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DILATATION AND ORDER OF CONTACT FOR HOLOMORPHIC SELF-MAPS OF STRONGLY CONVEX DOMAINS.

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INTRODUCTION

Let f be a holomorphic self-map of the unit disc Δ. It is well known that f has at most one fixed point in Δ. Moreover if f is not an automorphism and has a fixed point in Δ then such a point is *attractive*, *i.e*, the sequence of iterates $\{f^k\}$ converges to the constant map which shrinks the disc to that point. If f has no fixed points in Δ then there exists a unique boundary point, say $x \in \partial\Delta$, such that $\{f^k\}$ converges uniformly on compacta to x. Such a point is called *the Wolff point* of f. Therefore the functional $\mathrm{Hol}(\Delta,\Delta) - \{Id\} \to \overline{\Delta}$ associating to any f its inner fixed point (if any) or its Wolff point is well defined. Moreover Heins [8] proved that this function is continuous (where one endowes $\overline{\mathrm{Hol}(\Delta,\Delta)}$ with the topology of uniform convergence on compacta). Heins' result has been generalized to strongly convex domains by Joseph and Kwack [10] who used Abate's results on Wolff points (see [1]) to define the functional.

Suppose $f \in \mathrm{Hol}(\Delta,\Delta)$ has no fixed points in Δ and let $x \in \partial\Delta$ be its Wolff point. Let $\alpha(f)$ be the real number given by

$$\alpha(f) := \sup_{z\in\Delta} \left\{ \frac{|x-f(z)|^2}{1-|f(z)|^2} / \frac{|x-z|^2}{1-|z|^2} \right\}.$$

The number $\alpha(f)$ is called the *boundary dilatation coefficient* of f at x. The boundary dilatation coefficient owes its name to the following interpretation: an *horocycle* $E(x,R)$ of center x and radius $R>0$ is given by

$$E(x,R) := \left\{ z \in \Delta | \frac{|x-z|^2}{1-|z|^2} < R \right\}.$$

The horocycle $E(x,R)$ is a euclidean disc contained in Δ and tangent to $\partial\Delta$ at x. Then by the very definition it follows that $f(E(x,R)) \subseteq E(x,\alpha(f)R)$ for any $R>0$. Hence $\alpha(f)$

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measures how much f "shrinks" any horocycle. From the classical Julia-Wolff-Carathéodory Theorem one has that $\alpha(f) = f'(x)$, where $f'(x)$ indicates the non-tangential limit of f' at x. This happens just because $x \in \partial\Delta$. Indeed if f has a fixed point $z_0 \in \Delta$ then $f'(z_0)$ does not control how much f shrinks the Poincaré discs (the relatives of horocycles inside the disc) since it could be zero. Therefore in this case we are led to define $\alpha(f)$ as the maximum dilatation of f on any Poincaré disc of center z_0 (see Section 1 for a precise definition). Then $\alpha(f) = 0$ if and only if f is constant, and the fact that $\alpha(f) = 1$ does not necessarily imply that f is an automorphism. Moreover in case f has a fixed point in Δ the dilatation $\alpha(f)$ turns out to be an intrinsic measure of the "order of contact" of $f(\Delta)$ to $\partial\Delta$, where the order of contact is roughly defined as the real number k such that $(1 - |f(z)|) = O((1 - |z|)^k)$ (see Definition 2.3). In general the functional $\alpha : \overline{\mathrm{Hol}(\Delta, \Delta)} \to [0, 1]$ is lower semicontinuous. This allows to set up conditions for describing the limit of a sequence of holomorphic self-maps.

Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with regular boundary (say at least C^3) and endow $\mathrm{Hol}(D, D)$ with the topology of the uniform convergence on compacta. Abate [1], [2] proved that if $f \in \mathrm{Hol}(D, D)$ has no fixed points in D then there exists a unique boundary point, say $x \in \partial D$, such that the sequence of iterates $\{f^k\}$ converges uniformly on compacta to x. Call the point x the *Wolff point* of f. Then a boundary dilatation coefficient $\alpha(f)$ can be defined at x (see Section 3). If f has fixed points in D then one can define a dilatation $\alpha(f)$ similarly as we did in the disc using Kobayashi balls instead of Poincaré's ones. Even in this case the dilatation is a measure of the order of contact of $f(D)$ to the boundary. As in the disc case it is possible to prove that $\alpha : \overline{\mathrm{Hol}(D, D)} \to [0, 1]$ is lower semicontinuous and again have conditions for studying the limits of a sequence of holomorphic self-maps. The proof of this result exploits a tool discovered by Lempert (see [13]) and developed by Abate (see [1]): the *complex geodesic projection device* (see Section 2). As a spin off result of our work we give a different proof of Joseph-Kwack extension of Heins' result (see Theorem 3.10).

The method of "reduction to complex geodesics" also suggests to associate to any direction a "directional dilatation" which measures the "shrinking" of f along that direction. More in details let $\mathcal{G}$ be the (closure of the) space of closed complex geodesics endowes with the structure of complete metric space coming from the Hausdorff distance on compacta of $\overline{D}$ (see Definition 5.2). The functional

$$\alpha_G : \mathcal{G} \times \overline{\mathrm{Hol}(D, D)} \to [0, 1],$$

associating to (G, f) the dilatation $\alpha_G(f)$ of the restriction of f to G (see Section 5) is lower semicontinuous. In Theorem 5.10 we prove that the functional which maps a point $z \in \overline{D}$ and a vector $v \in \mathbb{C}^n - \{0\}$ to the element $G_{z,v} \in \mathcal{G}$ containing z and parallel to v at z is continuous. Therefore the *directional dilatation* $\alpha_{z,v}(f)$ of a map $f \in \mathrm{Hol}(D, D)$, defined as $\alpha_{G_{z,v}}(f)$, is a lower semicontinuous function.

As applications we study the relationships among fixed points (also boundary fixed points in the sense of non-tangential limits) of sequences of holomorphic self-maps with respect to the limit function.

1. The dilatation in the unit disc

Let Δ be the unit disc in $\mathbb{C}$, and endow $\mathrm{Hol}(\Delta,\Delta)$ with the topology of the uniform convergence on compacta. As the theory of holomorphic self-maps of the unit disc is well known, we avoid to recall it here in detail. Instead we refer tacitly the reader to, *e.g.*, [1], [5], [15].

Let us start this section with the following examples:

Example 1.1. Let $f_0 \in \mathrm{Hol}(\Delta,\Delta)$ be given by $f_0(z) := z^2$. Then f_0 fixes 0 and $f'(0) = 0$. Let $\eta_t(z) := (t-z)/(1-tz)$ for $t \in (0,1)$. Then η_t is a automorphism of Δ such that $\eta_t(0) = t$ and $\eta_t^{-1} = \eta_t$. Let $f_t := \eta_t \circ f_0 \circ \eta_t$. Then $f_t \in \mathrm{Hol}(\Delta,\Delta)$ for any t and moreover $f_t(t) = t$ and $f_t'(t) = 0$. A straightforward computation shows that the sequence $\{f_t\}$ converges to the hyperbolic automorphism

$$\gamma : z \mapsto \frac{3z+1}{z+3}.$$

The map γ has no fixed points in Δ, its Wolff point is 1 and $\gamma'(1) = 1/8 > 0$.

Example 1.2. Let $f \in \mathrm{Hol}(\Delta,\Delta)$ be a parabolic automorphism with Wolff point 1. Then $f'(1) = 1$. The sequence of iterates $\{f^k\}$ converges (by the Wolff-Denjoy Lemma [20], [7]) to the constant map $z \mapsto 1$ whose derivative is, of course, equals to 0.

The two previous examples show that the functional "first derivative at the Wolff/fixed point" is by no means continuous. However the two examples are of different nature, as it will be clear later.

Let ω be the Poincaré distance on the unit disc Δ.

Definition 1.3. Let $f \in \overline{\mathrm{Hol}(\Delta,\Delta)}$. If f is non-constant and has no fixed points in Δ then the *dilatation*, denoted by $\alpha(f)$, of f is its boundary dilatation coefficient at its Wolff point.

If f has a fixed point $z_0 \in \Delta$, then the *dilatation* of f is given by

$$\alpha(f) := \sup_{z \in \Delta - \{z_0\}} \frac{\omega(f(z), z_0)}{\omega(z, z_0)}.$$

If f is a constant such that $f(\Delta) = \tau \in \partial\Delta$ then we let $\alpha(f) := 0$.

Note that in the previous definition we allow f to be the identity and that $\alpha(Id) = 1$.

Remark 1.4. By the Schwarz Lemma and the Wolff Lemma it follows that $\alpha(f) \leq 1$. Moreover it is clear that $\alpha(f) = 0$ if and only if f is a constant map.

If γ is an automorphism of Δ then $\alpha(\gamma \circ f \circ \gamma^{-1}) = \alpha(f)$, for the automorphisms are isometries for the Poincaré distance.

