To appear in *Annals of Mathematics*

Index theorems for holomorphic self-maps

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0. Introduction

The usual index theorems for holomorphic self-maps, like for instance the classical holomorphic Lefschetz theorem (see, e.g., [GH]), assume that the fixed points set contains only isolated points. The aim of this paper, on the contrary, is to prove index theorems for holomorphic self-maps having a positive dimensional fixed points set.

The origin of our interest in this problem lies in holomorphic dynamics. A main tool for the complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity achieved in [A2] was an index theorem for holomorphic self-maps of a complex surface fixing pointwise a smooth complex curve S. This theorem (later generalized in [BT] to the case of a singular S) presented uncanny similarities with the Camacho-Sad index theorem for invariant leaves of a holomorphic foliation on a complex surface (see [CS]). So we started to investigate the reasons for these similarities; and this paper contains what we have found.

The main idea is that the simple fact of being pointwise fixed by a holomorphic self-map f induces a lot of structure on a (possibly singular) subvariety S of a complex manifold M. First of all, we shall introduce (in Section 3) a canonically defined holomorphic section X_f of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$, where N_S is the normal bundle of S in M (here we are assuming S smooth; however, we can also define X_f as a section of a suitable sheaf even when S is singular — see Remark 3.3 — but it turns out that only the behavior on the regular part of S is relevant for our index theorems), and ν_f is a positive integer, the *order of contact* of f with S, measuring how close f is to be the identity in a neighborhood S (see Section 1). Roughly speaking, the section X_f describes the directions in which S is pushed by f; see Proposition 8.1 for a more precise description of this phenomenon when S is a hypersurface.

The canonical section X_f can also be seen as a morphism from $N_S^{\otimes \nu_f}$ to $TM|_S$; its image Ξ_f is the *canonical distribution.* When Ξ_f is contained in TS (we shall say that f is *tangential*) and integrable (this happens for instance if S is a hypersurface), then on S we get a singular holomorphic foliation induced by f — and this is a first concrete connection between our discrete dynamical theory and the continuous dynamics studied in foliation theory. We stress, however, that we get a well-defined foliation on S *only,* while in the continuous setting one usually assumes that S is invariant under a foliation defined in a *whole neighborhood* of S. Thus even in the tangential codimension one case our results will not be a direct consequence of foliation theory.

As we shall momentarily discuss, to get index theorems it is important to have a section of $TS \otimes (N_S^*)^{\otimes \nu_f}$ (as it is the case when f is tangential) instead of merely a section of $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$. In Section 3, when f is not tangential (which is a situation akin to dicriticality for foliations; see Propositions 1.4 and 8.1) we shall define other holomorphic sections $H_{\sigma,f}$ and $H^1_{\sigma,f}$ of $TS \otimes (N_S^*)^{\otimes \nu_f}$ which are as good as X_f when S satisfies a geometric condition which we call to be *comfortably embedded* in M, meaning, roughly speaking, that S is a first-order approximation of the zero section of a vector bundle (see Section 2 for the precise definition, amounting to the vanishing of two sheaf cohomology classes — or, in other terms, to the triviality of two canonical extensions of N_S).

The canonical section is not the only object we are able to associate to S. Having a section X of $TS \otimes F^*$, where F is any vector bundle on S, is equivalent to having an F^*-valued derivation $X^\#$ of the sheaf of holomorphic functions $\mathcal{O}_S$ (see Section 5). If E is another vector bundle on S, a *holomorphic action of F on E along* X is a morphism $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ (where $\mathcal{E}$ and $\mathcal{F}$ are the sheafs of germs of holomorphic sections of E and F) satisfying $\tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s)$ for any $g \in \mathcal{O}_S$ and $s \in E$; it is a generalization of the notion of $(1,0)$-connection on E (see Example 5.1).

In Section 5 we shall show that when S is a hypersurface and f is tangential (or S is comfortably embedded in M) there is a natural way to define a holomorphic action of $N_S^{\otimes\nu_f}$ on N_S along X_f (or along $H_{\sigma,f}$ or $H^1_{\sigma,f}$). And this will allow us to bring into play the general theory developed by Lehmann and Suwa (see, e.g., [Su]) on a cohomological approach to index theorems. So, exactly as Lehmann and Suwa generalized to any dimension the Camacho-Sad index theorem, we are able to generalize the index theorems of [A2] and [BT] in the following form (see Theorem 6.2):

Theorem 0.1: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f\colon M \to M$, $f \not\equiv \mathrm{id}_M$, be a holomorphic self-map of M fixing pointwise S, and denote by $\mathrm{Sing}(f)$ the zero set of X_f. Assume that*

(a) *f is tangential to S, and then set $X = X_f$, or that*
(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into M, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

Assume moreover $X \not\equiv O$ (condition always satisfied when f is tangential), and denote by $\mathrm{Sing}(X)$ the zero set of X. Let $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ in connected components. Finally, let $[S]$ be the line bundle on M associated to the divisor S. Then there exist complex numbers $\mathrm{Res}(X, S, \Sigma_\lambda) \in \mathbb{C}$ depending only on the local behavior of X and $[S]$ near Σ_λ such that

$$\sum_\lambda \mathrm{Res}(X, S, \Sigma_\lambda) = \int_S c_1^{n-1}([S]),$$

where $c_1([S])$ is the first Chern class of $[S]$.

Furthermore, when Σ_λ is an isolated point $\{x_\lambda\}$, we have explicit formulas for the computation of the residues $\mathrm{Res}(X, S, \{x_\lambda\})$; see Theorem 6.5.

Since X is a global section of $TS \otimes (N_S^*)^{\otimes\nu_f}$, if S is smooth and X has only isolated zeroes it is well-known that the top Chern class $c_{n-1}\big(TS \otimes (N_S^*)^{\otimes\nu_f}\big)$ counts the zeroes of X. Our result shows that $c_1^{n-1}(N_S)$ is related in a similar (but deeper) way to the zero set of X. See also Section 8 for examples of results one can obtain using both Chern classes together.

If the codimension of S is greater than one, and S is smooth, we can blow-up M along S; then the exceptional divisor E_S is a hypersurface, and we can apply to it the previous theorem. In this way we get (see Theorem 7.2):

Theorem 0.2: *Let S be a compact complex submanifold of codimension $1 < m < n$ in an n-dimensional complex manifold M. Let $f\colon M \to M$, $f \not\equiv \mathrm{id}_M$, be a holomorphic self-map of M fixing pointwise S, and assume that f is tangential, or that $\nu_f > 1$ (or both). Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions (see Section 7 for the definition) for f in E_S. Then there exist complex numbers $\mathrm{Res}(f, S, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of f and S near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}(f, S, \Sigma_\lambda) = \int_S \pi_* c_1^{n-1}([E_S]),$$

where π_* denotes the integration along the fibers of the bundle $E_S \to S$.

Theorems 0.1 and 0.2 are only two of the index theorems we can derive using this approach. Indeed, we are also able to obtain versions for holomorphic self-maps of two other main index theorems of foliation theory, the Baum-Bott index theorem and the Lehmann-Suwa-Khanedani (or variation) index theorem: see Theorems 6.3, 6.4, 6.6, 7.3 and 7.4. In other words, it turns out that the existence of holomorphic actions of suitable complex vector bundles defined only on S is an efficient tool to get index theorems, both in our setting and (under slightly different assumptions) in foliation theory; and this is another reason for the similarities noticed in [A2].

Finally, in Section 8 we shall present a couple of applications of our results to the discrete dynamics of holomorphic self-maps of complex surfaces, thus closing the circle and coming back to the arguments that originally inspired our work.

1. The order of contact

Let M be an n-dimensional complex manifold, and $S \subset M$ an irreducible subvariety of codimension m. We shall denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M, and by $\mathcal{I}_S$ the subsheaf of functions vanishing on S. With a slight abuse of notations, we shall use the same symbol to denote both a germ at p and any representative defined in a neighborhood of p. We shall denote by TM the holomorphic tangent bundle of M, and by $\mathcal{T}_M$ the sheaf of germs of local holomorphic sections of TM. Finally, we shall denote by $\mathrm{End}(M,S)$ the set of (germs about S of) holomorphic self-maps of M fixing S pointwise.

Let $f \in \mathrm{End}(M,S)$ be given, $f \not\equiv \mathrm{id}_M$, and take $p \in S$. For every $h \in \mathcal{O}_{M,p}$ the germ $h\circ f$ is well-defined, and we have $h \circ f - h \in \mathcal{I}_{S,p}$.

Definition 1.1: The *f-order of vanishing* at p of $h \in \mathcal{O}_{M,p}$ is given by

$$\nu_f(h;p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}^{\mu}_{S,p}\},$$

and the *order of contact* $\nu_f(p)$ of f at p with S by

$$\nu_f(p) = \min\{\nu_f(h;p) \mid h \in \mathcal{O}_{M,p}\};$$

we shall momentarily prove that $\nu_f(p)$ does not depend on p.

Let $(z^1,\ldots,z^n)$ be local coordinates in a neighborhood of p. If h is any holomorphic function defined in a neighborhood of p, setting $f^j = z^j \circ f$ the definition of order of contact yields the important relation

$$h \circ f - h = \sum_{j=1}^{n} (f^j - z^j)\,\frac{\partial h}{\partial z^j} \qquad (\mathrm{mod}\ \mathcal{I}_{S,p}^{2\nu_f(p)}). \tag{1.1}$$

As a consequence we have

Lemma 1.1: (i) *Let $(z^1,\ldots,z^n)$ be any set of local coordinates at $p \in S$. Then*

$$\nu_f(p) = \min_{j=1,\ldots,n}\{\nu_f(z^j;p)\}.$$

(ii) *For any $h \in \mathcal{O}_{M,p}$ the function $p \mapsto \nu_f(h;p)$ is constant in a neighborhood of p.*
(iii) *The function $p \mapsto \nu_f(p)$ is constant.*

Proof: (i) Clearly, $\nu_f(p) \le \min_{j=1,\ldots,n}\{\nu_f(z^j;p)\}$. The opposite inequality follows from (1.1).

(ii) Let $h \in \mathcal{O}_{M,p}$, and choose a set $\{\ell^1,\ldots,\ell^k\}$ of generators of $\mathcal{I}_{S,p}$. Then we can write

$$h \circ f - h = \sum_{|I| = \nu_f(h;p)} \ell^I g_I, \tag{1.2}$$

where $I = (i_1,\ldots,i_k) \in \mathbb{N}^k$ is a k-multi-index, $|I| = i_1 + \cdots + i_k$, $\ell^I = (\ell^1)^{i_1}\cdots(\ell^k)^{i_k}$ and $g_I \in \mathcal{O}_{M,p}$. Furthermore, there is a multi-index I_0 such that $g_{I_0} \notin \mathcal{I}_{S,p}$. By the coherence of the sheaf of ideals of S, the relation (1.2) holds for the corresponding germs at all points $q \in S$ in a neighborhood of p. Furthermore, $g_{I_0} \notin \mathcal{I}_{S,p}$ means that $g_{I_0}|_S \not\equiv 0$ in a neighborhood of p, and thus $g_{I_0} \notin \mathcal{I}_{S,q}$ for all $q \in S$ close enough to p. Putting these two observations together we get the assertion.

(iii) By (i) and (ii) we see that the function $p \mapsto \nu_f(p)$ is locally constant. Being S connected, it is constant everywhere. □

We shall then denote by ν_f the *order of contact* of f with S, computed at any point $p \in S$.

As we shall see, it is important to compare the order of contact of f with the f-order of vanishing of germs in $\mathcal{I}_{S,p}$.

Definition 1.2: We say that f is *tangential* at p if

$$\min\{\nu_f(h;p) \mid h \in \mathcal{I}_{S,p}\} > \nu_f.$$

Lemma 1.2: *Let $\{\ell^1, \dots, \ell^k\}$ be a set of generators of $\mathcal{I}_{S,p}$. Then*

$$\forall h \in \mathcal{I}_{S,p} \qquad \nu_f(h;p) \geq \min\{\nu_f(\ell^1;p), \dots, \nu_f(\ell^k;p), \nu_f + 1\}.$$

In particular, f is tangential at p if and only if

$$\min\{\nu_f(\ell^1;p), \dots, \nu_f(\ell^k;p)\} > \nu_f.$$

Proof: Let us write $h = g_1\ell^1 + \cdots + g_k\ell^k$ for suitable $g_1, \dots, g_k \in \mathcal{O}_{M,p}$. Then

$$h \circ f - h = \sum_{j=1}^{k} \big[(g_j \circ f)(\ell^j \circ f - \ell^j) + (g_j \circ f - g_j)\ell^j\big],$$

and the assertion follows. □

Corollary 1.3: *If f is tangential at one point $p \in S$, then it is tangential at all points of S.*

Proof: The coherence of the sheaf of ideals of S implies that if $\{\ell^1, \dots, \ell^k\}$ are generators of $\mathcal{I}_{S,p}$ then the corresponding germs are generators of $\mathcal{I}_{S,q}$ for all $q \in S$ close enough to p. Then Lemmas 1.1.(ii) and 1.2 imply that both the set of points where f is tangential and the set of points where f is not tangential are open; hence the assertion follows because S is connected. □

Of course, we shall then say that f is *tangential* along S if it is tangential at any point of S.

EXAMPLE 1.1: Let p be a smooth point of S, and choose local coordinates $z = (z^1, \dots, z^n)$ defined in a neighborhood U of p, centered at p and such that $S \cap U = \{z^1 = \cdots = z^m = 0\}$. We shall write $z' = (z^1, \dots, z^m)$ and $z'' = (z^{m+1}, \dots, z^n)$, so that z'' yields local coordinates on S. Take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$; then in local coordinates the map f can be written as $(f^1, \dots, f^n)$ with

$$f^j(z) = z^j + \sum_{h \geq 1} P_h^j(z', z''),$$

where each P_h^j is a homogeneous polynomial of degree h in the variables z', with coefficients depending holomorphically on z''. Then Lemma 1.1 yields

$$\nu_f = \min\{h \geq 1 \mid \exists\, 1 \leq j \leq n : P_h^j \not\equiv 0\}.$$

Furthermore, $\{z^1, \dots, z^m\}$ is a set of generators of $\mathcal{I}_{S,p}$; therefore by Lemma 1.2 the map f is tangential if and only if

$$\min\{h \geq 1 \mid \exists\, 1 \leq j \leq m : P_h^j \not\equiv 0\} > \min\{h \geq 1 \mid \exists\, m+1 \leq j \leq n : P_h^j \not\equiv 0\}.$$

Remark 1.1: When S is smooth, the differential of f acts linearly on the normal bundle N_S of S in M. If S is an hypersurface, N_S is a line bundle, and the action is the multiplication by a holomorphic function b; if S is compact, this function is a constant. It is easy to check that in local coordinates chosen as in the previous example the expression of the function b is exactly $1 + P_1^1(z)/z^1$; therefore we must have $P_1^1(z) = (b_f - 1)z^1$ for a suitable constant $b_f \in \mathbb{C}$. In particular, if $b_f \neq 1$ then necessarily $\nu_f = 1$ and f is not tangential along S.

Remark 1.2: The number μ introduced in [BT, (2)] is, by Lemma 1.1, our order of contact; therefore our notion of tangential is equivalent to the notion of non-degeneracy defined in [BT] when $n = 2$ and $m = 1$.

On the other hand, as already remarked in [BT], a non-degenerate map in the sense defined in [A2] when $n=2$, $m=1$ and S is smooth is tangential iff $b_f=1$ (which was the case mainly considered in that paper).

Example 1.2: A particularly interesting example (actually, the one inspiring this paper) of map $f\in \mathrm{End}(M,S)$ is obtained blowing up a map tangent to the identity. Let f_o be a (germ of) holomorphic self-map of $\mathbb{C}^n$ (or of any complex n-manifold) fixing the origin (or any other point) and *tangent to the identity*, that is, such that $d(f_o)_O=\mathrm{id}$. If $\pi\colon M\to\mathbb{C}^n$ denotes the blow-up of the origin, let $S=\pi^{-1}(O)\cong\mathbb{P}^{n-1}(\mathbb{C})$ be the exceptional divisor, and $f\in\mathrm{End}(M,S)$ the lifting of f_o, that is, the unique holomorphic self-map of M such that $f_o\circ\pi=\pi\circ f$ (see, e.g., [A1] for details). If

$$f_o^j(w)=w^j+\sum_{h\ge2}Q_h^j(w)$$

is the expansion of f_o^j in series of homogeneous polynomials (for $j=1,\dots,n$), then in the canonical coordinates centered in $p=[1:0:\cdots:0]$ the map f is given by

$$f^j(z)=\begin{cases} z^1+\displaystyle\sum_{h\ge2}Q_h^1(1,z'')(z^1)^h & \text{for } j=1,\\ z^j+\dfrac{\sum_{h\ge2}\left[Q_h^j(1,z'')-z^jQ_h^1(1,z'')\right](z^1)^{h-1}}{1+\sum_{h\ge2}Q_h^1(1,z'')(z^1)^{h-1}} & \text{for } j=2,\dots,n,\end{cases}$$

where $z''=(z^2,\dots,z^n)$. Therefore $b_f=1$, $\nu_f(z^1;p)=\min\{h\ge2\mid Q_h^1(1,z'')\not\equiv0\}$, and

$$\nu_f=\min\bigl\{\nu_f(z^1;p),\min\{h\ge1\mid \exists\, 2\le j\le n: Q_{h+1}^j(1,z'')-z^jQ_{h+1}^1(1,z'')\not\equiv0\}\bigr\}.$$

Let $\nu(f_o)\ge2$ be the order of f_o, that is the minimum h such that $Q_h^j\not\equiv0$ for some $1\le j\le n$. Clearly, $\nu_f(z^1;p)\ge\nu(f_o)$ and $\nu_f\ge\nu(f_o)-1$. More precisely, if there is a $2\le j\le n$ such that $Q^j_{\nu(f_o)}(1,z'')\not\equiv z^jQ^1_{\nu(f_o)}(1,z'')$, then $\nu_f=\nu(f_o)-1$ and f is tangential. If on the other hand we have $Q^j_{\nu(f_o)}(1,z'')\equiv z^jQ^1_{\nu(f_o)}(1,z'')$ for all $2\le j\le n$, then necessarily $Q^1_{\nu(f_o)}(1,z'')\not\equiv0$, $\nu_f(z^1;p)=\nu(f_o)=\nu_f$, and f is not tangential.

