HOLOMORPHIC DEPENDENCE OF THE HEINS MAP

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ABSTRACT. Let D be a bounded convex domain and $\mathrm{Hol}_c(D,D)$ the set of holomorphic maps from D to $\mathbb{C}^n$ with image relatively compact in D. Consider $\mathrm{Hol}_c(D,D)$ as a open set in the complex Banach space $H^\infty_n(D)$ of bounded holomorphic maps from D to $\mathbb{C}^n$. We show that the map $\tau : \mathrm{Hol}_c(D,D) \to D$ (called the *Heins map* for D equals to the unit disc of $\mathbb{C}$) which associates to $f \in \mathrm{Hol}_c(D,D)$ its unique fixed point $\tau(f) \in D$ is holomorphic and its differential is given by $d\tau_f(v) = (Id - df_{\tau(f)})^{-1} v(\tau(f))$ for $v \in H^\infty_n(D)$.

1. Introduction

Let $\Delta : \{z \in \mathbb{C} : |z| < 1\}$. By the Schwarz Lemma and the Wolff-Denjoy Theorem (see, *e.g.*, [Ab]) for any $f \in \mathrm{Hol}(\Delta,\Delta)$, $f \neq Id$, there exists a distinguished point $\tau(f) \in \overline{\Delta}$ such that the sequence of iterates of f converges uniformly on compacta to the constant map $z \mapsto \tau(f)$. If $\tau(f) \in \Delta$ then $\tau(f)$ is the only fixed point of f (in Δ). In 1941 M. H. Heins [He] proved that the application—that we call *the Heins map*

$$\tau : \overline{\mathrm{Hol}(\Delta,\Delta)} - \{Id\} \to \overline{\Delta}$$

which maps $f \in \mathrm{Hol}(\Delta,\Delta)$ to $\tau(f)$ and $z \mapsto c \in \partial\Delta$ to c, is continuous when $\mathrm{Hol}(\Delta,\Delta)$ is endowed with the topology of the uniform convergence on compacta (which turns out to be equivalent to the topology induced by $H^\infty(\Delta)$). In 2000 J. E. Joseph and M. H. Kwack [JK] generalized Heins' result to holomorphic self-maps of bounded strongly convex domains with smooth boundary (see also [Br]). In this paper we deal with other properties of regularity of the Heins map (in the unit disc) and its generalization to (not necessarily strongly) convex domains in $\mathbb{C}^n$. Let $D \subset\subset \mathbb{C}^n$ be a convex domain. Let $H^\infty(D)$ be the Banach space of bounded holomorphic function on D (with values in $\mathbb{C}$) and norm given by the supremum norm. We denote by $H^\infty_n(D) := \oplus_n H^\infty(D)$ the Banach space of bounded holomorphic maps from D to $\mathbb{C}^n$. The set

$$\mathrm{Hol}_c(D,D) := \{f \in H^\infty_n(D) : f(D) \subset\subset D\}$$

is a open set in $H^\infty_n(D)$, dense in the closed set $\overline{\mathrm{Hol}(D,D)}$. A result of J.-P. Vigué [Vi] implies that if $f \in \mathrm{Hol}_c(D,D)$ then there exists a unique point $\tau(f) \in D$ such that $f(\tau(f)) = \tau(f)$. The main result of this paper is the following:

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Theorem 1.1. *The map $\tau : Hol_c(D, D) \to D$, defined as $\tau : f \to \tau(f)$, is holomorphic. The differential of τ at $f \in Hol_c(D, D)$ for $v \in H_n^\infty(D)$ is given by*

$$d\tau_f(v) = (Id - df_{\tau(f)})^{-1} v(\tau(f)).$$

Note that $\|df_{\tau(f)}\| < 1$ as a consequence of the *Cartan-Carathéodory Theorem* (see, *e.g.*, [Ab]) and therefore $I - df_{\tau(f)}$ is invertible. The proof of Theorem 1.1 is based on the Cauchy formula and on the continuity of the map $\tau : \mathrm{Hol}_c(D, D) \to D$, which we prove for bounded (not necessarily strongly) convex domain. In the first section we prove the result for the *Heins map* $\tau : \mathrm{Hol}_c(\Delta, \Delta) \to \Delta$. In the second section we give the proof of Theorem 1.1.