Remark 1.5. The dilatation in the case of a inner fixed point has the following interpretation. Suppose $f \in \mathrm{Hol}(\Delta,\Delta)$ and $f(z_0) = z_0$. Let $B(z_0, R)$ be a Poincaré disc of center z_0 and radius $R > 0$. then $f(B(z_0,R)) \subseteq B(z_0, \alpha(f)R)$. Actually one could equivalently have defined $\alpha(f)$ as

$$\alpha(f) := \sup_{R>0}\{\inf\{q > 0 | f(B(z_0,R)) \subseteq B(z_0, qR)\}\}.$$

Example 1.6. Let us consider again the situation in Example 1.1 and 1.2. In Example 1.1 we have $\alpha(f_0) = 1$ (as $z^2 \approx z$ as $|z| \to 1$). Therefore $\alpha(f_t) = \alpha(f_0) = 1$ for any t and hence

$$\liminf_{t\to 1} \alpha(f_t) = 1 > 1/8 = \alpha(\gamma).$$

In Example 1.2 we have $\alpha(f_k) = 1$ for any k (by the Julia-Wolff-Carathéodory Theorem [5]) and then the limit of $\alpha(f_k)$ is 1 and the dilatation of f is 0.

The situation of the previous example is a general one:

Theorem 1.7. *The function* $\alpha : \overline{Hol(\Delta, \Delta)} \to [0,1]$ *is lower semicontinuous.*

Proof. Let $f_k, f \in \overline{\mathrm{Hol}(\Delta, \Delta)}$ be such that f_k converges uniformly on compacta to f. If f is constant then the result is trivially true. Now we divide the proof in three cases, which we can always reduce to.

1) Suppose $f_k, f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f_k(z_k) = z_k$ for some $z_k \in \Delta$ and that $z_k \to z_0 \in \Delta$. We can *w.l.o.g.* suppose that $\alpha(f_k) \to \beta$. Then for any $z \in \Delta$ we have

$$\omega(z_k, f_k(z)) \le \alpha(f_k)\omega(z_k, z),$$

and taking the limit we find $\omega(z_0, f(z)) \le \beta\omega(z_0, z)$. therefore $f(z_0) = z_0$ and $\alpha(f) \le \beta$.

2) Suppose $f_k, f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f_k(z_k) = z_k$ for some $z_k \in \Delta$ and that $z_k \to x \in \partial\Delta$. We can as well suppose that $\alpha(f_k) \to \beta$. We want to show that x is the Wolff point of f and that $\beta \ge \alpha(f)$. The point x is the Wolff point of f by Heins' Theorem [8]. However this is not necessary and actually it will also follow from our considerations. Let $R > 0$ and $R_k \to \infty$ be such that

$$\lim_{k\to\infty} \frac{1 - |z_k|}{1 - \tanh R_k} = R. \tag{1.1}$$

By *Proposition 1.2.1 of* [1] (see also [11]) if $z \in E(x, R)$ for some $R > 0$ then $z \in B(z_k, R_k)$ eventually and therefore $f_k(z) \in B(z_k, \alpha(f_k)R_k)$. Equivalently

$$\frac{|1 - \overline{z_k} f_k(z)|^2}{1 - |f_k(z)|^2} < \frac{1 - |z_k|^2}{1 - \tanh^2(\alpha(f_k)R_k)}. \tag{1.2}$$

Now for $k \to \infty$ the left-hand term tends to

$$\frac{|x - f(z)|^2}{1 - |f(z)|^2}.$$

As for the right-hand term we have

$$\frac{1 - |z_k|^2}{1 - \tanh^2(\alpha(f_k)R_k)} = \frac{1 - |z_k|^2}{1 - \tanh^2 R_k} \frac{1 - \tanh^2 R_k}{1 - \tanh^2(\alpha(f_k)R_k)}. \tag{1.3}$$

By equation (1.1) the first term on the right-hand side tends to R. As for the second factor we first note that it is bounded below from a positive constant. If not then the right-hand term of equation (1.3) would tend to 0 and hence taking the limit in equation (1.2) we would find

$$\frac{|x - f(z)|^2}{1 - |f(z)|^2} = 0,$$

implying $f \equiv x$, against our assumption that $f \in \mathrm{Hol}(\Delta, \Delta)$. Therefore for any k large

$$0 < C \leq \frac{1 - \tanh^2 R_k}{1 - \tanh^2(\alpha(f_k) R_k)} \approx e^{2(\alpha(f_k)-1)R_k}. \tag{1.4}$$

This forces

$$\beta = \lim_{k\to\infty} \alpha(f_k) = 1. \tag{1.5}$$

Since $\alpha(f) \leq 1$ this proves the assertion in this case. Note that since $r \mapsto (1 - \tanh^2 r)$ is decreasing in $r > 0$, then equation (1.3) and the Wolff Lemma imply that x is the Wolff point of f or that f is the identity.

3) Suppose $f_k \in \mathrm{Hol}(\Delta, \Delta)$ be without fixed points in Δ for any k. Suppose that $\tau_k \in \partial\Delta$ is the Wolff point of f_k for any k, and that $\tau_k \to \tau \in \partial\Delta$. Up to conjugation we can suppose $\tau = 1$. Let $\eta_k(z) := \tau_k z$. Then $\{\eta_k\}$ is a sequence of rotations which tends to the identity. Moreover $g_k := \eta_k^{-1} \circ f_k \circ \eta_k$ is such that g_k has Wolff point 1 and $g_k \to f$. Since the boundary dilatation coefficient is invariant under conjugation it follows that $\alpha(g_k) = \alpha(f_k)$. Now let $z \in E(1, R)$. Then $g_k(z) \in E(1, \alpha(g_k)R)$. If $\alpha(g_k)$ converges (as we may suppose) to β, then $f(z) \in \overline{E(1, \beta R)}$, and (if $f \not\equiv 1$ as we may suppose) by the open mapping theorem it is actually $f(z) \in E(1, \beta R)$, implying that $\alpha(f) \leq \beta$. This and the Wolff Lemma imply that 1 is the Wolff point of f unless f is the identity. □

Remark 1.8. In the previous proof we did not assume Heins' Theorem, but actually we reproved it. Namely we showed that if $\{f_k\} \subset \mathrm{Hol}(\Delta, \Delta)$ converges to a map $f \in \overline{\mathrm{Hol}(\Delta, \Delta)}$ then the sequence of fixed points (or Wolff points) of f_k converges to the fixed point (or the Wolff point) of f, unless f is the identity.

Remark 1.9. The previous proof shows that for a sequence $\{f_k\} \subset \mathrm{Hol}(\Delta, \Delta)$ with fixed points $z_k \in \Delta$ to converge to a non-constant map f without fixed points in Δ is necessary that $\alpha(f_k) \to 1$.

A better estimate is given by the following:

Proposition 1.10. *Let $\{f_k\} \subset Hol(\Delta, \Delta)$ be such that for any k there exists $z_k \in \Delta$ such that $f(z_k) = z_k$ and $z_k \to \tau$ for some $\tau \in \partial\Delta$. Suppose that $f_k \to f$ for some $f \in \overline{Hol(\Delta, \Delta)}$. Then*

$$\alpha(f) \leq \liminf_{k\to\infty} e^{-(1-\alpha(f_k))\cdot\omega(0,z_k)}. \tag{1.6}$$

In particular if

$$\lim_{k\to\infty} (1 - \alpha(f_k)) \cdot \omega(0, z_k) = \infty,$$

then f is the constant map $z \mapsto \tau$.

Example 1.11. The inequality in (1.6) could be strict and therefore the condition is by no means necessary. For instance let $f_0(z) = iz$, $\varphi_t(z) = \frac{t-z}{1-tz}$ for $t \in (0, 1)$ and $f_t(z) := \varphi_t \circ f_0 \circ \varphi_t$. Then it is easy to see that $\alpha(f_t) = 1$ for any t and that f_t converges to the constant map 1 with dilatation equals to 0.

Proof of Proposition 1.10. We argue as in the proof of part (2) of Theorem 1.7, choosing R, R_k as there and assuming that $\alpha(f_k) \to \beta$. As before if $\beta < 1$ then f is constant and equation (1.6) follows. Otherwise we found equation (1.4). Now

$$e^{2(\alpha(f_k)-1)R_k} = e^{-(1-\alpha(f_k))\omega(0,z_k)} \cdot e^{2(\alpha(f_k)-1)(R_k-\omega(0,z_k)/2)}.$$

Since $\alpha(f_k) \to \beta = 1$ then equation (1.6) will follow as soon as we show that $R_k - \omega(0,z_k)/2$ is bounded above as k increases. But

$$R_k - \omega(0,z_k)/2 = R_k + \frac{1}{2}\log(1-|z_k|) - \frac{1}{2}\log(1+|z_k|),$$

and $R_k + \frac{1}{2}\log(1-|z_k|) \le C < \infty$ for all k since by equation (1.1) we have

$$\frac{1-|z_k|}{2}e^{2R_k} \to R.$$

This implies that f maps any horocycle of center τ and radius $R > 0$ into an horocycle of center τ and radius θR for $\theta := \liminf_{k\to\infty} e^{-(1-\alpha(f_k))\cdot\omega(0,z_k)}$, as wanted. □

2. The order of contact in the disc

In this section we give a definition of "order of contact" to the boundary. The usual order of contact of a map $f \in \mathrm{Hol}(\Delta,\Delta)$ at a given point $x \in \partial\Delta$ is measured by the ratio $(1-|f(z)|)/(1-|z|)$ as $z \to x$ which comes out naturally from Julia's Lemma and (whenever less than 1) is very important in the study of compactness of composition operators (see [15]). Here we generalize such an idea and introduce a precise definition of order of contact. Then we relate it to the previously introduced dilatation.

