any other fixed point of $\partial\mathbb{H}$. Section four deals with resonances and somehow explains the reason for the uniqueness of the Schröder map. Finally in section five we study and characterize families of commuting holomorphic maps.

In a forthcoming paper, we shall discuss the Abel's equation in several complex variables.

1. THE CLASS OF DIFFERENTIABLE MAPS AND THE CLASS OF EXPANDABLE MAPS

In this section we define the classes of maps we are working with and study their properties of regularity.

As a matter of notation, if $v = (v_1, \ldots, v_n, v_{n+1}) \in \mathbb{C}^{n+1}$, we use the symbol v' to denote the first n-components of v, namely $v' = (v_1, \ldots, v_n) \in \mathbb{C}^n$. Sometimes we decompose $\mathbb{C}^{n+1} \ni Z = (z, w) = (z_1, \ldots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$, that is $z = Z'$. Moreover we denote by $\langle \cdot, \cdot \rangle$ the canonical hermitian product in $\mathbb{C}^{n+1}$. If $P \in \partial\mathbb{H}$ we denote by $T_P^{\mathbb{C}}\partial\mathbb{H}$ the complex tangent space to $\partial\mathbb{H}$ at P. When we refer to metric properties we will always tacitly assume $T_P^{\mathbb{C}}\partial\mathbb{H}$ to be endowed with the standard hermitian structure induced by $\mathbb{C}^{n+1}$.

Definition 1.1. Let $P \in \partial\mathbb{H}$. Let $\varphi : \mathbb{H} \to \mathbb{C}^{n+1}$ be holomorphic. We say that φ belongs to the class $\mathcal{D}^1(P)$ if

$$\varphi(Z) - P = A_P(Z-P)^t + \gamma(Z), \tag{1.1}$$

where A_P is a $(n+1)\times(n+1)$ matrix and $\|\gamma(Z)\| = \|(\gamma'(Z), \gamma_{n+1}(Z))\|) = o(\|Z-P\|)$ for Z near P.

We let $\mathcal{D}^1_{\mathbb{H}}(P) = \mathcal{D}^1(P) \cap \mathsf{Hol}\,(\mathbb{H}, \mathbb{H})$.

Therefore $\varphi \in \mathcal{D}^1(P)$ if and only if φ is differentiable at P, and, if φ is given by (1.1), then A_P is its differential at P. In particular if φ extends C^1 to $\mathbb{H} \cup \{P\}$ then $\varphi \in \mathcal{D}^1(P)$.

By conjugating $\varphi \in \mathcal{D}^1_{\mathbb{H}}(P)$ with suitable automorphisms of $\mathbb{H}$ of the form $(z, w) \mapsto (z + a', w + a_{n+1} + 2i\langle z, a'\rangle)$ with $(a', a_{n+1}) \in \partial\mathbb{H}$, one can always suppose without loss of generality that $P = O$.

We begin with the following simple result:

Lemma 1.2. *Let $\varphi \in \mathcal{D}^1(O)$ be given by* (1.1). *Then φ is continuous on $\mathbb{H} \cup \{O\}$, $\varphi(O) = O$ and $d\varphi_Z \to A_O$ for $Z \to O$ non-tangentially. In particular, if A_O is invertible, for any non-tangential region V with vertex O there exists $T = T(V) > 0$ such that φ is injective on $V \cap \{Z \in \mathbb{H} : \|Z\| < T\}$.*

Moreover if $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$, i.e., $\varphi(\mathbb{H}) \subseteq \mathbb{H}$, then

$$A_O = \begin{pmatrix} A'_O & B^t \\ O & \alpha_O \end{pmatrix} \tag{1.2}$$

for some $n \times n$ matrix A'_O, $B \in \mathbb{C}^n$ and $\alpha_O > 0$. In particular, A_O is invertible if and only if A'_O is.

Proof. The first assertion is obvious. Let $\{Z_m\} \subset \mathbb{H}$ be a sequence converging to O non-tangentially. Using the Cauchy representation formula one shows that if $\delta_m > 0$ is such that the ball of radius δ_m and center Z_m is contained in $\mathbb{H}$ then

$$\|d\varphi_{Z_m} - A_O\| \leq \frac{1}{2\pi} \max_{\|X - Z_m\| = \delta_m} \frac{\|\gamma(Z_m)\|}{\delta_m}.$$

Since $Z_m \to O$ non-tangentially one can choose $\delta_m = O(|w_m|)$ (where w_m is the $(n+1)$-th component of Z_m) and then $d\varphi_{Z_m} \to A_O$ as wanted.

If A_O is invertible, since φ is C^1 on the closure of every non-tangential region of vertex O, then it is injective on V for $\|Z\|$ small enough.

The form (1.2) in case $\varphi(\mathbb{H}) \subseteq \mathbb{H}$ follows directly from Rudin's version of the Julia-Wolff-Carathéodory theorem for the unit ball (see [1] or [20]). $\square$

Many times it is useful to move the base point O to infinity. To do that one can exploit the following transformation

$$G(z, w) = \left(\frac{z}{w}, -\frac{1}{w}\right). \tag{1.3}$$

It follows easily that G is an automorphism of $\mathbb{H}$ with inverse $G^{-1}(z,w) = (-z/w, -1/w)$ and G maps O to ∞. If $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ let $F = G \circ \varphi \circ G^{-1}$. A straightforward calculation shows that

$$F(z,w) = \left(\frac{A'_0 z^t + B - w\gamma'(-z/w,-1/w)}{\alpha_O - w\gamma_{n+1}(-z/w,-1/w)}, \frac{w}{\alpha_O - w\gamma_{n+1}(-z/w,-1/w)} \right). \tag{1.4}$$

We know that $|\gamma_{n+1}| = o(\|(1/z, 1/w)\|)$ but, in several variables, this does not mean in general that $|w\gamma_{n+1}| < 1$. Suppose now that $\lim_{w\to\infty} |w\gamma_{n+1}| = 0$, *i.e.*, $|\gamma_{n+1}(v', v_{n+1})| = o(|v_{n+1}|)$ as $v \to 0$. Then one can expand the denominator of (1.4) as

$$\frac{1}{\alpha_O - w\gamma_{n+1}} = \frac{1}{\alpha_O}\left[1 + \frac{w\gamma_{n+1}}{\alpha_O}(1 + S(z,w))\right],$$

where $S(z,w) \to 0$ for $w \to \infty$. The first n components of F become then

$$F'(z,w) = \frac{1}{\alpha_O} A'_0 z^t + \Gamma'(z,w),$$

where $\|\Gamma'(z,w)\| = o(\|z\|) = o(|w|^{1/2})$ for $\|z\|^2 < \mathsf{Im}\, w < |w|$ in $\mathbb{H}$ and $(z,w) \to \infty$. Note that this allows to well define the "linear part" $A'_\infty = A'_O/\alpha_O$. For the $(n+1)$-th component of F we have

$$F_{n+1}(z,w) = \frac{1}{\alpha_O} w + \Gamma_{n+1}(z,w),$$

where this time $|\Gamma_{n+1}(z,w)| = o(|w|)$ for $w \to \infty$. Therefore the "linear part" in z, if well defined, is contained into $\Gamma_{n+1}(z,w)$. Hence

$$F(z,w) = (A'_\infty z^t + \Gamma'(z,w), \alpha_\infty w + \Gamma_{n+1}(z,w)), \tag{1.5}$$

where, $\alpha_\infty = 1/\alpha_O$, $A_\infty = A'_O/\alpha_O$, $\|\Gamma'\| = o(|w|^{1/2})$ and $|\Gamma_{n+1}| = o(|w|)$ for $|w| \to \infty$.