Borrowing a term from continuous dynamics, we say that a map f_o tangent to the identity at the origin is *dicritical* if $w^hQ^k_{\nu(f_o)}(w)\equiv w^kQ^h_{\nu(f_o)}(w)$ for all $1\le h,k\le n$. Then we have proved that:

Proposition 1.4: *Let $f_o\in\mathrm{End}(\mathbb{C}^n,O)$ be a (germ of) holomorphic self-map of $\mathbb{C}^n$ tangent to the identity at the origin, and let $f\in\mathrm{End}(M,S)$ be its blow-up. Then f is not tangential if and only if f_o is dicritical. Furthermore, $\nu_f=\nu(f_o)-1$ if f_o is not dicritical, and $\nu_f=\nu(f_o)$ if f_o is dicritical.*

In particular, most maps obtained with this procedure are tangential.

2. Comfortably embedded submanifolds

Up to now S was any complex subvariety of the manifold M. However, some of the proofs in the following sections do not work in this generality; so this section is devoted to describe the kind of properties we shall (sometimes) need on S.

Let E' and E'' be two vector bundles on the same manifold S. We recall (see, e.g., [Ati], section 1) that an *extension* of E'' by E' is an exact sequence of vector bundles

$$O\longrightarrow E'\stackrel{\iota}{\longrightarrow}E\stackrel{\pi}{\longrightarrow}E''\longrightarrow O.$$

Two extensions are *equivalent* if there is an isomorphism of exact sequences which is the identity on E' and E''.

A *splitting* of an extension $O\longrightarrow E'\stackrel{\iota}{\longrightarrow}E\stackrel{\pi}{\longrightarrow}E''\longrightarrow O$ is a morphism $\sigma\colon E''\to E$ such that $\pi\circ\sigma=\mathrm{id}_{E''}$. In particular, $E=\iota(E')\oplus\sigma(E'')$, and we shall say that the extension *splits*. We explicitly remark that an extension splits if and only if it is equivalent to the trivial extension $O\to E'\to E'\oplus E''\to E''\to O$.

Let now S be a complex submanifold of a complex manifold M. We shall denote by $TM|_S$ the restriction to S of the tangent bundle of M, and by $N_S=TM|_S/TS$ the normal bundle of S into M. Furthermore, $\mathcal{T}_{M,S}$ will be the sheaf of germs of holomorphic sections of $TM|_S$ (which is different from the restriction $\mathcal{T}_M|_S$ to S of the sheaf of holomorphic sections of TM), and $\mathcal{N}_S$ the sheaf of germs of holomorphic sections of N_S.

Definition 2.1: Let S be a complex submanifold of codimension m in a n-dimensional complex manifold M. A chart (U_α, z_α) of M is *adapted* to S if either $S \cap U_\alpha = \varnothing$ or $S \cap U_\alpha = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$, where $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$. In particular, $\{z_\alpha^1, \ldots, z_\alpha^m\}$ is a set of generators of $\mathcal{I}_{S,p}$ for all $p \in S \cap U_\alpha$. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of M is *adapted* to S if all charts in $\mathfrak{U}$ are. If $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ is adapted to S we shall denote by $\mathfrak{U}_S = \{(U''_\alpha, z''_\alpha)\}$ the atlas of S given by $U''_\alpha = U_\alpha \cap S$ and $z''_\alpha = (z_\alpha^{m+1}, \ldots, z_\alpha^n)$, where we are clearly considering only the indices such that $U_\alpha \cap S \neq \varnothing$. If (U_α, z_α) is a chart adapted to S, we shall denote by $\partial_{\alpha,r}$ the projection of $\partial/\partial z_\alpha^r|_{S\cap U_\alpha}$ in N_S, and by ω_α^r the local section of N_S^* induced by $dz_\alpha^r|_{S\cap U_\alpha}$; thus $\{\partial_{\alpha,1}, \ldots, \partial_{\alpha,m}\}$ and $\{\omega_\alpha^1, \ldots, \omega_\alpha^m\}$ are local frames for N_S and N_S^* respectively over $U_\alpha \cap S$, dual to each other.

From now on, every chart and atlas we consider on M will be adapted to S.

Remark 2.1: We shall use Einstein convention on the sum over repeated indices. Furthermore, indices like j, h, k will run from 1 to n; indices like r, s, t, u, v will run from 1 to m; and indices like p, q will run from $m+1$ to n.

Definition 2.2: We shall say that S *splits into* M if the extension $O \to TS \to TM|_S \to N_S \to O$ splits.

Example 2.1: It is well-known that if S is a rational smooth curve in M, then S splits into M.

Proposition 2.1: *Let S be a complex submanifold of codimension m in a n-dimensional complex manifold M. Then S splits into M if and only if there is an atlas $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ adapted to S such that*

$$\left.\frac{\partial \hat{z}_\beta^p}{\partial \hat{z}_\alpha^r}\right|_S \equiv 0, \tag{2.1}$$

for all $r = 1, \ldots, m$, $p = m+1, \ldots, n$ and indices α and β.

Proof: It is well known (see, e.g., [Ati, Proposition 2]) that there is a 1-to-1 correspondence between equivalence classes of extensions of N_S by TS and the cohomology group $H^1\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$, and an extension splits if and only if it corresponds to the zero cohomology class.

The class corresponding to the extension $O \to TS \to TM|_S \to N_S \to O$ is the class $\delta(\mathrm{id}_{N_S})$, where $\delta: H^0\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S)\big) \to H^1\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$ is the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence obtained by applying the functor $\mathrm{Hom}(\mathcal{N}_S, \cdot)$ to the extension sequence. More precisely, if $\mathfrak{U}$ is an atlas adapted to S, we get a local splitting morphism $\sigma_\alpha: N_{U''_\alpha} \to TM|_{U''_\alpha}$ by setting $\sigma_\alpha(\partial_{r,\alpha}) = \partial/\partial z_\alpha^r$, and then the element of $H^1\big(\mathfrak{U}_S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$ associated to the extension is $\{\sigma_\beta - \sigma_\alpha\}$. Now,

$$(\sigma_\beta - \sigma_\alpha)(\partial_{r,\alpha}) = \left.\frac{\partial z_\beta^s}{\partial z_\alpha^r}\right|_S \frac{\partial}{\partial z_\beta^s} - \frac{\partial}{\partial z_\alpha^r} = \left.\frac{\partial z_\beta^s}{\partial z_\alpha^r}\frac{\partial z_\alpha^p}{\partial z_\beta^s}\right|_S \frac{\partial}{\partial z_\alpha^p}.$$

So, if (2.1) holds, then S splits into M. Conversely, assume that S splits into M; then we can find an atlas $\mathfrak{U}$ adapted to S and a 0-cochain $\{c_\alpha\} \in H^0(\mathfrak{U}_S, \mathcal{T}_S \otimes \mathcal{N}_S^*)$ such that

$$\left.\frac{\partial z_\beta^s}{\partial z_\alpha^r}\frac{\partial z_\alpha^p}{\partial z_\beta^s}\right|_S = (c_\beta)_s^q \left.\frac{\partial z_\beta^s}{\partial z_\alpha^r}\frac{\partial z_\alpha^p}{\partial z_\beta^q}\right|_S - (c_\alpha)_r^p \tag{2.2}$$

on $U_\alpha \cap U_\beta \cap S$. We claim that the coordinates

$$\begin{cases} \hat{z}_\alpha^r = z_\alpha^r, \\ \hat{z}_\alpha^p = z_\alpha^p + (c_\alpha)_s^p(z''_\alpha) z_\alpha^s, \end{cases} \tag{2.3}$$

satisfy (2.1) when restricted to suitable open sets $\hat{U}_\alpha \subseteq U_\alpha$. Indeed, (2.2) yields

$$\begin{aligned}\frac{\partial \hat{z}_\beta^p}{\partial \hat{z}_\alpha^r} &= \frac{\partial \hat{z}_\beta^p}{\partial z_\alpha^s}\frac{\partial z_\alpha^s}{\partial \hat{z}_\alpha^r} + \frac{\partial \hat{z}_\beta^p}{\partial z_\alpha^q}\frac{\partial z_\alpha^q}{\partial \hat{z}_\alpha^r} = \frac{\partial \hat{z}_\beta^p}{\partial z_\alpha^r} - (c_\alpha)_r^q \frac{\partial \hat{z}_\beta^p}{\partial z_\alpha^q} + R_1 \\ &= \frac{\partial z_\beta^p}{\partial z_\alpha^r} + (c_\beta)_s^p \frac{\partial z_\beta^s}{\partial z_\alpha^r} - (c_\alpha)_r^q \frac{\partial z_\beta^p}{\partial z_\alpha^q} + R_1 = R_1,\end{aligned}$$

where R_1 denotes terms vanishing on S, and we are done. □

Definition 2.3: Assume that S splits into M. An atlas $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ adapted to S and satisfying (2.1) will be called a *splitting atlas* for S. It is easy to see that for any splitting morphism $\sigma\colon N_S\to TM|_S$ there exists a splitting atlas $\mathfrak{U}$ such that $\sigma(\partial_{r,\alpha})=\partial/\partial z^r_\alpha$ for all $r=1,\dots m$ and indices α; we shall say that $\mathfrak{U}$ is *adapted* to σ.

EXAMPLE 2.2: A *local holomorphic retraction* of M onto S is a holomorphic retraction $\rho\colon W\to S$, where W is a neighborhood of S in M. It is clear that the existence of such a local holomorphic retraction implies that S splits into M.

EXAMPLE 2.3: Let $\pi\colon M\to S$ be a rank m holomorphic vector bundle on S. If we identify S with the zero section of the vector bundle, π becomes a (global) holomorphic retraction of M on S. The charts given by the trivialization of the bundle clearly give a splitting atlas. Furthermore, if (U_α,z_α) and (U_β,z_β) are two such charts, we have $z''_\beta=\varphi_{\beta\alpha}(z''_\alpha)$ and $z'_\beta=a_{\beta\alpha}(z''_\alpha)z'_\alpha$, where $a_{\beta\alpha}$ is an invertible matrix depending only on z''_α. In particular, we have

$$\frac{\partial z^p_\beta}{\partial z^r_\alpha}\equiv 0\qquad\text{and}\qquad \frac{\partial^2 z^r_\beta}{\partial z^s_\alpha\partial z^t_\alpha}\equiv 0$$

for all r, s, $t=1,\dots,m$, $p=m+1,\dots,n$ and indices α and β.

The previous example, compared with (2.1), suggests the following

Definition 2.4: Let S be a codimension m complex submanifold of a n-dimensional complex manifold M. We say that S is *comfortably embedded* in M if S splits into M and there exists a splitting atlas $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ such that

$$\left.\frac{\partial^2 z^r_\beta}{\partial z^s_\alpha\partial z^t_\alpha}\right|_S\equiv 0 \tag{2.4}$$

for all r, s, $t=1,\dots,m$.

An atlas satisfying the previous condition is said *comfortable* for S. Roughly speaking, then, a comfortably embedded submanifold is like a first-order approximation of the zero section of a vector bundle.

Let us express condition (2.4) in a different way. If (U_α,z_α) and (U_β,z_β) are two charts about $p\in S$ adapted to S, we can write

$$z^r_\beta=(a_{\beta\alpha})^r_s\,z^s_\alpha \tag{2.5}$$

for suitable $(a_{\beta\alpha})^r_s\in\mathcal{O}_{M,p}$. The germs $(a_{\beta\alpha})^r_s$ (unless $m=1$) are not uniquely determined by (2.5); indeed, all the other solutions of (2.5) are of the form $(a_{\beta\alpha})^r_s+e^r_s$, where the e^r_s's are holomorphic and satisfy

$$e^r_s z^s_\alpha\equiv 0. \tag{2.6}$$

Differentiating with respect to z^t_α we get

$$e^r_t+\frac{\partial e^r_s}{\partial z^t_\alpha}\,z^s_\alpha\equiv 0; \tag{2.7}$$

in particular, $e^r_t|_S\equiv 0$, and so the restriction of $(a_{\beta\alpha})^r_s$ to S is uniquely determined — and it indeed gives the 1-cocycle of the normal bundle N_S with respect to the atlas $\mathfrak{U}_S$.

Differentiating (2.7) we obtain

$$\frac{\partial e^r_t}{\partial z^s_\alpha}+\frac{\partial e^r_s}{\partial z^t_\alpha}+\frac{\partial^2 e^r_u}{\partial z^s_\alpha\partial z^t_\alpha}\,z^u_\alpha\equiv 0; \tag{2.8}$$

in particular,

$$\left.\left[\frac{\partial e^r_t}{\partial z^s_\alpha}+\frac{\partial e^r_s}{\partial z^t_\alpha}\right]\right|_S\equiv 0,$$

and so the restriction of

$$\frac{\partial(a_{\beta\alpha})^r_t}{\partial z^s_\alpha}+\frac{\partial(a_{\beta\alpha})^r_s}{\partial z^t_\alpha}$$

to S is uniquely determined for all r, s, $t=1,\dots,m$.

With these notations, we have

$$\frac{\partial^2 z^r_\beta}{\partial z^s_\alpha \partial z^t_\alpha} = \frac{\partial (a_{\beta\alpha})^r_s}{\partial z^t_\alpha} + \frac{\partial (a_{\beta\alpha})^r_t}{\partial z^s_\alpha} + \frac{\partial^2 (a_{\beta\alpha})^r_u}{\partial z^s_\alpha \partial z^t_\alpha} z^u_\alpha;$$

therefore (2.4) is equivalent to requiring

$$\left.\left(\frac{\partial (a_{\beta\alpha})^r_t}{\partial z^s_\alpha} + \frac{\partial (a_{\beta\alpha})^r_s}{\partial z^t_\alpha}\right)\right|_S \equiv 0 \tag{2.9}$$

for all r, s, $t=1,\dots,m$, and indices α and β.

Example 2.4: It is easy to check that the exceptional divisor S in Example 1.2 is comfortably embedded into the blow-up M.

Then the main result of this section is

Theorem 2.2: *Let S be a codimension m complex submanifold of a n-dimensional complex manifold M. Assume that S splits into M, and let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a splitting atlas. Define a 1-cochain $\{h_{\beta\alpha}\}$ of $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$ by setting*

$$\begin{aligned} h_{\beta\alpha} &= \frac{1}{2} \left.\frac{\partial z^r_\alpha}{\partial z^u_\beta} \frac{\partial^2 z^u_\beta}{\partial z^s_\alpha \partial z^t_\alpha}\right|_S \partial_{\alpha,r} \otimes \omega^s_\alpha \otimes \omega^t_\alpha \\ &= \frac{1}{2} (a_{\alpha\beta})^r_u \left.\left(\frac{\partial (a_{\beta\alpha})^u_s}{\partial z^t_\alpha} + \frac{\partial (a_{\beta\alpha})^u_t}{\partial z^s_\alpha}\right)\right|_S \partial_{\alpha,r} \otimes \omega^s_\alpha \otimes \omega^t_\alpha. \end{aligned} \tag{2.10}$$

Then:

(i) *$\{h_{\beta\alpha}\}$ defines an element $[h] \in H^1(S, \mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*)$ independent of $\mathfrak{U}$;*
(ii) *S is comfortably embedded in M if and only if $[h]=0$.*

Proof: (i) Let us first prove that $\{h_{\beta\alpha}\}$ is a 1-cocycle with values in $\mathcal{N}_S \otimes \mathcal{N}_S^* \otimes \mathcal{N}_S^*$. We know that

$$(a_{\alpha\beta})^r_u (a_{\beta\alpha})^u_s = \delta^r_s + e^r_s,$$

where δ^r_s is Kronecker's delta, and the e^r_s's satisfy (2.6). Differentiating we get

$$\frac{\partial (a_{\alpha\beta})^r_u}{\partial z^t_\alpha} (a_{\beta\alpha})^u_s + (a_{\alpha\beta})^r_u \frac{\partial (a_{\beta\alpha})^u_s}{\partial z^t_\alpha} = \frac{\partial e^r_s}{\partial z^t_\alpha};$$

therefore (2.8) yields

$$(a_{\beta\alpha})^u_s \left.\frac{\partial (a_{\alpha\beta})^r_u}{\partial z^t_\alpha}\right|_S + (a_{\beta\alpha})^u_t \left.\frac{\partial (a_{\alpha\beta})^r_u}{\partial z^s_\alpha}\right|_S = -(a_{\alpha\beta})^r_u \left.\left(\frac{\partial (a_{\beta\alpha})^u_s}{\partial z^t_\alpha} + \frac{\partial (a_{\beta\alpha})^u_t}{\partial z^s_\alpha}\right)\right|_S.$$

Hence

$$\begin{aligned} h_{\alpha\beta} &= \frac{1}{2} (a_{\beta\alpha})^r_u \left.\left(\frac{\partial (a_{\alpha\beta})^u_s}{\partial z^t_\beta} + \frac{\partial (a_{\alpha\beta})^u_t}{\partial z^s_\beta}\right)\right|_S \partial_{\beta,r} \otimes \omega^s_\beta \otimes \omega^t_\beta \\ &= \frac{1}{2} (a_{\beta\alpha})^r_u (a_{\alpha\beta})^{r_1}_r (a_{\beta\alpha})^s_{s_1} (a_{\beta\alpha})^t_{t_1} \left.\left((a_{\alpha\beta})^{t_2}_t \frac{\partial (a_{\alpha\beta})^u_s}{\partial z^{t_2}_\alpha} + (a_{\alpha\beta})^{s_2}_s \frac{\partial (a_{\alpha\beta})^u_t}{\partial z^{s_2}_\alpha}\right)\right|_S \partial_{\alpha,r_1} \otimes \omega^{s_1}_\alpha \otimes \omega^{t_1}_\alpha \\ &= \frac{1}{2} \left.\left((a_{\beta\alpha})^s_{s_1} \frac{\partial (a_{\alpha\beta})^{r_1}_s}{\partial z^{t_1}_\alpha} + (a_{\beta\alpha})^t_{t_1} \frac{\partial (a_{\alpha\beta})^{r_1}_t}{\partial z^{s_1}_\alpha}\right)\right|_S \partial_{\alpha,r_1} \otimes \omega^{s_1}_\alpha \otimes \omega^{t_1}_\alpha \\ &= -h_{\beta\alpha}, \end{aligned}$$

where in the second equality we used (2.1). Analogously one proves that $h_{\alpha\beta}+h_{\beta\gamma}+h_{\gamma\alpha}=0$, and thus $\{h_{\beta\alpha}\}$ is a 1-cocycle as claimed.