Let us start with the following definition:

Definition 2.1. Let $f \in \mathrm{Hol}(\Delta,\Delta)$, $x \in \partial\Delta$ and $k \in \mathbb{R}$. We set

$$\mathcal{L}_{x,k}(f) := \liminf_{z\to x} \frac{1-|f(z)|}{(1-|z|)^k}.$$

Remark 2.2. Note that

$$\mathcal{L}_{x,m}(f) = \mathcal{L}_{x,k}(f) \cdot \liminf_{z\to x}(1-|z|)^{k-m}.$$

Therefore if $\mathcal{L}_{x,k}(f) > 0$ then $\mathcal{L}_{x,m}(f) = \infty$ for any $m > k$ and if $\mathcal{L}_{x,k}(f) < \infty$ then $\mathcal{L}_{x,m}(f) = 0$ for any $m < k$.

Moreover by Julia's Lemma [12], $\mathcal{L}_{x,1}(f) > 0$, hence $\mathcal{L}_{x,m}(f) = \infty$ for $m > 1$.

Definition 2.3. Let $f \in \mathrm{Hol}(\Delta,\Delta)$, $x \in \partial\Delta$ and $k \in \mathbb{R}$. We say that f has *order of contact* k at x, briefly $\mathrm{O}_x(f) = k$, if for any $\epsilon > 0$

$$\mathcal{L}_{x,k+\epsilon}(f) = \infty \text{ and } \mathcal{L}_{x,k-\epsilon}(f) = 0.$$

The *(global) order of contact* $\mathrm{O}_{\partial\Delta}(f)$ is defined as

$$\mathrm{O}_{\partial\Delta}(f) := \sup_{x\in\partial\Delta} \mathrm{O}_x(f).$$

By Remark 2.2 the order of contact is well defined and $0 \le \mathrm{O}_{\partial\Delta}(f) \le 1$.

Example 2.4. Let $f_\theta(z)$ be a *lens map*. Namely let $\sigma(z) := (1+z)/(1-z)$ be the Cayley transform which maps Δ onto the right half-plane. Then squeeze the half-plane onto the sector $\{|\arg w| < \theta\pi/2\}$ by means of $w \mapsto w^\theta$ (for $0 < \theta < 1$) and go back to the unit disc with σ^{-1}. The result is a holomorphic self-map of Δ whose image looks like a lens. Analytically

$$f_\theta(z) := \frac{(\sigma(z))^\theta - 1}{(\sigma(z))^\theta + 1}.$$

The function f_θ maps $\overline{\Delta} - \{\pm 1\}$ into Δ and $f_\theta(\pm 1) = \pm 1$. We claim that $\mathrm{O}_{\partial\Delta}(f) = \theta$. It is clear that we have only to check that $\mathrm{O}_{\pm 1}(f) = \theta$. For $k > \theta$.

$$\mathcal{L}_{\pm 1,k}(f) := \liminf_{z \to \pm 1} \frac{1 - |f_\theta(z)|}{(1-|z|)^k} = \liminf_{z \to \pm 1} \frac{|(1+z)^\theta + (1-z)^\theta| - |(1+z)^\theta - (1-z)^\theta|}{2^\theta (1-|z|)^k}$$

$$\geq \liminf_{z \to \pm 1} \frac{|(1+z)^\theta + (1-z)^\theta| - |(1+z)^\theta - (1-z)^\theta|}{2^\theta (|1 \mp z|)^k}.$$

Taking the square it is easy to see that this term goes to ∞. On the other hand it is easy to see that for $r \in (0,1)$

$$\lim_{r \to \pm 1} \frac{1 - |f_\theta(r)|}{(1-|r|)^\theta} = 2^{1-\theta}.$$

Therefore $\mathcal{L}_{\pm 1,\theta}(f) < \infty$ which implies that $\mathcal{L}_{\pm 1,k}(f) = 0$ for $k < \theta$ and hence $\mathrm{O}_{\pm 1}(f) = \theta$.

Remark 2.5. The (global) order of contact is invariant under composition with automorphisms. Indeed if φ and θ are automorphisms of Δ then

$$\frac{1 - |\varphi(f(\theta^{-1}(z)))|}{(1-|z|)^k} = \frac{1 - |\varphi(f(w))|}{1 - |f(w)|} \cdot \frac{1 - |f(w)|}{(1-|w|)^k} \cdot \frac{(1-|w|)^k}{(1-|\theta(w)|)^k}$$

where $w = \theta^{-1}(z)$. Since $(1 - |\varphi(\zeta)|)/(1-|\zeta|)$ and $(1-|\theta(\zeta)|)/(1-|\zeta|)$ tends to a positive number as $|\zeta| \to 1$, then

$$\mathcal{L}_{x,k}(f) = \mathcal{L}_{\theta(x),k}(\varphi \circ f \circ \theta^{-1}),$$

and therefore $\mathrm{O}_{\partial\Delta}(f) = \mathrm{O}_{\partial\Delta}(\varphi \circ f \circ \theta^{-1})$. In particular $\mathrm{O}_{\partial\Delta}(f)$ is invariant under conjugation.

By Remark 2.5 if $f \in \mathrm{Hol}(\Delta,\Delta)$ has a fixed point in Δ and there exists an automorphism θ such that $f \circ \theta$ has no fixed point in Δ then $\mathrm{O}_{\partial\Delta}(f) = 1$.

We want to show now that the dilatation is an intrinsic way of measuring the order of contact previously introduced.

Lemma 2.6. *Let $f \in Hol(\Delta,\Delta)$ be such that $f(0) = 0$ and $x \in \partial\Delta$. Let*

$$\alpha_x(f) := \limsup_{z \to x} \frac{\omega(0, f(z))}{\omega(0,z)}. \tag{2.1}$$

Then $O_x(f) = \alpha_x(f)$.

Proof. We can suppose $x = 1$. First of all note that the assertion is clearly true if $|f(z)|$ doesn't accumulate at $\partial\Delta$ for $z \to 1$. Therefore we can suppose that $|f(z)| \to 1$ for $z \to 1$. By the very definition of ω we have

$$\frac{\omega(0, f(z))}{\omega(0, z)} = \frac{\log\frac{(1+|f(z)|)}{(1-|f(z)|)}}{\log\frac{(1+|z|)}{(1-|z|)}}.$$

Therefore for any $a \in \mathbb{R}$ such that $\alpha_1(f) < a$ we can find a suitable neighborhood U of 1 such that for any $z \in \Delta \cap U$

$$\log\frac{1+|f(z)|}{1-|f(z)|} < a\log\frac{1+|z|}{1-|z|}. \tag{2.2}$$

Since $t \mapsto \log(1+t)/(1-t)$ is increasing, this is, for $z \in \Delta \cap U$, equivalent to

$$\frac{1+|f(z)|}{(1+|z|)^a} < \frac{1-|f(z)|}{(1-|z|)^a}. \tag{2.3}$$

Taking the liminf on both sides we get

$$\liminf_{z\to 1}\frac{1-|f(z)|}{(1-|z|)^a} \geq 2^{1-a} > 0.$$

Since $a > \alpha_1(f)$ was arbitrary this implies that $\mathcal{L}_{1,a}(f) = \infty$ and $\mathrm{O}_1(f) \leq \alpha_1(f)$. On the other hand if $L_{1,a}(f) = \infty$ for some $a < \alpha_1(f)$ then equation (2.3) as well as equation (2.2) holds for z close to 1. Hence $\limsup_{z\to 1}\omega(0, f(z))/\omega(0, z) \leq a$ and then $\alpha_1(f) \leq \mathrm{O}_1(f)$. □

Theorem 2.7. *Let $f \in Hol(\Delta, \Delta)$ be such that there exists $z_0 \in \Delta$ with $f(z_0) = z_0$. Then $O_{\partial\Delta}(f) \leq \alpha(f)$. Moreover $\alpha(f) = 1$ if and only if $O_{\partial\Delta}(f) = 1$. In particular if $\alpha(f) < 1$ then f has no finite angular derivative at any boundary point and if $\alpha(f) = 1$ then $f(\Delta)$ is not relatively compact in Δ.*

Proof. Since both $\alpha(f)$ and $\mathrm{O}_{\partial\Delta}(f)$ are invariant under conjugation we can suppose that $z_0 = 0$. Suppose $\alpha(f) = 1$. Let $\{z_m\} \subset \Delta$ be such that

$$\lim_{m\to\infty}\frac{\omega(0, f(z_m))}{\omega(0, z_m)} = \alpha(f).$$

If $\{z_m\}$ accumulates to $x \in \Delta$ then $x \neq 0$ and $\omega(f(0), f(x)) = \omega(0, x)$, implying that f is an automorphism of Δ and hence $\mathrm{O}_{\partial\Delta}(f) = 1$. If $\{z_m\}$ accumulates at $x \in \partial\Delta$ then $\mathrm{O}_x(f) = 1$ by Lemma 2.6.

In general for any $x \in \partial\Delta$ by Lemma 2.6 we have that $\mathrm{O}_x(f) \leq \alpha(f)$ and therefore $\mathrm{O}_{\partial\Delta}(f) \leq \alpha(f)$. The last part follows easily from the very definition and from the Julia-Wolff-Carathéodory Theorem [5] (see also [1] and [15]). □

Remark 2.8. The order of contact $\mathrm{O}_{\partial\Delta}(f)$ could be strictly less than the dilatation $\alpha(f)$, *e.g.*, think of $z \mapsto \frac{1}{2}z$.

It is remarkable that in case of a (inner) fixed point the dilatation $\alpha(f)$ measures the *order of contact* of $f(\Delta)$ to $\partial\Delta$ but *not* the contact. That is if $f(\Delta)$ is not compactly contained in Δ then $\alpha(f)$ is not necessarily 1. For instance the lens map f_θ built in Example 2.4 has dilatation < 1 by Theorem 2.7, but $f_\theta(\pm 1) = \pm 1$.