2. The proof of Theorem 1.1 for the unit disc

In this section we give the proof of Theorem 1.1 in the case $D = \Delta$, the unit disc of $\mathbb{C}$. The general case is proved in the next section. The proof of the general case follows the same outline of the one-dimensional proof but the underlying ideas become less transparent as the dimension increases. Therefore we decided to single out here the one dimensional case.

Theorem 2.1. *The Heins map $\tau : Hol_c(\Delta, \Delta) \to \Delta$ is holomorphic. The differential of τ at $f \in Hol_c(\Delta, \Delta)$ for $v \in H^\infty(\Delta)$ is given by*

$$d\tau_f(v) = \frac{v(\tau(f))}{1 - f'(\tau(f))}. \tag{2.1}$$

Note that since $f \in \mathrm{Hol}_c(\Delta, \Delta)$ then by the Schwarz Lemma $|f'(\tau(f))| < 1$ and then the right hand side of equation (2.1) makes sense.

Proof. By a well known result τ is holomorphic if we show that it is complex differentiable at all points. Let then $f \in \mathrm{Hol}_c(\Delta, \Delta)$. We need to show that

$$\lim_{\|h\|_\infty \to 0} \frac{|\tau(f+h) - \tau(f) - \frac{h(\tau(f))}{1 - f'(\tau(f))}|}{\|h\|_\infty} = 0. \tag{2.2}$$

Let $h \in H^\infty(\Delta)$ so that $f + h \in \mathrm{Hol}_c(\Delta, \Delta)$. Therefore there exists $0 < r < 1$ such that $\tau(f), \tau(f+h) \in \Delta_r := \{z \in \Delta : |z| < r\}$. Since $(f+h)(\tau(f+h)) = \tau(f+h)$ and $f(\tau(f)) = \tau(f)$ we apply the Cauchy formula to get

$$\begin{aligned}
\tau(f+h) - \tau(f) &= \frac{1}{2\pi i} \int_{\partial\Delta_r} \left[\frac{(f+h)(\zeta)}{\zeta - \tau(f+h)} - \frac{f(\zeta)}{\zeta - \tau(f)} \right] d\zeta \\
= \frac{[\tau(f+h) - \tau(f)]}{2\pi i} &\int_{\partial\Delta_r} \frac{f(\zeta)}{[\zeta - \tau(f+h)][\zeta - \tau(f)]} d\zeta + \frac{1}{2\pi i} \int_{\partial\Delta_r} \frac{h(\zeta)}{\zeta - \tau(f+h)} d\zeta \\
&= \frac{[\tau(f+h) - \tau(f)]}{2\pi i} \int_{\partial\Delta_r} \frac{f(\zeta)}{[\zeta - \tau(f+h)][\zeta - \tau(f)]} d\zeta + h(\tau(f+h)).
\end{aligned}$$

¿From this we get

$$\tau(f+h) - \tau(f) = \frac{h(\tau(f+h))}{1 - \frac{1}{2\pi i} \int_{\partial\Delta_r} \frac{f(\zeta)}{[\zeta - \tau(f+h)][\zeta - \tau(f)]} d\zeta}.$$

Therefore

$$\left|\tau(f+h)-\tau(f)-\frac{h(\tau(f))}{1-f'(\tau(f))}\right| = \left|\frac{h(\tau(f+h))}{1-\frac{1}{2\pi i}\int_{\partial\Delta_r}\frac{f(\zeta)}{[\zeta-\tau(f+h)][\zeta-\tau(f)]}d\zeta}-\frac{h(\tau(f))}{1-f'(\tau(f))}\right|.$$

Let us denote

$$I_h := \frac{1}{2\pi i}\int_{\partial\Delta_r}\frac{f(\zeta)}{[\zeta-\tau(f+h)][\zeta-\tau(f)]}d\zeta,$$

and

$$I_0 := \frac{1}{2\pi i}\int_{\partial\Delta_r}\frac{f(\zeta)}{[\zeta-\tau(f)]^2}d\zeta.$$