Now we have to prove that the cohomology class $[h]$ is independent of the atlas $\mathfrak{U}$. Let $\hat{\mathfrak{U}}=\{(\hat U_\alpha,\hat z_\alpha)\}$ be another splitting atlas; up to taking a common refinement we can assume that $U_\alpha=\hat U_\alpha$ for all α. Choose $(A_\alpha)^r_s\in\mathcal{O}(U_\alpha)$ so that $\hat z^r_\alpha=(A_\alpha)^r_s z^s_\alpha$; as usual, the restrictions to S of $(A_\alpha)^r_s$ and of

$$\frac{\partial (A_\alpha)^r_s}{\partial z^t_\alpha}+\frac{\partial (A_\alpha)^r_t}{\partial z^s_\alpha}$$

are uniquely defined. Set then

$$C_\alpha=\frac{1}{2}(A_\alpha^{-1})^r_u\left[\frac{\partial (A_\alpha)^u_s}{\partial z^t_\alpha}+\frac{\partial (A_\alpha)^u_t}{\partial z^s_\alpha}\right]\bigg|_S\,\partial_{\alpha,r}\otimes\omega^s_\alpha\otimes\omega^t_\alpha;$$

then it is not difficult to check that

$$h_{\beta\alpha}-\hat h_{\beta\alpha}=C_\beta-C_\alpha,$$

where $\{\hat h_{\beta\alpha}\}$ is the 1-cocycle built using $\hat{\mathfrak{U}}$, and this means exactly that both $\{h_{\beta\alpha}\}$ and $\{\hat h_{\beta\alpha}\}$ determine the same cohomology class.

(ii) If S is comfortably embedded, using a comfortable atlas we immediately see that $[h]=0$. Conversely, assume that $[h]=0$; therefore we can find a splitting atlas $\mathfrak{U}$ and a 0-cochain $\{c_\alpha\}$ of $\mathcal{N}_S\otimes\mathcal{N}_S^*\otimes\mathcal{N}_S^*$ such that $h_{\beta\alpha}=c_\alpha-c_\beta$. Writing

$$c_\alpha=(c_\alpha)^r_{st}\,\partial_{\alpha,r}\otimes\omega^s_\alpha\otimes\omega^t_\alpha,$$

we define $\hat z_\alpha$ by setting

$$\begin{cases}\hat z^r_\alpha=z^r_\alpha+(c_\alpha)^r_{st}(z''_\alpha)\,z^s_\alpha z^t_\alpha & \text{for } r=1,\dots,m,\\ \hat z^p_\alpha=z^p_\alpha & \text{for } p=m+1,\dots,n,\end{cases}$$

on a suitable $\hat U_\alpha\subseteq U_\alpha$. Then $\hat{\mathfrak{U}}=\{(\hat U_\alpha,\hat z_\alpha)\}$ clearly is a splitting atlas; we claim that it is comfortable too. Indeed, by definition the functions

$$(\hat a_{\beta\alpha})^r_s=[\delta^r_u+(c_\beta)^r_{uv}(a_{\beta\alpha})^v_t z^t_\alpha](a_{\beta\alpha})^u_{u_1}d^{u_1}_s$$

satisfy (2.5) for $\hat{\mathfrak{U}}$, where the $d^{u_1}_s$'s are such that $z^{u_1}_\alpha=d^{u_1}_s\hat z^s_\alpha$. Hence

$$\left(\frac{\partial (\hat a_{\beta\alpha})^r_s}{\partial \hat z^t_\alpha}+\frac{\partial (\hat a_{\beta\alpha})^r_t}{\partial \hat z^s_\alpha}\right)\bigg|_S=2(c_\beta)^r_{uv}(a_{\beta\alpha})^u_s(a_{\beta\alpha})^v_t|_S+\left(\frac{\partial (a_{\beta\alpha})^r_s}{\partial z^t_\alpha}+\frac{\partial (a_{\beta\alpha})^r_t}{\partial z^s_\alpha}\right)\bigg|_S+(a_{\beta\alpha})^r_u\left(\frac{\partial d^u_s}{\partial z^t_\alpha}+\frac{\partial d^u_t}{\partial z^s_\alpha}\right)\bigg|_S.$$

Now, differentiating

$$z^u_\alpha=d^u_v\big(z^v_\alpha+(c_\alpha)^v_{rs}z^r_\alpha z^s_\alpha\big)$$

we get

$$\delta^u_t=\frac{\partial d^u_v}{\partial z^t_\alpha}\big(z^v_\alpha+(c_\alpha)^v_{rs}z^r_\alpha z^s_\alpha\big)+d^u_v\big(\delta^v_t+2(c_\alpha)^v_{rt}z^r_\alpha\big)$$

and

$$0=\left(\frac{\partial d^u_s}{\partial z^t_\alpha}+\frac{\partial d^u_t}{\partial z^s_\alpha}\right)\bigg|_S+2(c_\alpha)^u_{st}.$$

Recalling that $h_{\beta\alpha}=c_\alpha-c_\beta$ we then see that $\hat{\mathfrak{U}}$ satisfies (2.9), and we are done. □

Remark 2.2: Since $N_S\otimes N_S^*\otimes N_S^*\cong\mathrm{Hom}\big(N_S,\mathrm{Hom}(N_S,N_S)\big)$, the previous theorem asserts that to any submanifold S splitting into M we can canonically associate an extension

$$O\to\mathrm{Hom}(N_S,N_S)\to E\to N_S\to O$$

of N_S by $\mathrm{Hom}(N_S,N_S)$, and S is comfortably embedded in M if and only if this extension splits.

3. The canonical sections

Our next aim is to associate to any $f \in \mathrm{End}(M,S)$ different from the identity a section of a suitable vector bundle, indicating (very roughly speaking) how f would move S if it did not keep it fixed. To do so, in this section we still assume that S is a smooth complex submanifold of a complex manifold M; however, in Remark 3.3 we shall describe the changes needed to avoid this assumption.

Given $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, it is clear that $df|_{TS} = \mathrm{id}$; therefore $df - \mathrm{id}$ induces a map from N_S to $TM|_S$, or, more precisely, a holomorphic section over S of the bundle $TM|_S \otimes N_S^*$. If (U,z) is a chart adapted to S, we can define germs g^h_r for $h = 1,\dots,n$ and $r = 1,\dots,m$ by writing

$$z^h \circ f - z^h = z^1 g^h_1 + \cdots + z^m g^h_m.$$

It is easy to check that the germ of the section of $TM|_S \otimes N_S^*$ defined by $df - \mathrm{id}$ is locally expressed by

$$g^h_r|_{U\cap S}\,\frac{\partial}{\partial z_h} \otimes \omega^r,$$

where we are again indicating with ω^r the germ of section of the conormal bundle induced by the 1-form dz^r restricted to S.

A problem with this section is that it vanishes identically if (and only if) $\nu_f > 1$. The solution consists in expanding f at a higher order.

Definition 3.1: Given a chart (U,z) adapted to S, set $f^j = z^j \circ f$, and write

$$f^j - z^j = z^{r_1} \cdots z^{r_{\nu_f}}\, g^j_{r_1\ldots r_{\nu_f}}, \tag{3.1}$$

where the $g^j_{r_1\ldots r_{\nu_f}}$'s are symmetric in $r_1,\dots,r_{\nu_f}$ and do not all vanish restricted to S. Let us then define

$$\mathcal{X}_f = g^h_{r_1\ldots r_{\nu_f}}\,\frac{\partial}{\partial z^h} \otimes dz^{r_1} \otimes \cdots \otimes dz^{r_{\nu_f}}. \tag{3.2}$$

This is a local section of $TM \otimes (T^*M)^{\otimes \nu_f}$, defined in a neighborhood of a point of S; furthermore, when restricted to S, it induces a local section of $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$.

Remark 3.1: When $m > 1$ the $g^j_{r_1\ldots r_{\nu_f}}$'s are *not* uniquely determined by (3.1). Indeed, if $e^j_{r_1\ldots r_{\nu_f}}$ are such that

$$e^j_{r_1\ldots r_{\nu_f}} z^1 \cdots z^{r_{\nu_f}} \equiv 0 \tag{3.3}$$

then $g^j_{r_1\ldots r_{\nu_f}} + e^j_{r_1\ldots r_{\nu_f}}$ still satisfies (3.1). This means that the section (3.2) is not uniquely determined too; but, as we shall see, this will not be a problem. For instance, (3.3) implies that $e^j_{r_1\ldots r_{\nu_f}} \in \mathcal{I}_S$; therefore $\mathcal{X}_f|_{U\cap S}$ is always uniquely determined — though a priori it might depend on the chosen chart. On the other hand, when $m = 1$ both the $g^j_{r_1\ldots r_{\nu_f}}$'s and $\mathcal{X}_f$ *are* uniquely determined; this is one of the reasons making the codimension one case simpler than the general case.

We have already remarked that when $\nu_f = 1$ the section $\mathcal{X}_f$ restricted to $U \cap S$ coincides with the restriction of $df - \mathrm{id}$ to S. Therefore when $\nu_f = 1$ the restriction of $\mathcal{X}_f$ to S gives a globally well-defined section. Actually, this holds for any $\nu_f \geq 1$:

Proposition 3.1: *Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. Then the restriction of $\mathcal{X}_f$ to S induces a global holomorphic section X_f of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$.*

Proof: Let (U,z) and $(\hat{U},\hat{z})$ be two charts about $p \in S$ adapted to S. Then we can find holomorphic functions a^r_s such that

$$\hat{z}^r = a^r_s\, z^s; \tag{3.4}$$

in particular,

$$\frac{\partial \hat{z}^r}{\partial z^s} = a^r_s \pmod{\mathcal{I}_S} \qquad \text{and} \qquad \frac{\partial \hat{z}^r}{\partial z^p} = 0 \pmod{\mathcal{I}_S}. \tag{3.5}$$

Now set $f^j = z^j \circ f$, $\hat f^j = \hat z^j \circ f$, and define $g^j_{r_1\cdots r_{\nu_f}}$ and $\hat g^j_{r_1\cdots r_{\nu_f}}$ using (3.1) with (U,z) and $(\hat U,\hat z)$ respectively. Then (3.4) and (1.1) yield

$$\begin{aligned}a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\,\hat g^j_{r_1\ldots r_{\nu_f}} z^{s_1}\cdots z^{s_{\nu_f}} &= \hat g^j_{r_1\ldots r_{\nu_f}} \hat z^{r_1}\cdots \hat z^{r_{\nu_f}}\\ &= \hat f^j - \hat z^j = (f^h - z^h)\frac{\partial \hat z^j}{\partial z^h} + R_{2\nu_f} = g^h_{s_1\ldots s_{\nu_f}}\frac{\partial \hat z^j}{\partial z^h} z^{s_1}\cdots z^{s_{\nu_f}} + R_{2\nu_f},\end{aligned}$$

where the remainder terms $R_{2\nu_f}$ belong to $\mathcal{I}_S^{2\nu_f}$. Therefore we find

$$a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\,\hat g^j_{r_1\ldots r_{\nu_f}} = \frac{\partial \hat z^j}{\partial z^h}\, g^h_{s_1\ldots s_{\nu_f}} \pmod{\mathcal{I}_S}. \tag{3.6}$$

Recalling (3.5) we then get

$$\begin{aligned}\hat g^j_{r_1\ldots r_{\nu_f}}\frac{\partial}{\partial \hat z^j}\otimes d\hat z^{r_1}\otimes\cdots\otimes d\hat z^{r_{\nu_f}} &= \frac{\partial z^k}{\partial \hat z^j}\frac{\partial \hat z^{r_1}}{\partial z^{k_1}}\cdots\frac{\partial \hat z^{r_{\nu_f}}}{\partial z^{k_{\nu_f}}}\hat g^j_{r_1\ldots r_{\nu_f}}\frac{\partial}{\partial z^h}\otimes dz^{k_1}\otimes\cdots\otimes dz^{k_{\nu_f}}\\ &= a^{r_1}_{s_1}\cdots a^{r_{\nu_f}}_{s_{\nu_f}}\hat g^j_{r_1\ldots r_{\nu_f}}\frac{\partial z^k}{\partial \hat z^j}\frac{\partial}{\partial z^h}\otimes dz^{s_1}\otimes\cdots\otimes dz^{s_{\nu_f}} \pmod{\mathcal{I}_S}\\ &= g^h_{s_1\ldots s_{\nu_f}}\frac{\partial}{\partial z^h}\otimes dz^{s_1}\otimes\cdots\otimes dz^{s_{\nu_f}} \pmod{\mathcal{I}_S},\end{aligned}$$

and we are done. □

Remark 3.2: For later use, we explicitly notice that when $m=1$ the germs a^r_s are uniquely determined, and (3.6) becomes

$$(a^1_1)^{\nu_f}\hat g^j_{1\ldots 1} = \frac{\partial \hat z^j}{\partial z^h}\, g^h_{1\ldots 1} \pmod{\mathcal{I}_S^{\nu_f}}. \tag{3.7}$$

Definition 3.2: Let $f\in \mathrm{End}(M,S)$, $f\not\equiv \mathrm{id}_M$. The *canonical section* $X_f \in H^0\big(S, \mathcal{T}_{M,S}\otimes(\mathcal{N}_S^*)^{\otimes\nu_f}\big)$ *associated to* f is defined by setting

$$X_f = g^h_{s_1\ldots s_{\nu_f}}|_S\,\frac{\partial}{\partial z^h}\otimes\omega^{s_1}\otimes\cdots\otimes\omega^{s_{\nu_f}} \tag{3.8}$$

in any chart adapted to S. Since $(N_S^*)^{\otimes\nu_f} = (N_S^{\otimes\nu_f})^*$, we can also think of X_f as a holomorphic section of $\mathrm{Hom}(N_S^{\otimes\nu_f}, TM|_S)$, and introduce the *canonical distribution* $\Xi_f = X_f(N_S^{\otimes\nu_f})\subseteq TM|_S$.

In particular we can now justify the term "tangential" previously introduced:

Corollary 3.2: *Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. Then f is tangential if and only if the canonical distribution is tangent to S, that is if and only if $\Xi_f\subseteq TS$.*

Proof: It follows from Lemma 1.2. □

Example 3.1: Using the notations introduced in Example 1.2, if f is obtained by blowing up a map f_o tangent to the identity, then the canonical coordinates centered in $p=[1:0:\cdots:0]$ are adapted to S. In particular, if f_o is non-dicritical (that is if f is tangential) then in a neighborhood of p we have

$$X_f = \big[Q^q_{\nu(f_o)}(1,z'') - z^q Q^1_{\nu(f_o)}(1,z'')\big]\,\frac{\partial}{\partial z^q}\otimes(\omega^1)^{\otimes(\nu(f_o)-1)}.$$

Remark 3.3: The sheaf $\mathcal{N}_S^*$ is isomorphic to $\mathcal{I}_S/\mathcal{I}_S^2$, and it is known that $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f}$ is isomorphic to $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$. This allows us to define X_f as a global section of the coherent sheaf $\mathcal{T}_{M,S}\otimes(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f}$ even when S is singular. Indeed, if (U,z) is a local chart adapted to S, for $j=1,\ldots,n$ the functions f^j-z^j determine local sections $[f^j-z^j]$ of $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1}$. But, since for any other chart $(\hat U,\hat z)$ we have

$$\hat f^j - \hat z^j = (f^h - z^h)\frac{\partial \hat z^j}{\partial z^h} + R_{2\nu_f},$$

then $(\partial/\partial z^j)\otimes[f^j-z^j]$ is a well-defined global section of $\mathcal{T}_{M,S}\otimes(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes\nu_f}$ which coincides with X_f when S is smooth.

Remark 3.4: When f is tangential and Ξ_f is involutive as a sub-distribution of TS — for instance when $m=1$ — we thus get a holomorphic singular foliation on S canonically associated to f. As already remarked in [Br], this possibly is the moral reason explaining the similarities discovered in [A2] between the local dynamics of holomorphic maps tangent to the identity and the dynamics of singular holomorphic foliations.

Definition 3.3: A point $p\in S$ is *singular* for f if there exists $v\in(N_S)_p$, $v\neq O$, such that $X_f(v\otimes\cdots\otimes v)=O$. We shall denote by $\mathrm{Sing}(f)$ the set of singular points of f.

In Section 7 it will become clear why we choose this definition for singular points. In Section 8 we shall describe a dynamical interpretation of X_f at non-singular points in the codimension one case: see Proposition 8.1.

Remark 3.5: If S is an hypersurface, the normal bundle is a line bundle. Therefore Ξ_f is a 1-dimensional distribution, and the singular points of f are the points where Ξ_f vanishes. Recalling (3.8), we then see that $p\in\mathrm{Sing}(f)$ if and only if $g^1_{1\ldots1}(p)=\cdots=g^n_{1\ldots1}(p)=0$ for any adapted chart, and thus both the strictly fixed points of [A2] and the singular points of [BT, Br] are singular in our case as well.

As we shall see later on, our index theorems will need a section of $TS\otimes(N_S^*)^{\otimes\nu_f}$; so it will be natural to assume f tangential. When f is not tangential but S splits in M we can work too.