On the other hand, starting from a map F of the form (1.5) then a calculation analogous to the previous one shows that $\varphi = G^{-1} \circ F \circ G$, for G as in (1.3), is given by

$$\varphi(z,w) = (A'_O z^t + \gamma'(z,w), \alpha_O w + \gamma_{n+1}(z,w)), \tag{1.6}$$

with $\alpha_O = 1/\alpha_\infty$, $A'_O = A'_\infty/\alpha_\infty$, $\|\gamma'\| = o(|w|^{1/2})$ and $|\gamma_{n+1}| = o(|w|)$ for $|w| \to 0$. Notice that $|w| = o(|w|^{1/2})$ for $(z,w) \to 0$ and thus "linear terms in w", if well defined, are contained entirely in γ'.

Therefore we are led to define a new class $\mathcal{E}^1$ as follows

Definition 1.3. Let $F : \mathbb{H} \to \mathbb{C}^{n+1}$ be holomorphic. We say that $F \in \mathcal{E}^1(\infty)$ (*respect.* $F \in \mathcal{E}^1(O)$) if F has an expansion as in (1.5) (*respect.* as in (1.6)) where $\|\Gamma'(z,w)\| = o(|w|^{1/2})$ and $|\Gamma_{n+1}(z,w)| = o(|w|)$ for $(z,w) \to \infty$ (*respect.* $\|\gamma'(z,w)\| = o(|w|^{1/2})$ and $|\gamma_{n+1}(z,w)| = o(|w|)$ for $(z,w) \to 0$). We let $\mathcal{E}^1_{\mathbb{H}}(\infty) = \mathcal{E}^1(\infty) \cap \mathsf{Hol}\,(\mathbb{H},\mathbb{H})$ and $\mathcal{E}^1_{\mathbb{H}}(O) = \mathcal{E}^1(O) \cap \mathsf{Hol}\,(\mathbb{H},\mathbb{H})$.

The previous considerations lead to the following result.

Proposition 1.4. *There exists a one-to-one correspondence between $\mathcal{E}^1_{\mathbb{H}}(O)$ and $\mathcal{E}^1_{\mathbb{H}}(\infty)$. The correspondence is obtained by conjugation using the automorphism G of $\mathbb{H}$ given by* (1.3).

Moreover if a map $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ given by (1.5) *corresponds to $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ given by* (1.6) *then $\alpha_\infty = 1/\alpha_O$ and $A'_\infty = A'_O/\alpha_\infty$.*

Remark 1.5. 1. In the one-dimensional case, *i.e.* for $n = 0$, $\mathcal{D}^1_{\mathbb{H}}(O) = \mathcal{E}^1_{\mathbb{H}}(O)$, whereas this is not the case in several variables. For instance the map $\mathbb{C}^2 \ni (z, w) \mapsto (0, w + z^2) \in \mathbb{C}^2$ is in $\mathcal{D}^1_{\mathbb{H}}(O)$ but not in $\mathcal{E}^1_{\mathbb{H}}(O)$. Actually if $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ has an expansion at O up to the second order, then $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ if and only if $\gamma_{n+1}(z, w) = Kz^t w + bw^2 + o(\|(z, w)\|^2)$, for some $n \times n$ matrix K and $b \in \mathbb{C}$. Namely γ_{n+1} does not contain pure terms in $z_j z_k$.

2. In [10] (cfr. pag. 50) the class $\mathcal{D}^1_{\mathbb{H}}(\infty) = \mathcal{E}^1_{\mathbb{H}}(\infty)$ for $n = 0$ is denoted by $C^1(\infty)$. In fact in that paper it is claimed that $\varphi \in C^1(\infty)$ if and only if φ' extends continuously to $\mathbb{H} \cup \{\infty\}$. Unfortunately this assertion is not true. Indeed $\varphi(w) = aw + i + e^{iw}$, for $a \geq 0$, belongs to $\mathcal{E}^1_{\mathbb{H}}(\infty)$ but the limit of $\varphi'(w)$ does not exist if $w \to \infty$ tangentially.

3. The class $\mathcal{E}^1_{\mathbb{H}}(O)$ (*respect.* $\mathcal{E}^1_{\mathbb{H}}(\infty)$) is closed under conjugation with automorphisms of $\mathbb{H}$ fixing O (*respect.* ∞). This fact follows easily from the explicit form of such automorphisms (see, *e.g.*, [1] or [20]).

4. In case $\varphi \in \mathcal{D}^1_{\mathbb{H}}(O)$ (*respect.* $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$) then the number α_O appearing in (1.2) (*respect.* (1.6)) is such that $\alpha_O > 0$ and it coincides with the *boundary dilatation coefficient* of φ at O, see [7] for results at this concern.

5. By the Julia-Wolff-Carathéodory Theorem for the ball (see [20] or [1]) it follows that, if $\varphi \in \mathsf{Hol}(\mathbb{H}, \mathbb{H})$ and O is a fixed point for φ, then φ has an expression as in (1.6) with $\|\gamma'\| = o(|w|^{1/2})$ and $|\gamma_{n+1}| = o(|w|)$ for $(z, w) \to O$ *non-tangentially*. Thus the class $\mathcal{E}^1(O)$ contains maps which well approximate a self-map of $\mathbb{H}$ near O.

More generally we give the following definition.

Definition 1.6. We say that $\varphi \in \mathcal{E}^1(P)$, for $P \in \partial\mathbb{H}$, if there exists an automorphism η of $\mathbb{H}$ mapping O to P and fixing ∞ such that $\eta^{-1} \circ \varphi \circ \eta \in \mathcal{E}^1(O)$. As usual we let $\mathcal{E}^1_{\mathbb{H}}(P) = \mathcal{E}^1(P) \cap \mathsf{Hol}(\mathbb{H}, \mathbb{H})$.

Recall that if η is an automorphism of $\mathbb{H}$ fixing ∞ then it has one of the following forms (see, *e.g.*, [1] or [20]):

$$\mathbb{C}^n \times \mathbb{C} \ni (z, w) \mapsto (Uz^t + c', w + c_{n+1} + 2i\langle Uz^t, c'\rangle), \tag{1.7}$$

for $U \in \mathsf{U}(n)$ and $c = (c', c_{n+1}) \in \partial\mathbb{H}$, or

$$\mathbb{C}^n \times \mathbb{C} \ni (z, w) \mapsto (kUz^t, k^2 w), \tag{1.8}$$

for $U \in \mathsf{U}(n)$ and $k > 0$.

