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LINEAR FRACTIONAL MAPS OF THE UNIT BALL: A GEOMETRIC STUDY

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ABSTRACT. We classify up to conjugation with automorphisms the linear fractional self-maps of the unit ball of $\mathbb{C}^n$ $(n > 1)$. Then we give some applications of these normal forms to the study of composition operators.

1. INTRODUCTION

In a recent paper [4] C. Cowen and B. MacCluer introduced a class of holomorphic maps of the unit ball $\mathbb{B}^n$ into itself which generalize the automorphisms and can be represented as $(n + 1) \times (n + 1)$-matrices in a Kreĭn space. Therefore they named these maps *linear fractional maps* of the ball. There are many good reasons (at least in the opinion of the authors) for studying such maps. First of all they provide a large class of easy to handle examples of holomorphic self-maps of the unit ball which are not automorphisms. Secondly, although they present analogies with the usual linear fractional maps of $\mathbb{C}$, they also have marked differences with their one-dimensional relatives. Third, they can be used to understand composition operators on $\mathbb{B}^n$. Finally, they seem to be the maps one expects to find once fractional linear models of fixed points free holomorphic self-maps of $\mathbb{B}^n$ will be discovered (see [7] for a survey on fractional linear models in the unit disc).

In [4] the focus is on the basic properties of linear fractional maps of $\mathbb{B}^n$ obtained mainly using their matrix representation. Here we adopt a more geometric point of view. We study the connections between the "normal forms" of a linear fractional map up to conjugation, the set and the distribution of its fixed points and its invariant subspaces.

After a brief review of previous results—for some of which we give new proofs in our setting—we prove the first main fact (Theorem 3.1): if a linear fractional map has more than two fixed points on $\partial\mathbb{B}^n$ then it must have fixed points in $\mathbb{B}^n$ (actually a complex geodesic of fixed points does exist). This is the first step toward a complete classification of linear fractional maps up to conjugation with automorphisms of $\mathbb{B}^n$. Indeed, similarly to the classical setting of linear fractional maps of the unit disc Δ of $\mathbb{C}$, the main classification depends on the number (and, in the multidimensional case, the displacement) of the boundary fixed points of the

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map (see Theorem 3.4). In $\mathbb{C}^n$ $(n > 1)$ there are basically four classes of linear fractional maps according to the number of boundary fixed points: those having no boundary fixed points, only one boundary fixed point, two boundary fixed points or more than two. For each of these four cases there are different subclasses of maps. We give a (sub-)classification based on a geometric tool developed by the second author in [3]. Roughly speaking, we determine the behavior of a map by studying the behavior of its differential at a fixed point. Hence, given a linear fractional map f we prove the existence of an automorphism g of $\mathbb{B}^n$ such that $g^{-1} \circ f \circ g$ is of a (in some sense) unique prescribed form—the "normal form" of f—depending only on the geometry of f. A normal form together with the intertwining automorphism can also be thought of as a "model" for the map.

Once we have established the existence of normal forms for linear fractional maps, we give some applications of these forms to the study of composition operators. In particular we prove that the composition operator stemming from a linear fractional map is non-cyclic if the map has more than two fixed points in the ball, and it is hypercyclic if the map has exactly two boundary fixed points and its differential is injective at some point.

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2. Preliminary Results

Definition 2.1. Let $A = (a_{jk})$ be an $n \times n$-matrix, $B = (b_j)$ an n-column vector, $C = (c_j)$ an n-row vector and d a complex number. A *linear fractional map* is a map of the form

$$f(z) := \frac{Az + B}{< z, \overline{C} > + d} \tag{2.1}$$

where $< .,. >$ indicates the usual Hermitian product in $\mathbb{C}^n$. The map f is said to be a *linear fractional map of* $\mathbb{B}^n$, where $\mathbb{B}^n := \{z \in \mathbb{C}^n : ||z||^2 < 1\}$, whenever f is defined on a neighborhood of $\mathbb{B}^n$ and $f(\mathbb{B}^n) \subseteq \mathbb{B}^n$. By definition, we will also always assume throughout the paper that f is non-constant.

Any linear fractional map f is associated with a $(n+1) \times (n+1)$ matrix M_f given by

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix} \tag{2.2}$$

Embed $\mathbb{C}^n$ into $\mathbb{C}^{n+1}$ with $z \mapsto (z, 1)$ and consider the standard Hermitian product of signature $(n, 1)$ given by the matrix

$$J := \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Namely $(z, w) := \langle z, Jw \rangle$. The couple $(\mathbb{C}^{n+1}, J)$ is called a *Kreĭn space*. In [4], Cowen and MacCluer prove that a linear fractional map f maps $\mathbb{B}^n$ into itself if and only if M_f is a contraction *up to multiples* for the Hermitian product of signature $(n, 1)$ in $\mathbb{C}^{n+1}$. Due to the words "up to multiples" this condition is unfortunately very difficult to check. For a study of linear fractional maps from the point of view of Kreǐn spaces we refer the reader to [4].

Recall that an *m-dimensional affine subset* of $\mathbb{B}^n$ (or an *m-slice*) is the intersection of $\mathbb{B}^n$ with an affine m-dimensional subspace of $\mathbb{C}^n$. In [4] it is proven that a linear fractional map takes m-dimensional affine subspaces into m-dimensional affine subspaces. A *complex geodesic* of $\mathbb{B}^n$ is a (injective) holomorphic parameterization of a (non-empty) one-dimensional affine subset of $\mathbb{B}^n$. As is customary we will call a *complex geodesic* also the image of such a map. In the sequel we will also say that a complex geodesic G *passes through* some point $x \in \overline{\mathbb{B}^n}$ if $x \in \overline{G}$. Moreover we will say that G passes through x *with direction* $v \in \mathbb{C}^n \setminus \{0\}$ if $x \in \overline{G}$ and G is parallel to v. In [3] a holomorphic map $f : \mathbb{B}^n \to \mathbb{B}^n$ holomorphic is said to be *rigid* if the image under f of any complex geodesic is contained in a complex geodesic. The cited result by Cowen and MacCluer then implies:

Proposition 2.2. *Let f be a linear fractional map of $\mathbb{B}^n$ and let $D(f)$ be its domain. Let G be an m-dimensional affine subspace of $\mathbb{C}^n$. Then $f(G \bigcap D(f))$ is contained in an m-dimensional affine subspace of $\mathbb{C}^n$. In particular f is rigid.*

For future reference we state here some well-known results about automorphisms of $\mathbb{B}^n$ (a proof can be found *e.g.* in [1] or [6]):

Theorem 2.3.
(1) *The group of automorphisms of $\mathbb{B}^n$ acts transitively on $\mathbb{B}^n$ and double transitively on $\partial\mathbb{B}^n$.*
(2) *An automorphism of $\mathbb{B}^n$ is a linear fractional map of $\mathbb{B}^n$.*

Remark 2.4. Let G be a one-dimensional affine subset of $\mathbb{B}^n$ (*i.e.* the image of a complex geodesic or, simply, a complex geodesic). Consider the "standard" complex geodesic $\varphi_0 : \Delta \to \mathbb{B}^n$ defined by

$$\varphi_0 : \zeta \mapsto (\zeta, 0, \ldots, 0).$$

By property 1 of Theorem 2.3 there exists $g \in \mathrm{Aut}(\mathbb{B}^n)$ such that $g \circ \varphi_0(\Delta) = G$. Therefore $g \circ \varphi_0$ is a complex geodesic whose image is G. If $\varphi : \Delta \to \mathbb{B}^n$ is another complex geodesic such that $\varphi(\Delta) = G$ then $\zeta \mapsto \varphi^{-1} \circ g \circ \varphi_0(\zeta)$ is an automorphism of Δ and therefore $g \circ \varphi_0$ is "essentially" (*i.e.* up to automorphisms of Δ) the only parameterization of G. It follows that if f is a linear fractional map of $\mathbb{B}^n$ such that $f(G) \subseteq G_1$, where G_1 is the image of the complex geodesic φ_1, then $\zeta \mapsto \varphi_1^{-1} \circ f \circ \varphi(\zeta)$ is a linear fractional map of Δ. In particular if there exist $x, y \in \partial G$ with $x \neq y$ such that $f(x) \neq f(y)$ and $f(x), f(y) \in \partial G_1$, then $\zeta \mapsto \varphi_1^{-1} \circ f \circ \varphi(\zeta)$ is an automorphism of Δ. Sometimes in what follows we shall refer to this just saying that f *acts as* an automorphism on G.