Remark 2.9. We do not know if $\alpha(f) < 1$ is equivalent to f having no angular derivative at any boundary point. Actually the open question is whether f fixes 0 and has order of contact k at $x \in \partial\Delta$ imply that $\mathcal{L}_{x,k}(f) \neq 0, \infty$.

3. THE DILATATION IN STRONGLY CONVEX DOMAINS

Let D be a bounded strongly convex domain with smooth boundary (at least C^3). Let k_D be the Kobayashi distance in D. For all this section fix a point $z_0 \in D$ (sometimes we refer to such a point as the *base point* of D). For all the unproved statement and terminology we refer to [1] or [4]. Let us recall (see [2]):

Theorem 3.1 (Abate)**.** *Let $f \in Hol(D, D)$. If f has no fixed points in D then there exists a unique point $x \in \partial D$, the* Wolff point *of f, such that the sequence of iterates $\{f^k\}$ converges to the constant map $z \mapsto x$.*

Recall that if $f \in \mathrm{Hol}(D, D)$ then the *boundary dilatation coefficient* of f at $x \in \partial D$ is the (strictly) positive real number $\beta_x(f)$ given by

$$\frac{1}{2}\log\beta_x(f) := \liminf_{w\to x}[k_D(z_0, w) - k_D(z_0, f(w))].$$

Remark 3.2. The number $\beta_x(f)$ is finite or infinite independently of z_0. Namely if $w_0 \in D$ and $\beta'_x(f)$ is the boundary dilatation coefficient with base point w_0 then $\beta_x(f) < \infty$ if and only if $\beta'_x(f) < \infty$. This is a simple application of the triangle inequality. In Lemma 6.1 we will show that actually $\beta_x(f) = \beta'_x(f)$.

Similarly to the disc case, even in the strongly convex domain D it is possible to define horospheres. This definition due to Abate [2] exploits the Kobayashi distance and turns out to be the right tool to study iteration theory. In particular the boundary dilatation coefficient $\beta_x(f)$ measures how f acts on horospheres centered at $x \in \partial D$. Using the explicit expression of ω it is possible to show that in the disc the notion of boundary dilatation coefficient given in the Introduction coincides with that given here (see [1]), and it is independent of the base point.

We also need (see *Proposition 1.6 in* [4]):

Proposition 3.3. *Let $f \in Hol(D, D)$ have no fixed points. A point $x \in \partial D$ is the Wolff point of f if and only if f has limit x at x along some non-tangential path and $\beta_x(f) \leq 1$.*

Motivated by the work in the unit disc we give:

Definition 3.4. Let $f \in \overline{\mathrm{Hol}(D, D)}$. If f is non-constant and has no fixed points in D then the *dilatation* $\alpha(f)$ of f is the boundary dilatation coefficient of f at its Wolff point. If f is a non-constant map with at least one fixed point in D then the *dilatation* of f is given by:

$$\alpha(f) := \sup_{z\in D-\{w_0\}} \frac{k_D(w_0, f(z))}{k_D(w_0, z)},$$

where $f(w_0) = w_0$. We let $\alpha(f) := 0$ if f is constant.

Remark 3.5. If $f \in \overline{\text{Hol}(D,D)}$ then $\alpha(f) \leq 1$ by Proposition 3.3 and the decreasing property of $\text{Hol}(D,D)$ with respect to k_D. If $\text{Fix}(f)$, the set of fixed points of f, contains two points (and then it is a submanifold of dimension greater or equal to one, see [17]) then $\alpha(f) = 1$. Moreover $\alpha(f) = 0$ if and only if f is constant.

Theorem 3.6. *The function* $\alpha : \overline{Hol(D,D)} \to [0,1]$ *is lower semicontinuous.*

Before proving the theorem we need to introduce the *Lempert projection device* and prove some facts about it.

By Lempert's work (see [13] and [1]) given any point $z \in \overline{D}$ there exists a unique complex geodesic $\varphi : \Delta \to D$, *i.e.*, a holomorphic isometry between ω and k_D, such that φ extends smoothly past the boundary, $\varphi(0) = z_0$ and $\varphi(t) = z$, with $t \in (0,1)$ if $z \in D$ and $t = 1$ if $z \in \partial D$. Moreover for any such complex geodesic there exists a holomorphic retraction $p : D \to \varphi(\Delta)$, *i.e.* p is a holomorphic self-map of D such that $p \circ p = p$ and $p(z) = z$ for any $z \in \varphi(\Delta)$. We call such a p the *Lempert projection* associated to φ. Furthermore we let $\tilde{p} := \varphi^{-1} \circ p$ and call it the *left inverse* of φ, for $\tilde{p} \circ \varphi = Id_\Delta$. The triple $(\varphi, p, \tilde{p})$ is the so-called *Lempert projection device*.

Lemma 3.7. *Suppose that* $f \in Hol(D,D)$ *has no fixed points in D and let* $\tau \in \partial D$ *be the Wolff point of* f. *Let* $\varphi : \Delta \to D$ *be the complex geodesic such that* $\varphi(0) = z_0$ *and* $\varphi(1) = \tau$ *and let p be the Lempert projection associated to* φ. *Let* $\tilde{f} := \tilde{p} \circ f \circ \varphi$. *Then* $\alpha(f) = \alpha(\tilde{f})$.

Proof. Using Abate's version of the Julia-Wolff-Carathéodory Theorem for strongly convex domains (actually we just need a maimed version of it, coming from the classical Julia-Wolff-Carathéodory Theorem, see *Theorem 2.4.(i) in* [4]) we get

$$\alpha(f) = \lim_{r \to 1} \frac{1 - \tilde{f}(r)}{1 - r}.$$

Then the classical Julia-Wolff-Carathéodory Theorem implies that $\alpha(\tilde{f}) = \alpha(f)$. $\square$

Remark 3.8. The map $\tilde{f}$ in the previous Lemma cannot be the identity on Δ. For if this were so then for any $\zeta, \xi \in \Delta$

$$\begin{aligned} k_D(\varphi(\zeta), \varphi(\xi)) &\geq k_D(f(\varphi(\zeta)), f(\varphi(\xi))) \geq k_D(p(f(\varphi(\zeta))), p(f(\varphi(\xi)))) \\ &= k_D(\varphi(\tilde{f}(\zeta)), \varphi(\tilde{f}(\xi))) = k_D(\varphi(\zeta), \varphi(\xi)), \end{aligned}$$

forcing equality at all the steps. In particular $f \circ \varphi$ would be a complex geodesic and $f(\varphi(\Delta)) = \varphi(\Delta)$—as sets—for the uniqueness of complex geodesics. But

$$f \circ \varphi(\zeta) = \varphi \circ \tilde{p} \circ f \circ \varphi(\zeta) = \varphi(\zeta)$$

for any $\zeta \in \Delta$, which would imply that $f_{|\varphi(\Delta)} = id$, against our hypothesis.

Lemma 3.9. *Suppose* $f \in Hol(D,D)$ *be such that* $f(x) = x$ *for exactly one* $x \in D - \{z_0\}$. *Let* $\varphi : \Delta \to D$ *be the complex geodesic such that* $\varphi(0) = z_0$ *and* $\varphi(t) = x$ *for some* $t > 0$ *and let p be the associated Lempert projection. Let* $\tilde{f} := \tilde{p} \circ f \circ \varphi$. *Then* $\alpha(\tilde{f}) \leq \alpha(f)$.

Proof. Note that $\tilde{f}$ maps Δ into Δ and that $\tilde{f}(t) = t$. Now

$$\omega(t, \tilde{f}(\xi)) = k_D(\varphi(t), \varphi(\tilde{f}(\xi)) = k_D(x, p \circ f \circ \varphi(\xi))$$
$$\leq k_D(x, f(\varphi(\xi))) \leq \alpha(f) k_D(x, \varphi(\xi)) = \alpha(f)\omega(t, \xi).$$

Since this holds for any $\xi \in \Delta$ then it follows that $\alpha(\tilde{f}) \leq \alpha(f)$. □

Proof of Theorem 3.6. Let $\{f_k\} \subset \overline{\mathrm{Hol}(D, D)}$ be such that $f_k \to f$ for some $f \in \overline{\mathrm{Hol}(D, D)}$. We can suppose that $\alpha(f_k) \to \beta$. If f is constant then there is nothing to prove. Also if $\mathrm{Fix}(f_k)$ has positive dimension for any k then $\alpha(f_k) = 1$ and then the limit is 1 which is certainly greater than or equal to $\alpha(f)$. Therefore we can suppose that $f, f_k \in \mathrm{Hol}(D, D)$ and that f_k has at most one fixed point in D for any k.

Suppose first that for any k there exists $z_k \in D$ such that $f(z_k) = z_k$. If $z_k \to x \in D$ then x is a fixed point for f. Now

$$k_D(z_k, f_k(z)) \leq \alpha(f_k) k_D(z_k, z)$$

for any $z \in D$, and passing to the limit we get for any $z \in D$

$$k_D(x, f(z)) \leq \beta k_D(x, z),$$

which means that $\beta \geq \alpha(f)$.