Let $O\longrightarrow TS\stackrel{\iota}{\longrightarrow}TM|_S\stackrel{\pi}{\longrightarrow}N_S\longrightarrow O$ be the usual extension. Then we can associate to any splitting morphism $\sigma\colon N_S\to TM|_S$ a morphism $\sigma'\colon TM|_S\to TS$ such that $\sigma'\circ\iota=\mathrm{id}_{TS}$, by $\sigma'=\iota^{-1}\circ(\sigma\circ\pi-\mathrm{id}_{TM|_S})$. Conversely, if there is a morphism $\sigma'\colon TM|_S\to TS$ such that $\sigma'\circ\iota=\mathrm{id}_{TS}$, we get a splitting morphism by setting $\sigma=(\pi|_{\mathrm{Ker}\,\sigma'})^{-1}$. Then

Definition 3.4: Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$, and assume that S splits in M. Choose a splitting morphism $\sigma\colon N_S\to TM|_S$ and let $\sigma'\colon TM|_S\to TS$ be the induced morphism. We set

$$H_{\sigma,f}=(\sigma'\otimes\mathrm{id})\circ X_f\in H^0\big(S,\mathcal{T}_S\otimes(N_S^*)^{\otimes\nu_f}\big).$$

Since the differential of f induces a morphism from N_S into itself, we have a dual morphism $(df)^*\colon N_S^*\to N_S^*$. Then if $\nu_f=1$ we also set

$$H^1_{\sigma,f}=\big(\mathrm{id}\otimes(df)^*\big)\circ H_{\sigma,f}\in H^0\big(S,\mathcal{T}_S\otimes N_S^*\big).$$

Remark 3.6: We defined $H^1_{\sigma,f}$ only for $\nu_f=1$ because when $\nu_f>1$ one has $(df)^*=\mathrm{id}$. On the other hand, when $\nu_f=1$ one has $(df)^*=\mathrm{id}$ if and only if f is tangential. Finally, we have $X_f\equiv H_{\sigma,f}$ if and only if f is tangential, and $H_{\sigma,f}\equiv O$ if and only if $\Xi_f\subseteq\mathrm{Im}\,\sigma=\mathrm{Ker}\,\sigma'$.

Finally, if (U,z) is a chart in an atlas adapted to the splitting σ, locally we have

$$H_{\sigma,f}=g^p_{s_1\ldots s_{\nu_f}}|_S\,\frac{\partial}{\partial z^p}\otimes\omega^{s_1}\otimes\cdots\otimes\omega^{s_{\nu_f}},$$

and, if $\nu_f=1$,

$$H^1_{\sigma,f}=(\delta^s_r+g^s_r)g^p_s|_S\,\frac{\partial}{\partial z^p}\otimes\omega^r.$$

4. Local extensions

As we have already remarked, while X_f is well-defined, its extension $\mathcal{X}_f$ in general is not. However, we shall now derive formulas showing how to control the ambiguities in the definition of $\mathcal{X}_f$, at least in the cases we are mostly interested in.

In this section we assume $m=1$, i.e., that S has codimension one in M. To simplify notations we shall write g^j for $g^j_{1\ldots1}$ and a for a^1_1. We shall also use the following notations:

- T_1 will denote any sum of terms of the form $g\,\frac{\partial}{\partial z^p}\otimes dz^{h_1}\otimes\cdots\otimes dz^{h_{\nu_f}}$ with $g\in\mathcal{I}_S$;
- R_k will denote any local section with coefficients in $\mathcal{I}_S^k$.

For instance, if (U,z) and $(\hat U,\hat z)$ are two charts adapted to S, we have

$$\begin{aligned}\frac{\partial}{\partial\hat z^h}\otimes(d\hat z^1)^{\otimes\nu_f}&=a^{\nu_f}\frac{\partial z^k}{\partial\hat z^h}\frac{\partial}{\partial z^k}\otimes(dz^1)^{\otimes\nu_f}\\&\quad+\frac{\partial z^1}{\partial\hat z^h}a^{\nu_f-1}z^1\sum_{\ell=1}^{\nu_f}\frac{\partial a}{\partial z^{j_\ell}}\frac{\partial}{\partial z^1}\otimes dz^1\otimes\cdots\otimes dz^{j_\ell}\otimes\cdots\otimes dz^1+T_1+R_2,\end{aligned}\tag{4.1}$$

where

$$T_1=\frac{\partial z^p}{\partial\hat z^h}a^{\nu_f-1}z^1\sum_{\ell=1}^{\nu_f}\frac{\partial a}{\partial z^{j_\ell}}\frac{\partial}{\partial z^p}\otimes dz^1\otimes\cdots\otimes dz^{j_\ell}\otimes\cdots\otimes dz^1.$$

Assume now that f is tangential, and let (U,z) be a chart adapted to S. We know that $f^1-z^1\in\mathcal{I}_S^{\nu_f+1}$, and thus we can write

$$f^1-z^1=h^1(z^1)^{\nu_f+1},$$

where h^1 is uniquely determined. Now, if $(\hat U,\hat z)$ is another chart adapted to S we have

$$\begin{aligned}a^{\nu_f+1}\hat h^1(z^1)^{\nu_f+1}&=\hat f^1-\hat z^1=(a\circ f)f^1-az^1\\&=a(f^1-z^1)+(a\circ f-a)z^1+(a\circ f-a)(f^1-z^1)\\&=a(f^1-z^1)+\frac{\partial a}{\partial z^p}(f^p-z^p)z^1+R_{\nu_f+2}\\&=\left[ah^1+\frac{\partial a}{\partial z^p}g^p\right](z^1)^{\nu_f+1}+R_{\nu_f+2}.\end{aligned}$$

Therefore

$$a^{\nu_f+1}\hat h^1=ah^1+\frac{\partial a}{\partial z^p}g^p+R_1.\tag{4.2}$$

Since $g^1=h^1z^1$ we then get

$$a^{\nu_f}\hat g^1=ag^1+\frac{\partial a}{\partial z^p}g^pz^1+R_2,\tag{4.3}$$

that generalizes (3.6) when f is tangential and $m=1$.

Putting (4.3), (3.6) and (4.1) into (3.2) we then get

Lemma 4.1: *Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. Assume that f is tangential, and that S has codimension 1. Let $(\hat U,\hat z)$ and (U,z) be two charts about $p\in S$ adapted to S, and let $\hat{\mathcal{X}}_f$, $\mathcal{X}_f$ be given by (3.2) in the respective coordinates. Then*

$$\hat{\mathcal{X}}_f=\mathcal{X}_f+T_1+R_2.$$

When S is comfortably embedded in M and of codimension one we shall also need nice local extensions of $H_{\sigma,f}$ and $H^1_{\sigma,f}$, and to know how they behave under change of (comfortable) coordinates.

Definition 4.1: Let S be comfortably embedded in M and of codimension 1, and take $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. Let (U,z) be a chart in a comfortable atlas, and set $b^1(z)=g^1(O,z'')$; notice that f is tangential if and only if $b^1\equiv O$. Write $g^1=b^1+h^1z^1$ for a well-defined holomorphic function h^1; then set

$$\mathcal{H}_{\sigma,f}=h^1z^1\frac{\partial}{\partial z^1}\otimes(dz^1)^{\otimes\nu_f}+g^p\frac{\partial}{\partial z^p}\otimes(dz^1)^{\otimes\nu_f},\tag{4.4}$$

and if $\nu_f=1$ set

$$\mathcal{H}^1_{\sigma,f}=h^1z^1\frac{\partial}{\partial z^1}\otimes dz^1+g^p(1+b^1)\frac{\partial}{\partial z^p}\otimes dz^1.\tag{4.5}$$

Notice that $\mathcal{H}_{\sigma,f}$ (respectively, $\mathcal{H}^1_{\sigma,f}$) restricted to S yields $H_{\sigma,f}$ (respectively, $H^1_{\sigma,f}$).

Proposition 4.2: *Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$. Assume that S is comfortably embedded in M, and of codimension one. Fix a comfortable atlas $\mathfrak{U}$, and let (U,z), $(\hat U,\hat z)$ be two charts in $\mathfrak{U}$ about $p\in S$. Then if $\nu_f=1$ we have*

$$\hat{\mathcal{H}}^1_{\sigma,f}=\mathcal{H}^1_{\sigma,f}+T_1+R_2,\tag{4.6}$$

while if $\nu_f > 1$ *we have*

$$\hat{\mathcal{H}}_{\sigma,f} = \mathcal{H}_{\sigma,f} + T_1 + R_2, \tag{4.7}$$

where $T_1 = T_1^o + T_1^1$ *with*

$$T_1^o = \frac{1}{a} g^q z^1 \sum_{\ell=1}^{\nu_f} \frac{\partial a}{\partial z^{p_\ell}} \frac{\partial}{\partial z^q} \otimes dz^1 \otimes \cdots \otimes dz^{p_\ell} \otimes \cdots \otimes dz^1, \qquad T_1^1 = -a g^1 \frac{\partial z^q}{\partial \hat{z}^1} \frac{\partial}{\partial z^q} \otimes (dz^1)^{\otimes \nu_f}.$$

Proof: First of all, from (3.7) it follows that $a^{\nu_f} \hat{b}^1 = a b^1 (\mathrm{mod}\ \mathcal{I}_S)$. But since we are using a comfortable atlas we get

$$\frac{\partial (a^{\nu_f} \hat{b}^1 - a b^1)}{\partial z^1} = (\nu_f a^{\nu_f - 1} \hat{b}^1 - b^1) \frac{\partial a}{\partial z^1} + R_1 = R_1,$$

and thus

$$a^{\nu_f} \hat{b}^1 = a b^1 \pmod{\mathcal{I}_S^2}. \tag{4.8}$$

If $\nu_f > 1$ then by (3.7) and (4.8) we have

$$a^{\nu_f} \hat{h}^1 \hat{z}^1 = (a h^1 + \frac{\partial a}{\partial z^p} g^p) z^1 \pmod{\mathcal{I}_S^2},$$

which implies

$$a^{\nu_f + 1} \hat{h}^1 = a h^1 + \frac{\partial a}{\partial z^p} g^p \pmod{\mathcal{I}_S}. \tag{4.9}$$

If $\nu_f = 1$, using (2.4) we can write

$$\begin{aligned} \hat{b}^1 \hat{z}^1 + \hat{h}^1 (\hat{z}^1)^2 &= \hat{f}^1 - \hat{z}^1 \\ &= \frac{\partial \hat{z}^1}{\partial z^j} (f^j - z^j) + \frac{1}{2} \frac{\partial^2 \hat{z}^1}{\partial z^h \partial z^k} (f^h - z^h)(f^k - z^k) + R_3 \\ &= a b^1 z^1 + \left[a h^1 + \frac{\partial a}{\partial z^p} g^p (1 + b^1) \right] (z^1)^2 + R_3, \end{aligned}$$

and by (4.8) we have

$$a^2 \hat{h}^1 = a h^1 + \frac{\partial a}{\partial z^p} g^p (1 + b^1) \pmod{\mathcal{I}_S}. \tag{4.10}$$

So if we compute $\hat{\mathcal{H}}_{\sigma,f}$ for $\nu_f > 1$ (respectively, $\hat{\mathcal{H}}^1_{\sigma,f}$ for $\nu_f = 1$) using (3.7), (4.1) and (4.9) (respectively, (3.7), (4.1), (4.8) and (4.10)), we get the assertions. □

5. Holomorphic actions

The index theorems we are going to discuss depend on actions of vector bundles. This concept has been introduced by Baum and Bott in [BB], and later generalized in [CL], [LS], [LS2] and [Su]. Let us recall here the relevant definitions.

Let again S be a submanifold of codimension m in a n-dimensional complex manifold M, and let $\pi_F \colon F \to S$ be a holomorphic vector bundle on S. We shall denote by $\mathcal{F}$ the sheaf of germs of holomorphic sections of F, by $\mathcal{T}_S$ the sheaf of germs of holomorphic sections of TS, and by Ω^1_S (respectively, Ω^1_M) the sheaf of holomorphic 1-forms on S (respectively, on M).

A section X of $\mathcal{T}_S \otimes \mathcal{F}^*$ (or, equivalently, a holomorphic section of $TS \otimes F^*$) can be interpreted as a morphism $X \colon \mathcal{F} \to \mathcal{T}_S$. Therefore it induces a derivation $X^\# \colon \mathcal{O}_S \to \mathcal{F}^*$ by setting

$$X^\#(g)(u) = X(u)(g) \tag{5.1}$$

for any $p \in S$, $g \in \mathcal{O}_{S,p}$ and $u \in \mathcal{F}_p$. If $\{f_1^*, \ldots, f_k^*\}$ is a local frame for F^* about p, and X is locally given by $X = \sum_j v_j \otimes f_j^*$, then

$$X^\#(g) = \sum_j v_j(g) f_j^*. \tag{5.2}$$

Notice that if $X^*\colon \Omega^1_S \to \mathcal{F}^*$ denotes the dual morphism of $X\colon \mathcal{F} \to \mathcal{T}_S$, by definition we have

$$X^*(\omega)(u) = \omega\bigl(X(u)\bigr)$$

for any $p \in S$, $\omega \in (\Omega^1_S)_p$ and $u \in \mathcal{F}_p$, and so

$$X^\#(g) = X^*(dg).$$

Definition 5.1: Let $\pi_E\colon E \to S$ be another holomorphic vector bundle on S, and denote by $\mathcal{E}$ the sheaf of germs of holomorphic sections of E. Let X be a section of $\mathcal{T}_S \otimes \mathcal{F}^*$. A *holomorphic action of F on E along X* (or a *X-connection* on E) is a morphism $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\tilde{X}(gs) = X^\#(g) \otimes s + g\tilde{X}(s) \tag{5.3}$$

for any $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$.

Example 5.1: If $F = TS$, and the section X is the identity $\mathrm{id}\colon TS \to TS$, then $X^\#(g) = dg$, and a holomorphic action of TS on E along X is just a (1,0)-connection on E.

Definition 5.2: A point $p \in S$ is a *singularity* of a holomorphic section X of $\mathcal{T}_S \otimes \mathcal{F}^*$ if the induced map $X_p\colon F_p \to T_pS$ is not injective. The set of singular points of X will be denoted by $\mathrm{Sing}(X)$, and we shall set $S^0 = S \setminus \mathrm{Sing}(X)$ and $\Xi_X = X(F|_{S^0}) \subseteq TS^0$. Notice that Ξ_X is a holomorphic subbundle of TS^0.

The canonical section previously introduced suggests the following definition:

Definition 5.3: A *Camacho-Sad action* on S is a holomorphic action of $N_S^{\otimes\nu}$ on N_S along a section X of $\mathcal{T}_S \otimes (N_S^{\otimes\nu})^*$, for a suitable $\nu \geq 1$.

Remark 5.1: The rationale behind the name is the following: as we shall see, the index theorem in [A2] is induced by a holomorphic action of $N_S^{\otimes\nu_f}$ on N_S along X_f when f is tangential, and this index theorem was inspired by Camacho-Sad index theorem [CS].

Let us describe a way to get Camacho-Sad actions. Let $\pi\colon TM|_S \to N_S$ be the canonical projection; we shall use the same symbol for all other projections naturally induced by it. Let X be any global section of $TS \otimes (N_S^{\otimes\nu})^*$. Then we might try to define an action $\tilde{X}\colon \mathcal{N}_S \to (\mathcal{N}_S^{\otimes\nu})^* \otimes \mathcal{N}_S = \mathrm{Hom}(\mathcal{N}_S^{\otimes\nu}, \mathcal{N}_S)$ by setting

$$\tilde{X}(s)(u) = \pi([\mathcal{X}(\tilde{u}), \tilde{s}]|_S) \tag{5.4}$$

for any $s \in \mathcal{N}_S$ and $u \in \mathcal{N}_S^{\otimes\nu}$, where: $\tilde{s}$ is any element in $\mathcal{T}_M|_S$ such that $\pi(\tilde{s}|_S) = s$; $\tilde{u}$ is any element in $\mathcal{T}_M|_S^{\otimes\nu_f}$ such that $\pi(\tilde{u}|_S) = u$; and $\mathcal{X}$ is a *suitably chosen* local section of $\mathcal{T}_M \otimes (\Omega^1_M)^{\otimes\nu}$ that restricted to S induces X.

Surprisingly enough, we can make this definition work in the cases interesting to us:

Theorem 5.1: *Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that S has codimension one in M and that*

(a) *f is tangential to S, or that*
(b) *S is comfortably embedded into M.*

Then we can use (5.4) *to define a Camacho-Sad action on S along X_f in case* (a), *along $H_{\sigma,f}$ in case* (b) *when $\nu_f > 1$, and along $H^1_{\sigma,f}$ in case* (b) *when $\nu_f = 1$.*

Proof: We shall denote by X the section X_f, $H_{\sigma,f}$ or $H^1_{\sigma,f}$ depending on the case we are considering. Let $\mathfrak{U}$ be an atlas adapted to S, comfortable and adapted to the splitting morphism σ in case (b), and let $\mathcal{X}$ be the local extension of X defined in a chart belonging to $\mathfrak{U}$ by Definition 3.1 (respectively, Definition 4.1). We first prove that the right-hand side of (5.4) does not depend on the chart chosen. Take (U,z), $(\hat{U},\hat{z}) \in \mathfrak{U}$ local charts about $p \in S$. Using Lemma 4.1 and Proposition 4.2 we get

$$[\hat{\mathcal{X}}(\tilde{u}), \tilde{s}] = [(\mathcal{X} + T_1 + R_2)(\tilde{u}), \tilde{s}] = [\mathcal{X}(\tilde{u}) + T_1 + R_2, \tilde{s}] = [\mathcal{X}(\tilde{u}), \tilde{s}] + T_0 + R_1,$$

where T_0 represents a local section of TM that restricted to S is tangent to it. Thus

$$\pi\big([\hat{\mathcal{X}}(\tilde u),\tilde s]|_S\big)=\pi\big([\mathcal{X}(\tilde u),\tilde s]|_S\big),$$

as desired.