Property 2 of Theorem 2.3 allows us to conjugate linear fractional maps with automorphisms, *i.e.* if $\gamma \in \mathrm{Aut}(\mathbb{B}^n)$ and f is linear fractional then $\gamma^{-1} \circ f \circ \gamma$ is a linear fractional map (this follows from a straightforward calculation, or see [4]). In the sequel we classify linear fractional maps up to conjugation with $\mathrm{Aut}(\mathbb{B}^n)$.

We recall now some facts about iteration of holomorphic functions in $\mathbb{B}^n$ as well as their boundary behavior, adapted to the linear fractional setting. For a better and general exposition we refer the reader to [1] and [3]. As a matter of notation, for $x \in \partial\mathbb{B}^n$ we indicate the complex tangent space of $\partial\mathbb{B}^n$ at x by $T_x^{\mathbb{C}}(\partial\mathbb{B}^n)$, *i.e.*

$$T_x^{\mathbb{C}}(\partial\mathbb{B}^n) = \{z \in \mathbb{C}^n : \langle z, x\rangle = 0\}.$$

The first and the second parts of the following Theorem are a version of a Julia-Wolff-Carathéodory Theorem for $\mathbb{B}^n$ due to Rudin [6]. For completeness, here we include their simple proof in our case.

Theorem 2.5. *Let $g : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic.*

(1) *Suppose g extends holomorphically past $x \in \partial\mathbb{B}^n$ and $g(x) = y$, with $y \in \partial\mathbb{B}^n$. Then $dg_x(T_x^{\mathbb{C}}(\partial\mathbb{B}^n)) \subseteq T_y^{\mathbb{C}}(\partial\mathbb{B}^n)$.*
(2) *Suppose g extends holomorphically past $\partial\mathbb{B}^n$. If $x \in \partial\mathbb{B}^n$ and $g(x) = x$ then $\langle dg_x(x), x\rangle > 0$.*
(3) *If $g(x) \neq x$ for all $x \in \mathbb{B}^n$ then there exists a unique point $\tau \in \partial\mathbb{B}^n$ such that $g(\tau) = \tau$ and $\langle dg_\tau(\tau), \tau\rangle = \alpha$ with $0 < \alpha \leq 1$.*

Proof. For the first part it is enough to show that the real tangent space $T_x\partial\mathbb{B}^n$ is mapped into $T_y\partial\mathbb{B}^n$, since then the assertion follows from df_x being $\mathbb{C}$-linear. Let ρ be a smooth defining function for $\mathbb{B}^n$ near x. If $\gamma : (-1, 1) \to \overline{\mathbb{B}^n}$ is a C^1-curve such that $\gamma((-1, 1) \setminus \{0\})) \subset \mathbb{B}^n$, $\gamma(0) = x$ and $\gamma'(0) \in T_x\partial\mathbb{B}^n$, then $t \mapsto \rho \circ f \circ \gamma(t)$ has a maximum at 0 and therefore $(\rho \circ f \circ \gamma)'(0) = 0$, *i.e.* $d\rho_y(df_x(\gamma'(0))) = 0$, showing that $df_x(\gamma'(0)) \in T_y\partial\mathbb{B}^n$.

For the second part consider $\eta(\zeta) := \langle g(\zeta x), x\rangle$, $\zeta \in \Delta$. The function η is a holomorphic self-map of Δ, extending holomorphically past $\partial\Delta$ and $\eta(1) = 1$. Since $\Psi : \zeta \mapsto |\eta(\zeta)|^2 - 1$ is subharmonic and has a maximum at 1, then Hopf's Lemma implies that

$$\lim_{r\to 1^-} \Psi'(r) > 0.$$

Namely $\Re e\, \eta'(1) > 0$. Now

$$\frac{d}{d\theta}\eta(e^{i\theta})_{|\theta=0} = i\eta'(1)$$

is a basis of $T_1\partial\Delta$, and therefore $\eta'(1) \in \mathbb{R}$. Hence $\eta'(1) > 0$. This means that for any boundary fixed points of f, we get $\langle dg_x(x), x\rangle = \alpha > 0$.

If f has no fixed point in $\mathbb{B}^n$, the existence of at least one fixed point for g on the boundary follows from Brouwer's Theorem . The fact that there should be at least one -and only one- with $\alpha \leq 1$ is a consequence of iteration theory (see *e.g.* [1] or [3]). □

Remark 2.6. The first part of Theorem 2.5 is particularly useful whenever f is a linear fractional map of the ball, $x = y$ and, after conjugation, $x = e_1$. It says that $\langle df_{e_1} e_j, e_1\rangle = 0$ for $j = 2, \ldots, n$, giving conditions on A, B, C, d.

A well-known result by Alexander (see *e.g.* [6]) states that the only proper holomorphic self-maps of $\mathbb{B}^n$ $(n > 1)$ are automorphisms. Here, using only Theorem 2.5.2 and the rigidity of linear fractional maps, we characterize the automorphisms of $\mathbb{B}^n$ as the only proper maps among the linear fractional maps.

Theorem 2.7. *Let f be a linear fractional map of $\mathbb{B}^n$ (any n). Then f is an automorphism of $\mathbb{B}^n$ if and only if $f(\partial\mathbb{B}^n) \subseteq \partial\mathbb{B}^n$.*

Proof. One direction is obvious. Suppose then that $f(\partial\mathbb{B}^n) \subseteq \partial\mathbb{B}^n$. We first prove that if $x \in \overline{\mathbb{B}^n}$ and $f(x) = x$ then df_x is injective. Seeking for a contradiction we assume $x \in \overline{\mathbb{B}^n}$, $f(x) = x$ and df_x is not injective. Namely there exist two linearly independent vectors $v, w \in \mathbb{C}^n \setminus \{0\}$ such that $df_x(w) = df_x(v)$. If $x \in \partial\mathbb{B}^n$ clearly we can assume $v, w \notin T_x^{\mathbb{C}}\partial\mathbb{B}^n$. Up to composing f with an automorphism of $\mathbb{B}^n$ fixing x, we can also assume that v is an eigenvector for df_x (*i.e.* using a terminology to be introduced later, $v \in \mathbb{A}(f)$ if $x \in \partial\mathbb{B}^n$). Let G_v, G_w be the complex geodesics passing through x with directions v and w, respectively. Let $a \in \partial G_v \setminus \{x\}$ and $b \in \partial G_w \setminus \{x\}$ and let $G_{a,b}$ be the complex geodesic whose closure contains a and b. We can assume $f(a) \neq f(b)$, for if not then the restriction of f to $G_{a,b}$ would be an automorphism which is not injective on $\partial G_{a,b}$. Since f is rigid and maps a, b on the boundary of G_v then $f(\overline{G_{a,b}}) \subseteq \overline{G_v}$. Moreover by hypothesis $f(\partial G_{a,b}) \subseteq \partial G_v$. Therefore, if $\varphi_v : \Delta \to \mathbb{B}^n$ is a complex geodesic whose image is G_v and $\varphi_{a,b} : \Delta \to \mathbb{B}^n$ is a complex geodesic whose image is $G_{a,b}$, then $\eta \in \mathrm{Hol}(\Delta,\Delta)$ given by

$$\eta : \zeta \mapsto \varphi_v^{-1} \circ f \circ \varphi_{a,b}(\zeta)$$

is an automorphism of Δ (see Remark 2.4). Hence there exists $y \in \partial G_{a,b}$ such that $f(y) = x$. Now the complex geodesic $G_{x,y}$ through x and y is mapped by f onto another complex geodesic G whose closure contains x. Reasoning as above we see that f acts on $G_{x,y}$ to G as an automorphism, but x, y are mapped both to x. Contradiction. Therefore df_x is injective.