Suppose now that z_k converges to the point $\tau \in \partial D$. For any k let $\varphi_k : \Delta \to D$ be the complex geodesic such that $\varphi_k(0) = z_0$ and $\varphi_k(t_k) = z_k$ with $t_k \in (0, 1)$. Since the family $\{\varphi_k\}$ is normal we can suppose that, up to subsequences, $\varphi_k \to \varphi$. Since for any k and $\xi, \zeta \in \Delta$

$$\omega(\xi, \zeta) = k_D(\varphi_k(\xi), \varphi_k(\zeta)),$$

it follows that φ is a complex geodesic. Moreover t_k tends to 1. Indeed if $t_k \to r$ for $r < 1$ then

$$\infty > \omega(0, r) = \lim_{k\to\infty} \omega(0, t_k) = \lim_{k\to\infty} k_D(z_0, \varphi_k(t_k)) = \infty.$$

In the same way we can suppose that the Lempert projections p_k's converge to a holomorphic map p which is easily seen to be the Lempert projection associated to φ. Now let $\tilde{f}_k := \tilde{p}_k \circ f_k \circ \varphi_k$ and $\tilde{f} := \tilde{p} \circ f \circ \varphi$. Then $\tilde{f}_k, \tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ and moreover $\tilde{f}_k$ converges to $\tilde{f}$. By Lemma 3.7 and Lemma 3.9 it follows that $\alpha(\tilde{f}_k) \leq \alpha(f_k)$ and that $\alpha(f) = \alpha(\tilde{f})$. Hence the result follows from Theorem 1.7.

Finally suppose that each f_k has no fixed points in D and let $\tau_k \in \partial D$ be the Wolff point of f_k. Let $\tau_k \to \tau \in \partial D$. As before let φ_k be the family of complex geodesics such that $\varphi_k(0) = z_0$ and $\varphi_k(1) = \tau_k$. Up to subsequences we can suppose that $\{\varphi_k\}$ converges to the complex geodesic φ such that $\varphi(0) = z_0$ and $\varphi(1) = \tau$ and that the Lempert projections p_k converge to the Lempert projection p associated to φ. Let $\tilde{f}_k := \tilde{p_k} \circ f \circ \varphi_k$ and $\tilde{f} := \tilde{p} \circ f \circ \varphi$. Then $\tilde{f}_k \to \tilde{f}$, and the result follows from Theorem 1.7 and Lemma 3.7. □

In the proof of Theorem 3.6 we have not used the fact that "the Wolff (or fixed) points of f_k converge to the Wolff point of f". This is true thanks to a theorem due to Joseph and Kwack [10]. However it follows directly from our method of reduction to the one variable case. We briefly decribe this point.

Theorem 3.10 (Joseph-Kwack)**.** *Let $\{f_k\} \subset Hol(D, D)$ and let $f \in Hol(D, D)$. Suppose that $f_k \to f$ and f has no fixed points in D. Then*

(1) *if $f_k(z_k) = z_k$ and $z_k \to x \in \partial D$ then x is the Wolff point of f.*
(2) *if f_k has no fixed points for any k and $x_k \in \partial D$ is the Wolff point of f_k such that $x_k \to x \in \partial D$ then x is the Wolff point of f.*

Proof. In the first case let $\varphi_k : \Delta \to D$ be the complex geodesic such that $\varphi_k(0) = z_0$ and $\varphi_k(t_k) = x_k$ for some $t_k \in (0, 1)$. In the second case let $\varphi_k : \Delta \to D$ be the complex geodesic such that $\varphi_k(0) = z_0$ and $\varphi_k(1) = x_k$. In both cases let p_k be the Lempert projection associated to φ_k. As in the proof of Theorem 3.6 we suppose that φ_k converges to the complex geodesic $\varphi : \Delta \to D$ such that $\varphi(0) = z_0$ and $\varphi(1) = x$. Also we suppose p_k converging to the Lempert projection p associated to φ. We set $\tilde{f}_k := \tilde{p}_k \circ f_k \circ \varphi_k$ and $\tilde{f} := \tilde{p} \circ f \circ \varphi_k$. Then $\tilde{f}_k \to \tilde{f}$. By Lemma 3.7 $\alpha(f) = \alpha(\tilde{f})$. If $\tilde{f}$ is not the identity then by Heins' Theorem (see Remark 1.8) $\tilde{f}$ has Wolff point 1. Therefore, no matter whether $\tilde{f}$ is the identity, $\alpha(f) \leq 1$ and $f \circ \varphi : \Delta \to D$ has non-tangential limit x at 1. Since a complex geodesic is transverse to ∂D by Hopf's Lemma then f has limit x at x along a non-tangential path and by Proposition 3.3 it follows that x is the Wolff point of f. □

As an application of the previous results we have:

Corollary 3.11. *Suppose $\{f_k\} \subset Hol(D, D)$ and that for any k there exists $z_k \in D$ such that $f(z_k) = z_k$. Let $\beta := \limsup_{k\to\infty} \alpha(f_k)$. If $\beta < 1$ then a limit of $\{f_k\}$ is either a constant map $z \mapsto \tau \in \partial D$ where τ is in the cluster set of $\{z_k\}$ or it has a unique fixed point in $w_0 \in D$ such that w_0 belongs to the cluster set of $\{z_k\}$.*

Proof. Suppose that, up to subsequences, $\{z_k\}$ converges to a point $\tau \in \partial D$. Using the notation as in the proof of Theorem 3.6, the one-dimensional maps $\tilde{f}_k$ converge to $\tilde{f}$ and $\alpha(\tilde{f}) = \alpha(f) < 1$. By Remark 1.9 $\tilde{f}$ is the constant map $\zeta \mapsto 1$ and therefore $f(D) = \tau$.

If $\{z_k\}$ converges to $w_0 \in D$ then clearly $f(w_0) = w_0$. Moreover $\alpha(f) < 1$ implies that the dimension of $\mathrm{Fix}(f)$ is zero and hence by Vigué [17] w_0 is the only fixed point of f. □

4. The order of contact in strongly convex domains

In this section we describe what is the "order of contact" for a self-map of a strongly convex domain and how it is related to the dilatation. In all this section D is a bounded strongly convex domain with C^2 boundary and $z_0 \in D$ is its base point.

We need to recall the following lemma (see *Theorem (2.3.51) and (2.3.52) in* [1] *and* [19]).

Lemma 4.1 (Abate, Vormoor)**.** *Let $d(\cdot, \cdot)$ denote the euclidean distance in $\mathbb{C}^n$. Then there are two constant $C_1 > 0$ and $C_2 > 0$ depending only on z_0 such that for all $z \in D$*

$$-C_1 - \frac{1}{2}\log d(z, \partial D) \leq k_D(z_0, z) \leq C_2 - \frac{1}{2}\log d(z, \partial D).$$

Let $x \in \partial D$ and $f \in \mathrm{Hol}(D, D)$. By Lemma 4.1 it follows easily that

$$\liminf_{z \to x} \frac{d(f(z), \partial D)}{d(z, \partial D)} > 0. \tag{4.1}$$

Now we define the order of contact in analogy to the disc case:

Definition 4.2. Let $f \in \mathrm{Hol}(D, D)$, $x \in \partial D$ and $k \in \mathbb{R}$. We set

$$\mathcal{L}_{x,k}(f) := \liminf_{z \to x} \frac{d(f(z), \partial D)}{d(z, \partial D)^k}.$$

A similar argument as in Remark 2.2 allows one to define the *order of contact of f at x*, $\mathrm{O}_x(f)$, as the unique real number k such that $\mathcal{L}_{x,m}(f) = \infty$ for $m > k$ and $\mathcal{L}_{x,t}(f) = 0$ for $t < k$. The *(global) order of contact* is defined as

$$\mathrm{O}_{\partial D}(f) = \sup_{x \in \partial D} \mathrm{O}_x(f).$$

By equation (4.1) it follows that the order of contact is at most 1. We start to study the relationship between the order of contact and the dilatation:

Lemma 4.3. *Let $f \in Hol(D, D)$ be such that $f(w_0) = w_0$ for some $w_0 \in D$ and let $x \in \partial D$. Let*

$$\alpha_x(f) := \limsup_{z \to x} \frac{k_D(w_0, f(z))}{k_D(w_0, z)}.$$

Then $O_x(f) = \alpha_x(f)$.