From (1.7) and (1.8) one sees that given any $P \in \partial\mathbb{H}$, there always exists an automorphism η of $\mathbb{H}$ which fixes ∞ and maps O to P. Moreover, even if $\varphi : \mathbb{H} \to \mathbb{C}^{n+1}$ is such that $\varphi(\mathbb{H}) \not\subseteq \mathbb{H}$, the composition $\eta^{-1} \circ \varphi \circ \eta$ appearing in Definition 1.6 is well given. Finally, Remark 1.5.3 assures that Definition 1.6 is well posed, that is $\varphi \in \mathcal{E}^1(P)$ if and only if $\eta^{-1} \circ \varphi \circ \eta \in \mathcal{E}^1(O)$ for every automorphism η of $\mathbb{H}$ fixing ∞.

Remark 1.7. It is a direct but tedious calculation to see that $\varphi \in \mathcal{E}^1(P)$ if and only if φ has an expression as in (1.1) with $|\langle \gamma, \nu_P\rangle| = o(|\langle Z-P, \nu_P\rangle|)$, where ν_P is the complex normal to $T_P^{\mathbb{C}}\partial\mathbb{H}$ and $|\langle \gamma, \tau\rangle| = o(|\langle Z-P, \nu_P\rangle|^{1/2})$ for any $\tau \in T_P^{\mathbb{C}}\partial\mathbb{H}$.

Let $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ and define an action τ_φ of φ on $T_O^{\mathbb{C}}\partial\mathbb{H}$ by

$$\tau_\varphi : T_O^{\mathbb{C}}\partial\mathbb{H} \ni (v', 0) \mapsto (A'_O(v')^t, 0) \in T_O^{\mathbb{C}}\partial\mathbb{H}.$$

If η is an automorphism of $\mathbb{H}$ and $\eta(O) = P$ then the action $\tau_{\tilde\varphi}$ of $\tilde\varphi = \eta\circ\varphi\circ\eta^{-1} \in \mathcal{E}^1_{\mathbb{H}}(P)$ on $T_P^{\mathbb{C}}\partial\mathbb{H}$ can be defined by $\tau_{\tilde\varphi} = d\eta_O \circ \tau_\varphi \circ d\eta_P^{-1}$, since, as it is well known, $d\eta_O, d\eta_P^{-1}$ are isometries of the complex tangent spaces to $\partial\mathbb{H}$.

Definition 1.8. Let $\varphi \in \mathcal{E}^1_{\mathbb{H}}(P)$. We call the induced action τ_φ of φ on $T_P^{\mathbb{C}}\partial\mathbb{H}$ a *normal action* if τ_φ commutes with the adjoint action τ_φ^*.

Note that the normality of the action of a map depends only on its class of conjugacy. For a map $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ of the form (1.6) it follows that τ_φ is normal if and only if $A'_O(A'_O)^* = (A'_O)^* A'_O$.

Let us define the *boundary complex tangent space* $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ *of* $\partial\mathbb{H}$ *at* ∞ by

$$T_\infty^{\mathbb{C}}\partial\mathbb{H} := \{v \in \mathbb{C}^{n+1} : v_{n+1} = 0\},$$

and let us give it a hermitian structure by restricting to $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ the classical hermitian structure of $\mathbb{C}^{n+1}$.

If $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$, the action τ_F of F on $T_\infty^{\mathbb{C}}\partial\mathbb{H} := \{v \in \mathbb{C}^{n+1} : v_{n+1} = 0\}$ given by

$$\tau_F : T_\infty^{\mathbb{C}}\partial\mathbb{H} \ni (v', 0) \mapsto (v' A'_\infty, 0)$$

is well defined and τ_F is conjugated by an isometry of $T_\infty^{\mathbb{C}}\partial\mathbb{H}$ to $\tau_{\tilde F}$, if $\tilde F$ is conjugated to F by means of an automorphism fixing ∞.

By Proposition 1.4, if F is conjugated to $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ then τ_F is normal if and only if τ_φ is so.

When the action is normal one can "diagonalize" F at the first order:

Lemma 1.9. *Let $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$. If τ_F is normal then, up to conjugations with automorphisms of $\mathbb{H}$ fixing ∞, we can write*

$$F(z_1, \ldots, z_n, w) = (\mu_1^\infty z_1 + \Gamma_1(z,w), \ldots, \mu_n^\infty z_n + \Gamma_n(z,w), \alpha_\infty w + \Gamma_{n+1}(z,w)), \tag{1.9}$$

where $\mu_j^\infty \in \mathbb{C}$ are the eigenvalues of A'_∞ and $|\mu_j^\infty| \leq \sqrt{\alpha_\infty}$ for $j = 1, \ldots, n$.

Proof. Since A'_∞ commutes with $(A'_\infty)^*$, up to conjugation with an automorphism of type $(z,w) \mapsto (Uz^t, w)$ for some $U \in \mathsf{U}(n)$ we may assume that F is given by (1.5) with A'_∞ a diagonal matrix with entries $\mu_1^\infty, \ldots, \mu_n^\infty \in \mathbb{C}$. Since $F(\mathbb{H}) \subseteq \mathbb{H}$, for any $(z,w) = (z_1, \ldots, z_n, w) \in \mathbb{H}$ with $\|z\|$ big enough, we have (taking into account that $\|\Gamma'\| = o(|w|^{1/2})$)

$$\alpha_\infty \mathsf{Im}\, w + |\Gamma_{n+1}| \geq \alpha_\infty \mathsf{Im}\, w + \mathsf{Im}\,\Gamma_{n+1} > \|A'_\infty z^t + \Gamma'\|^2 \geq (\|A'_\infty z^t\| - \|\Gamma'\|)^2,$$

that is

$$\alpha_\infty \mathsf{Im}\, w + |\Gamma_{n+1}| > (\|A'_\infty z^t\| - \|\Gamma'\|)^2. \tag{1.10}$$

Let $\delta > 0$. Setting $(z,w) = (z, \|z\|^2 + i(1+\delta)\|z\|^2) \in \mathbb{H}$, dividing (1.10) by $\|z\|^2$, letting $z_2 = \dots z_n = 0$ and taking the limit for $|z_1| \to \infty$, we obtain $\alpha_\infty(1+\delta) \geq |\mu_1^\infty|^2$. Since this holds for any $\delta > 0$ this gives $\alpha_\infty \geq |\mu_1^\infty|^2$. Similarly for $\mu_2^\infty, \dots, \mu_n^\infty$. □

Lemma 1.9 and Proposition 1.4 imply

Corollary 1.10. *Let $\varphi \in \mathcal{E}^1_{\mathbb{H}}(O)$ be given by* (1.6) *and assume τ_φ is normal. Moreover assume φ corresponds by conjugation to $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ given by* (1.5). *Let μ_j^O be the eigenvalues of A'_O and μ_j^∞ the eigenvalues of A'_∞, $j = 1, \dots, n$. Then, up a reordering, we have $\mu_j^O = \mu_j^\infty/\alpha_\infty$ for $j = 1, \dots, n$. In particular $|\mu_j^O| \leq \sqrt{\alpha_O}$.*

Remark 1.11. The hypothesis about the normality of τ_φ in the previous corollary is superfluous: by slightly refining the argument in the proof of Lemma 1.9 one can show the relations between the μ_j^∞'s and α_∞ even if A'_∞ is not diagonalizable.

In our construction we need some more regularity on the remainder of the expansion of $F \in \mathcal{E}^1(\infty)$.