Now we assume $f(\partial\mathbb{B}^n) \subseteq \partial\mathbb{B}^n$, $f(x) = x$ for some $x \in \partial\mathbb{B}^n$ and df_x invertible (the case $x \in \mathbb{B}^n$ is similar and we omit it). Let $v \in \mathbb{C}^n \setminus T_x^{\mathbb{C}}\partial\mathbb{B}^n$. Then there exists a unique $w \in \mathbb{C}^n$ such that $df_x(w) = v$. By Theorem 2.5.2 $w \notin T_x^{\mathbb{C}}\partial\mathbb{B}^n$. Therefore by Proposition 2.2 f maps the complex geodesic G_1 passing for x and with direction w to the complex geodesic G_2 for x and with direction v. If one parameterizes G_1 and G_2 with φ_1 and φ_2 respectively, then

$$\zeta \mapsto \varphi_2^{-1} \circ f \circ \varphi_1(\zeta)$$

is a linear fractional map of Δ which maps $\partial\Delta$ into $\partial\Delta$, therefore it is an automorphism of Δ. In particular this implies that $f : G_1 \to G_2$ is injective and surjective. Therefore for any $p \in \mathbb{B}^n$ there exists a unique $q \in \mathbb{B}^n$ such that $f(q) = p$, hence f is an automorphism of $\mathbb{B}^n$. ☐

Remark 2.8. Reasoning as in the proof of Theorem 2.7 one can show that a linear fractional map f of $\mathbb{B}^n$ is injective if and only if df_x is injective at some—and hence any—$x \in \overline{\mathbb{B}^n}$. Roughly speaking, this is so because f is determined by knowing $f(x)$ and df_x at some point $x \in \overline{\mathbb{B}^n}$.

The number α given by Theorem 2.5 part 3, is often referred to as *boundary dilatation coefficient* of f at τ. It turns out that it is always an eigenvalue of df_τ (see *Theorem 5.1* in [3]). By Proposition 2.2 an eigenvector of df_τ gives rise to a one-dimensional affine subset of $\mathbb{C}^n$ containing τ and fixed (as a set) by f. If this eigenvector is "pointing toward" the ball, then the intersection of such one-dimensional affine space and the ball itself is fixed (as a set) for f, *i.e.* f has a fixed complex geodesic. In [3] a tool has been developed to make precise the ideas described above. Here we recall the main definitions and properties, adapted to our needs:

Definition 2.9. Let f be a linear fractional map of $\mathbb{B}^n$.

(1) A complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ is said to be a *cut complex geodesic* for f if $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$.
(2) The set of fixed points of f is $\mathrm{Fix}(f) := \{x \in \mathbb{B}^n : f(x) = x\}$.
(3) If $\mathrm{Fix}(f) = \emptyset$ and τ is the point given by Theorem 2.5 part 3, then τ is called the *Wolff point* of f. The number $0 < \alpha \leq 1$ is the *boundary dilatation coefficient* of f at τ.

In the following Theorem we collect the results we need from [3].

Theorem 2.10. *Let f be a linear fractional map, $Fix(f) = \emptyset$, $x \in \partial\mathbb{B}^n$ the Wolff point of f and let α be the boundary dilatation coefficient of f at x. Then α is an eigenvalue of df_x. Moreover if $df_x(v) = \lambda v$ and $\langle v, x\rangle \neq 0$, then $\lambda = \alpha$. Let*

$$\mathbb{A}(f) := span\{v \in \mathbb{C}^n : df_x(v) = \alpha v, \langle v, x\rangle \neq 0\},$$

and

$$\mathcal{AG}(f) := \bigcup_{j=1}^{\infty} \ker(df_x - \alpha I)^j.$$

The spaces $\mathbb{A}(f)$ and $\mathcal{AG}(f)$ are called the inner space *and the* generalized inner space *of f respectively. Then*

1. *The point x belongs to the closure of any cut complex geodesic of f.*
2. *Let $\varphi : \Delta \to \mathbb{B}^n$ be a complex geodesic such that φ extends C^1 up to the boundary, $\varphi(1) = x$ and $\varphi'(1) = v$. Then $\varphi(\Delta)$ is a cut complex geodesic for f if and only if $v \in \mathbb{A}(f)$.*

Remark 2.11. As a consequence of Theorem 2.10 we have that $\mathcal{AG}(f)$ is invariant for df_x and by Proposition 2.2 it follows that f maps $\mathbb{B}^n \bigcap(\mathcal{AG}(f) + x)$ into itself. Moreover, as explained in *Theorem 5.3* of [3], there is a df_x-invariant decomposition of $\mathbb{C}^n = \mathcal{AG}(f) \bigoplus V_f$ such that $V_f \subseteq T_x^{\mathbb{C}}(\partial\mathbb{B}^n)$. Therefore $\mathbb{B}^n \bigcap(\mathcal{AG}(f) + x)$ is the maximum (maybe proper) invariant set of f in the ball. Any other invariant set of f is obtained as $\mathbb{B}^n \bigcap(W + x)$ for $W \subset \mathcal{AG}(f)$ and $df_x(W) \subseteq W$. This answers a question raised at the end of *section 4* of [4].

3. Fixed Points of Linear Fractional Maps of the Ball

In this section we generalize a result of Hayden and Suffridge [5] (see also [6]) on $\mathrm{Aut}(\mathbb{B}^n)$ to linear fractional maps of $\mathbb{B}^n$. Our proof seems to be new also for the case of automorphisms of $\mathbb{B}^n$.

Theorem 3.1. *Let f be a linear fractional map of $\mathbb{B}^n$. If f has more than two fixed points on $\partial\mathbb{B}^n$ then f has fixed points in $\mathbb{B}^n$.*

Proof. Suppose that f has three fixed points on $\partial\mathbb{B}^n$. Up to conjugation we can suppose $f(e_1) = e_1$, $f(-e_1) = -e_1$. Theorem 2.5 implies then $< df_{e_1}(e_j), e_1 >= 0$ and $< df_{-e_1}(e_j), e_1 >= 0$ for $j = 2, \ldots, n$. Recall that f is of the form (2.1). Writing down all these conditions we have $b_j = a_{j1} = 0$, $c_j = a_{1j} = 0$ for $j = 2, \ldots, n$, and $a := a_{11} = d$, $b := b_1 = c_1$. In other words

$$f(z) = \frac{(az_1 + b, A_1 z')}{bz_1 + a}, \tag{3.1}$$

where A_1 is a $(n-1) \times (n-1)$ matrix and $z' = (z_2, \ldots, z_n)$ as usual. Therefore f fixes $\mathbb{C}e_1$ (as a set). Now $\eta(\zeta) := f_1(\zeta, 0, \ldots, 0)$ is a linear fractional map of $\mathbb{C}$ with the properties that $\eta(\Delta) \subseteq \Delta$ and $\eta(\pm 1) = \pm 1$. Using the conformality of linear fractional maps of $\mathbb{C}$ is then easy to see that η has to be an automorphism of Δ or the identity. Namely $a = \cosh t$, $b = \sinh t$, for $t \in \mathbb{R}$ ($t = 0$ if and only if η is the identity). Suppose now that f fixes the point $v = (v_1, \ldots, v_n)$ different from $\pm e_1$. Since $v \in \overline{\mathbb{B}^n}$ and $v \neq \pm e_1$, then $v_1 \neq \pm 1$. Hence $f(v) = v$ implies $f_1(v) = v_1$. But f_1 depends only on z_1, and then $\eta(v_1) = v_1$. By the Schwarz lemma $\eta(\zeta) = \zeta$ and hence f fixes $(z_1, 0, \ldots, 0)$ for any z_1. So f has fixed points in $\mathbb{B}^n$ as wanted. $\square$

Remark 3.2. The proof of Theorem 3.1 says actually that if f has more than two boundary fixed points then it has a whole complex geodesic of fixed points.

We want to give now a classification theorem of linear fractional maps based on their fixed point sets. Before that we need some definitions:

Definition 3.3. Let

$$\mathcal{P}_0 := \mathrm{span}_{\mathbb{C}}\,\{x \in \partial\mathbb{B}^n : f(x) = x\}$$

and $p_0 := \dim_{\mathbb{C}}\mathcal{P}_0$. If $p_0 > 0$ and $f(x_0) = x_0$, $x_0 \in \partial\mathbb{B}^n$, let

$$\mathcal{P}_1 := \mathrm{span}_{\mathbb{C}}\,\{x - x_0 : f(x) = x, x \in \partial\mathbb{B}^n\}$$

and $p_1 := \dim_{\mathbb{C}}\mathcal{P}_1$. Finally, let

$$\mathcal{P}_1^{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\,\{x - x_0 : f(x) = x, x \in \partial\mathbb{B}^n\}$$

and ${p^{\mathbb{R}}}_1 := \dim_{\mathbb{R}}\mathcal{P}_1^{\mathbb{R}}$.