Proof. If $\alpha_x(f) \leq k$ then by Lemma 4.1 it follows that

$$\limsup_{z \to x} \frac{\log d(f(z), \partial D)}{\log(d(z, \partial D))^k} \leq 1.$$

Therefore for a fixed $\epsilon > 0$ there exists a neighborhood U of x such that for any $z \in U \cap D$ it holds

$$\log d(f(z), \partial D) \geq \log(d(z, \partial D))^{k(1+\epsilon)}.$$

Since $r \mapsto \log r$ is increasing for $r > 0$ then

$$d(f(z), \partial D) \geq d(z, \partial D))^{k(1+\epsilon)},$$

and therefore $\mathcal{L}_{x,k(1+\epsilon)}(f) \geq 1$. Since ϵ is arbitrary this implies that $\mathrm{O}_x(f) \leq k = \alpha_x(f)$. On the other hand, if $\mathcal{L}_{x,k}(f) = \infty$ for some k then there exists a neighborhood U of x such that for any $z \in U \cap D$ it follows

$$d(f(z), \partial D) \geq d(z, \partial D)^k,$$

which is equivalent to

$$\frac{\log d(f(z), \partial D)}{\log d(z, \partial D)^k} \leq k,$$

implying that $\alpha_x(f) \leq k$ and hence $\alpha_x(f) \leq \mathrm{O}_x(f)$. $\square$

Theorem 4.4. *Let $f \in Hol(D, D)$ be such that there exists $w_0 \in D$ with $f(w_0) = w_0$. Then $O_{\partial D}(f) \leq \alpha(f)$. Moreover $\alpha(f) = 1$ if and only if $O_{\partial D}(f) = 1$.*

Proof. For any $x \in \partial D$ we get $\mathrm{O}_x(f) \leq \alpha(f)$ by Lemma 4.3, and hence $\mathrm{O}_{\partial D}(f) \leq \alpha(f)$. Suppose $\alpha(f) = 1$ and let $\{z_m\} \subset D$ be such that

$$\lim_{m\to\infty} \frac{k_D(w_0, f(z_m))}{k_D(w_0, z_m)} = 1.$$

We can suppose that $z_m \to x \in \overline{D}$. If $x \in \partial D$ then $\mathrm{O}_x(f) = 1$ by Lemma 4.3 and we are done. If $x \in D$ then $k_D(w_0, f(x)) = k_D(w_0, x)$. Let $\varphi_x : \Delta \to D$ be the complex geodesic such that $\varphi_x(0) = w_0$ and $\varphi_x(t) = x$ for some $t > 0$. Let $\varphi_{f(x)} : \Delta \to D$ be the complex geodesic such that $\varphi_{f(x)}(0) = w_0$ and $\varphi_{f(x)}(r) = f(x)$ for some $r > 0$. Let $p_{f(x)} : D \to D$ be the Lempert projection associated to $\varphi_{f(x)}$. Define $\tilde{f} := \varphi_{f(x)}^{-1} \circ p_{f(x)} \circ f \circ \varphi_x$. Since

$$\omega(0, r) = k_D(w_0, f(x)) = k_D(w_0, x) = \omega(0, t),$$

then $r = t$. Therefore

$$\omega(0, t) = k_D(w_0, x) = k_D(w_0, f(x)) = \omega(\tilde{f}(0), \tilde{f}(t)),$$

implying that $\tilde{f}$ is an automorphism of Δ. Namely f maps $\varphi_x(\Delta)$ onto $\varphi_{f(x)}(\Delta)$ acting as an automorphism. In particular

$$\limsup_{z\to\varphi_x(1)} \frac{k_D(w_0, f(z))}{k_D(w_0, z)} \geq \lim_{\zeta\to 1} \frac{\omega(0, \tilde{f}(\zeta))}{\omega(0, \zeta)} = 1,$$

and by Lemma 4.3 $\mathrm{O}_{\varphi_x(1)}(f) = 1$ which gives the assertion. $\square$

5. The directional dilatation

Let us start with an example.

Example 5.1. Let $f \in \mathrm{Hol}(\mathbb{B}^2, \mathbb{B}^2)$, where $\mathbb{B}^2$ is the unit ball in $\mathbb{C}^n$, given by $(z_1, z_2) \mapsto (z_1, 0)$. Then $f((0, 0)) = (0, 0)$ and $\alpha(f) = 1$. However $f((0, \zeta)) = (0, 0)$ for any $\zeta \in \Delta$. Therefore the map $\tilde{f}_1(\zeta) := f_1(\zeta, 0)$ has dilatation $1 = \alpha(f)$ while $\tilde{f}_2(\zeta) := f_2(0, \zeta)$ has dilatation 0.

The previous example suggests that one could associate to f a family of "directional dilatations" measuring in some sense the behavior of the map along these directions. To make precise this idea we need to recall and to prove some facts.

Let D be a bounded strongly convex domani with C^3 boundary. Let $\mathcal{K}$ be the set of compacta of $\overline{D}$. We can endow $\mathcal{K}$ with a structure of complete metric space by defining the *Hausdorff distance* between two elements $A, B \in \mathcal{K}$ as

$$d_H(A, B) := \max\{d(A, B), d(B, A)\},$$

where $d(A, B) = \max\{d(x, B) | x \in A\}$ and $d(\cdot, \cdot)$ is the euclidean distance. We denote by $\mathcal{H}(\overline{D})$ the metric space $(\mathcal{K}, d_H)$. Note that by the very definition $\{A_k\} \subset \mathcal{H}(\overline{D})$ is (Hausdorff-)converging to $A \in \mathcal{H}(\overline{D})$ if and only if both any point of A is in the cluster set of some sequence $\{z_k\}$ for $z_k \in A_k$ and any sequence $\{z_k\}$ such that $z_k \in A_k$ accumulates only at points of A. We define and topologize the "space of (images of) complex geodesics":

Definition 5.2. Let

$$\mathcal{G} := \{G \subset \overline{D} \mid \exists \varphi : \Delta \to D \text{ complex geodesic} : G = \varphi(\overline{\Delta})\} \cup \{x \mid x \in \partial D\}.$$

We endow $\mathcal{G}$ with the topology induced by $\mathcal{H}(\overline{D})$.

Note that an element of $\mathcal{G}$ is either the closed image of a complex geodesic or a single boundary point. Before proving some interesting properties of $\mathcal{G}$ we need to look closely at complex geodesics.

We recall (see [13]) that a complex geodesic $\varphi : \Delta \to D$ is defined to be a solution of an Euler-Lagrange equation in the sense that there exists a positive function P defined on $\partial\Delta$ so that $\varphi^*(\zeta) := P(\zeta)\zeta\overline{n(\varphi(\zeta))}$, defined on $\partial\Delta$, extends holomorphically to Δ (here $n(x)$ is the outer unit normal to ∂D at $x \in \partial D$). The map φ^* is called a *dual map* of φ. Then $\tilde{p}(z)$ is defined to be the (unique) solution to the following equation in the unknown $\zeta \in \Delta$ (here $\langle z, w\rangle = \sum z_j w_j$):

$$\langle z - \varphi(\zeta), \varphi^*(\zeta)\rangle = 0. \tag{5.1}$$

It is a basic fact in Lempert's theory (see [13] and [1]) that φ^* is unique up to a positive constant and extends C^1 up to $\partial\Delta$. Therefore $\tilde{p}$ is uniquely determined by φ and extends C^1 up to $\partial\Delta$. Hence if $\eta := \varphi \circ \theta$ with θ an automorphism of Δ it follows that $\theta^{-1} \circ \tilde{p}$ is the left inverse associated to η. The Lempert projection associated to φ is $\varphi \circ \tilde{p}$ which therefore turns out to depend only on the image $\varphi(\Delta)$ and to extend C^1 up to ∂D. Therefore if $G \in \mathcal{G}$ then there exists a *unique* C^1 map $p : \overline{D} \to \overline{D}$ such that p is holomorphic on D, $p^2 = p$ and $p(\overline{D}) = G$ (if $G = x \in \partial D$ then $p(\overline{D}) := x$). Conversely (see [14]) if $p : D \to D$ is holomorphic, $p^2 = p$ and $\tilde{G} = p(D)$ is one-dimensional then $\tilde{G}$ is the image of a complex geodesic $\varphi : \Delta \to D$, and therefore p actually is the Lempert projection associated to a complex geodesic and in particular extends C^1 up to the boundary. Therefore the correspondence:

$$F : \mathcal{G} \to \mathcal{P} := \{p | p \in \overline{\mathrm{Hol}(D, D)}, p^2 = p, \dim p(D) \leq 1\},$$

$$G \mapsto p_G$$

is one-to-one and onto. Moreover if we endow $\mathcal{P}$ with the topology induced by $\overline{\mathrm{Hol}(D, D)}$ we have

Lemma 5.3. *The space $\mathcal{P}$ is closed in $\overline{Hol(D, D)}$ and the map $F : \mathcal{G} \to \mathcal{P}$ is a homeomorphism.*

Proof. Suppose that $\{G_k\} \subset \mathcal{G}$ converges to $G \in \mathcal{H}(\overline{D})$, and let $p_k := F(G_k)$. Since $\{p_k\}$ is a normal family it follows that it contains some converging subsequences. Suppose that $p_{k_m} \to p \in \overline{\mathrm{Hol}(D, D)}$. Then taking the limit to $p_{k_m}^2 = p_{k_m}$ we find $p^2 = p$. We have to show that $\dim p(D) \leq 1$. Assume not. Then there exist $z_0, z_1, z_2 \in p(D)$ which are not contained in any complex geodesic. Let $z_m^j := p_{k_m}(z_j) \in D$ for $j = 0, 1, 2$. Then $z_m^j \in p_{k_m}(D) = \varphi_m(\Delta)$ for some complex geodesic φ_m and $j = 0, 1, 2$. Up to subsequences, $\varphi_m \to \varphi$, with φ being a complex geodesic. Therefore $\varphi_m(\Delta) \ni z_m^j \to z_j \in \varphi(\Delta)$ for $j = 0, 1, 2$ and thus $\varphi(\Delta)$ contains z_0, z_1, z_2, a contradiction. Then $p \in \mathcal{P}$ (which is thus closed). By the previous discussion p

extends C^1 up to the boundary. Now if $z \in G$ then there exists $\{z_k\} \subset \overline{D}$ such that $z_k \in G_k$ and $z_k \to z$. Therefore

$$p(z) = \lim_{m\to\infty} p_{k_m}(z_{k_m}) = \lim_{m\to\infty} z_{k_m} = z,$$

and hence $G \subseteq p(\overline{D})$. On the other hand if $z \in p(\overline{D})$ then

$$z = p(z) = \lim_{m\to\infty} p_{k_m}(z).$$

Let $z_m := p_{k_m}(z) \in G_{k_m}$. Then $z = \lim_{m\to\infty} z_m$ and therefore $z \in G$, implying that $p(\overline{D}) \subseteq G$ and actually $G = p(\overline{D})$. Since this holds for any converging subsequences of $\{p_k\}$ then it follows that p_k actually converges to p, holomorphic retraction and that $G = p(\overline{D})$ (and hence $\mathcal{G}$ is closed in $\mathcal{H}(\overline{D})$). This also shows that F is bi-continuous and therefore it is a homeomorphism. □

The previous Lemma allows one to move freely from maps to sets. Before going ahead we need to introduce some more notation. If $G \in \mathcal{G}$ and $G \cap D \neq \emptyset$ we indicate by $\varphi_G : \Delta \to D$ a complex geodesic such that $\varphi_G(\overline{\Delta}) = G$ (note that φ_G is unique only up to composition by automorphisms of Δ on the right). If $G = x \in \partial D$ then we indicate by φ_G the constant map $\zeta \mapsto x$ for $\zeta \in \Delta$.