Definition 1.12. Let $F \in \mathcal{E}^1(\infty)$ be given by

$$F(z,w) = (A'_\infty z^t + \Gamma'(z,w), \langle z, B\rangle + \alpha_\infty w + \Gamma_{n+1}(z,w)), \tag{1.11}$$

with $B \in \mathbb{C}^n$. We say that $F \in \mathcal{E}^{1+\epsilon}(\infty)$ if $|\Gamma_{n+1}(Z)| \leq M|w|^{1-\epsilon}$ and $\|\Gamma'(Z)\| \leq M|w|^{(1-\epsilon)/2}$ for some $M > 0$, $\epsilon > 0$ and $Z \to \infty$. We set $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty) = \mathcal{E}^{1+\epsilon}(\infty) \cap \mathsf{Hol}\,(\mathbb{H}, \mathbb{H})$.

Remark 1.13. Since $|\langle z, B\rangle| \leq \|B\|\|z\| \leq \|B\||w|^{1/2}$, the term $\langle z, B\rangle$ in (1.11) is actually contained in Γ_{n+1} if $\epsilon \leq 1/2$, while it is well defined for $\epsilon > 1/2$. Therefore from now on we tacitly set $B = 0$ whenever $\epsilon \leq 1/2$.

Using (1.7) and (1.8) one can check that $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty)$ is closed under conjugation with automorphisms of $\mathbb{H}$ fixing ∞.

Definition 1.14. Let $F \in \mathcal{E}^{1+\epsilon}(O)$ be given by (1.11). We call *linear part of F at ∞* the matrix

$$A_\infty = \begin{pmatrix} A'_\infty & O \\ B & \alpha_\infty \end{pmatrix}. \tag{1.12}$$

We say that the linear part is *invertible* provided $\alpha_\infty \det(A'_\infty) \neq 0$.

Again we recall that $B = 0$ if $\epsilon \leq 1/2$.

We could define the class $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(O)$ by conjugating $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty)$ using the automorphism G given by (1.3). A straightforward computation shows then that $\varphi \in \mathcal{E}^{1+\epsilon}_{\mathbb{H}}(O)$ if and only if

$$\varphi(z,w) = (A'_O z^t + Cw + \gamma'(z,w), \alpha_O w + \gamma_{n+1}(z,w)), \tag{1.13}$$

where $C \in \mathbb{C}^n$—we set $C = 0$ if $\epsilon \leq 1/2$—$\|\gamma'\| \leq M'|w|^{(1+\epsilon)/2}$ and $|\gamma_{n+1}| \leq M'|w|^{1+\epsilon}$ for some $M' > 0$. Using this last fact we define the class $\mathcal{E}^{1+\epsilon}(O)$. Again, for $\varphi \in \mathcal{E}^{1+\epsilon}(O)$ given by (1.13), we call *linear part of φ at O* the matrix

$$A_O = \begin{pmatrix} A'_O & C^t \\ O & \alpha_O \end{pmatrix}. \tag{1.14}$$

The linear part is *invertible* provided $\alpha_O \det(A'_O) \neq 0$.

Similarly one can define the classes $\mathcal{E}^{1+\epsilon}(P)$ and $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(P)$. One can also define the *linear part* A_P at $P \in \partial\mathbb{H}$ for a map $\varphi \in \mathcal{E}^{1+\epsilon}(P)$. We leave the details to the reader.

In the hyperbolic case, *i.e.*, $\alpha_\infty > 1$, one can always dispose of the term B in (1.11):

Proposition 1.15. *Let $F \in \mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty)$ given by* (1.11)*. Assume $\alpha_\infty > 1$ and τ_F normal. Then, up to conjugation with automorphisms of $\mathbb{H}$, we can write*

$$F(z_1, \ldots, z_n, w) = (\mu_1^\infty z_1 + \Gamma_1(z, w), \ldots, \mu_n^\infty z_n + \Gamma_n(z, w), \alpha_\infty w + \Gamma_{n+1}(z, w)), \tag{1.15}$$

where $\mu_j^\infty \in \mathbb{C}$ are the eigenvalues of A'_∞, $|\mu_j^\infty| \leq \sqrt{\alpha_\infty}$ for $j = 1, \ldots, n$, $\|\Gamma'\| \leq M^{1/2}|w|^{(1-\epsilon)/2}$ and $|\Gamma_{n+1}| \leq M|w|^{1-\epsilon}$ for some $M > 0$.

Proof. If $\epsilon \leq 1/2$ then the result is contained in Lemma 1.9 (and there is no need of the condition on α_∞).

Suppose $\epsilon > 1/2$. By Lemma 1.9 we can assume A'_∞ to be diagonal with entries μ_j^∞, $j = 1, \ldots, n$ satisfying the appropriate bounds. In particular if $\alpha_\infty > 1$ then $\mu_j^\infty \neq \alpha_\infty$ for all j's. Suppose now $B \neq 0$ and let η be an automorphism of type (1.7) (with $U = \mathsf{id}$ since A'_∞ is already diagonal). A direct calculation shows that

$$\eta^{-1} \circ F \circ \eta(z, w) = (A'_\infty z^t + \Gamma', \alpha_\infty w + \langle z, B\rangle + 2i\langle (A'_\infty - \alpha_\infty \mathsf{id})z^t, c'\rangle + \Gamma_{n+1}),$$

where, with some abuse of notations, we also denoted by Γ the remainder of $\eta \circ F \circ \eta^{-1}$. Now consider the equation

$$\langle z, B\rangle + 2i\langle (A'_\infty - \alpha_\infty \mathsf{id})z^t, c'\rangle \equiv 0, \quad \forall z \in \mathbb{C}^n.$$

This has a solution c'_0 because $\det(A'_\infty - \alpha_\infty \mathsf{id}) \neq 0$. Then we are allowed to take (c', c_{n+1}) in (1.7) such that $c' = c'_0$. Hence $\eta \circ F \circ \eta^{-1}$ has the desired form (1.15). $\square$

2. The Schröder equation at infinity

In this section we solve the Schröder equation for hyperbolic elements of the class $\mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty)$ whose eigenvalues satisfy some relations. To simplify the notation in this section we avoid to write the subscripts and superscripts ∞, that is, we set $\alpha = \alpha_\infty$, $\mu_j = \mu_j^\infty$.

Recall that a *horosphere* of center ∞ and radius $R > 0$ in $\mathbb{H}$ is given by

$$\mathbb{H}_R := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathsf{Im}\, w > \|z\|^2 + R\}.$$

By the well known Wolff Lemma (see, *e.g.*, [1]) it follows that for any $F \in \mathcal{E}^1_{\mathbb{H}}(\infty)$ given by (1.5) and $R > 0$

$$F(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha R}. \tag{2.1}$$

By (2.1) it follows that if F is hyperbolic, *i.e.*, $\alpha > 1$, then F has no fixed points in $\mathbb{H}$ and ∞ is its *Wolff point*, in particular $F^k(Z) \to \infty$ as $k \to \infty$ for any $Z \in \mathbb{H}$.