Now we must show that the right-hand side of (5.4) does not depend on the extensions of s and u chosen. Now, if $\tilde s'$ and $\tilde u'$ are other extensions of s and u respectively, we have $(\tilde s'-\tilde s)|_S=T_0$, while $(\tilde u'-\tilde u)|_S$ is a sum of terms of the form $V_1\otimes\cdots\otimes V_{\nu_f}$ with at least one V_ℓ tangent to S. Therefore $\mathcal{X}(\tilde u'-\tilde u)|_S=O$ and

$$[\mathcal{X}(\tilde u'),\tilde s']|_S=[\mathcal{X}(\tilde u),\tilde s]|_S+[\mathcal{X}(\tilde u),\tilde s'-\tilde s]|_S+[\mathcal{X}(\tilde u'-\tilde u),\tilde s]|_S+[\mathcal{X}(\tilde u'-\tilde u),\tilde s'-\tilde s]|_S=[\mathcal{X}(\tilde u),\tilde s]|_S+T_0,$$

so that $\pi\big([\mathcal{X}(\tilde u'),\tilde s']|_S\big)=\pi\big([\mathcal{X}_f(\tilde u),\tilde s]|_S\big)$, as wanted.

We are left to show that $\tilde X$ is actually an action. Take $g\in\mathcal{O}_S$, and let $\tilde g\in\mathcal{O}_M|_S$ be any extension. First of all,

$$\tilde X(s)(gu)=\pi\big([\mathcal{X}(\tilde g\tilde u),\tilde s]|_S\big)=g\tilde X(s)(u)-\tilde s(\tilde g)|_S\pi\big(X(u)\big)=g\tilde X(s)(u),$$

and so $\tilde X$ is a morphism. Finally, (5.1) yields

$$\mathcal{X}(\tilde u)(\tilde g)|_S=X^\#(g)(u),$$

and so

$$\tilde X(gs)(u)=\pi\big([\mathcal{X}(\tilde u),\tilde g\tilde s]|_S\big)=g\tilde X(s)(u)+\mathcal{X}(\tilde u)(\tilde g)|_S\,s=g\tilde X(s)(u)+X^\#(g)(u)s,$$

and we are done. □

Remark 5.2: If $\nu_f=1$ and f is not tangential then (5.4) with $\mathcal{X}=\mathcal{H}_{\sigma,f}$ does not define an action. This is the reason we introduced the new section $H^1_{\sigma,f}$ and its extension $\mathcal{H}^1_{\sigma,f}$.

Later it will be useful to have an expression of $\tilde X_f,\tilde H_{\sigma,f}$ and $\tilde H^1_{\sigma,f}$ in local coordinates. Let then (U,z) be a local chart belonging to a (comfortable, if necessary) atlas adapted to S, so that $\{\partial_1\}$ is a local frame for N_S, and $\{(\omega^1)^{\otimes\nu_f}\otimes\partial_1\}$ is a local frame for $(N_S^{\otimes\nu_f})^*\otimes N_S$. There is a holomorphic function M_f such that

$$\tilde X_f(\partial_1)(\partial_1^{\otimes\nu_f})=M_f\partial_1.$$

Now, recalling (3.2),

$$\tilde X_f(\partial_1)(\partial_1^{\otimes\nu_f})=\pi\left(\left[\mathcal{X}_f\left((\frac{\partial}{\partial z^1})^{\otimes\nu_f}\right),\frac{\partial}{\partial z_1}\right]\bigg|_S\right)=\pi\left(\left[g^j\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^1}\right]\bigg|_S\right)=-\left.\frac{\partial g^1}{\partial z^1}\right|_S\partial_1,$$

and so

$$M_f=-\left.\frac{\partial g^1}{\partial z^1}\right|_S. \tag{5.5}$$

In particular, recalling that f is tangential we can write $g^1=z^1h^1$, and hence (5.5) yields

$$M_f=-h^1|_S. \tag{5.6}$$

Similarly, if we write $\tilde H_{\sigma,f}(\partial_1)(\partial_1^{\otimes\nu_f})=M_{\sigma,f}\partial_1$ and $\tilde H^1_{\sigma,f}(\partial_1)(\partial_1)=M^1_{\sigma,f}\partial_1$, we obtain

$$M_{\sigma,f}=M^1_{\sigma,f}=-h^1|_S, \tag{5.7}$$

where h^1 is defined by $f^1-z^1=b^1(z^1)^{\nu_f}+h^1(z^1)^{\nu_f+1}$.

Following ideas originally due to Baum and Bott (see [BB]), we can also introduce a holomorphic action on the virtual bundle $TS-N_S^{\otimes\nu_f}$. But let us first define what we mean by a holomorphic action on such a bundle.

Definition 5.4: Let S^0 be an open dense subset of a complex manifold S, F a vector bundle on S, $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$, W a vector bundle over S^0 and $\tilde{W}$ any extension of W over S in K-theory. Then we say that F *acts holomorphically on* $\tilde{W}$ along X if $F|_{S^0}$ acts holomorphically on W along $X|_{S^0}$.

Let S be a codimension one submanifold of M and take $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, as usual. If f is tangential set $X = X_f$. If not, assume that S is comfortably embedded in M and set $X = H_{\sigma,f}$ or $X = H^1_{\sigma,f}$ according to the value of ν_f; in this case, we shall also assume that $X \not\equiv O$. Set $S^0 = S \setminus \mathrm{Sing}(X)$, and let $\mathcal{Q}_f = \mathcal{T}_S / X(\mathcal{N}_S^{\otimes \nu_f})$. The sheaf $\mathcal{Q}_f$ is a coherent analytic sheaf which is locally free over S^0. The associated vector bundle (over S^0) is denoted by Q_f and it is called the *normal bundle of* f. Then the virtual bundle $TS - N_S^{\otimes \nu_f}$, represented by the sheaf $\mathcal{Q}_f$, is an extension (in the sense of K-theory) of Q_f.

Definition 5.5: A *Baum-Bott action* on S is a holomorphic action of $N_S^{\otimes \nu}$ on the virtual bundle $TS - N_S^{\otimes \nu}$ along a section X of $\mathcal{T}_S \otimes N_S^{\otimes \nu}$, for a suitable $\nu \geq 1$.

Theorem 5.2: *Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that S has codimension one in M, and that either f is tangential to S (and then set $X = X_f$) or S is comfortably embedded into M (and then set $X = H_{\sigma,f}$ or $X = H^1_{\sigma,f}$ according to the value of ν_f). Assume moreover that $X \not\equiv 0$. Then we can define a Baum-Bott action $\tilde{B}: \mathcal{Q}_f \to (\mathcal{N}_S^{\otimes \nu_f})^* \otimes \mathcal{Q}_f$ of $N_S^{\otimes \nu_f}$ on $TS - N_S^{\otimes \nu_f}$ along X by setting*

$$\tilde{B}(s)(u) = \pi_f([X(u), \tilde{s}]) \tag{5.8}$$

where $\pi_f: \mathcal{T}_S \to \mathcal{Q}_f$ is the natural projection, and $\tilde{s} \in \mathcal{T}_S$ is any section such that $\pi_f(\tilde{s}) = s$.

Proof: If $\hat{s} \in \mathcal{T}_S$ is another section such that $\pi_f(\hat{s}) = s$ we have $\hat{s} - \tilde{s} \in X(\mathcal{N}_S^{\otimes \nu_f})$; hence $\pi_f([X(u), \hat{s} - \tilde{s}]) = O$, and (5.8) does not depend on the choice of $\tilde{s}$. Finally, one can easily check that we get a holomorphic action on S^0. □

Remark 5.3: Since S has codimension one, $X: N_S^{\otimes \nu_f} \to TS$ yields a (possibly singular) holomorphic foliation on S, and the previous action coincides with the one used in [BB] for the case of foliations.

We can also define a third holomorphic action, on the virtual bundle $TM|_S - N_S^{\otimes \nu_f}$. Assume that f is tangential, and let $S^0 = S \setminus \mathrm{Sing}(X_f)$, as before. Then the sheaf $\mathcal{W}_f = \mathcal{T}_{M,S} / X_f(\mathcal{N}_S^{\otimes \nu_f})$ is a coherent analytic sheaf, locally free over S^0; let $W_f = TM|_{S^0} / X_f(N_S^{\otimes \nu_f}|_{S^0})$ be the associated vector bundle over S^0. Then the virtual bundle $TM|_S - N_S^{\otimes \nu_f}$, represented by the sheaf $\mathcal{W}_f$, is an extension (in the sense of K-theory) of W_f.

Definition 5.6: A *Lehmann-Suwa action* on S is a holomorphic action of $N_S^{\otimes \nu}$ on $TM|_S - N_S^{\otimes \nu}$ along a section X of $\mathcal{T}_S \otimes N_S^{\otimes \nu}$, for a suitable $\nu \geq 1$.

Again, the name is chosen to honor the ones who first discovered the analogous action for holomorphic foliations in any dimension; see [LS], [LS2] (and [KS] for dimension two).

To present an example of such an action we first need a definition.

Definition 5.7: Let S be a codimension one comfortably embedded submanifold of M, and choose a comfortable atlas $\mathfrak{U}$ adapted to a splitting morphism $\sigma: N_S \to TM|_S$. If $v \in (\mathcal{N}_S^{\otimes \nu})_p$ and $(U, \varphi) \in \mathfrak{U}$ is a chart about $p \in S$, we can write $v = \lambda(z'')\partial_1^{\otimes \nu}$ for a suitable $\lambda \in \mathcal{O}(U \cap S)$. Then the *local extension of* v *along the fibers of* σ is the local section $\tilde{v} = \lambda(z'')(\partial/\partial z^1)^{\otimes \nu} \in (\mathcal{T}_M|_S^{\otimes \nu})_p$.

If $(\hat{U}, \hat{z})$ is another chart in $\mathfrak{U}$ about p, and $v \in (\mathcal{N}_S^{\otimes \nu})_p$, we can also write $v = \hat{\lambda}\hat{\partial}_1^{\otimes \nu}$, and we clearly have $\hat{\lambda} = (a|_S)^\nu \lambda$. But since S is comfortably embedded in M we also have

$$\left. \frac{\partial(\hat{\lambda} - a^\nu \lambda)}{\partial z^1} \right|_S \equiv 0,$$

and thus

$$a^\nu \lambda = \hat{\lambda} + R_2.$$

Therefore if $\hat v$ denotes the local extension of v along the fibers of σ in the chart $(\hat U,\hat\varphi)$ we have

$$\hat v=\hat\lambda\left(\frac{\partial}{\partial\hat z^1}\right)^{\otimes\nu}=a^\nu\lambda\frac{\partial z^{h_1}}{\partial\hat z^1}\cdots\frac{\partial z^{h_\nu}}{\partial\hat z^1}\frac{\partial}{\partial z^{h_1}}\otimes\cdots\otimes\frac{\partial}{\partial z^{h_\nu}}+R_2=\tilde v+T_1+R_2, \tag{5.9}$$

where

$$T_1=a\lambda\sum_{\ell=1}^{\nu}\frac{\partial z^p_\ell}{\partial\hat z^1}\frac{\partial}{\partial z^1}\otimes\cdots\otimes\frac{\partial}{\partial z^{p_\ell}}\otimes\cdots\otimes\frac{\partial}{\partial z^1}.$$

Hence:

Theorem 5.3: *Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$, be given. Assume that S is of codimension one and comfortably embedded in M, and that f is tangential with $\nu_f>1$. Let $\rho_f\colon\mathcal{T}_{M,S}\to\mathcal{W}_f$ be the natural projection. Then we can define a Lehmann-Suwa action $\tilde V\colon\mathcal{W}_f\to(\mathcal{N}_S^{\otimes\nu_f})^*\otimes\mathcal{W}_f$ of $N_S^{\otimes\nu_f}$ on $TM|_S-N_S^{\otimes\nu_f}$ along X_f by setting*

$$\tilde V(s)(v)=\rho_f([\mathcal{X}_f(\tilde v),\tilde s]|_S), \tag{5.10}$$

for $s\in\mathcal{W}_f$ and $v\in\mathcal{N}_S^{\otimes\nu}$, where $\tilde s$ is any element in $\mathcal{T}_M|_S$ such that $\rho_f(\tilde s|_S)=s$, and $\tilde v\in\mathcal{T}_M|_S^{\otimes\nu_f}$ is an extension of v constant along the fibers of a splitting morphism σ.

Proof: Since $\mathcal{X}_f(\tilde v)|_S\in\mathcal{T}_S$ then clearly (5.10) does not depend on the extension $\tilde s$ chosen. Using (5.9) and (4.7), since f tangential implies $\mathcal{X}_f=\mathcal{H}_{\sigma,f}$ and $T_1^1=R_2$, we have

$$[\hat{\mathcal{X}}_f(\hat v),\tilde s]=[(\mathcal{X}_f+T_1^o+R_2)(\tilde v+T_1+R_2),\tilde s]=[\mathcal{X}_f(\tilde v),\tilde s]+R_1,$$

and therefore (5.10) does not depend on the comfortable coordinates chosen to define it. Finally, arguing as in Theorem 5.1 we can show that $\tilde V$ actually is a holomorphic action, and we are done. □

6. Index theorems for hypersurfaces

Let S be a compact, globally irreducible, possibly singular hypersurface in a complex manifold M, and set $S'=S\setminus\mathrm{Sing}(S)$. Assume we have the following data:

(a) a line bundle F over S';
(b) a holomorphic section X of $TS'\otimes F^*$;
(c) a vector bundle E defined on M;
(d) a holomorphic action $\tilde X$ of $F|_{S'}$ on $E|_{S'}$ along X.

From these data we can recover a partial connection (in the sense of Bott) on E restricted to $S^0=S'\setminus\mathrm{Sing}(X)$ as follows: since, by definition of S^0, the dual map $X^*\colon\Xi_X^*\to F^*|_{S^0}$ is an isomorphism, we can define a partial connection (in the sense of Bott [Bo]) $D\colon\Xi_X\times H^0(S^0,E|_{S^0})\to H^0(S^0,E|_{S^0})$ by setting

$$D_v(s)=(X^*\otimes\mathrm{id})^{-1}\big(\tilde X(s)\big)(v)$$

for $p\in S^0$, $v\in(\Xi_X)_p$ and $s\in H^0(S^0,E|_{S^0})$. Furthermore, we can always extend this partial connection D to a $(1,0)$-connection on $E|_{S^0}$, for instance using a partition of unity (see, e.g., [BB]). Any such connection (which is a Ξ_X*-connection* in the terminology of [Bo, Su]) will be said *induced* by the holomorphic action $\tilde X$.

We can then apply the general theory developed by Lehmann and Suwa for foliations (see in particular Theorem 1′ and Proposition 4 of [LS], as well as [Su]) to get the following

Theorem 6.1: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M, and set $S'=S\setminus\mathrm{Sing}(S)$. Let F be a line bundle over S' admitting an extension to M, and X a holomorphic section of $TS'\otimes F^*$. Set $S^0=S'\setminus\mathrm{Sing}(X)$, and let $\mathrm{Sing}(S)\cup\mathrm{Sing}(X)=\bigcup_\lambda\Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S)\cup\mathrm{Sing}(X)$ in connected components. Finally, let E be a vector bundle defined on M. Then for any holomorphic action $\tilde X$ of $F|_{S'}$ on $E|_{S'}$ along X and any homogeneous symmetric polynomial φ of degree $n-1$ we can define complex numbers $\mathrm{Res}_\varphi(\tilde X,E,\Sigma_\lambda)\in\mathbb{C}$, depending only on the local behavior of $\tilde X$ and E near Σ_λ, such that*

$$\sum_\lambda\mathrm{Res}_\varphi(\tilde X,E,\Sigma_\lambda)=\int_S\varphi(E),$$

where $\varphi(E)$ is the evaluation of φ on the Chern classes of E.

Recalling the results of the previous section, we then get the following index theorem for holomorphic self-maps:

Theorem 6.2: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that*

(a) *f is tangential to S, and then set $X = X_f$, or that*
(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into M, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

Assume moreover $X \not\equiv O$. Let $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ in connected components. Finally, let $[S]$ be the line bundle on M associated to the divisor S. Then there exist complex numbers $\mathrm{Res}(X,S,\Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of X and $[S]$ near Σ_λ, such that

$$\sum_\lambda \mathrm{Res}(X,S,\Sigma_\lambda) = \int_S c_1^{n-1}([S]).$$

Proof: By Theorem 5.1 we have a Camacho-Sad action on S along X on N_{S^0}. Since $[S]$ is an extension to M of N_{S^0}, we can apply Theorem 6.1. □

Remark 6.1: If M has dimension two, and S has at least one singularity or X_f has at least one zero, then $S' \setminus \mathrm{Sing}(f)$ is *always* comfortably embedded in M. Indeed, it is an open Riemann surface; so $H^1(S' \setminus \mathrm{Sing}(f), \mathcal{F}) = O$ for any coherent analytic sheaf $\mathcal{F}$, and the result follows from Proposition 2.1 and Theorem 2.2.

In a similar way, applying [Su, Theorem IV.5.6], Theorem 5.3, and recalling that $\varphi(H-L) = \varphi(H \otimes L^*)$ for any vector bundle H, line bundle L and homogeneous symmetric polynomial φ, we get

Theorem 6.3: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that $S' = S \setminus \mathrm{Sing}(S)$ is comfortably embedded into M, and that f is tangential to S with $\nu_f > 1$. Let $\mathrm{Sing}(S) \cup \mathrm{Sing}(X_f) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X_f)$ in connected components. Finally, let $[S]$ be the line bundle on M associated to the divisor S. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of X_f and $TM|_S - [S]^{\otimes \nu_f}$ near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \Sigma_\lambda) = \int_S \varphi\big(TM|_S \otimes ([S]^*)^{\otimes \nu_f}\big).$$

Finally, applying the Baum-Bott index theorem (see, [Su, Theorem III.7.6]) and Theorem 5.2 we get

Theorem 6.4: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that*

(a) *f is tangential to S, and then set $X = X_f$, or that*
(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into M, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

Assume moreover $X \not\equiv O$. Let $\mathrm{Sing}(S) \cup \mathrm{Sing}(X) = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$ in connected components. Finally, let $[S]$ be the line bundle on M associated to the divisor S. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $\mathrm{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of X and $TS - [S]^{\otimes \nu_f}$ near Σ_λ, such that

$$\sum_\lambda \mathrm{Res}_\varphi(X, TS - [S]^{\otimes \nu_f}, \Sigma_\lambda) = \int_S \varphi\big(TS \otimes ([S]^*)^{\otimes \nu_f}\big).$$

So we have recovered three main index theorems of foliation theory in the setting of holomorphic self-maps fixing pointwise a hypersurface.