Theorem 3.4. *Let f be a linear fractional map of $\mathbb{B}^n$. One and only one of the following cases is possible:*

(1) $p_0 = 0$ *if and only if f has only one (isolated) fixed point in $\mathbb{B}^n$ and no fixed points on $\partial\mathbb{B}^n$.*

(2) $p_0 > 0$ *if and only if f has at least one fixed point on the boundary. In this case:*

(i) $p_1 = 0$ *if and only if f has only one fixed point on the boundary. In this case it is the unique fixed point of f in $\overline{\mathbb{B}^n}$ if and only if the boundary dilatation coefficient of f at that point is less than or equal to* 1. *Otherwise f has also an isolated fixed point inside $\mathbb{B}^n$.*

(ii) $p_1 = 1$ *if and only if one (and only one) of the two holds:*

(a) ${p^{\mathbb{R}}}_1 = 1$, *f has only two fixed points on $\partial\mathbb{B}^n$, and f is conjugate to a map which has a hyperbolic automorphism (different from the identity) as first coordinate,* i.e. *f is conjugate to a map of the form*

$$z \mapsto \left(\frac{az_1 + b}{bz_1 + a}, \frac{A_1 z'}{bz_1 + a}\right),$$

where $a = \cosh t$, $b = \sinh t$ *with* $t \in \mathbb{R} - \{0\}$ *and A_1 is a $(n-1)\times(n-1)$ matrix with $\|A_1\| \le 1$.*

(b) ${p^{\mathbb{R}}}_1 = 2$, *f is conjugate to a map of the form*

$$z \mapsto (z_1, A_1 z'),$$

where A_1 is a $(n-1)\times(n-1)$ matrix with $\|A_1\| \le 1$.

(iii) $p_1 > 1$ *if and only if f is conjugate to a map of the form*

$$z \mapsto (z_1, \ldots, z_{p_1}, A_{p_1} z^{(p_1)}),$$

where A_{p_1} is an $(n - p_1 - 1)\times(n - p_1 - 1)$ matrix with $\|A_{p_1}\| \le 1$ and $z^{(p_1)} = (z_{p_1+1}, \ldots, z_n)$.

Proof. Suppose $p_0 = 0$ and f fixes two different points $x, y \in \mathbb{B}^n$. Since f is rigid, then f fixes (as a set) the complex geodesic G passing through x and y. Therefore f restricted to G is a self-map of the unit disc with two fixed points. By the Schwarz Lemma it has to be the identity and so $f(z) = z$ for all z belonging to G. In particular f fixes all the points on $\partial\mathbb{B}^n \bigcap \overline{G}$, contradicting $p_0 = 0$.

Suppose now $p_0 > 0$. Then it is clear that this is possible if and only if f has some boundary fixed points. Let $p_1 = 0$. Therefore f has only one fixed point $x \in \partial\mathbb{B}^n$. Let α be the boundary dilatation coefficient of f at x. If f has no other fixed points in $\mathbb{B}^n$ then x is the Wolff point of f and $\alpha \leq 1$ (by Theorem 2.5 part 3). If f has another fixed point $y \in \mathbb{B}^n$, then reasoning as before it is easy to see that y is the only fixed point of f in $\mathbb{B}^n$. Now, since f is rigid, f fixes (as a set) the complex geodesic for x and y, and in particular f restricted to such a geodesic is a self-map of the unit disc. Therefore its boundary dilatation coefficient is > 1 by the classical Julia's lemma.

Let $p_1 = 1$. In the proof of Theorem 3.1 it is shown that f has exactly two boundary fixed points if and only if f is conjugate to a map of the form (3.1) with $t \neq 0$. It is clear that the condition "f has exactly two boundary fixed points" is equivalent to ${p^{\mathbb{R}}}_1 = 1$. Also ${p^{\mathbb{R}}}_1 > 1$ if and only if f has more than two boundary fixed points. Again looking at the proof of Theorem 3.1, this turns out to be equivalent to f being conjugate to a map of the form (3.1) with $t = 0$.

For the case $p_1 > 1$, we reason as follow. We have already shown that (up to conjugation) $f_1(z) = z_1$ and $f'(z) = A_1 z'$. Suppose $p_1 = 2$. Therefore there must be a vector $v \in \partial\mathbb{B}^n$ such that $f(v) = v$ and $v \neq e^{i\theta} e_1$ for any $\theta \in \mathbb{R}$. Hence $A_1 v' = v'$ and f fixes $(0, \zeta v')$ for $\zeta \in \Delta$. Conjugating f with a unitary transformation fixing e_1, we can suppose that $(0, v') = e_2$ and f has the form $z \mapsto (z_1, z_2, A_2 z'')$. If $p_1 > 2$, reasoning similarly, after a finite number of steps we get the claimed form.

For ending the proof we have to show that the matrices A_1 in the cases *(ii) (a)* and *(b)* and A_{p_1} in the case *(iii)* are contractions. This follows easily by setting up the condition $f(\mathbb{B}^n) \subseteq \mathbb{B}^n$ and using the identity $a^2 - b^2 = 1$. $\square$

In the next two sections we provide further classifications for the cases 1 and $2(i)$ of Theorem 3.4.

4. Linear fractional maps with a unique fixed point on the boundary and non-trivial inner space.

In this section we study linear fractional maps with no fixed points in $\mathbb{B}^n$ and at least one cut complex geodesic. Before that we briefly recall the one-dimensional classification (see *e.g.* [7]);

Proposition 4.1. *Let $\gamma : \Delta \to \Delta$ be a linear fractional map (not the identity) that fixes* 1. *Then*

- **(i)** *γ is called* of hyperbolic type *if $\gamma'(1) < 1$. In this case γ is an automorphism of Δ if and only if there exists $x \in \partial\Delta$ such that $x \neq 1$ and $\gamma(x) = x$. This is also the case if and only if there is a point -and hence any- $x \in \partial\Delta - \{1\}$ such that $\gamma(x) \in \partial\Delta$.*
- **(ii)** *γ is called* of parabolic type *if $\gamma'(1) = 1$. In this case γ is an automorphism of Δ if and only if there is a point—and hence any—$x \in \partial\Delta - \{1\}$ such that $\gamma(x) \in \partial\Delta$. This is also the case if and only if $\Re e(\gamma''(1)) = 0$.*
- **(iii)** *γ is called* of dilation type *if $\gamma'(1) > 1$. This happens if and only if γ has a fixed point in Δ.*

Remark 4.2. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a linear fractional map with a unique fixed point on $\partial\mathbb{B}^n$ that, up to conjugation, we can suppose to be e_1. Moreover suppose that f has at least one cut complex geodesic (*i.e.* $\dim\mathbb{A}(f) \geq 1$). Again, conjugating f if necessary, we can suppose that one of the cut complex geodesics is the standard geodesic $\varphi_0 : \zeta \mapsto (\zeta, 0, \ldots, 0)$ for $\zeta \in \Delta$. The holomorphic self-map η of Δ defined as

$$\eta : \zeta \mapsto \varphi_0^{-1} \circ f \circ \varphi_0(\zeta) = f_1(\zeta, 0, \ldots, 0),$$

is a linear fractional map with 1 as unique fixed point (see Remark 2.4). Therefore according to Proposition 4.1 η could be a (non-automorphism) map of hyperbolic type or a (non-automorphism) map of parabolic type or an automorphism (necessary a parabolic one). Simple geometric considerations (or see [3]) show that

$$\alpha := \eta'(1) = \langle df_{e_1} e_1, e_1 \rangle,$$

is invariant under conjugation. Therefore the *boundary dilatation coefficient* α controls if f restricted to one—and hence any—cut complex geodesic is of (non-automorphism) hyperbolic type (the independence of the cut complex geodesic follows from Theorem 2.10).