Note that by the Cauchy formula we have $I_0 = f'(\tau(f))$, and that, since $\tau(f+h)\to\tau(f)$ as $\|h\|_\infty\to 0$ by [He], then $I_h\to I_0$ as $\|h\|_\infty\to 0$. Hence

$$\begin{aligned}&\left|\frac{h(\tau(f+h))}{1-I_h}-\frac{h(\tau(f))}{1-I_0}\right|\\ &\quad\le\left|\frac{h(\tau(f+h))}{1-I_h}-\frac{h(\tau(f+h))}{1-I_0}\right|+\left|\frac{h(\tau(f+h))}{1-I_0}-\frac{h(\tau(f))}{1-I_0}\right|\\ &\quad\le\|h\|_\infty\left|\frac{1}{1-I_h}-\frac{1}{1-I_0}\right|+\frac{1}{|1-I_0|}|h(\tau(f+h))-h(\tau(f))|\\ &\quad= o(\|h\|_\infty)+C|h(\tau(f+h))-h(\tau(f))|.\end{aligned}$$

Therefore to get equation (2.2) it is enough to prove that

$$|h(\tau(f+h))-h(\tau(f))| = o(\|h\|_\infty).$$

Again we use the Cauchy formula:

$$\begin{aligned}|h(\tau(f+h))-h(\tau(f))| &\le \frac{1}{2\pi}\int_{\partial\Delta_r}\left|\frac{h(\zeta)}{\zeta-\tau(f+h)}-\frac{h(\zeta)}{\zeta-\tau(f)}\right|d|\zeta|\\ &\le\frac{\|h\|_\infty|\tau(f+h)-\tau(f)|}{2\pi}\int_{\partial\Delta_r}\frac{d|\zeta|}{|\zeta-\tau(f+h)||\zeta-\tau(f)|}\\ &= C\|h\|_\infty|\tau(f+h)-\tau(f)| = o(\|h\|_\infty).\end{aligned}$$

And the assertion is proved. ☐

3. The proof of Theorem 1.1

First we need to prove the following result:

Lemma 3.1. *The map $\tau : Hol_c(D,D)\to D$ is continuous.*

Proof. Let $\{f_k\}\subset \mathrm{Hol}_c(D,D)$ be such that $f_k\to f\in\mathrm{Hol}_c(D,D)$. We have to show that $\tau(f_k)\to\tau(f)$. Let x be an accumulation point of $\{\tau(f_k)\}$. Since the limit f has image relatively compact in D then $x\in D$. Now let $\tau(f_{k_j})\to x$. Then

$$f(x)=\lim_{j\to\infty} f_{k_j}(\tau(f_{k_j}))=\lim_{j\to\infty}\tau(f_{k_j})=x.$$

Since by [Vi] the map f has only one fixed point in D then $\tau(f)=x$ and actually $\tau(f_k)\to x$. ☐

Let $f \in \mathrm{Hol}_c(D,D)$. Now let $h \in H_n^\infty(D)$ be such that $f+h \in \mathrm{Hol}_c(D,D)$. Let us denote by $p := \tau(f) \in D$ and $q := \tau(f+h) \in D$. In what follows if $x \in \mathbb{C}^n$ we write x_j for the j-th coordinate of x. Let P be a polydisc such that $\overline{P} \subset D$ and $p, q \in P$ (such a polydisc exists for $\|h\|_\infty$ small since τ is continuous by Lemma 3.1). Since $f(p) = p$ and $(f+h)(q) = q$ from the Cauchy formula we get

$$q - p = \frac{1}{(2\pi i)^n} \int_{\partial P} \left[\frac{f(\zeta)}{\zeta - q} - \frac{f(\zeta)}{\zeta - p} \right] d\zeta + h(q), \tag{3.1}$$

where, as usual, $(\zeta - x) = \prod_{j=1}^n (\zeta_j - x_j)$, $d\zeta = d\zeta_1 \wedge \ldots d\zeta_n$ and the integrals are vector valued. Let us examine the integrand in (3.1):

$$\frac{f(\zeta)}{\zeta - q} - \frac{f(\zeta)}{\zeta - p} = \frac{\left[\prod_{j=1}^n (\zeta_j - p_j) - \prod_{j=1}^n (\zeta_j - q_j) \right] f(\zeta)}{(\zeta - p)(\zeta - q)}.$$