Clearly, these index theorems are as useful as explicit are the formulas for the computation of the residues; the rest of this section is devoted to deriving such formulas in many important cases.

Let us first describe the general way these residues are defined in Lehmann-Suwa theory. Assume we are in the hypothesis of Theorem 6.1. Let $\tilde U_0$ be a tubular neighborhood of S^0 in M, and denote by $\rho\colon \tilde U_0 \to S^0$

the associated retraction. Given any connection D on $E|_{S^0}$ induced by the holomorphic action $\tilde{X}$ of F along X, set $D^0 = \rho^*(D)$. Next, choose an open set $\tilde{U}_\lambda \subset M$ such that $\tilde{U}_\lambda \cap \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(X)\big) = \Sigma_\lambda$, and a compact real $2n$-dimensional manifold $\tilde{R}_\lambda \subset \tilde{U}_\lambda$ with C^∞ boundary containing Σ_λ in its interior and such that $\partial\tilde{R}_\lambda$ intersects S transversally. Let D^λ be any connection on $E|_{\tilde{U}_\lambda}$, and denote by $B\big(\varphi(D^0), \varphi(D^\lambda)\big)$ the Bott difference form of $\varphi(D^0)$ and $\varphi(D^\lambda)$ on $\tilde{U}_0 \cap \tilde{U}_\lambda$. Then (see [LS] and [Su, Chapter IV])

$$\mathrm{Res}_\varphi(\tilde{X}, E, \Sigma_\lambda) = \int_{R_\lambda} \varphi(D^\lambda) - \int_{\partial R_\lambda} B\big(\varphi(D^0), \varphi(D^\lambda)\big), \tag{6.1}$$

where $R_\lambda = \tilde{R}_\lambda \cap S$. A similar formulas holds for virtual vector bundles too; see again [Su, Chapter IV].

Remark 6.2: When $\Sigma_\lambda = \{x_\lambda\}$ is an isolated singularity of S, the second integral in (6.1) is taken on the link of x_λ in S. In particular if S is not irreducible at x_k then the residue is the sum of several terms, one for each irreducible component of S at x_k.

We now specialize (6.1) to our situations. Let us begin with the Camacho-Sad action: we shall compute the residues for connected components Σ_λ reduced to an isolated point x_λ. Let again $[S]$ be the line bundle associated to the divisor S, and choose an open set $\tilde{U}_\lambda \subset M$ containing x_λ so that $\tilde{U}_\lambda \cap \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(X)\big) = \{x_\lambda\}$ and $[S]$ is trivial on $\tilde{U}_\lambda$; take then as D^λ the trivial connection for $[S]$ on W with respect to some frame. In particular, then $\varphi(D^\lambda) = O$ on R_λ. By (6.1) the residue is then obtained simply by integrating $B\big(\varphi(D^0), \varphi(D^\lambda)\big)$ over ∂R_λ. Notice furthermore that since $[S]$ is a line bundle there is only one non-trivial φ to consider: the $(n-1)$-th power of the linear symmetric function, so that $\varphi(D) = c_1^{n-1}([S])$.

Let η^j be a connection one-form of D^j, for $j = 0$, λ; with respect to a suitable frame for $[S]$ we can assume that $\eta^\lambda \equiv O$. Let

$$\tilde{\eta} := t\eta^0 + (1-t)\eta^\lambda = t\eta^0,$$

and let $\tilde{K} := d\tilde{\eta} + \tilde{\eta} \wedge \tilde{\eta} = d\tilde{\eta}$. From the very definition of the Bott difference form, it follows that

$$B\big(\varphi(D^0), \varphi(D^\lambda)\big) = \left(\frac{1}{2\pi i}\right)^{n-1} \int_0^1 \tilde{K}^{n-1}.$$

A straightforward computation shows that

$$\tilde{K}^{n-1} = (n-1)t^{n-2} dt \wedge \eta^0 \wedge \overbrace{d\eta^0 \wedge \cdots \wedge d\eta^0}^{n-2} + N,$$

where N is a term not containing dt. Therefore

$$B\big(\varphi(D^0), \varphi(D^\lambda)\big) = \left(\frac{1}{2\pi i}\right)^{n-1} \eta^0 \wedge \overbrace{d\eta^0 \wedge \cdots \wedge d\eta^0}^{n-2}. \tag{6.2}$$

Assume now that $x_\lambda \in \mathrm{Sing}(X)$ and S is smooth at x_λ. Up to shrinking $\tilde{U}_\lambda$ we may assume that $\tilde{U}_\lambda$ is the domain of a chart z adapted to S (and belonging to a comfortable atlas if necessary), so that $\{\partial_1\}$ is a local frame for N_{S^0}, and $\{dz^2, \ldots, dz^n\}$ is a local frame for T^*S^0. Then any connection D induced by the Camacho-Sad action is locally represented by the (1,0)-form η^0 such that $D(\partial_1) = \eta^0 \otimes \partial_1$. To compute η^0, we first of all notice that $X = g^p \frac{\partial}{\partial z^p} \otimes (\omega^1)^{\otimes \nu_f}$, if $X = X_f$ or $X = H_{\sigma,f}$, and $X = (1+b^1) g^p \frac{\partial}{\partial z^p} \otimes \omega^1$ if $X = H^1_{\sigma,f}$. Then, when X is X_f or $H_{\sigma,f}$ we have

$$(X^*)^{-1}((\omega^1)^{\otimes \nu_f}) = \frac{1}{g^p}\, dz^p,$$

while when $X = H^1_{\sigma,f}$ we have

$$(X^*)^{-1}((\omega^1)^{\otimes \nu_f}) = \frac{1}{(1+b^1) g^p}\, dz^p.$$

Therefore, recalling formulas (5.6) and (5.7), we can choose D so that when X is X_f or $H_{\sigma,f}$ we have

$$\eta^0 = (X^* \otimes \mathrm{id})^{-1}\bigl(\tilde{X}(\partial_1)\bigr) = -\left.\frac{h^1}{g^p}\right|_S dz^p, \tag{6.3}$$

while when $X = H^1_{\sigma,f}$ we have

$$\eta^0 = (X^* \otimes \mathrm{id})^{-1}\bigl(\tilde{H}^1_{\sigma,f}(\partial_1)\bigr) = -\left.\frac{h^1}{(1+b^1)g^p}\right|_S dz^p. \tag{6.4}$$

Remark 6.3: When $n = 2$ and $X = X_f$ we recover the connection form obtained in [Br]. The form η introduced in [A2], which is the opposite of η^0, is the connection form of the dual connection on $N^*_{S^0}$, by [A2, (1.7)]. Since the definition of Chern class implicitly used in [A2] is the opposite of the one used in [Br] everything is coherent. Finally, when $n = 2$ and $X = H^1_{\sigma,f}$ we have obtained the correct multiple of the form η introduced in [A2] when S was the smooth zero section of a line bundle (notice that $1 + b^1$ is constant because S is compact, and that the form η of [A2] must be divided by $b = 1 + b^1$ to get a connection form).

Now we can take $R_1 = \{|g^p(x)| \le \varepsilon \mid p = 2, \ldots, n\}$ for a suitable $\varepsilon > 0$ small enough. In particular, if we set $\Gamma = \{|g^p(x)| = \varepsilon \mid p = 2, \ldots, n\} \cap S$, oriented so that $d\theta^2 \wedge \cdots \wedge d\theta^n > 0$ where $\theta^p = \arg(g^p)$, then arguing as in [L, Section 5] or [LS, Section 4] (see also [Su, pp.105–107]) we obtain

$$\mathrm{Res}(X, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n, \tag{6.5}$$

when $X = X_f$ or $H_{\sigma,f}$, while when $X = H^1_{\sigma,f}$ we have

$$\mathrm{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{(1+b^1)^{n-1} g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n. \tag{6.6}$$

Remark 6.4: For $n = 2$, formulas (6.5) and (6.6) give the indices defined in [A2]. Thus, if S is smooth, Theorem 6.2 implies the index theorem of [A2], because $c_1([S]) = c_1(N_S)$. In an analogous way, Lehmann and Suwa (see [L, LS, LS2]) proved that the Camacho-Sad index theorem also is a consequence of Theorem 6.1.

When x_λ is an isolated singular point of S the computation of the residue is more complicated, because one cannot apply directly the results in [LS] as before for in general there is no natural extension of Ξ_X and the Camacho-Sad action to $\mathrm{Sing}(S)$. However we are able to compute explicitly the index in this case too when $n = 2$, and when $n > 2$ and f is tangential with $\nu_f > 1$.

If $n = 2$ we can choose local coordinates $\{(w^1, w^2)\}$ in $\tilde{U}_\lambda$ so that $S \cap \tilde{U}_\lambda = \{l(w^1, w^2) = 0\}$ for some holomorphic function l, and $dl \wedge dw^2 \ne 0$ on $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$. In particular (l, w^2) are local coordinates adapted to S^0 near $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$ and $\frac{\partial}{\partial l}$ can be chosen as a local frame for N_{S^0} on ∂R_1.

Remark 6.5: When S^0 is comfortably embedded in M the chart (l, w^2) should belong to a comfortable atlas. Studying the proofs of Propositions 2.1 and of Theorem 2.2 one sees that this is possible up to replacing l by a function of the form $\hat{l} = \bigl(1 + c(w^2)l\bigr)l$, where c is a holomorphic function defined on $S \cap \tilde{U}_\lambda \setminus \{x_\lambda\}$. Since to compute the residues we only need the behavior of l and w^2 near ∂R_1, it is easy to check that using $\hat{l}$ or l in the following computations yield the same results. So for the sake of simplicity we shall not distinguish between l and $\hat{l}$ in the sequel.

Up to shrinking $\tilde{U}_\lambda$, we can again assume that $[S]$ is trivial on $\tilde{U}_\lambda$. The function l is a local generator of $\mathcal{I}_S$ on $\tilde{U}_\lambda$. Then the dual of $[l] \in \mathcal{I}_S/\mathcal{I}_S^2$, denoted by s, is a holomorphic frame of $[S]$ on $\tilde{U}_\lambda$ which extends the holomorphic frame $\frac{\partial}{\partial l}$ of $N_{S'}$ (see [Su, p.86]). In particular $s|_{\partial R_1} = \frac{\partial}{\partial l}$. We then choose on $[S]|_{\tilde{U}_\lambda}$ the trivial connection with respect to s, so that $\eta^\lambda = O$. We are left with the computation of the form η^0 nearby ∂R^1. But if $X = X_f$ or $X = H_{\sigma,f}$ we can apply (6.3) to get

$$\eta^0|_{\partial R_1} = -\left.\frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)}\right|_{\partial R_1} dw^2,$$

where

$$b^1 = \left.\frac{l\circ f - l}{l^{\nu_f}}\right|_S$$

is identically zero when f is tangential. On the other hand, when $X = H^1_{\sigma,f}$ applying (6.4) we get

$$\eta^0|_{\partial R_1} = -\left.\frac{(l\circ f - l) - b^1 l}{(l + (l\circ f - l))(w^2\circ f - w^2)}\right|_{\partial R_1} dw^2.$$

Hence the residue is

$$\mathrm{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i}\int_{\partial R_1} \left.\frac{(l\circ f - l) - b^1 l^{\nu_f}}{l\cdot(w^2\circ f - w^2)}\right|_S dw^2, \tag{6.7}$$

when $X = X_f$ or $X = H_{\sigma,f}$, while when $X = H^1_{\sigma,f}$ we have

$$\mathrm{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \frac{1}{2\pi i}\int_{\partial R_1} \left.\frac{(l\circ f - l) - b^1 l}{(l + (l\circ f - l))(w^2\circ f - w^2)}\right|_S dw^2. \tag{6.8}$$

Remark 6.6: When f is tangential we have $b^1 \equiv 0$; therefore the formula (6.7) gives the index defined in [BT], and Theorem 6.2 implies the index theorem of [BT].

When $n > 2$, f is tangential and $\nu_f > 1$, we can define a local vector field $\tilde v_f$ which generates the Camacho-Sad action $\tilde X_f$ and compute explicitly the residue even at a singular point x_λ of S. To see this, assume $(w^1, \dots, w^n)$ are local coordinates in $\tilde U_\lambda$ so that $S\cap \tilde U_\lambda = \{l(w^1,\dots,w^n) = 0\}$ for some holomorphic function l. Define the vector field $\tilde v_f$ on $\tilde U_\lambda$ by

$$\tilde v_f = \frac{w^1\circ f - w^1}{l^{\nu_f}}\frac{\partial}{\partial w^1} + \dots + \frac{w^n\circ f - w^n}{l^{\nu_f}}\frac{\partial}{\partial w^n}. \tag{6.9}$$

We claim that the "holomorphic action" $\theta_{\tilde v_f}$ in the sense of Bott [Bo] of $\tilde v_f$ on $N_{S'}$ as defined in [LS, p.177] coincides with our Camacho-Sad action, and thus we can apply [LS, Theorem 1] to compute the residue. To prove this we consider $W_1 = \{x\in \tilde U_\lambda | \frac{\partial l}{\partial w^1}(x) \neq 0\}$. On this open set we make the following change of coordinates:

$$\begin{cases} z^1 = l(w^1,\dots,w^n), & \\ z^p = w^p & \text{for } p = 2,\dots,n. \end{cases}$$

The new coordinates $(z^1,\dots,z^n)$ are adapted to S on W_1. If $f^j = z^j + g^j (z^1)^{\nu_f}$ as usual, we have

$$w^p\circ f - w^p = g^p (z^1)^{\nu_f}, \tag{6.10}$$

and

$$w^1\circ f - w^1 = \frac{\partial w^1}{\partial z^j} g^j (z^1)^{\nu_f} + R_{2\nu_f} = \left(\frac{\partial l}{\partial w^1}\right)^{-1}\left[g^1 - \frac{\partial l}{\partial w^p} g^p\right](z^1)^{\nu_f} + R_{2\nu_f}. \tag{6.11}$$

Therefore, from (6.10) and (6.11), taking into account that $\nu_f > 1$, we get

$$\tilde v_f = \left(\frac{w^1\circ f - w^1}{(z^1)^{\nu_f}}\frac{\partial l}{\partial w^1} + \frac{w^p\circ f - w^p}{(z^1)^{\nu_f}}\frac{\partial l}{\partial w^p}\right)\frac{\partial}{\partial z^1} + \frac{w^q\circ f - w^q}{(z^1)^{\nu_f}}\frac{\partial}{\partial z^q} = \mathcal{X}_f(\partial_1^{\otimes \nu_f}) + R_2, \tag{6.12}$$

which gives the claim on W_1. Since the same holds on each $W_j = \{x\in\tilde U_\lambda | \frac{\partial l}{\partial w^j}(x)\neq 0\}$, $j = 1,\dots,n$ and $(\tilde U_\lambda\cap S)\setminus\{x_\lambda\} = \bigcup_j W_j$, it follows that the Bott holomorphic action induced by $\tilde v_f$ is the same as the Camacho-Sad action given by $\tilde X_f$. Thus, if we choose — as we can — the coordinates $(w^1,\dots,w^n)$ as in [LS, Theorem 2], that is so that $\{l, (w^p\circ f - w^p)/l^{\nu_f}\}$ form a regular sequence at x_λ, the residue is expressed by the formula after [LS, Theorem 2]. Taking into account that, for f is tangent to S and by (6.12), the function l divides $dl(\tilde v_f)$, we get

$$\mathrm{Res}(X_f, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi i}\right)^{n-1}\int_\Gamma \frac{\left[\sum_{j=1}^n \frac{\partial l}{\partial w^j}(w^j\circ f - w^j)\right]^{n-1}}{l^{n-1}\prod_{p=2}^n (w^p\circ f - w^p)}\, dw^2\wedge\dots\wedge dw^n, \tag{6.13}$$

where this time

$$\Gamma = \left\{ w \in \tilde U_\lambda \;\middle|\; \left| \frac{w^p \circ f - w^p}{l^{\nu_f}}(w) \right| = \epsilon,\ l(w) = 0 \right\},$$

for a suitable $0 < \epsilon << 1$, and Γ is oriented as usual (in particular $\Gamma = (-1)^{[\frac{n-1}{2}]} R_{u_0}$ where R_{u_0} is the set as defined in [LS, Theorem 2]).

Note that for $n = 2$ we recover, when $\nu_f > 1$, formula (6.7). On the other hand, if x_λ is non-singular for S then the previous argument with $l = w^1$ works for $\nu_f = 1$ as well, and we get formula (6.5).