Proposition 2.1. *Let $F \in \mathcal{E}^{1+\epsilon}_{\mathbb{H}}(\infty)$ be given by* (1.15). *Assume $\alpha > 1$ and $|\mu_j| > \alpha^{(1-\epsilon)/2}$ for $j = 1, \dots, n$. Then there exists a (non-constant) holomorphic map $\sigma = (\sigma^1, \dots, \sigma^n, \sigma^{n+1}) \in \mathcal{E}^{1+\epsilon}(\infty)$ such that, for any $(z, w) \in \mathbb{H}$,*

$$\sigma(F(z,w)) = (\mu_1\sigma^1(z,w), \dots, \mu_n\sigma^n(z,w), \alpha\sigma^{n+1}(z,w)), \tag{2.2}$$

and

$$\sigma(z,w) = (z + \Lambda'(z,w), w + \Lambda_{n+1}(z,w)). \tag{2.3}$$

Moreover σ is the unique solution of (2.2) *in the class $\mathcal{E}^{1+\epsilon}(\infty)$ with the asymptotic expansion* (2.3).

Remark 2.2. For $n = 0$, *i.e.* F a holomorphic self-map of the unit disc of $\mathbb{C}$, we recover Bourdon-Shapiro's Theorem (see [10], *Theorem 4.9*, p. 58) under weaker hypotheses. In fact in Proposition 2.2, contrary to what happens in Bourdon-Shapiro's Theorem, we do not require that the boundary $\partial\mathbb{H}$ is mapped to $\mathbb{H}$ by F, nor that F is continuous on $\partial\mathbb{H}$ and univalent in $\mathbb{H}$.

For the rest of this section we assume F as in the hypothesis of Proposition 2.1. Moreover we write the k-th iterate of F as

$$F^k(z,w) = (f^1_k(z,w), \dots, f^n_k(z,w), g_k(z,w)),$$

and set $f_k = (f^1_k, \dots, f^n_k)$.

We begin with the following lemma:

Lemma 2.3. *For the k-th iterate of F it holds*

$$\begin{aligned} f^h_k(z,w) &= \mu^k z + \sum_{j=0}^{k-1} \mu^{k-1-j}\Gamma_h(F^j) \quad \textit{for } h = 1, \dots, n \textit{ and} \\ g_k(z,w) &= \alpha^k w + \sum_{j=0}^{k-1} \alpha^{k-1-j}\Gamma_{n+1}(F^j). \end{aligned} \tag{2.4}$$

Moreover

$$|g_k(z,w)| \le |w| \prod_{j=0}^{k-1} \left(\alpha + \frac{M}{|g_j(z,w)|^\epsilon}\right). \tag{2.5}$$

Proof. Equation (2.4) follows from (1.15) by a simple induction. We point out that

$$|g(z,w)| \le |w|(\alpha + M|w|^{-\epsilon}),$$

and then (2.5) follows by induction, noting that $g_k = g_{k-1} \circ F$. □

Now we need the following estimates on the images of compact subsets of $\mathbb{H}$ under iterates of F.

Lemma 2.4. *There exist $\delta > 0$ and $t > 1$ such that, setting $E_\delta := \{(z,w) \in \mathbb{H} : |w| > \delta\}$, then $F(E_\delta) \subset E_\delta$. Moreover for any $k \in \mathbb{N}$ and $(z,w) \in E_\delta$ it follows*

$$|g_k(z,w)| \geq \delta t^k. \tag{2.6}$$

Proof. By the triangle inequality it follows

$$|g(z,w)| \geq |w|(\alpha - M|w|^{-\epsilon}).$$

In particular for $\alpha > t > 1$ there exists $\delta > 0$ such that for any $|w| > \delta$ it follows

$$|g(z,w)| > t|w| > t\delta > \delta. \tag{2.7}$$

Thus $F(E_\delta) \subset E_\delta$ and, by iterating such inequality, we obtain

$$|g_k(z,w)| > t^k|w| > t^k\delta, \tag{2.8}$$

for any $k \in \mathbb{N}$ and $(z,w) \in E_\delta$. □

Lemma 2.5. *Let $K \subset\subset \mathbb{H}$ be a compact set. Let $\delta > 0, t > 1$ be given by Lemma 2.4. Then there exists $k_0 = k_0(K) \in \mathbb{N}$ such that for any $(z,w) \in K$ and $k \geq k_0$ it holds*

$$|g_k(z,w)| \geq \delta t^{k-k_0}. \tag{2.9}$$

Proof. If $K \subset E_\delta$ then the statement is proved setting $k_0 = 0$ and invoking Lemma 2.4. Thus assume $K \not\subset E_\delta$. We claim that there exists $k_0 \in \mathbb{N}$ such that

$$F^j(z,w) \in E_\delta, \ \ \forall (z,w) \in K, j \geq k_0. \tag{2.10}$$

Assuming (2.10), by (2.8) we have, for any $k \in \mathbb{N}$,

$$|g_{k+k_0}(z,w)| = |g_k(F^{k_0}(z,w))| > t^k\delta.$$

Thus we are left to prove (2.10). Since K is compact, there exists $R > 0$ such that $K \subset \mathbb{H}_R$. By (2.1), $F(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha R}$, and $F^j(\mathbb{H}_R) \subseteq \mathbb{H}_{\alpha^j R}$. Then taking $k_0 \in \mathbb{N}$ such that $\alpha^{k_0}R \geq \delta$ we have, for any $(z,w) \in K$,

$$|g_{k_0}(z,w)| \geq \mathsf{Im}\, g_{k_0}(z,w) > \|f_{k_0}(z,w)\|^2 + \delta > \delta,$$

and thus (2.10) follows. □

Now we are in the position to prove the existence of σ.

Proof of the existence part in Prop. 2.1. For $k \in \mathbb{N}$ let $\sigma_k(z,w) = (\sigma_k^1(z,w), \ldots, \sigma_k^{n+1}(z,w)) \in \mathbb{C}^{n+1}$ be defined as

$$\sigma_k^h(z,w) = \frac{f_k^h(z,w)}{\mu_h^k} \quad \text{for } h = 1, \ldots, n \quad \text{and}$$
$$\sigma_k^{n+1}(z,w) = \frac{g_k(z,w)}{\alpha^k}.$$

A direct computation shows that

$$\sigma_{k-1} \circ F = (\mu_1\sigma_k^1, \ldots, \mu_n\sigma_k^n, \alpha\sigma_k^{n+1}),$$

and thus, if $\{\sigma_k\}$ converges to a non-constant holomorphic map σ, then σ verifies the functional equation (2.2). We prove that $\{\sigma_k\}$ converges uniformly on compacta. Fix K a compact subset of $\mathbb{H}$ and let $(z,w)\in K$. We start looking at σ_k^h, $h=1,\dots,n$. By (2.4)

$$\sigma_k^h(z,w)=\frac{f_k^h}{\mu_h^k}=z+\sum_{j=1}^k \mu_h^{-j}\Gamma_h(F^{j-1}(z,w)).$$