Lemma 5.4. *Let $\{G_k\} \subset \mathcal{G}$, $G \in \mathcal{G}$. Then $G_k \to G$ if and only if there exist φ_{G_k} and φ_G such that $\varphi_{G_k} \to \varphi_G$.*

Proof. If $\varphi_{G_k} \to \varphi_G$ then it is easy to see using the definition that $p_k \to p$ and by Lemma 5.3 it follows that $G_k \to G$.

On the other hand if $G_k \to G$ then we have two cases. If $G \cap D \neq \emptyset$ (which implies that $G_k \cap D \neq \emptyset$ eventually) then let $z_0 \in G \cap D$. There exists $\{z_k\} \subset D$ such that $z_k \in G_k$ and $z_k \to z_0$. Let φ_{G_k} be the complex geodesic such that $\varphi_{G_k}(0) = z_k$. Since the family $\{\varphi_{G_k}\}$ is normal, we can extract a converging subsequence $\varphi_{G_{k_m}} \to \varphi$, where $\varphi : \Delta \to D$ is a complex geodesic for $\varphi(0) = z_0$. The sequence of Lempert's projections associated to $\varphi_{G_{k_m}}$ converges to a holomorphic retraction p that by Lemma 5.3 must be $F(G)$, therefore $\varphi = \varphi_G$. The same token shows that any other subsequence of $\{\varphi_{G_k}\}$ must converge to φ and therefore φ_{G_k} converges to φ_G.

The second case is when G is a point $x \in \partial D$. If G_k is a point for all k then the result is trivial. Suppose then that the G_k's are not all reduced to a point. As before $\{\varphi_{G_k}\}$ contains a converging subsequence. If the limit map φ is a constant $\zeta \mapsto y \in \partial D$ then by Lemma 5.3 it must be $y = x$. If φ were non-constant then the Lempert projection p_k associated to the converging subsequence would converge to a non-constant holomorphic retraction p, and by Lemma 5.3 $G = F^{-1}(p)$ would not be a single point. Therefore the entire sequence φ_{G_k} must converge to φ_G. □

Lemma 5.5. *Let $f \in Hol(D, D)$. Let $\varphi : \Delta \to D$ be a complex geodesic and let $\tilde{p}$ be its left inverse. Then $\alpha(\tilde{p} \circ f \circ \varphi)$ depends only on $G := \varphi(\overline{\Delta})$.*

Proof. By the discussion on complex geodesics at the beginning of this section any other parameterization of the geodesic disc whose closure is G has the form $\varphi \circ \theta$ for some automorphism θ of Δ. The associated left inverse is therefore $\theta^{-1} \circ \tilde{p}$. Thus the result follows from Remark 2.5. $\square$

Definition 5.6. Let $f \in \mathrm{Hol}(D, D)$ and $G \in \mathcal{G}$, with $G \cap D \neq \emptyset$. We define the *directional dilatation* $\alpha_G(f)$ as

$$\alpha_G(f) := \alpha(\tilde{p}_G \circ f \circ \varphi_G).$$

If G is a (boundary) point then we set $\alpha_G(f) := 0$.

Remark 5.7. The directional dilatation is well defined by Lemma 5.5.

The main result for the directional dilatation is the following:

Theorem 5.8. *The map* $\mathcal{G} \times \overline{Hol(D, D)} \mapsto [0, 1]$ *defined as*

$$(G, f) \mapsto \alpha_G(f)$$

is lower semicontinuous.

Proof. The result is trivially true if G is reduced to a boundary point. Suppose that $G \cap D \neq \emptyset$. By Lemma 5.4 if $G_k \to G$ there exist φ_{G_k} such that $\varphi_{G_k} \to \varphi_G$. Therefore the left-inverses $\tilde{p}_{G_k}$ associated to φ_{G_k} converge to the left-inverse $\tilde{p}_G$ associated to φ_G and hence $\tilde{p}_{G_k} \circ f_k \circ \varphi_{G_k}$ converges to $\tilde{p} \circ f \circ \varphi_G$. the result follows then from Theorem 1.7. $\square$

Let us indicate by $S := \partial \mathbb{B}^n = \{v \in \mathbb{C}^n : \|v\| = 1\}$. Let $z_0 \in D$. For any $v \in S$ let $\varphi_v : \Delta \to D$ be the complex geodesic such that $\varphi_v(0) = z_0$ and $\varphi_v'(0) = \lambda v$ for some $\lambda \in \mathbb{C} - \{0\}$. Lempert's theory (see [13]) asserts that such a geodesic does exist, unique up to automorphisms of the disc, and extends smoothly through the boundary. If $\tau \in \partial D$ let us indicate by $A_\tau := S \cap (\mathbb{C}^n - T^{\mathbb{C}}_\tau \partial D)$. Then for any $v \in A_\tau$ there exists a complex geodesic $\varphi_v : \Delta \to D$ such that $\varphi_v(1) = \tau$ and $\varphi_v'(1) = \lambda v$ with $\lambda \in \mathbb{C} - \{0\}$ (see [6] for ∂D of class C^{14} and [9] for ∂D of class C^3). Even in this case the geodesic is uniquely determined up to right composition with automorphims of the unit disc.

Therefore there are two different ways of indicating a complex geodesic: by means of its closed image G or by giving a point and a (non complex tangent if the point is on the boundary) direction. One could therefore define a *directional dilatation* $\alpha_{z,v}(f) := \alpha_G(f)$ for $z \in \overline{D}$ and $v \in S$, according to whether G is the (closure of the) image of φ_v (and as usual $\alpha_{z,v}(f) := 0$ if $z \in \partial D$ and $v \in T^{\mathbb{C}}_z \partial D$). Even in this case the result is a lower semicontinuous function:

Proposition 5.9. *The map* $\overline{D} \times S \times \overline{Hol(D, D)} \to [0, 1]$ *defined as*

$$(z, v, f) \mapsto \alpha_{z,v}(f)$$

is lower semicontinuous.

The proof is a consequence of Theorem 5.8 and the following Theorem:

Theorem 5.10. *The map* $\overline{D} \times S \mapsto \mathcal{G}$ *which associates to* (z, v) *the element* $G_{z,v} \in \mathcal{G}$ *according to whether* $G_{z,v}$ *contains* z *and is parallel to* v *at* z *(or* $G_{z,v} := z$ *if* $z \in \partial D$ *and* $v \in T^{\mathbb{C}}_z \partial D$*) is continuous.*

Proof. Let $\{v_k\} \subset S$ and $\{z_k\} \subset \overline{D}$. Suppose that $v_k \to v$, $z_k \to z$ with, respectively, $v \in S$ and $z \in \overline{D}$.

First suppose that $z \in D$. Then $z_k \in D$ eventually. Let $\{\varphi_{v_k}\}$ be the family of complex geodesics such that $\varphi_{v_k}(0) = z_k$ and $\varphi'_{v_k}(0) = \lambda_k v_k$ for $\lambda_k > 0$. Since this family is normal we can extract a subsequence $\{\varphi_m\}$ which converges to a map $\varphi : \Delta \to D$. Using the continuity of k_D it is not difficult to show that φ is a complex geodesic such that $\varphi(0) = z$. Moreover $\varphi'_m(0) \to \lambda v$ for some $\lambda > 0$. Therefore $G_{z_m,v_m} \to G_{z,v}$ by Lemma 5.4. Note that the same reasoning applies to any subsequence and therefore actually we have $G_{z_k,v_k} \to G_{z,v}$.