Summing up, we have proved the following

Theorem 6.5: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given. Assume that*

(a) *f is tangential to S, and then set $X = X_f$, or that*
(b) *$S^0 = S \setminus \big(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\big)$ is comfortably embedded into M, and then set $X = H_{\sigma,f}$ if $\nu_f > 1$, or $X = H^1_{\sigma,f}$ if $\nu_f = 1$.*

Assume $X \not\equiv O$. Let $x_\lambda \in S$ be an isolated point of $\mathrm{Sing}(S) \cup \mathrm{Sing}(X)$. Then the number $\mathrm{Res}(X, S, \{x_\lambda\}) \in \mathbb{C}$ introduced in Theorem 6.2 is given

(i) *if $x_\lambda \in \mathrm{Sing}(X) \cap (S \setminus \mathrm{Sing}(S))$, and f is tangential or S^0 is comfortably embedded in M and $\nu_f > 1$, by*

$$\mathrm{Res}(X, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n;$$

(ii) *if $x_\lambda \in \mathrm{Sing}(X) \cap (S \setminus \mathrm{Sing}(S))$, S^0 is comfortably embedded in M and $\nu_f = 1$, by*

$$\mathrm{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi}\right)^{n-1} \int_\Gamma \frac{(h^1)^{n-1}}{(1+b^1)^{n-1} g^2 \cdots g^n}\, dz^2 \wedge \cdots \wedge dz^n;$$

(iii) *if $n = 2$, $x_\lambda \in \mathrm{Sing}(S)$, and f is tangential or S^0 is comfortably embedded in M and $\nu_f > 1$, by*

$$\mathrm{Res}(X, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l^{\nu_f}}{l \cdot (w^2 \circ f - w^2)} \right|_S dw^2;$$

(iv) *if $n = 2$, $x_\lambda \in \mathrm{Sing}(S)$, S^0 is comfortably embedded in M and $\nu_f = 1$, by*

$$\mathrm{Res}(H^1_{\sigma,f}, S, \{x_\lambda\}) = \frac{1}{2\pi i} \int_{\partial R_1} \left. \frac{(l \circ f - l) - b^1 l}{(l + (l \circ f - l))(w^2 \circ f - w^2)} \right|_S dw^2;$$

(v) *if $n > 2$, $x_\lambda \in \mathrm{Sing}(S)$, f is tangential and $\nu_f > 1$, by*

$$\mathrm{Res}(X_f, S, \{x_\lambda\}) = \left(\frac{-i}{2\pi i}\right)^{n-1} \int_\Gamma \frac{\left[\sum_{j=1}^n \frac{\partial l}{\partial w^j}(w^j \circ f - w^j)\right]^{n-1}}{l^{n-1} \prod_{p=2}^n (w^p \circ f - w^p)}\, dw^2 \wedge \cdots \wedge dw^n.$$

Our next aim is to compute the residue for the Lehmann-Suwa action, at least for an isolated smooth point $x_\lambda \in \mathrm{Sing}(X_f)$. Let (W, w) be a local chart about x_λ belonging to a comfortable atlas. Set $l = w^1$ and define $\tilde v_f$ as in (6.9). By (6.12) the Lehmann-Suwa action $\tilde V$ is given by the holomorphic action in the sense of Bott of $\tilde v_f$ on $TM|_S - [S]^{\otimes \nu_f}$. Therefore we can apply [L], [LS] (see also [Su, Theorems IV.5.3, IV.5.6], and [Su, Remark IV.5.7]) to obtain

$$\mathrm{Res}_\varphi(X_f, TM|_S - [S]^{\otimes \nu_f}, \{x_\lambda\}) = \mathrm{Res}_\varphi(X_f, TM|_S, \{x_\lambda\}),$$

where $\mathrm{Res}_\varphi(X_f, TM|_S, \{x_\lambda\})$ is the residue for the local Lie derivative action of $\tilde v_f$ on $TM|_S$ given by

$$\tilde V_l(s)(\tilde v_f) = [\tilde v_f, \tilde s]|_S,$$

where s is a section of $TM|_S$ and $\tilde{s}$ is a local extension of s constant along the fibers of σ.

We can write an expression of $\tilde{V}_l$ in local coordinates. Let (U,z) be a local chart belonging to a comfortable atlas. Then $\{\frac{\partial}{\partial z^1},\ldots,\frac{\partial}{\partial z^n}\}$ is a local frame for TM, and $\{(\omega^1)^{\otimes\nu_f}\otimes\frac{\partial}{\partial z^1}|_S,\ldots,(\omega^1)^{\otimes\nu_f}\otimes\frac{\partial}{\partial z^n}|_S\}$ is a local frame for $(N_S^{\otimes\nu_f})^*\otimes TM|_S$. Thus there exist holomorphic functions V_j^k (for j, $k=1,\ldots,n$) so that

$$\tilde{V}_l(\frac{\partial}{\partial z^j})(\partial_1^{\otimes\nu_f}) = V_j^k\frac{\partial}{\partial z^k}.$$

Now, from (4.4) we get

$$\tilde{V}_l(\frac{\partial}{\partial z^j})(\partial_1^{\otimes\nu_f}) = \left[\mathcal{X}_f\left((\frac{\partial}{\partial z^1})^{\otimes\nu_f}\right),\frac{\partial}{\partial z^j}\right]\Bigg|_S = \left[h^1z^1\frac{\partial}{\partial z^1}+g^p\frac{\partial}{\partial z^p},\frac{\partial}{\partial z^j}\right]\Bigg|_S = -h^1|_S\delta_j^1\frac{\partial}{\partial z^1}-\left.\frac{\partial g^p}{\partial z^j}\right|_S\frac{\partial}{\partial z^p},$$

and hence

$$V_1^1=-h^1|_S,\qquad V_p^1\equiv 0,\qquad V_j^p=-\left.\frac{\partial g^p}{\partial z^j}\right|_S. \tag{6.14}$$

Therefore [Su, Theorem IV.5.3] yields

Theorem 6.6: *Let S be a compact, globally irreducible, possibly singular hypersurface in an n-dimensional complex manifold M. Let $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$, be given. Assume that $S'=S\setminus\mathrm{Sing}(S)$ is comfortably embedded into M, and that f is tangential to S with $\nu_f>1$. Let $x_\lambda\in\mathrm{Sing}(X_f)$ be an isolated smooth point of $\mathrm{Sing}(S)\cup\mathrm{Sing}(X_f)$. Then for any homogeneous symmetric polynomial φ of degree $n-1$ the complex number $\mathrm{Res}_\varphi(X_f,TM|_S-[S]^{\otimes\nu_f},\{x_\lambda\})$ introduced by Theorem 6.3 is given by*

$$\mathrm{Res}_\varphi(X_f,TM|_S-[S]^{\otimes\nu_f},\{x_\lambda\}) = \int_\Gamma\frac{\varphi(V)\,dz^2\wedge\cdots\wedge dz^n}{g^2\cdots g^n}, \tag{6.15}$$

where $V=(V_j^k)$ is the matrix given by (6.14) and Γ is as in (6.5).

Remark 6.7: We adopt here the convention that if V is an $n\times n$ matrix then $c_j(V)$ is the j-th symmetric function of the eigenvalues V multiplied by $(i/2\pi)^j$, and $\varphi(V)=\varphi\big(c_1(V),\ldots,c_{n-1}(V)\big)$.

Finally, if x_λ is an isolated point in $\mathrm{Sing}(S)\cup\mathrm{Sing}(X)$, the complex numbers $\mathrm{Res}_\varphi(X,TS-[S]^{\otimes\nu_f},\{x_\lambda\})$ appearing in Theorem 6.4 can be computed exactly as in the foliation case using a Grothendieck residue with a formula very similar to (6.15); see [BB], [Su, Theorem III.5.5].

7. Index theorems in higher codimension

Let $S\subset M$ be a complex submanifold of codimension $1<m<n$ in a complex n-manifold M. A way to get index theorems for holomorphic self-maps of M fixing pointwise S is to blow-up S and then apply the index theorems for hypersurfaces; this is what we plan to do in this section.

We shall denote by $\pi\colon M_S\to M$ the blow-up of M along S, and by $E_S=\pi^{-1}(S)$ the exceptional divisor, which is an hypersurface in M_S isomorphic to the projectivized normal bundle $\mathbb{P}(N_S)$.

Remark 7.1: If S is singular, the blow-up M_S is in general singular too. So this approach works only for smooth submanifolds.

If (U,z) is a chart adapted to S centered in $p\in S$, in M_S we have m charts $(\tilde{U}_r,w_r)$ centered in $[\partial_1],\ldots,[\partial_m]$ respectively, where if $v\in N_{S,p}$, $v\neq O$, we denote by $[v]$ its projection in $\mathbb{P}(N_S)$. The coordinates z^j and w_r^h are related by

$$z^j(w_r)=\begin{cases} w_r^j & \text{if } j=r,m+1,\ldots,n,\\ w_r^rw_r^j & \text{if } j=1,\ldots,r-1,r+1,\ldots,m.\end{cases}$$

Remark 7.2: We have $\tilde{U}_r\cap E_S=\{w_r^r=0\}$, and thus $(\tilde{U}_r,w_r)$ is adapted to E_S up to a permutation of the coordinates.

Now take $f\in\mathrm{End}(M,S)$, $f\not\equiv\mathrm{id}_M$, and assume that df acts as the identity on N_S (this is automatic if $\nu_f>1$, while if $\nu_f=1$ it happens if and only if f is tangential). Then we can lift f to a unique map

$\tilde f \in \mathrm{End}(M_S, E_S)$, $\tilde f \not\equiv \mathrm{id}_{M_S}$, such that $f \circ \pi = \pi \circ \tilde f$ (see, e.g., [A1] for details). If (U, z) is a chart adapted to S and we set $f^j = z^j \circ f$ and $\tilde f^j_r = w^j_r \circ \tilde f$, we have

$$\tilde f^j_r(w_r) = \begin{cases} f^j\bigl(z(w_r)\bigr) & \text{if } j = r, m+1, \ldots, n, \\ \dfrac{f^j\bigl(z(w_r)\bigr)}{f^r\bigl(z(w_r)\bigr)} & \text{if } j = 1, \ldots, r-1, r+1, \ldots, m. \end{cases} \tag{7.1}$$

If f is tangential we can find holomorphic functions $h^r_{r_1 \ldots r_{\nu_f+1}}$ symmetric in the lower indices such that

$$f^r - z^r = h^r_{r_1 \ldots r_{\nu_f+1}} z^{r_1} \cdots z^{r_{\nu_f+1}} + R_{\nu_f+2}; \tag{7.2}$$

as usual, only the restriction to S of each $h^r_{r_1 \ldots r_{\nu_f+1}}$ is uniquely defined. Set then

$$Y = h^r_{r_1 \ldots r_{\nu_f+1}}|_S\, \partial_r \otimes \omega^{r_1} \otimes \cdots \otimes \omega^{r_{\nu_f+1}};$$

it is a local section of $N_S \otimes (N_S^*)^{\otimes(\nu_f+1)}$.

On the other hand, if f is not tangential set $B = (\pi \otimes \mathrm{id})_* \circ X_f$, where $\pi \colon TM|_S \to N_S$ is the canonical projection. In this way we get a global section of $N_S \otimes (N_S^*)^{\otimes \nu_f}$, not identically zero if and only if f is not tangential, and given in local adapted coordinates by

$$B = g^r_{r_1 \ldots r_{\nu_f}}|_S\, \partial_r \otimes \omega^{r_1} \otimes \cdots \otimes \omega^{r_{\nu_f}}.$$

Definition 7.1: Take $p \in S$. If f is tangential, a non-zero vector $v \in (N_S)_p$ is a *singular direction* for f at p if $X_f(v \otimes \cdots \otimes v) = O$ and $Y(v \otimes \cdots \otimes v) \wedge v = O$. If f is not tangential, v is a *singular direction* if $B(v \otimes \cdots \otimes v) \wedge v = O$.

Remark 7.3: The condition $Y(v \otimes \cdots \otimes v) \wedge v = O$ is equivalent to requiring $Y(v \otimes \cdots \otimes v) = \lambda v$ for some $\lambda \in \mathbb{C}$.

Of course, in the tangential case we must check that this definition is well-posed, because the morphism Y depends on the local coordinates chosen to define it. First of all, if (U, z) is a chart adapted to S and centered at p then $X_f(v \otimes \cdots \otimes v) = O$ when f is tangential means

$$g^p_{r_1 \ldots r_{\nu_f}}(O)\, v^{r_1} \cdots v^{r_{\nu_f}} \frac{\partial}{\partial z^p} = O, \tag{7.3}$$

where $v = v^r \partial_r$. Now let $(\hat U, \hat z)$ be another chart adapted to S centered in p. Then we can find holomorphic functions a^r_s and $\hat a^r_s$ such that $\hat z^r = a^r_s z^s$ and $z^r = \hat a^r_s \hat z^s$. Arguing as in the proof of (4.2) we get

$$a^{r_1}_{s_1} \cdots a^{r_{\nu_f+1}}_{s_{\nu_f+1}} \hat h^r_{r_1 \ldots r_{\nu_f+1}} = a^r_s h^s_{s_1 \ldots s_{\nu_f+1}} + \sum_{\ell=1}^{\nu_f+1} \frac{\partial a^r_{s_\ell}}{\partial z^p} g^p_{s_1 \ldots \hat{s_\ell} \ldots s_{\nu_f+1}} + R_1,$$

where the index with the hat is missing from the list. Therefore

$$\hat Y = Y + \hat a^s_r \sum_{\ell=1}^{\nu_f+1} \frac{\partial a^r_{s_\ell}}{\partial z^p} g^p_{s_1 \ldots \hat{s_\ell} \ldots s_{\nu_f+1}}\Bigg|_S\, \partial_s \otimes \omega^{s_1} \otimes \cdots \otimes \omega^{s_{\nu_f+1}};$$

in particular if $X_f(v \otimes \cdots \otimes v) = O$ equation (7.3) yields

$$\hat Y(v \otimes \cdots \otimes v) = Y(v \otimes \cdots \otimes v),$$

and the notion of singular direction when f is tangential is well-defined.

Proposition 7.1: *Let $S \subset M$ be a complex submanifold of codimension $1 < m < n$ of a complex n-manifold M, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, such that df acts as the identity on N_S (that is f is tangential, or $\nu_f > 1$, or both). Denote by $\pi \colon M_S \to M$ the blow-up of M along S with exceptional divisor E_S, and let $\tilde{f} \in \mathrm{End}(M_S, E_S)$ be the lifted map. Then:*

(i) *if S is comfortably embedded in M then E_S is comfortably embedded in M_S, and the choice of a splitting morphism for S induces a splitting morphism for E_S;*
(ii) *$d\tilde{f}$ acts as the identity on N_{E_S};*
(iii) *$\tilde{f}$ is always tangential; furthermore $\nu_{\tilde{f}} = \nu_f$ if f is tangential, $\nu_{\tilde{f}} = \nu_f - 1$ otherwise;*
(iv) *a direction $[v] \in E_S$ is a singular point for $\tilde{f}$ iff it is a singular direction for f.*

Proof: (i) Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a comfortable atlas adapted to S; we claim that $\tilde{\mathfrak{U}} = \{(\tilde{U}_{\alpha,r}, w_{\alpha,r})\}$ is a comfortable atlas adapted to E_S (and in particular determines a splitting morphism for E_S). Let us first prove that it is a splitting atlas, that is that

$$\left.\frac{\partial w^j_{\beta,s}}{\partial w^r_{\alpha,r}}\right|_{E_S} \equiv 0$$

for every r, s, $j \neq s$ and indices α and β. We have

$$z^j_\beta = z^j_\beta|_S + \left.\frac{\partial z^j_\beta}{\partial z^s_\alpha}\right|_S z^s_\alpha + \frac{1}{2}\left.\frac{\partial^2 z^j_\beta}{\partial z^u_\alpha \partial z^v_\alpha}\right|_S z^u_\alpha z^v_\alpha + R_3.$$

Since $w^r_{\alpha,r} = z^r_\alpha$, if $j = p > m$ we immediately get

$$\left.\frac{\partial w^p_{\beta,s}}{\partial w^r_{\alpha,r}}\right|_{E_S} = \left.\frac{\partial z^p_\beta}{\partial z^r_\alpha}\right|_S \equiv 0,$$

because $\mathfrak{U}$ is a splitting atlas. If $j = t \le m$ we have

$$z^t_\beta = \left.\frac{\partial z^t_\beta}{\partial z^s_\alpha}\right|_S z^s_\alpha + \frac{1}{2}\left.\frac{\partial^2 z^t_\beta}{\partial z^u_\alpha \partial z^v_\alpha}\right|_S z^u_\alpha z^v_\alpha + R_3 = \left[\left.\frac{\partial z^t_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u \neq r}\left.\frac{\partial z^t_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r}\right] w^r_{\alpha,r} + O\big((w^r_{\alpha,r})^3\big), \tag{7.4}$$

because $\mathfrak{U}$ is a comfortable atlas. Therefore if $t \neq s$ we find

$$w^t_{\beta,s} = \frac{z^t_\beta}{z^s_\beta} = \frac{\left.\frac{\partial z^t_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u \neq r}\left.\frac{\partial z^t_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r} + O\big((w^r_{\alpha,r})^2\big)}{\left.\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right|_S + \sum_{u \neq r}\left.\frac{\partial z^s_\beta}{\partial z^u_\alpha}\right|_S w^u_{\alpha,r} + O\big((w^r_{\alpha,r})^2\big)},$$

and so

$$\frac{\partial w^t_{\beta,s}}{\partial w^r_{\alpha,r}} = O(w^r_{\alpha,r}),$$

as required.

Finally, since $w^s_{\beta,s} = z^s_\beta$, equation (7.4) yields

$$\frac{\partial^2 w^s_{\beta,s}}{\partial (w^r_{\alpha,r})^2} = O(w^r_{\alpha,r}),$$

and $\tilde{\mathfrak{U}}$ is a comfortable atlas, as claimed.

(ii) Since df acts as the identity on N_S, in local coordinates we can write

$$f^j(z) = z^j + g^j_{r_1 \ldots r_{\nu_f+1}} z^{r_1} \cdots z^{r_{\nu_f}} + R_{\nu_f+1},$$

with $g^s_{r_1}|_S \equiv 0$ if $\nu_f = 1$. Then (7.1) yields

$$\tilde f^j_r(w_r) = w^j_r + (w^r_r)^{\nu_f} g^j_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr) w_r^{\hat r_1}\cdots w_r^{\hat r_{\nu_f}} + O\bigl((w^r_r)^{\nu_f+1}\bigr) \tag{7.5}$$

if $j = r, m+1, \dots, n$, and

$$\tilde f^j_r(w_r) = w^j_r + (w^r_r)^{\nu_f-1}\bigl[g^j_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr) - w^j_r g^r_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr)\bigr] w_r^{\hat r_1}\cdots w_r^{\hat r_{\nu_f}} + O\bigl((w^r_r)^{\nu_f}\bigr) \tag{7.6}$$

if $j = 1, \dots, r-1, r+1, \dots, m$, where $w_r^{\hat s} = w^s_r$ if $s \ne r$, and $w_r^{\hat r} = 1$. In particular, $d\tilde f$ acts as the identity on N_{E_S}.