For calculating this last expression we need to define some sets of indices. For $m \in \{0, \ldots, n\}$ let $I_m := \{(k_1, \ldots, k_m) \in \mathbb{N}^m | 1 \leq k_1 < \ldots < k_m \leq n\}$. If $\mathbf{k} \in I_m$ we say $j \notin \mathbf{k}$ if $\mathbf{k} = (k_1, \ldots, k_m)$ and $j \neq k_l$ for $l = 1, \ldots, m$. For $\mathbf{k} \in I_m$ we let $I'_{\mathbf{k}} := \{l_1, \ldots, l_{n-m} | l_j \notin \mathbf{k}\}$. Moreover we write $\zeta_{\mathbf{k}} := \zeta_{k_1} \cdots \zeta_{k_m}$. With this notation

$$\begin{aligned}
\prod_{j=1}^n (\zeta_j - p_j) - \prod_{j=1}^n (\zeta_j - q_j) &= \sum_{m=0}^n (-1)^{n-m} \sum_{\mathbf{k} \in I_m} \zeta_{\mathbf{k}} \Big(\prod_{j \in I'_{\mathbf{k}}} p_j - \prod_{j \in I'_{\mathbf{k}}} q_j \Big) \\
&= \sum_{m=0}^n (-1)^{n-m} \sum_{\mathbf{k} \in I_m} \zeta_{\mathbf{k}} \left(\sum_{r \in I'_{\mathbf{k}}} (p_r - q_r) \prod_{s \in I'_{\mathbf{k}}, s > r} p_s \prod_{t \in I'_{\mathbf{k}}, t < r} q_t \right) \\
&= \sum_{\mathbf{k} \in I_{n-1}, j \notin I_{\mathbf{k}}} \zeta_{\mathbf{k}} (q_j - p_j) + \sum_{m=0}^{n-2} (-1)^{n-m} \sum_{\mathbf{k} \in I_m} \zeta_{\mathbf{k}} \left(\sum_{r \in I'_{\mathbf{k}}} (p_r - q_r) \prod_{s \in I'_{\mathbf{k}}, s > r} p_s \prod_{t \in I'_{\mathbf{k}}, t < r} q_t \right) \\
&= \sum_{l=1}^n \zeta_1 \cdots \widehat{\zeta_l} \cdots \zeta_n (q_l - p_l) + \sum_{l=1}^n \tilde{C}_l^h (q_l - p_l),
\end{aligned} \tag{3.2}$$

where we set

$$\tilde{C}_l^h = \sum_{m=0}^{n-2} (-1)^{n-m+1} \sum_{\mathbf{k} \in I_m : l \notin \mathbf{k}} \zeta_{\mathbf{k}} \prod_{r \in I'_{\mathbf{k}}, r < l} p_r \prod_{s \in I'_{\mathbf{k}}, s > l} q_s.$$

Therefore for $j = 1, \ldots, n$

$$q_j - p_j = \sum_{l=1}^n B_{l,j}^h (q_l - p_l) + \sum_{l=1}^n C_{l,j}^h (q_l - p_l) + h_j(q),$$

where

$$B_{l,j}^h := \frac{1}{(2\pi i)^n} \int_{\partial P} \frac{\zeta_1 \cdots \widehat{\zeta_l} \cdots \zeta_n f_j(\zeta) d\zeta}{(\zeta - p)(\zeta - q)},$$

and

$$C_{l,j}^h := \frac{1}{(2\pi i)^n} \int_{\partial P} \frac{\tilde{C}_l^h f_j(\zeta) d\zeta}{(\zeta - p)(\zeta - q)}.$$

Note that the $C^h_{l,j}$'s are sums of terms of the type

$$p_{j_1}\dots p_{j_s} q_{l_1}\dots q_{l_t}\frac{1}{2\pi i}\int_{\partial P}\frac{\zeta_{r_1}\dots\zeta_{r_u} f_j(\zeta)d\zeta}{(\zeta-p)(\zeta-q)}.$$