Suppose now that f is as in Remark 4.2 and moreover that $\alpha = 1$. We want to show that if f is an automorphism (of parabolic type) restricted to a cut complex geodesic then it is so when restricted to any other cut complex geodesics. After that we can give a well-posed classification based on the behavior of f on a cut complex geodesic.

The linear fractional map f has the form given by equation (2.1). After conjugation f fixes e_1 and $df_{e_1} e_1 = e_1$ (since f has $\mathbb{C}e_1 \cap \mathbb{B}^n$ as a cut complex geodesic). Setting up these conditions and those given by Theorem 2.5 we get information on the matrix A and the vectors B, C and d. Namely:

$$\begin{cases} b_j = a_{j1} = 0 & \text{for } j = 2, \ldots, n. \\ a_{1j} = c_j & \text{for } j = 2, \ldots, n \end{cases} \tag{4.1}$$

Lemma 4.3. *Let f be a linear fractional map of $\mathbb{B}^n$ such that $f(e_1) = e_1$, $df_{e_1} e_1 = e_1$ and $\dim \mathbb{A}(f) = k$, for $1 \leq k \leq n$. Then there exists a unitary transformation U such that $Ue_1 = e_1$ and $\mathbb{A}(U^* \circ f \circ U)$ is spanned by $e_1, e_2, \ldots e_k$.*

Proof. If $k = 1$ there is nothing to prove. Suppose that $k > 1$. By hypothesis e_1 is an (inner) eigenvector for df_{e_1}. Reasoning by induction we can suppose that $e_2, \ldots, e_{k-1}$ belong to $\mathbb{A}(f)$. Since $\dim \mathbb{A}(f) = k$ there exists $v \in \mathbb{C}^n$ such that

$$v \neq \sum_{j=1}^{k-1} \lambda_j e_j,$$

with $\lambda_j \in \mathbb{C}$ for $j = 2, \ldots, k-1$ and $df_{e_1} v = v$. Hence

$$u := v - \sum_{j=1}^{k-1} \langle v, e_j \rangle e_j$$

is still an eigenvector for df_{e_1} with eigenvalue 1. Moreover u belongs to the orthogonal complement of $\oplus_{j=1,\ldots,k-1} \mathbb{C}e_j$. Therefore there exists a unitary transformation U given by

$$U := \begin{pmatrix} I_{k-1} & 0 \\ 0 & T_k \end{pmatrix}$$

where I_{k-1} is the identity matrix on $\mathbb{C}^{k-1}$ and T_k is a unitary transformation of $\mathbb{C}^{n-k+1}$, such that $Ue_k = u/\|u\|$. Notice that also $Ue_j = e_j$ for $j = 1, \ldots, k-1$. $\square$

Lemma 4.4. *Let f be a linear fractional map of $\mathbb{B}^n$ such that $f(e_1) = e_1$, $df_{e_1} e_1 = e_1$ and $\dim \mathbb{A}(f) = k$, for $2 \leq k \leq n$. If the restriction of f to $\mathbb{C}e_1 \cap \mathbb{B}^n$ is a parabolic automorphism then $c_2 = \cdots = c_k = 0$.*

Proof. Suppose first that f restricted to $\mathbb{C}e_1 \cap \mathbb{B}^n$ is a parabolic automorphism. Using an (hyperbolic) automorphism of the form

$$(z_1, \ldots, z_n) \mapsto \frac{(\cosh s z_1 + \sinh s, z_2, \ldots, z_n)}{\sinh s z_1 + \cosh s}$$

for $s \in \mathbb{R}$, we can conjugate f to a map such that

$$f_1(z_1, 0, \ldots, 0) = \frac{(1+it)z_1 + it}{-itz_1 + 1 + it},$$

for some $t \in \mathbb{R} - \{0\}$. Now by Lemma 4.3 we can suppose that $\mathbb{A}(f)$ is spanned by $\{e_1, \ldots, e_k\}$ (this is compatible with the previous operation since it leaves fixed the first component restricted to $\mathbb{C}e_1$). Hence:

$$df_{e_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_{k-2} & 0 & * & \cdots & * \\ 0 & \cdots & 1 & \frac{a_{k,k+1}}{c+d} & \cdots & \frac{a_{k,n}}{c+d} \\ 0 & \cdots & 0 & \frac{a_{k+1,k+1}}{c+d} & \cdots & \frac{a_{k+1,n}}{c+d} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{a_{n,k+1}}{c+d} & \cdots & \frac{a_{n,n}}{c+d} \end{pmatrix}$$

and

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= \frac{(1-it)z_1 + c_2 z_2 + \cdots + c_n z_n + it}{(-it)z_1 + c_2 z_2 + \cdots c_n z_n + (1+it)} \\ f_2(z_1, \ldots, z_n) &= \frac{z_2 + a_{2,k+1} z_{k+1} + \cdots + a_{2n} z_n}{(-it)z_1 + c_2 z_2 + \cdots + c_n z_n + (1+it)} \\ &\cdots \\ f_k(z_1, \ldots, z_n) &= \frac{z_k + a_{k,k+1} z_{k+1} + \cdots + a_{kn} z_n}{(-it)z_1 + c_2 z_2 + \cdots + c_n z_n + (1+it)} \\ f_{k+1}(z_1, \ldots, z_n) &= \frac{a_{k+1,k+1} z_{k+1} + \cdots + a_{k+1,n} z_n}{(-it)z_1 + c_2 z_2 + \cdots + c_n z_n + (1+it)} \\ &\cdots \\ f_n(z_1, \ldots, z_n) &= \frac{a_{n,k+1} z_{k+1} + \cdots + a_{nn} z_n}{(-it)z_1 + c_2 z_2 + \cdots + c_n z_n + (1+it)}. \end{aligned}$$

Now fix $j \in \{2, \ldots, k\}$. Setting the condition $f(\zeta e_j) \in \mathbb{B}^n$ for $\zeta \in \Delta$, we find:

$$-2\Re e(c_j \zeta) < 1 - |\zeta|^2, \forall \zeta \in \Delta,$$

which is easily seen to be verified only for $c_j = 0$. □

Theorem 4.5. *Let f be a linear fractional map of $\mathbb{B}^n$ such that it has a unique fixed point on $\partial \mathbb{B}^n$ and $\dim \mathbb{A}(f) = k \geq 2$. If the restriction of f to a cut complex geodesic is a parabolic automorphism, then the restriction to any other cut complex geodesic is also a parabolic automorphism.*

Proof. By Lemma 4.3 and Lemma 4.4 we can suppose, up to conjugation, that

$$
\begin{aligned}
f_1(z_1,\dots,z_n) &= \frac{(1-it)z_1 + c_{k+1}z_{k+1} + \ldots + c_n z_n + it}{(-it)z_1 + c_{k+1}z_{k+1} + \cdots c_n z_n + (1+it)}\\
f_2(z_1,\dots,z_n) &= \frac{z_2 + a_{2,k+1}z_{k+1} + \ldots + a_{2n}z_n}{(-it)z_1 + c_{k+1}z_{k+1} + \ldots + c_n z_n + (1+it)}\\
&\dots\\
f_k(z_1,\dots,z_n) &= \frac{z_k + a_{k,k+1}z_{k+1} + \ldots + a_{kn}z_n}{(-it)z_1 + c_{k+1}z_{k+1} + \ldots + c_n z_n + (1+it)}\\
f_{k+1}(z_1,\dots,z_n) &= \frac{a_{k+1,k+1}z_{k+1} + \ldots + a_{k+1,n}z_n}{(-it)z_1 + c_{k+1}z_{k+1} + \ldots + c_n z_n + (1+it)}\\
&\dots\\
f_n(z_1,\dots,z_n) &= \frac{a_{n,k+1}z_{k+1} + \ldots + a_{nn}z_n}{(-it)z_1 + c_{k+1}z_{k+1} + \ldots + c_n z_n + (1+it)}.
\end{aligned}
\tag{4.2}
$$