(iii) We have

$$g^j_{r_1\dots r_{\nu_f}}|_{E_S}\bigl(z(w_r)\bigr) = g^j_{r_1\dots r_{\nu_f}}|_S(O, w''_r);$$

therefore if f is tangential w^r_r divides all $g^s_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr)$, while it does not divide some $g^p_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr)$. In particular, then, $\tilde f$ is tangential and $\nu_{\tilde f} = \nu_f$, by (7.5) and (7.6). On the other hand, if f is not tangential w^r_r does not divide some $g^s_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr)$; therefore

$$\bigl[g^s_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr) - w^s_r g^r_{r_1\dots r_{\nu_f}}\bigl(z(w_r)\bigr)\bigr]\big|_{E_S} = g^s_{r_1\dots r_{\nu_f}}(O, w''_r) - w^s_r g^r_{r_1\dots r_{\nu_f}}(O, w''_r) \not\equiv 0,$$

and thus $\nu_{\tilde f} = \nu_f - 1$ and $\tilde f$ is again tangential.

(iv) Take $v \in (N_S)_p$, $v \ne O$, and a chart (U, z) adapted to S centered in p. Then $v = v^s\partial_s$, with $v^r \ne 0$ for some r. Hence $[v] \in \tilde U_r$ has coordinates

$$w^j_r([v]) = \begin{cases} 0 & \text{if } j = r, m+1, \dots, n, \\ v^j/v^r & \text{if } j = 1, \dots, r-1, r+1, \dots, m. \end{cases}$$

If f is not tangential, then $[v]$ is a singular point for $\tilde f$ if and only if

$$[v^r g^s_{r_1\dots r_{\nu_f}}(O) - v^s g^r_{r_1\dots r_{\nu_f}}(O)] v^{r_1}\cdots v^{r_{\nu_f}} = 0$$

for all s, and thus if and only if $B(v\otimes\cdots\otimes v)\wedge v = O$, as claimed.

If f is tangential, writing $f^s - z^s$ as in (7.2) we get

$$\tilde f^s_r(w_r) = w^s_r + (w^r_r)^{\nu_f}\bigl[h^s_{r_1\dots r_{\nu_f+1}}\bigl(z(w_r)\bigr) - w^s_r h^r_{r_1\dots r_{\nu_f+1}}\bigl(z(w_r)\bigr)\bigr] w_r^{\hat r_1}\cdots w_r^{\hat r_{\nu_f+1}} + O\bigl((w^r_r)^{\nu_f+1}\bigr)$$

for $s \ne r$, and then it is clear that $[v]$ is a singular point for $\tilde f$ if and only if v is a singular direction for f. □

We therefore get index theorems in any codimension:

Theorem 7.2: *Let S be a compact complex submanifold of codimension $1 < m < n$ in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given, and assume that df acts as the identity on N_S. Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions for f in $\mathbb{P}(N_S)$. Then there exist complex numbers $\mathrm{Res}(f, S, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of f and S near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}(f, S, \Sigma_\lambda) = \int_{E_S} c_1^{n-1}([E_S]) = \int_S \pi_* c_1^{n-1}([E_S]),$$

where E_S is the exceptional divisor in the blow-up $\pi\colon M_S \to M$ of M along S, and π_* denotes the integration along the fibers of the bundle $\pi|_{E_S}\colon E_S \to S$.

Proof: It follows immediately from Theorem 6.2, Proposition 7.1, and the projection formula (see, e.g., [Su, Proposition II.4.5]). □

Remark 7.4: The restriction to E_S of the cohomology class $c_1([E_S])$ is the Chern class $\zeta = c_1(T)$ of the tautological bundle T on the bundle $\pi|_{E_S}: E_S \to S$ and it satisfies the relation

$$\zeta^{n-m} - \pi|_{E_S}^* c_1(N_S)\zeta^{n-m-1} + \pi|_{E_S}^* c_2(N_S)\zeta^{n-m-2} + \cdots + (-1)^{n-m}\pi|_{E_S}^* c_{n-m}(N_S) = 0$$

in the cohomology ring of the bundle (see, e.g., [GH, pp. 606-608]). This formula can sometimes be used to compute ζ in terms of the Chern classes of N_S and TM in specific examples.

Theorem 7.3: *Let S be a compact complex submanifold of codimension $1 < m < n$ in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given, and set $\nu = \nu_f$ if f is tangential, and $\nu = \nu_f - 1$ otherwise. Assume that S is comfortably embedded into M, and that $\nu > 1$. Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions for f in $\mathbb{P}(N_S)$. Finally, let $\pi: M_S \to M$ be the blow-up of M along S, with exceptional divisor E_S. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $\mathrm{Res}_\varphi(f, TM_S|_{E_S} - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of f and $TM_S|_{E_S} - N_{E_S}^{\otimes\nu}$ near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}_\varphi(f, TM_S|_{E_S} - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) = \int_S \pi_*\varphi\big(TM_S|_{E_S} \otimes (N_{E_S}^*)^{\otimes\nu}\big),$$

where π_* denotes the integration along the fibers of the bundle $E_S \to S$.

Proof: It follows immediately from Theorem 6.3, Proposition 7.1, and the projection formula. □

Theorem 7.4: *Let S be a compact complex submanifold of codimension $1 < m < n$ in an n-dimensional complex manifold M. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, be given, and assume that df acts as the identity on N_S. Set $\nu = \nu_f$ if f is tangential, and $\nu = \nu_f - 1$ otherwise. Let $\bigcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the set of singular directions for f in $\mathbb{P}(N_S)$. Finally, let $\pi: M_S \to M$ be the blow-up of M along S, with exceptional divisor E_S. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $\mathrm{Res}_\varphi(f, TE_S - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) \in \mathbb{C}$, depending only on the local behavior of f and $TE_S - N_{E_S}^{\otimes\nu}$ near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}_\varphi(f, TE_S - N_{E_S}^{\otimes\nu}, \Sigma_\lambda) = \int_S \pi_*\varphi\big(TE_S \otimes (N_{E_S}^*)^{\otimes\nu}\big),$$

where π_* denotes the integration along the fibers of the bundle $E_S \to S$.

Proof: It follows immediately from Theorem 6.4, Proposition 7.1, and the projection formula. □

8. Applications to dynamics

We conclude this paper with a couple of applications to the study of the dynamics of endomorphisms of complex manifolds.

We first recall a definition from [A2]:

Definition 8.1: Let $f \in \mathrm{End}(M, p)$ be a germ at $p \in M$ of holomorphic self map of a complex manifold M fixing p. A *parabolic curve* for f at p is a injective holomorphic map $\varphi: \Delta \to M$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous at the origin, and $\varphi(0) = p$;
(iii) $\varphi(\Delta)$ is invariant under f, and $(f|_{\varphi(\Delta)})^n \to p$ as $n \to \infty$.

Furthermore, we say that φ is *tangent to a direction* $v \in T_pM$ at p if for one (and hence any) chart (U, z) centered at p the direction of $z\big(\varphi(\zeta)\big)$ converges to the direction $dz_p(v)$ as $\zeta \to 0$.

Then we have the promised dynamical interpretation of X_f at nonsingular points:

Proposition 8.1: *Assume that S has codimension one in M, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. Let $p \in S$ be a regular point of X_f, that is $X_f(p) \neq O$. Then*

(i) *If f is tangential then no infinite orbit of f can stay arbitrarily close to p. More precisely, there is a neighborhood U of p such that for every $q \in U$ there is $n_0 \in \mathbb{N}$ such that $f^{n_0}(q) \notin U$ or $f^{n_0}(q) \in S$.*

(ii) *If $\Xi_f(p)$ is transversal to T_pS (so in particular f is non-tangential) and $\nu_f > 1$ then there exists at least one parabolic curve for f at p tangent to $\Xi_f(p)$.*

(iii) *If $\Xi_f(p)$ is transversal to T_pS, $\nu_f = 1$, and $|b(p)| \neq 0, 1$ or $b(p) = \exp(2\pi i\theta)$ where θ satisfies the Bryuno condition (and b is the function defined in Remark 1.1) then there exists an f-invariant one-dimensional holomorphic disk Δ passing through p tangent to $\Xi_f(p)$ such that $f|_\Delta$ is holomorphically conjugated to the multiplication by $b(p)$.*

Proof: In local adapted coordinates centered at $p \in S$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z),$$

so that

$$\Xi_f(p) = \mathrm{Span}\left(g^1(O) \left.\frac{\partial}{\partial z^1}\right|_p + \cdots + g^n(O) \left.\frac{\partial}{\partial z^n}\right|_p \right).$$

In case (i), we have $g^1 = z^1 h^1$ for a suitable holomorphic function h^1, and $g^{p_0}(O) \neq 0$ for some $2 \leq p_0 \leq n$, because p is not singular. Therefore we can apply [AT, Proposition 2.1] (see also [A2, Proposition 2.1]), and the assertion follows.

Now, $\Xi_f(p)$ is transversal to T_pS if and only if $g^1(O) \neq 0$. In case (ii) we can then write

$$f^j(z) = z^j + g^j(O)(z^1)^{\nu_f} + O(\|z\|^{\nu_f+1})$$

with $g^1(O) \neq 0$. Then $\Xi_f(p)$ is a non-degenerate characteristic direction of f at p in the sense of Hakim, and thus by [H1, 2] there exist at least $\nu_f - 1$ parabolic curves for f at p tangent to $\Xi_f(p)$.

If $\nu_f = 1$, it is easy to see that $b^1(p) = 1 + g^1(O)$, and $b^1(p) \neq 1$ because $\Xi_f(p)$ is transversal to T_pS. Therefore we can write

$$f^j(z) = \begin{cases} b^1(p)z^1 + O(\|z\|^2) & \text{if } j = 1, \\ z^j + g^j(O)z^1 + O(\|z\|^2) & \text{if } 2 \leq j \leq n, \end{cases}$$

and the assertion in case (iii) follows immediately from [Po] (see also [N]). □

In other words, X_f essentially dictates the dynamical behavior of f away from the singularities — or, from another point of view, we can say that the interesting dynamics is concentrated near the singularities of X_f.

Remark 8.1: If $\Xi_f(p)$ is transversal to T_pS, $\nu_f = 1$ and $b(p) = 0$ or $b(p) = \exp(2\pi i\theta)$ with θ irrational not satisfying the Bryuno condition, there might still be an f-invariant one-dimensional holomorphic disk passing through p and tangent to $\Xi_f(p)$. On the other hand, if $b(p) = \exp(2\pi i\theta)$ is a k-th root of unity, necessarily different from one, several things might happen. For instance, if $b(p) = -1$, up to a linear change of coordinates we can write

$$f^j(z) = \begin{cases} z^1 + z^1\bigl(-2 + (z^1)^{\mu_1}\hat{g}^1(z)\bigr) & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\hat{g}^j(z) & \text{if } j = 2, \ldots, n, \end{cases}$$

for suitable $\mu_1, \ldots, \mu_n \in \mathbb{N}$ and holomorphic functions $\hat{g}^j$ not divisible by z^1 and such that $\hat{g}^j(O) = 0$ if $\mu_j = 0$. Then if $\mu_1 = 0$ we get

$$(f \circ f)^j(z) = \begin{cases} z^1 - z^1\bigl[\hat{g}^1(z) + \hat{g}^1\bigl(f(z)\bigr) - \hat{g}^1(z)\hat{g}\bigl(f(z)\bigr)\bigr] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\bigl[\hat{g}^j(z) - \bigl(-1 + \hat{g}^1(z)\bigr)^{\mu_j+1}\hat{g}^j\bigl(f(z)\bigr)\bigr] & \text{if } j = 2, \ldots, n. \end{cases}$$

So $\nu_{f\circ f} = 1$, $f \circ f$ is non-tangential but p is singular for $f \circ f$. On the other hand, if $\mu_1 = 1$ we get

$$(f \circ f)^j(z) = \begin{cases} z^1 - (z^1)^2\bigl[\hat{g}^1(z) - \hat{g}^1\bigl(f(z)\bigr) + O(z^1)\bigr] & \text{if } j = 1, \\ z^j + (z^1)^{\mu_j+1}\bigl[\hat{g}^j(z) + (-1)^{\mu_j}\hat{g}^j\bigl(f(z)\bigr) + O(z^1)\bigr] & \text{if } j = 2, \ldots, n. \end{cases}$$

So if, for instance, $\mu_2 = 0$ we get $\nu_{f\circ f} = 1$, but $f \circ f$ is tangential and p is singular for $f \circ f$. But if $\mu_2 = 2$ and $\mu_j \geq 2$ for $j \geq 3$ we get $\nu_{f\circ f} = 3$ and p can be either singular or non-singular for $f \circ f$.

Remark 8.2: If $\nu_f = 1$, $\Xi_f(p) \subset T_pS$ and S is compact, necessarily f is tangential, because $b \equiv 1$ and then $g^1(0, z'') \equiv 0$. If S is not compact we might have an isolated point of tangency, and in that case we might have parabolic curves at p not tangent to $\Xi_f(p)$. For instance, the methods of [A1] show that this happens for the map

$$f^j(z) = \begin{cases} z^1 + z^1\bigl(az^2 + bz^3 + h_1(z'') + z^1 h_2(z)\bigr) & \text{if } j = 1, \\ z^2 + z^1\bigl(c + h_3(z)\bigr) & \text{if } j = 2, \\ z^3 + z^1 g^3(z) & \text{if } j = 3, \end{cases}$$

when a, $c \neq 0$.

Finally, we describe a couple of applications to endomorphisms of complex surfaces:

Corollary 8.2: *Let S be a smooth compact one-dimensional submanifold of a complex surface M, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. Assume that f is tangential, or that $S \setminus \mathrm{Sing}(f)$ is comfortably embedded in M, and let X denote X_f, $H_{\sigma,f}$ or $H^1_{\sigma,f}$ as usual; assume moreover that $X \not\equiv O$. Then*

(i) *if $c_1(N_S) \neq 0$ then $\chi(S) - \nu_f c_1(N_S) > 0$;*
(ii) *if $c_1(N_S) > 0$ then S is rational, $\nu_f = 1$ and $c_1(N_S) = 1$.*

Proof: The well-known theorem about the localization of the top Chern class at the zeroes of a global section (see, e.g., [Su, Theorem III.3.5]) yields

$$\sum_{x\in\mathrm{Sing}(X)} N(X; x) = \chi(S) - \nu_f\, c_1(N_S), \tag{8.1}$$

where $N(X; x)$ is the multiplicity of x as zero of X. Now, If $c_1(N_S) \neq 0$ then by Theorem 6.2 the set $\mathrm{Sing}(X)$ is not empty. Therefore the sum in (8.1) must be strictly positive, and the assertions follow. □

Definition 8.2: Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. We say that a point $p \in S$ is *weakly attractive* if there are infinite orbits arbitrarily close to p, that is if for every neighborhood U of p there is $q \in U$ such that $f^n(q) \in U \setminus S$ for all $n \in \mathbb{N}$. In particular, this happens if there is an infinite orbit converging to p.

Then we can prove the following

Proposition 8.3: *Let S be a smooth compact one-dimensional submanifold of a complex surface M, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. If f is tangential then there are at most $\chi(S) - \nu_f c_1(N_S)$ weakly attractive points for f on S.*

Proof: By (8.1) the sum of zeros of the section X_f (counting multiplicity) is equal to $\chi(S) - \nu_f c_1(N_S)$. Thus the number of zeros (not counting multiplicity) is at most $\chi(S) - \nu_f c_1(N_S)$. The assertion then follows from Proposition 8.1. □

Finally, the previous index theorems allow a classification of the smooth curves which are fixed by a holomorphic map and are dynamically trivial.

Theorem 8.4: *Let S be a smooth compact one-dimensional submanifold of a complex surface M, and take $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$. Moreover assume that $\mathrm{sp}(df_p) = \{1\}$ for some $p \in S$. If there are no weakly attractive points for f on S then only one of the following cases occur:*

(i) *$\chi(S) = 2$, $c_1(N_S) = 0$, or*
(ii) *$\chi(S) = 2$, $c_1(N_S) = 1$, $\nu_f = 1$, or*
(iii) *$\chi(S) = 0$, $c_1(N_S) = 0$.*

Proof: Since N_S is a line bundle over a compact curve S, the action of df on N_S is given by multiplication by a constant, and since df_p has only the eigenvalue 1 then this constant must be 1. If f were non-tangential then by Proposition 8.1.(ii) all but a finite number of points of S would be weakly attractive. Therefore f is tangential. By [A2, Corollary 3.1] (or [Br, Proposition 6.6]) if there is a point $q \in S$ so that $\mathrm{Res}(X_f, N_S, p) \notin \mathbb{Q}^+$ then q is weakly attractive. Thus the sum of the residues is non-negative and by Theorem 6.2 it follows that $c_1(N_S) \geq 0$. Thus (8.1) yields

$$\chi(S) \geq \nu_f c_1(N_S) \geq 0. \tag{8.2}$$

Therefore the only possible cases are $\chi(S) = 0, 2$. If $\chi(S) = 0$ then (8.2) implies $c_1(N_S) = 0$. Assume that $\chi(S) = 2$. Thus $c_1(N_S) = 0, 1, 2$. However if $c_1(N_S) = 1$ and $\nu_f = 2$ or if $c_1(N_S) = 2$ (and necessarily $\nu_f = 1$) then (8.1) would imply that X_f has no zeroes, and thus $c_1(N_S) = 0$ by Theorem 6.2. $\square$

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