We want to estimate the series on the right hand side. By (2.5) we have

$$\begin{aligned}\sum_{j=1}^k |\mu_h|^{-j}|\Gamma_h(F^{j-1}(z,w))| &\le \sum_{j=1}^k |\mu_h|^{-j}\|\Gamma'(F^{j-1}(z,w))\| \le \sum_{j=1}^k |\mu_h|^{-j}M^{1/2}|g_{j-1}(z,w)|^{(1-\epsilon)/2}\\ &\le M^{1/2}|w|^{(1-\epsilon)/2}\sum_{j=1}^k |\mu_h|^{-j}\prod_{l=0}^{j-1}\left(\alpha+\frac{M}{|g_l(z,w)|^\epsilon}\right)^{(1-\epsilon)/2}\\ &= M^{1/2}|w|^{(1-\epsilon)/2}\sum_{j=1}^k \Big(\frac{\alpha}{|\mu_h|}^{(1-\epsilon)/2}\Big)^j\prod_{l=0}^{j-1}\left(1+\frac{M}{\alpha|g_l(z,w)|^\epsilon}\right)^{(1-\epsilon)/2}.\end{aligned}$$

Now note that the product $\prod_{l=0}^{j-1}\left(1+\frac{M}{\alpha|g_l(z,w)|^\epsilon}\right)$ converges for $j\to\infty$ if $\sum_{l=0}^\infty \frac{1}{|g_l(z,w)|^\epsilon}<\infty$. Let k_0,t,δ be given by Lemma 2.5, then

$$\sum_{l=k_0}^\infty \frac{1}{|g_l(z,w)|^\epsilon}\le\frac{1}{\delta}\sum_{l=k_0}^\infty t^{-(l-k_0)\epsilon}<\infty, \tag{2.11}$$

and the product is actually converging uniformly on K. Since by hypothesis $\alpha^{(1-\epsilon)/2}<|\mu_h|$, the previous argument implies that there exists a constant $C_h>0$, depending on K, such that

$$\sum_{j=1}^\infty |\mu_h|^{-j}|\Gamma_h(F^{j-1}(z,w))|\le C_h|w|^{(1-\epsilon)/2}, \tag{2.12}$$

yielding that σ_k^h converges uniformly on compacta for $k\to\infty$, $h=1,\dots,n$. By Lemma 2.4, the previous argument works *for any* $(z,w)\in E_\delta$ and thus there exists $C_h>0$ such that (2.12) is valid on E_δ. In particular this shows that σ^h is not constant and gives a part of the asymptotic expansion claimed for σ.

Arguing similarly we can show that for

$$\sigma_k^{n+1}(z,w)=\frac{g_k}{\alpha^k}=w+\sum_{j=1}^k \alpha^{-j}\Gamma_{n+1}(F^{j-1}(z,w)),$$

we have

$$\sum_{j=1}^k \alpha^{-j}|\Gamma_{n+1}(F^{j-1}(z,w))|\le M|w|^{1-\epsilon}\sum_{j=1}^k \alpha^{-\epsilon j}\prod_{l=0}^{j-1}\left(1+\frac{M}{\alpha|g_l(z,w)|^\epsilon}\right)^{1-\epsilon},$$

and thus there exists $C > 0$ such that

$$\sum_{j=1}^{\infty} \alpha^{-j} |\Gamma_{n+1}(F^{j-1}(z,w))| \leq C|w|^{1-\epsilon}. \tag{2.13}$$

In particular σ^{n+1} is non-constant and has the required expansion. □

Our next aim is to show the uniqueness of the solution of equation (2.2). Since the solutions of (2.2) form a $\mathbb{C}$-vector space, the uniqueness claimed in Proposition 2.1 will follow at once from the following lemma.

Lemma 2.6. *Let $R \geq 0$. Let $h = (h_1, \ldots, h_{n+1}) : \mathbb{H}_R \to \mathbb{C}^{n+1}$ be a holomorphic solution of* (2.2).

(1) *If $|h_{n+1}(z,w)| = o(|w|)$ for $(z,w) \to \infty$ then $h_{n+1} \equiv 0$.*
(2) *If there exists $M > 0$ such that $|h_j(z,w)| \leq M|w|^{(1-\epsilon)/2}$ for $(z,w) \to \infty$, and for some $j = 1, \ldots, n$, then $h_j \equiv 0$.*

Proof. Assume $|h_{n+1}(z,w)| = o(|w|)$. We want to show that $h_{n+1} \equiv 0$. If not, by (2.2) and by (2.1) we have that for any $(z,w) \in \mathbb{H}_R$

$$h_{n+1}(F^m(z,w)) = \alpha^m h_{n+1}(z,w), \tag{2.14}$$

for all $m \in \mathbb{N}$. Dividing by $g_m(z,w)$ both sides of (2.14) and taking the limit for $m \to \infty$, we have

$$\lim_{m\to\infty} \frac{|h_{n+1}(F^m(z,w))|}{|g_m(z,w)|} = \lim_{m\to\infty} \frac{\alpha^m |h_{n+1}(z,w)|}{|g_m(z,w)|}. \tag{2.15}$$

Since $|g_m(z,w)| \to \infty$ by (2.1), the left hand side of (2.15) is such that

$$\lim_{m\to\infty} \frac{|h_{n+1}(F^m(z,w))|}{|g_m(z,w)|} = \lim_{m\to\infty} \frac{o(|g_m(z,w)|)}{|g_m(z,w)|} = 0.$$

As for the right hand side of (2.15), by (2.4) we have

$$\begin{aligned}
\lim_{m\to\infty} \frac{\alpha^m |h_{n+1}(z,w)|}{|g_m(z,w)|} &= |h_{n+1}(z,w)| \lim_{m\to\infty} \frac{1}{|w + \sum_{j=0}^{m-1} \alpha^{-(1+j)} \Gamma_{n+1}(F^j(z,w))|} \\
&\geq \frac{|h_{n+1}(z,w)|}{|w| + \sum_{j=0}^{\infty} \alpha^{-(1+j)} |\Gamma_{n+1}(F^j(z,w))|} \geq \frac{|h_{n+1}(z,w)|}{|w| + C|w|^{1-\epsilon}} > 0,
\end{aligned}$$

where in the last line we used (2.13). Thus $h_{n+1} \equiv 0$.

Now assume $|h_j(z,w)| \leq M|w|^{(1-\epsilon)/2}$ for some $j = 1, \ldots, n$. Let $\beta > (1-\epsilon)/2$ be such that $\mu_j > \alpha^\beta$. Let $(z,w) \in \mathbb{H}_R$. From (2.2) we have

$$\lim_{m\to\infty} \frac{|h_j(F^m(z,w))|}{|g_m(z,w)|^\beta} = \lim_{m\to\infty} \frac{|\mu_j|^m |h_j(z,w)|}{|g_m(z,w)|^\beta}. \tag{2.16}$$

The left hand side of (2.16) is zero for $|h_j(F^m(z,w))| = o(|g_m(z,w)|^\beta)$. As for the right hand side, if $h_j \not\equiv 0$, by (2.5) we have

$$\lim_{m\to\infty} \frac{|\mu_j|^m |h_j(z,w)|}{|g_m(z,w)|^\beta} \geq \lim_{m\to\infty} \frac{|h_j(z,w)|}{|w|^\beta |\mu_j|^{-m} \prod_{h=0}^{m-1} \left(\alpha + \frac{M}{|g_h(z,w)|^\epsilon}\right)^\beta}$$
$$= \lim_{m\to\infty} \frac{|h_j(z,w)|}{|w|^\beta \left(\frac{\alpha^\beta}{|\mu_j|}\right)^m \prod_{h=0}^{m-1} \left(1 + \frac{M}{\alpha |g_h(z,w)|^\epsilon}\right)^\beta} = +\infty,$$

for the infinite product is converging, as follows from (2.11). Therefore $h_j \equiv 0$ and this completes the proof. □