Therefore $f_1(z_1,0,\dots,0)$ is a parabolic automorphism of Δ (since $t \in \mathbb{R} - \{0\}$) and $\mathbb{A}(f)$ is spanned by $e_1,\dots,e_k$. Hence any cut complex geodesic is given by

$$\{\zeta \in \mathbb{C} : e_1 + \zeta v \in \mathbb{B}^n\},$$

for $v = \sum_{j=1}^k \lambda_j e_j$ with $\lambda_1 \in \mathbb{C} - \{0\}$, $\lambda_j \in \mathbb{C}$ for $j = 2,\dots,k$ and $\sum_{j=1}^k |\lambda_j|^2 = 1$. Fix such a v. The boundary of the cut complex geodesic G_v with direction v is given by

$$\{\zeta \in \mathbb{C} : ||e_1 + \zeta v||^2 = 1\}.$$

Namely

$$\partial G_v = \{\zeta : |\zeta|^2 + 2\Re e(\lambda_1 \zeta) = 0\}.$$

Now

$$f(e_1 + \zeta v) = \left(\frac{\zeta\lambda_1(1-it)+1}{\zeta\lambda_1(-it)+1}, \frac{\lambda_2\zeta}{\zeta\lambda_1(-it)+1}, \dots, \frac{\lambda_k\zeta}{\zeta\lambda_1(-it)+1}, 0, \dots, 0 \right).$$

Therefore

$$f(\partial G_v) = \{\zeta \in \mathbb{C} : \|f(e_1+\zeta v)\| = 1\} = \{\zeta : |\zeta|^2 + 2\Re e(\lambda_1\zeta) = 0\} = \partial G_v.$$

This means that f is a parabolic automorphism on G_v as claimed. □

Corollary 4.6. *Let f be a linear fractional map of $\mathbb{B}^n$ with a unique fixed point in $\overline{\mathbb{B}^n}$ and $\dim\mathbb{A}(f) = n$. Then f is a (parabolic) automorphism of $\mathbb{B}^n$ if and only if f restricted to a cut complex geodesic is a parabolic automorphism of such a geodesic.*

Now we are in the position to give the following:

Definition 4.7. Let f be a linear fractional map of the unit ball $\mathbb{B}^n$ with a unique fixed point on $\partial\mathbb{B}^n$ and $\dim\mathbb{A}(f) \geq 1$. Let G be an arbitrary cut complex geodesic for f. Then f is said of *hyperbolic type* if $f_{|G}$ is a hyperbolic linear fractional map of G, f is said of *parabolic automorphism type* if $f_{|G}$ is a parabolic automorphism of G and finally f is said of *parabolic non-automorphism type* if $f_{|G}$ is a parabolic non-automorphism of G.

Note that this definition is well posed by Remark 4.2 and Theorem 4.5. The last proposition of this section is a multi-dimensional analogue of the second derivative characterization for parabolic automorphisms of the disc:

Proposition 4.8. *Let f be a linear fractional map of $\mathbb{B}^n$ with only e_1 as a fixed point in $\overline{\mathbb{B}^n}$. Suppose that f has non-trivial inner space and boundary dilatation coefficient 1 (*i.e. *f is of parabolic type). Then f is of* parabolic automorphism *type if and only if for one—and hence for any—unitary vector $v \in \mathbb{A}(f)$ there holds*

$$\Re e \left(\overline{v}_1 \cdot \sum_{j,k,l=1}^{n} \frac{\partial^2 f_l}{\partial z_j \partial z_k}(e_1) v_j v_k \overline{v}_l \right) = 0. \tag{4.3}$$

Proof. Let $v \in \mathbb{A}(f)$ be a unitary vector. Therefore $\|v\| = 1$, $v_1 \neq 0$ and $df_{e_1}(v) = v$. The map $\zeta \to f(\zeta v + e_1)$ is well defined for $\zeta \in \mathbb{C}$ such that $\|\zeta v + e_1\| < 1$, i.e. for $\zeta \in D := \{\zeta \in \mathbb{C} | |\zeta + \overline{v}_1|^2 < |v_1|^2\}$. The set D is a disc in $\mathbb{C}$ of radius $|v_1|$ and center $-\overline{v}_1$. The affine transformation $\tau(\zeta) := \frac{1}{\overline{v}_1}\zeta + 1$ is such that $\tau(D) = \Delta$ and $\tau(0) = 1$. A simple computation shows that $\tau^{-1}(\zeta) = \overline{v}_1(\zeta - 1)$. Since the complex geodesic for e_1 and v is fixed (as a set) by f, then $f(\zeta v + e_1) = \tilde{f}(\zeta) v + e_1$ where $\tilde{f}(\zeta) := \langle f(\zeta v + e_1) - e_1, v\rangle$. The condition $\|f(\zeta v + e_1)\|^2 < 1$ for $\zeta \in D$ implies that $\tilde{f}(\zeta) \in D$, as well. Therefore a well-defined holomorphic map $k : \Delta \to \Delta$ is given by

$$k(\zeta) := \tau \circ \tilde{f} \circ \tau^{-1}(\zeta).$$

By definition k is a linear fractional map of the unit disc, with no fixed points in Δ and $k(1) = 1$. Moreover

$$k'(\zeta) = \frac{1}{\overline{v}_1} \cdot \langle df_{\tau^{-1}(\zeta)v+e_1}(\overline{v}_1 v), v\rangle = \langle df_{\tau^{-1}(\zeta)v+e_1}(v), v\rangle.$$

Hence $k'(1) = \langle df_{e_1}(v), v\rangle = \langle v, v\rangle = 1$ and k is of parabolic type. By Proposition 4.1 k is a (parabolic) automorphism if and only if $\Re e k''(1) = 0$. Now a simple calculation shows that

$$k''(1) = \frac{d}{d\zeta}_{|\zeta=1} \langle df_{\tau^{-1}(\zeta)v+e_1}(v), v\rangle = \overline{v}_1 \cdot \sum_{j,k,l=1}^{n} \frac{\partial^2 f_l}{\partial z_j \partial z_k}(e_1) v_j v_k \overline{v}_l.$$

Therefore equation (4.3) is equivalent to $\Re e k''(1) = 0$. Hence equation (4.3) holds for a unitary vector $v \in \mathbb{A}(f)$ if and only if f restricted to the cut complex geodesic for e_1 and v is a parabolic automorphism, and by Theorem 4.5 it must be of this type on any other cut complex geodesic (and in particular equation (4.3) holds for any other unitary element in $\mathbb{A}(f)$). □

Remark 4.9. If for a vector $v \in \mathbb{C}^n$ we denote by $Hf_j(v)$ the (holomorphic) Hessian matrix of the component f_j of f applied to (v, v), and by $\mathbb{H}f(v)$ the vector $(Hf_1(v), \ldots, Hf_n(v))$, then Proposition 4.8 can be rephrased saying that *f is of parabolic automorphism type if and only if there exists a unitary vector v such that $v_1 = \Re e v_1 > 0$, $df_{e_1}(v) = v$ and $\Re e\langle \mathbb{H}f_{e_1}(v), v\rangle = 0$.*

5. Linear Fractional Maps with Trivial Inner Space

In this section we let f be a linear fractional map of the unit ball with e_1 its Wolff point and $\mathbb{A}(f) = \{0\}$. A simple consequence of f being rigid is that e_1 is the only fixed point of f in $\overline{\mathbb{B}^n}$.

Example 5.1. Consider the following family of linear fractional maps indexed by $\beta \in \mathbb{C}$:

$$\eta_\beta(z) = \frac{((1-\beta)z_1 - sz_2 + \beta, sz_1 + z_2 - s, z_3, \cdots, z_n)}{-\beta z_1 - sz_2 + 1 + \beta} \tag{5.1}$$

where $s^2 = 2\Re e\beta \neq 0$. A straightforward calculation shows that $\eta_\beta(\partial\mathbb{B}^n) \subseteq \partial\mathbb{B}^n$ and hence by Theorem 2.7 η_β is an automorphism of $\mathbb{B}^n$. Moreover $\eta_\beta(e_1) = e_1$ and

$$d(\eta_\beta)_{e_1} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \tag{5.2}$$

Therefore the η_β's are parabolic automorphisms of $\mathbb{B}^n$ with Wolff point e_1, inner space $\mathbb{A}(\eta_\beta) = \{0\}$ and generalized inner space $\mathcal{AG}(\eta_\beta) = \mathbb{C}^n$.