Therefore by Lemma 3.1 and by the Cauchy formula for $\|h\|_\infty\to 0$ the $B^h_{l,j}$'s and $C^h_{l,j}$'s tend to sums of terms of the form

$$p_{j_1}\dots p_{j_s}\frac{\partial^m f_j(p)}{\partial\zeta_{k_1}\dots\partial\zeta_{k_r}}.$$

Moreover a careful hunting from the very definition of $B^h_{l,j}$ and $C^h_{l,j}$ shows that for $\|h\|_\infty\to 0$

$$B^h_{l,j}+C^h_{l,j}\to\frac{\partial f_j(p)}{\partial\zeta_l}.$$

Let $A^h_{l,j}:=B^h_{l,j}+C^h_{l,j}$ and let A^h be the $n\times n$ matrix with entries given by $A^h_{l,j}$. Then

$$(Id-A^h)(q-p)=h(q). \tag{3.3}$$

Moreover, since $A^h\to df_p$ as $\|h\|_\infty\to 0$ and $\|df_p\|<1$ by the Cartan-Carathéodory Theorem (see, *e.g.*, [Ab]), for $\|h\|_\infty$ small $(Id-A^h)$ is invertible. Hence

$$(q-p)=(Id-A^h)^{-1}h(q).$$

Therefore

$$\lim_{\|h\|_\infty\to 0}\frac{\|\tau(f+h)-\tau(f)\|}{\|h\|_\infty}=C<\infty. \tag{3.4}$$

It is well known that in order to prove that τ is holomorphic on $\mathrm{Hol}_c(D,D)$ we need only to show that it is complex differentiable at each point. For this—and to get the claimed form for $d\tau$—we need to show that

$$\lim_{\|h\|_\infty\to 0}\frac{\|\tau(f+h)-\tau(f)-(Id-df_{\tau(f)})^{-1}h(\tau(f))\|}{\|h\|_\infty}=0.$$

Since $Id-df_{\tau(f)}$ is injective, this is clearly equivalent to

$$\lim_{\|h\|_\infty\to 0}\frac{\|(Id-df_{\tau(f)})(\tau(f+h)\tau(f))-h(\tau(f))\|}{\|h\|_\infty}=0.$$

By (3.3) we get

$$\begin{aligned}&\frac{\|(Id-df_{\tau(f)})(\tau(f+h)-\tau(f))-h(\tau(f))\|}{\|h\|_\infty}\\ &\qquad=\frac{\|[(A^h-df_{\tau(f)})+(Id-A^h)](\tau(f+h)-\tau(f))-h(\tau(f))\|}{\|h\|_\infty}\\ &\qquad\quad\le\|A^h-df_{\tau(f)}\|\frac{\|\tau(f+h)-\tau(f))\|}{\|h\|_\infty}+\frac{\|h(\tau(f+h))-h(\tau(f))\|}{\|h\|_\infty}.\end{aligned}$$

By (3.4) and since $A^h\to df_{\tau(f)}$ it follows that the first summand in the last line of the previous equation goes to zero as $\|h\|_\infty\to 0$. It remains to show that

$$\|h(\tau(f+h))-h(\tau(f))\|=o(\|h\|_\infty). \tag{3.5}$$

Again by the Cauchy formula, and arguing as in (3.2), we have

$$\begin{aligned}\|h(\tau(f+h)) - h(\tau(f))\| &\leq \frac{\|h\|_\infty}{2\pi} \int_{\partial P} \left| \frac{1}{(\zeta - \tau(f))} - \frac{1}{(\zeta - \tau(f+h))} \right| d|\zeta| \\ &= \|h\|_\infty \sum_{l=1}^{n} |\tau(f+h)_l - \tau(f)_l| C_l \leq \|h\|_\infty \|\tau(f+h) - \tau(f)\| C,\end{aligned}$$

with C_l's, C bounded for $\|h\|_\infty \to 0$. By Lemma 3.1 this proves (3.5) and we are done.

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