Now we collect some properties of the solution of (2.2) given by Proposition 2.1:

Proposition 2.7. *Let σ be the solution of* (2.2) *given by Proposition 2.1. Then*

(1) $\mathsf{Im}\, \sigma^{n+1}(z,w) > 0$ *for any* $(z,w) \in \mathbb{H}$.
(2) *If* $|\mu_j| = \sqrt{\alpha}$ *for* $j = 1, \ldots, n$ *then* $\sigma(\mathbb{H}) \subseteq \mathbb{H}$. *In particular in this case* $\sigma \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(\infty)$.
(3) *If* $\mu_j = \mu_k$ *then* $\sigma^j = \sigma^k \circ U_{j,k}$ *where* $U_{j,k}$ *is a unitary matrix which swaps* z_j *for* z_k *and keeps fixed the other variables.*

Proof. For any $k \in \mathbb{N}$ we have

$$\mathsf{Im}\, \sigma_k^{n+1} = \frac{\mathsf{Im}\, g_k(z,w)}{\alpha^k} > \frac{\|f_k(z,w)\|^2}{\sqrt{\alpha^{2k}}} > 0,$$

and thus $\mathsf{Im}\, \sigma_{n+1} > 0$. Moreover if $|\mu_1| = \ldots = |\mu_n| = \sqrt{\alpha}$ then $\frac{\|f_k(z,w)\|^2}{\sqrt{\alpha^{2k}}} = \|(\sigma_k^1, \ldots, \sigma_k^n)\|^2$ and $\sigma_k(\mathbb{H}) \subseteq \mathbb{H}$, from which (2) follows. Point (3) follows easily from the uniqueness of σ. □

Looking at the proof of Proposition 2.1, without any assumption on the μ_j's, one can build a one-dimensional model along the normal direction. Note that in the proof of Lemma 2.6, the uniqueness along the normal direction is obtained in a slightly larger class. Thus

Proposition 2.8. *Let $F \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(\infty)$ be given by* (1.15) *and assume $\alpha > 1$. Then there exists $\theta : \mathbb{H} \to \{\zeta \in \mathbb{C} : \mathsf{Im}\,\zeta > 0\}$ holomorphic such that*

$$\theta(F(z,w)) = \alpha\theta(z,w) \tag{2.17}$$

and $\theta(z,w) = w + \Lambda(z,w)$ with $|\Lambda(z,w)| \leq M|w|^{1-\epsilon}$ for some $M > 0$. Moreover the map θ is the unique solution of (2.17) *such that $\theta(z,w) - w = o(|w|)$.*

We end this section finding all the (regular) solutions to the Schröder equation at ∞.

Proposition 2.9. *Let $F \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(\infty)$ be given by* (1.11). *Suppose τ_F is normal and $\alpha_\infty > 1$. Let $\mu_1^\infty, \ldots, \mu_n^\infty$ denote the eigenvalues of A'_∞ and suppose that $|\mu_j^\infty| > \alpha_\infty^{(1-\epsilon)/2}$ for every j. Then there exists a $\sigma_0 \in \mathcal{E}^{1+\epsilon}(\infty)$ with invertible linear part at ∞ which solves the Schröder equation*

$$\sigma \circ F = A_\infty \sigma^t.$$

If $\tilde{\sigma} \in \mathcal{E}^{1+\epsilon}(\infty)$ with invertible linear part at ∞ is a solution of the same Schröder equation then there exists a $(n+1)\times(n+1)$ invertible matrix E such that $[A_\infty, E]=0$ and $\tilde{\sigma} = E\sigma_0^t$.

Proof. By Proposition 1.15 there exists an automorphism η of $\mathbb{H}$ fixing ∞ such that $\tilde{F} = \eta \circ F \circ \eta^{-1}$ is given by (1.15). Then Proposition 2.1 gives a solution σ_D of the diagonal Schröder equation $\sigma_D \circ \tilde{F} = D\sigma_D^t$, where D is a diagonal matrix with entries $\mu_1, \ldots, \mu_n, \alpha_\infty$. Note that, even if in general $\sigma_D(\mathbb{H}) \not\subset \mathbb{H}$ the map $\sigma_0 = \eta^{-1}\circ\sigma_D\circ\eta$ is well defined (recall the possible forms (1.7) and (1.8) for η). Since $\eta^{-1}D\eta = A_\infty$ it is easy to see that $\sigma_0 \in \mathcal{E}^{1+\epsilon}(\infty)$ is one solution of the Schröder equation. Using the conjugation with η we see that any solution of the Schröder equation corresponds to a solution of (2.2). From this and from Lemma 2.6 the assertion follows. $\square$

3. THE SCHRÖDER EQUATION AT A FINITE BOUNDARY POINT

Theorem 3.1. *Let $\varphi \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(P)$ for some $P \in \partial\mathbb{H}$. Let $\alpha_P > 0$ be the boundary dilatation coefficient of φ at P and A_P the linear part of φ at P. Suppose that $\alpha_P < 1$, the action $\tau_\varphi : T_P^{\mathbb{C}}\partial\mathbb{H} \to T_P^{\mathbb{C}}\partial\mathbb{H}$ is normal and its eigenvalues are all of modulus strictly greater than $\alpha_P^{(1+\epsilon)/2}$. Then there exists $\sigma \in \mathcal{E}^{1+\epsilon}(P)$ with invertible linear part at P such that*

$$\sigma \circ \varphi = A_P \sigma^t. \tag{3.1}$$

Moreover $\tilde{\sigma} \in \mathcal{E}^{1+\epsilon}(P)$ with invertible linear part is another solution of (3.1) *if and only if there exists a $(n+1)\times(n+1)$ invertible matrix E such that $[A_P, E]=0$ and $\tilde{\sigma} = E\sigma^t$.*

Proof. Let G be an automorphism of $\mathbb{H}$ which maps P to ∞ and let $F = G\circ\varphi\circ G^{-1}$. Then $F \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(\infty)$ and one can easily check that it satisfies all the hypotheses of Proposition 2.9 and thus we have a solution σ_0 of the Schröder equation for F at ∞. Since $\sigma = G^{-1}\circ\sigma_0\circ G$ is well defined by Proposition 2.7.1, then it turns out that σ has all the required properties. It is obvious that if $[E, A_P] = 0$ then $E\sigma^t \in \mathcal{E}^{1+\epsilon}(P)$ is still a solution of (3.1). However one cannot use directly Proposition 2.9 for concluding that all the (regular) solutions of (3.1) are of that form $E\sigma^t$. The point is that if $\tilde{\sigma} \in \mathcal{E}^{1+\epsilon}(P)$ is a solution of (3.1), it might happen that $\tilde{\sigma}(\mathbb{H})$ contains the indeterminacy locus of any automorphism of $\mathbb{H}$ which maps P to ∞. However, using linear automorphisms as in (1.7), (1.8), one can assume $P = O$. Write $\tilde{\sigma}_{n+1}(z,w) = tw + \kappa_{n+1}$ with $t \in \mathbb{C}$ and $|\kappa_{n+1}| = o(|w|)$ for $(z,w)\to O$. Note that since $\tilde{\sigma}$ has invertible linear part at O then $t \neq 0$, and, by the triangle inequality, there exists $\delta > 0$ such that for $|w| < \delta$

$$|\tilde{\sigma}_{n+1}(z,w)| > r|w|,$$

for some $r > 0$. Thus for $(z,w) \in \mathbb{H}$ with $|w| < \delta$ it follows that $\tilde{\sigma}_{n+1} \neq 0$. Therefore there exists $R > 0$ such that $G\circ\tilde{\sigma}\circ G^{-1}$ is a well defined solution of the Schröder equation for F at ∞ on $\mathbb{H}_R$ with the appropriate bound on the remainder. Then an application of Lemma 2.6 shows that $\tilde{\sigma} = E\sigma$ for some E commuting with A_O, as desired. $\square$