Now we are going to prove that any linear fractional map with trivial inner space stems (up to conjugation) from the composition of one element of the family $\{\eta_\beta\}$ and a well-behaving map g:

Proposition 5.2. *Let f be a linear fractional map of $\mathbb{B}^n$ with e_1 its Wolff point, α the boundary dilatation coefficient of f at e_1 and $\mathbb{A}(f) = \{0\}$. Then there exist $T \in \mathrm{Aut}(\mathbb{B}^n)$ and a parabolic automorphism η_β of the family (5.1), such that $T^{-1} \circ f \circ T = \eta_\beta^{-1} \circ g$, where g is a linear fractional map of $\mathbb{B}^n$ with the following properties:*

1. *$Fix(g) = \emptyset$.*
2. *$g(e_1) = e_1$.*
3. *$dg_{e_1}(e_1) = \alpha e_1$, and in particular e_1 is the Wolff point of g and α is the boundary dilatation coefficient.*
4. *$dg_{|T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n} = dT^{-1}_{e_1} \circ df_{|T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n} \circ dT_{e_1}$.*
5. *If $\mathbb{C}^n = W_1 \oplus \ldots \oplus W_m$ is the Jordan decomposition of $\mathbb{C}^n$ in irreducible cyclic df_{e_1}-invariant subspaces, and $W_j \subset T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$ for $j \in \{2, \ldots, m\}$, then $\mathbb{C}^n = \mathbb{C}e_1 \oplus \tilde{W}_1 \oplus \ldots \oplus \tilde{W}_m$, with $\tilde{W}_j = dT^{-1}_{e_1}(W_j)$ for $j \in \{2, \ldots, m\}$ and $\tilde{W}_1 = dT^{-1}_{e_1}(W_1) \cap T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$, is the Jordan decomposition relative to dg_{e_1}.*
6. *$\mathcal{AG}(g) = dT^{-1}_{e_1}(\mathcal{AG}(f))$.*
7. *If $V_\alpha(f) := \{v \in \mathbb{C}^n | df_{e_1}(v) = \alpha v\}$, then $\mathbb{A}(g) = span\{dT^{-1}_{e_1}(V_\alpha(f)), e_1\}$. In particular $dim\mathbb{A}(g) = 1 + dimV_\alpha(f) = 1 + k$, with $k \geq 1$ the number of cyclic irreducible subspaces for the Jordan decomposition of $\mathcal{AG}(f)$ relative to df_{e_1}.*

Proof. Let $v \in \mathcal{AG}(f) \cap (\mathbb{C}^n - T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n)$. Then there exists an automorphism Φ which fixes e_1 and such that $d\Phi_{e_1}e_1 = \lambda v$, with $\lambda \neq 0$. Up to conjugating f with Φ we can therefore suppose that $e_1 \in \mathcal{AG}(f)$. By Theorem 2.5 it follows that $df_{e_1}(e_1) = \alpha e_1 + w$, with $\langle w, e_1\rangle = 0$. Let $r := \|w\|$. There exists a unitary $(n-1) \times (n-1)$-matrix H such that, if we set

$$U = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix}$$

then $Uw = re_2$ and, clearly, $Ue_1 = e_1$. Therefore up to conjugating f with U we can suppose $df_{e_1}(e_1) = \alpha e_1 + re_2$. The double conjugation with U and Φ (in the right order) will be the automorphism T in the statement. Since it is clear how the objects involved change passing from f to $T^{-1} \circ f \circ T$ (or see [3]) we assume for the rest of the proof that $e_1 \in \mathcal{AG}(f)$ and $df_{e_1}(e_1) = \alpha e_1 + re_2$ with $r > 0$ (*i.e.* we prove the assertions assuming $dT_{e_1} = Id$). Let η_β be a parabolic automorphism of the family (5.1) such that $s := -r/\alpha < 0$. Let $g := \eta_\beta \circ f$. Then g is a linear fractional map of the unit ball such that $g(e_1) = e_1$ and, looking at $d(\eta_\beta)_{e_1}$—see equation (5.2)—such that $dg_{e_1}(e_1) = \alpha e_1$. Therefore $\mathbb{C}e_1 \cap \mathbb{B}^n$ is a cut complex geodesic for g. Since we only needed conditions on $\Re e\beta$ for picking up η_β, we can choose $\Im m\beta$ in such a way that $g(z_1, 0, \dots, 0) \neq (z_1, 0, \dots, 0)$ (this is always the case if $\alpha < 1$). In this case $\mathrm{Fix}(g) = \emptyset$. Indeed if there were a fixed point in $\mathbb{B}^n$, then g would be the identity once restricted to the slice G joining e_1 and such a point. But then the dynamical behavior of g on G would be different to that on $\mathbb{C}e_1 \cap \mathbb{B}^n$, where the iterates of g form a compactly divergent sequence (see [1]). Therefore e_1 is the Wolff point of g and (by Theorem 2.5) α is its boundary dilatation coefficient. Now $d(\eta_\beta)_{e_1}$ is the identity on $T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$ (see equation (5.2)), and therefore $dg_{e_1} = df_{e_1}$ on $T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$. Since $T^{\mathbb{C}}_{e_1}\partial\mathbb{B}^n$ is dg_{e_1} (and df_{e_1})-invariant, then all the remaining assertions follow easily from this. □

Remark 5.3. (1) The previous proposition allows a classification of linear fractional maps with trivial inner space based on the classification given in the previous section for maps with non-trivial inner space. More precisely, let f, g be as in Proposition 5.2. One says that f is *of hyperbolic, parabolic automorphism* or *parabolic non-automorphism* type according to the type of g.

(2) Not all the linear fractional map of $\mathbb{B}^n$with non-trivial inner space give rise after composition with a η_β to a linear fractional map of $\mathbb{B}^n$ with trivial inner space. For example, if g is a linear fractional map of $\mathbb{B}^n$ with Wolff point e_1 and $\mathbb{A}(g)$ has dimension one then $\eta_\beta^{-1} \circ g$ has no chances to have trivial inner space, for Property 7 of Proposition 5.2 would imply that $\mathbb{A}(g)$ has dimension strictly greater than one.

6. Applications to Composition Operators

In this section we use the previous results to obtain information about the properties of composition operators whose symbols are linear fractional maps.

Let σ be the *rotation-invariant* positive Borel measure on $\partial\mathbb{B}^n$ for which $\sigma(\partial\mathbb{B}^n) = 1$. We say that a holomorphic map h defined on $\mathbb{B}^n$ belongs to the Hardy space $H^2(\mathbb{B}^n)$ provided that

$$\sup_{0<r<1} \int_{\partial\mathbb{B}^n} |h_r|^2 d\sigma < \infty,$$

where $h_r(z) := h(rz)$. The space $H^2(\mathbb{B}^n)$ is a Hilbert space. We refer the reader to [6] for the properties of Hardy spaces.

The *composition operator* with symbol $f \in \mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$ is the operator C_f on $H^2(\mathbb{B}^n)$ defined as

$$C_f(h) := h \circ f, \quad \text{for } h \in H^2(\mathbb{B}^n).$$

In general for $n > 1$ a composition operator is not bounded as an operator from $H^2(\mathbb{B}^n)$ into itself. However if f is a linear fractional map then Cowen and MacCluer showed that $C_f : H^2(\mathbb{B}^n) \to H^2(\mathbb{B}^n)$ is a bounded operator (see *Thm. 19* in [4]). Before going ahead we need an operative formula for the adjoint of C_f (see [4]).

Definition 6.1. If f is a linear fractional map of $\mathbb{B}^n$ of the form (2.1) then the *adjoint map* f^* is given by

$$f^*(z) := \frac{A^*z - C}{\langle z, -\overline{B}\rangle + \overline{d}},$$

where A^* is the adjoint matrix of A.