Remark 3.2. There exists $\sigma \in \mathcal{E}^{1+\epsilon}(P)$, unique in the class $\mathcal{E}^{1+\epsilon}(P)$, which is a solution of (3.1) with linear part at P equal to the identity. Indeed if $\sigma_0 \in \mathcal{E}^{1+\epsilon}(P)$ is any solution of (3.1) with invertible linear part E then clearly $[E, A_p] = 0$, and thus $\sigma = E^{-1}\sigma_0^t$ is the wanted Schröder map.

4. Resonances

Let $\varphi \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(O)$ be given by (1.13) and assume that φ satisfies the hypothesis of Theorem 3.1. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of A'_O (we omit to write the superscript ^{O}). Recall that, according to Poincaré-Dulac (see, *e.g.* [2]) the eigenvalue μ_j (for $j = 1, \ldots, n$) is *resonant* if there exist $k_1, \ldots, k_n, s \in \mathbb{N}$ with $k_1 + \ldots + k_n + s \geq 2$ such that

$$\mu_j = \mu_1^{k_1} \cdots \mu_n^{k_n} \alpha_O^s. \tag{4.1}$$

However, since $\alpha_O^{(1+\epsilon)/2} < |\mu_j| \leq \alpha_O^{1/2}$ and $\alpha_O < 1$ by hypothesis, it follows that μ_j cannot be resonant for $j = 1, \ldots, n$. Indeed if (4.1) were true, it would hold

$$|\mu_j| = |\mu_1|^{k_1} \cdots |\mu_n|^{k_n} \alpha_O^s \leq \alpha_O^{(k_1+\ldots+k_n)/2+s} \leq \alpha_O^{(1+\epsilon)/2}.$$

Thus under our hypothesis on the μ_j's, the eigenvalues μ_j's are not resonant.

Now we examine the eigenvalue α_O. Writing the resonant condition $\alpha_O = \mu_1^{k_1} \cdots \mu_n^{k_n} \alpha_O^s$ for $k_1 + \ldots + k_n + s \geq 2$ and arguing as above, it follows that α_O is resonant if and only if $s = 0$ and there exist $j_1, j_2 \in \{1, \ldots, n\}$ (possibly $j_1 = j_2$) such that $|\mu_{j_1}\mu_{j_2}| = \alpha_O$. Therefore the resonances of α_O are all of degree two. Let σ be one of the solutions of (3.1) given by Theorem 3.1. For $c_{jk} \in \mathbb{C}$, let us consider a map of the form

$$g(z, w) = (z, w + \sum_{j,k=1}^{n} c_{jk} z_j z_k)$$

where $c_{j,k} = 0$ if $|\mu_j \mu_k| < \alpha_O$. It is direct to see that $g \circ \sigma$ is still a solution of (3.1). However $g \circ \sigma \notin \mathcal{E}^1(O)$. This means that contrary to what happens to the case of a inner fixed point (see [14]), there exists (up to multiplication with matrices) a unique special solution of the Schröder equation.

5. Applications to commuting mappings

In this section we apply the theory developed so far to the study of commuting holomorphic mappings without fixed points in the ball. We will find analogous results to those stated in [16] for the case of maps with inner fixed points.

By [6] two commuting holomorphic maps of the ball have generically the same Wolff point. That is to say, two commuting holomorphic self-map of the ball have the same Wolff point unless they fix (as set) the complex slice joining the two different Wolff points and are hyperbolic automorphisms of such a slice.

Theorem 5.1. *Let $f, g \in \mathcal{E}_{\mathbb{H}}^{1+\epsilon}(P)$ for some $P \in \partial\mathbb{H}$. Assume that $f \circ g = g \circ f$ and that P is the Wolff point of both f and g. Moreover suppose that the boundary coefficient $\alpha_P(f)$ of f at P is strictly smaller than 1, the action $\tau_f : T_P^{\mathbb{C}}\partial\mathbb{H} \to T_P^{\mathbb{C}}\partial\mathbb{H}$ is normal and its eigenvalues are* *all of modulus strictly greater than* $\alpha_P(f)^{(1+\epsilon)/2}$. *If* g *has invertible linear part at* P *then* f, g *share a common Schröder map with invertible linear part at* P.

Proof. Let A_f and A_g denote respectively the linear part of f and g at P. Let $\sigma \in \mathcal{E}^{1+\epsilon}(P)$ be the solution of the Schröder equation $\sigma \circ f = A_f \sigma$ given by Remark 3.2. Then

$$(\sigma \circ g) \circ f = \sigma \circ f \circ g = A_f (\sigma \circ g)^t.$$

Therefore $\sigma \circ g \in \mathcal{E}^{1+\epsilon}(P)$ is another solution of the Schröder equation of f. Moreover the linear part of $\sigma \circ g$ at P is A_g which is invertible. Thus by the uniqueness statement of Theorem 3.1 it follows that $\sigma \circ g = A_g \sigma^t$, as wanted. □

The converse of Theorem 5.1 holds in this form:

Theorem 5.2. *Let* $f, g \in \mathcal{E}^1_{\mathbb{H}}(P)$ *and let* A_f, A_g *be the linear part of* f *and* g *respectively. Assume that* $[A_f, A_g] = 0$. *If there exists* $\sigma \in \mathcal{D}^1(P)$ *with invertible linear part at* P *such that both* $\sigma \circ f = A_f \sigma$ *and* $\sigma \circ g = A_g \sigma$ *hold, then* $f \circ g = g \circ f$.

Proof. Since

$$\sigma \circ (f \circ g) = A_f (\sigma \circ g)^t = A_f A_g \sigma^t = A_g A_f \sigma^t = A_g (\sigma \circ f)^t = \sigma \circ (g \circ f),$$

it follows that $\sigma \circ (f \circ g)(Z) = \sigma \circ (g \circ f)(Z)$ for any $Z \in \mathbb{H}$. By Lemma 1.2 the map σ is eventually injective in any nontangential region at P. Since $f, g \in \mathcal{E}^1_{\mathbb{H}}(P)$ then $f \circ g$ has finite boundary dilatation coefficient at P and (nontangentially) fixes P (see [7], [1]). Thus by the Julia-type Lemma for the ball (see [1], [20]) we can take an open set $W \subset \mathbb{H}$ such that $(f \circ g)(W)$ is contained in one region of injectivity of σ, and thus $f \circ g = g \circ f$. □

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