Using the adjoint map we have the following formula for the (Hilbert space) adjoint of C_f:

Lemma 6.2 (Cowen-MacCluer). *If f is a linear fractional map of the form (2.1), let*

$$h(z) := (\langle z, \overline{C}\rangle + d)^n,$$
$$g(z) := \frac{1}{(\langle z, -\overline{B}\rangle + \overline{d})^n}.$$

Let T_h, T_g be the multiplication operators in $H^2(\mathbb{B}^n)$ associated to h and g. Then the adjoint operator C_f^* of $C_f : H^2(\mathbb{B}^n) \to H^2(\mathbb{B}^n)$ is given by:

$$C_f^* := T_g \circ C_{f^*} \circ T_h^*.$$

For using the adjoint map with some profit we also need the following:

Lemma 6.3. *Let f be a linear fractional map of $\mathbb{B}^n$ and let $\gamma \in \operatorname{Aut}(\mathbb{B}^n)$. Then $(\gamma^{-1} \circ f \circ \gamma)^* = \gamma^{-1} \circ f^* \circ \gamma$.*

Proof. It is easy to see that $(\gamma^{-1} \circ f \circ \gamma)^* = \gamma^* \circ f^* \circ (\gamma^{-1})^*$. Therefore we only need $\gamma^* = \gamma^{-1}$ for $\gamma \in \operatorname{Aut}(\mathbb{B}^n)$. This follows from $M_\gamma M_{\gamma^*} = I$. □

Recall that an operator $T : H^2(\mathbb{B}^n) \to H^2(\mathbb{B}^n)$ is said to be *cyclic* if there exists $h \in H^2(\mathbb{B}^n)$, a *cyclic vector* for T, such that $\{p(T)h : p \text{ polynomial}\}$ is dense in $H^2(\mathbb{B}^n)$. Moreover T is called *hypercyclic* if the set $\{T^n h : n \in \mathbb{N}\}$ is dense in $H^2(\mathbb{B}^n)$. Our first result is about the (non-)cyclicity of composition operators whose symbols have more than two fixed points in the closed ball:

Theorem 6.4. *Let f be a linear fractional map of the unit ball. If f has more than two fixed points in $\overline{\mathbb{B}^n}$ then C_f is not cyclic.*

Proof. By Theorem 3.4 f is conjugate to a map of the form

$$z \mapsto (z_1, \dots, z_q, A_q z^{(q)}), \tag{6.1}$$

for $q > 0$. Since conjugation by invertible operators does not affect the cyclicity of an operator we can assume f to be of the form (6.1). Let

$$\mathcal{L}_q := \{h \in H^2(\mathbb{B}^n) : h \text{ depends only on } z_1, \dots z_q\}. \tag{6.2}$$

Obviously $\mathcal{L}_q$ is a closed subspace of $H^2(\mathbb{B}^n)$ and $C_f(\mathcal{L}_q) = \mathcal{L}_q$. If we prove that $C_f(\mathcal{L}_q^\perp) \subseteq \mathcal{L}_q^\perp$ then C_f is not cyclic, for if h is a cyclic vector for C_f then its projection on $\mathcal{L}_q$ must be a cyclic vector for $C_{f|\mathcal{L}_q}$, but $C_{f|\mathcal{L}_q} = Id_{|\mathcal{L}_q}$. To show this it is enough to prove that $\mathcal{L}_q$ is C_f^*-invariant. Now applying Lemma (6.2) we find $C_f^* = C_{f^*}$, where $f^*(z) = (z_1, \dots, z_q, A_q^* z^{(q)})$. Therefore $C_f^*(\mathcal{L}_q) = \mathcal{L}_q$, and we have the assertion. □

Remark 6.5. By Lemma 6.3 it follows from the previous proof that if f has more than two fixed points in $\overline{\mathbb{B}^n}$ then the adjoint operator $(C_f)^*$ is itself a composition operator. This again implies that it has at least one eigenvalue and hence C_f cannot be hypercyclic (see [2]).

Reasoning as in [2] *p.18* one can show that if the symbol f is not injective in $\mathbb{B}^n$ then C_f cannot be cyclic. As we pointed out in Remark 2.8 a linear fractional map is injective if and only if its differential is invertible everywhere. Keeping this in mind we can investigate the cyclicity of other kind of linear fractional maps:

Theorem 6.6. *Let f be a linear fractional map of $\mathbb{B}^n$ with exactly two boundary fixed points. Then C_f is hypercyclic if and only if df is invertible at one—and hence any—point of $\overline{\mathbb{B}^n}$.*

Proof. By the previous remark f is injective if and only if df is invertible at one -and hence any- point. Therefore if df is not invertible somewhere then f is not cyclic (and hypercyclic). On the other hand suppose df is invertible everywhere, implying that f is injective (see Remark 2.8). By Theorem 3.4 we can assume that up to conjugation f has the form

$$f(z_1, z') = \left(\frac{\cosh t z_1 + \sinh t}{\sinh t z_1 + \cosh t}, \frac{A_1 z'}{\sinh t z_1 + \cosh t} \right),$$

for $t > 0$ and $\|A_1\| \leq 1$ (if $t < 0$ then just change e_1 with $-e_1$ in the following reasoning). It is easy to see that f is injective if and only if A_1 is invertible. Let $B := A_1^{-1}$ and

$$g(z_1, z') := \left(\frac{\cosh t z_1 - \sinh t}{-\sinh t z_1 + \cosh t}, \frac{B z'}{-\sinh t z_1 + \cosh t} \right).$$

Notice that in general g is not a self-map of $\mathbb{B}^n$ (it is so if and only if f is a hyperbolic automorphism, *i.e.* $\|A_1\| = 1$). We want to use the *hypercyclicity criterion* (see [7] for a proof). In our case this criterion states: if there exist X, Y two dense subsets of $H^2(\mathbb{B}^n)$ and an operator $S : Y \to Y$ such that $C_f^k \to 0$ on X, $S^k \to 0$ on Y and $C_f \circ S = \mathrm{id}_Y$ then C_f is hypercyclic. Let

$$\mathcal{D}_p := \{h \text{ holomorphic in } \mathbb{C}^n : h(p) = 0\}.$$

We set $X := \mathcal{D}_{e_1}$, $Y := \mathcal{D}_{-e_1}$ and $S := C_g$. It is clear that $\mathcal{D}_{e_1}$ is dense in $H^2(\mathbb{B}^n)$, for $\{1 - z_1^k, z_2^k, \ldots, z_n^k\}_{k \in \mathbb{N}}$ are in $\mathcal{D}_{e_1}$ and then $1, z_1$ are in its closure. Similarly $\mathcal{D}_{-e_1}$ is dense in $H^2(\mathbb{B}^n)$. The operator C_g maps $\mathcal{D}_{-e_1}$ into itself since $g(-e_1) = -e_1$. Moreover $C_f \circ C_g = \mathrm{id}_Y$ for $f \circ g(z) = z$ for $z \in \mathbb{B}^n$ and $C_f^k \to 0$ on $\mathcal{D}_{e_1}$ since $f^k \to e_1$ uniformly on compact subsets of the unit ball. Now we want to show that also $g^k \to -e_1$ uniformly on compact subsets of the unit ball, proving that $C_g^k \to 0$ on $\mathcal{D}_{-e_1}$, which is the last condition for the hypercyclicity criterion. A straightforward calculation shows that

$$g^k(z) = \left(\frac{\cosh(2^{k-1}t) z_1 - \sinh(2^{k-1}t)}{\cosh(2^{k-1}t) - \sinh(2^{k-1}t) z_1}, \frac{B^k z'}{\cosh(2^{k-1}t) - \sinh(2^{k-1}t) z_1} \right)$$

The first component of $g^k(z)$ is easily seen going to -1 (also because $g_1(z) = g_1(z_1)$ is the inverse of $f_1(z_1)$). We want to see that the last $n-1$ components tend to zero as k tends to ∞. For $|z_1|^2 + \|z'\|^2 < 1$ we have

$$\begin{aligned} \frac{\|B^k z'\|^2}{|\cosh(2^{k-1}t) - \sinh(2^{k-1}t) z_1|^2} &< \frac{\|B\|^{2k}(1 - |z_1|)^2}{|\cosh(2^{k-1}t) - \sinh(2^{k-1}t) z_1|^2} \\ &\leq \frac{\|B\|^{2k}}{\frac{e^{2^{k-1}t}}{2}(1 - |z_1|) - e^{-2^{k-1}t}} \to 0, \end{aligned}$$

as $k \to \infty$. $\square$

Remark 6.7. Let f be a linear fractional map of $\mathbb{B}^n$ with exactly two boundary fixed points and df invertible somewhere. Then C_f is hypercyclic on $H^2(\mathbb{B}^n)$. However the closed subspace $\mathcal{L}_1 := \{h \in H^2(\mathbb{B}^n) : h \text{ depends only on } z_1\}$ is a infinite dimensional C_f-invariant space.

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