Suppose now that $z_k \in \partial D$ and then $z \in \partial D$. If $v_k \in T^{\mathbb{C}}_{\tau_k}\partial D$ for any k then $v \in T^{\mathbb{C}}_{\tau}\partial D$. Indeed if $\rho : \mathbb{C}^n \to \mathbb{R}$ is a local defining function for D near τ then $\partial\rho_{\tau_k}(v_k) = 0$ for k large, and taking the limit we get $\partial\rho_\tau(v) = 0$. In this case the assertion is verified as $G_{z_k,v_k} = z_k \to z = G_{z,v}$. Therefore we can suppose that $v_k \in A_{z_k}$ for any k. Let $\{\varphi_{v_k}\}$ be a family of complex geodesics such that $\varphi_{v_k}(1) = z_k$ and $\varphi'_{v_k}(1) = \lambda_k v_k$ for $\lambda_k \neq 0$. Up to subsequences we can suppose that $\varphi_{v_k} \to \varphi$ with $\varphi : \Delta \to \overline{D}$ holomorphic. If $\varphi(\Delta) \subset D$ then it is a complex geodesic. Moreover $\varphi(1) = \lim_{k\to\infty} \varphi_{v_k}(1) = z$ and $\varphi'(1) = \lim_{k\to\infty} \varphi'_{v_k}(1) = \lambda v$ for some $\lambda \neq 0$. If $v \in T^{\mathbb{C}}_z\partial D$ this is impossible, all the converging subsequences of $\{\varphi_k\}$ must therefore converge to a constant map $\zeta \mapsto w \in \partial D$, for $\zeta \in \Delta$, and it is easy to see that $w = z$. By Lemma 5.4 this implies that $G_{z_k,v_k} \to G_{z,v} = z$. If $v \in A_z$ then Lemma 5.4 implies that $G_{z_m,v_m} \to G_{z,v}$ for any subsequence $\{\varphi_m\}$ which converges to a (non-constant) complex geodesic. So we are led to prove that we can reparameterize φ_{v_k} in such a way that no subsequences of $\{\varphi_{v_k}\}$ can converge to a constant map. Since $v_k \to v$ and $v \notin T^{\mathbb{C}}_z\partial D$ there exists k_0 and $C > 0$ such that for any $k > k_0$

$$|(v_k)_N| > \frac{C}{2}|(v_k)_T|,$$

where if $x \in \partial D$ and $a \in \mathbb{C}^n - \{0\}$ then $|(a)_T|$ and $|(a)_N|$ indicate respectively the complex tangential and complex normal component of a. Hence for $k > k_0$ there exists $\tilde{t}_k \in (0,1)$ such that for any $t_k > \tilde{t}_k$ the point z_k is the nearest boundary point to $\varphi_{v_k}(t_k)$ (for $\varphi_{v_k}(\Delta)$ is transversal to ∂D by Hopf's Lemma) and

$$|(\varphi'_{v_k}(t_k))_N| > C|((\varphi'_{v_k}(t_k))_T|,$$

where this time $(\varphi'_{v_k}(t_k))_N$ and $(\varphi'_{v_k}(t_k))_T$ indicate, respectively, the complex tangential and the complex normal component of the vector $\varphi'_{v_k}(t_k)$ as if it were a vector of $T^{\mathbb{C}}_{z_k}\partial D$. Thus *Corollary 2* of [9] implies that the diameter of $\varphi_{v_k}(\Delta)$ is bounded from below independently of k and *Proposition 4* of [6] provides a compact $K \subset\subset D$ such that $\varphi_{v_k}(\Delta) \cap K \neq \emptyset$ for any k. Therefore for any k there exists $\zeta_k \in \Delta$ such that $\varphi_{v_k}(\zeta_k) \in K$. Then for any k there exists an automorphism θ_k of Δ such that $\theta_k(0) = \zeta_k$. Hence $\tilde{\varphi}_k := \varphi_k \circ \theta_k$ is a reparameterization of φ_k such that $\tilde{\varphi}_k(0) \in K$ for any k. Therefore any limit φ is such that $\varphi(0) \in K$ and so φ is a complex geodesic.

Now suppose that $z_k \in D$ for any k and $z \in \partial D$. Let $\{\varphi_{v_k}\}$ be the family of complex geodesics such that $\varphi_{v_k}(0) = z_k$ and $\varphi'_{v_k}(0) = \lambda_k v_k$ for some $\lambda_k > 0$. If $v \in T^{\mathbb{C}}_z\partial D$ then any limit of $\{\varphi_{v_k}\}$ is the constant map $\zeta \mapsto z$ and the result follows from Lemma 5.4. Indeed if there were a non-constant limit φ, then by *Corollary 2* of [9] the directions $v_m := \varphi'_{v_m}(0)/\|\varphi'_{v_m}(0)\|$ of such subsequence would be eventually of the form $|(v_m)_N| > C|(v_m)_T|$, and then $|v_N| \geq C|v_T|$ which is impossible (here we retain the notations of the previous step). Suppose now that $v \in A_z$. Again we want to show that it is possible to reparameterize φ_{v_k} in such a way that no subsequence is converging to a constant. Once we have this it is clear that any limit must be of the form $\varphi_{G_{z,v}}$—for the convergence is actually in $C^1(\overline{\Delta})$—and the result follows by Lemma 5.4. Since $v_k \to v$ and $v \notin T_z^{\mathbb{C}}\partial D$ then there exists $C > 0$ such that for any k it follows $|\varphi'_{v_k}(0))_N| \geq C|(\varphi'_{v_k}(0))_T|$. Arguing as before we can find a suitable reparameterization of $\{\varphi_k\}$ such that any convergent subsequence tends to a (non-constant) complex geodesic. $\square$

6. Applications

The main point in the previous section was that the restriction of $f \in \mathrm{Hol}(D, D)$ to a "geodesic disc" is well-defined up to conjugation. Namely if $\varphi : \Delta \to D$ and $\eta : \Delta \to D$ are complex geodesics such that $\varphi(\overline{\Delta}) = \eta(\overline{\Delta}) =: G$, and $\tilde{p}$, $\tilde{q}$ are the associated left-inverses, then $\tilde{p} \circ f \circ \varphi$ is conjugated to $\tilde{q} \circ f \circ \eta$. Therefore given $G \in \mathcal{G}$ with $G \cap D \neq \emptyset$ we can define f_G as

$$f_G := \{g \in \mathrm{Hol}(\Delta, \Delta) \mid \exists \theta \in \mathrm{Aut}(\Delta) : \theta^{-1} \circ g \circ \theta = \tilde{p_G} \circ f \circ \varphi_G\}.$$

Note that any two functions $h, g : \Delta \to \partial\Delta$ are conjugated, for $\mathrm{Aut}(\Delta)$ acts (doubly) transitively on $\partial\Delta$. Therefore if $G = x \in \partial D$ we can define f_G as the conjugation class of $\zeta \mapsto 1$. The map

$$\overline{\mathrm{Hol}(D, D)} \times \mathcal{G} \to \overline{\mathrm{Hol}(\Delta, \Delta)}/\mathrm{Aut}(\Delta),$$
$$(f, G) \mapsto f_G$$

is continuous by Lemma 5.4.

Roughly speaking the underlying philosophy is that if $\mathcal{W}$ is an *intrinsic property* of $\mathrm{Hol}(\Delta, \Delta)$ (*i.e.* depending only on the conjugation class of $\mathrm{Hol}(\Delta, \Delta)$) then $\mathcal{W}$ extends to a "directional property" on convex domains by $(G, f) \mapsto \mathcal{W}(f_G)$. We give some examples of this.

Recall that if $f \in \mathrm{Hol}(D, D)$ we indicate by $\beta_x(f)$ the boundary dilatation coefficient of f at $x \in \partial D$. We begin with the following lemma:

Lemma 6.1. *Let $f \in Hol(D, D)$, $\tau \in \partial D$ and $G, H \in \mathcal{G}$ with $G \cap D \neq \emptyset$, $H \cap D \neq \emptyset$ and $\tau \in G \cap H$. Let φ_G be a complex geodesic such that $\varphi_G(\overline{\Delta}) = G$ and $\varphi_G(1) = \tau$. Let φ_H be a complex geodesic such that $\varphi_H(\overline{\Delta}) = H$ and $\varphi_H(1) = \tau$. Let $\tilde{p}_G$ and $\tilde{p}_H$ be the left inverses of φ_G and φ_H respectively. Finally let $h_G := \tilde{p}_G \circ f \circ \varphi_G$ and $h_H := \tilde{p}_H \circ f \circ \varphi_H$. Then $\beta_1(h_G) = \beta_1(h_H) = \beta_\tau(f)$. Therefore the boundary dilatation coefficient does not depend on the base point chosen.*

Proof. Let us choose $z_0 := \varphi_H(0)$ as base point. Then *Theorem 2.7.14 of* [1] (see also [3]) and the classical Julia-Wolff-Carathéodory Theorem [5] imply that $\beta_\tau(f) = \beta_1(h_H)$. By Remark 3.2 $\beta_1(h_H) = \infty$ if and only if $\beta_1(h_G) = \infty$. Suppose then $\beta_1(h_H) < \infty$. The curve $t \mapsto \varphi_H(t)$ goes non-tangentially to τ by Hopf's Lemma. Therefore again *Theorem 2.7.14 of* [1] gives

$$\beta_1(h_G) = \lim_{t \to 1} \frac{1 - \tilde{p}_G \circ f \circ \varphi_H(t)}{1 - \tilde{p}_G \circ \varphi_H(t)}.$$

Up to subsequences we can suppose that $(\tau - f \circ \varphi_H(t))/\|\tau - f \circ \varphi_H(t)\|$ tends to some unit vector v. Therefore

$$\lim_{t\to 1} \frac{1-\tilde{p}_G \circ f \circ \varphi_H(t)}{1-\tilde{p}_G \circ \varphi_H(t)}$$
$$= \lim_{t\to 1} \frac{1-\tilde{p}_G \circ f \circ \varphi_H(t)}{\|\tau - f \circ \varphi_H(t)\|} \cdot \frac{\|\tau - f \circ \varphi_H(t)\|}{1-h_H(t)} \frac{1-h_H(t)}{1-t} \cdot \frac{1-t}{1-\tilde{p}_G \circ \varphi_H(t)}.$$

By *Lemma 2.6.44 of* [1] (or see [3]) it follows then

$$\beta_1(h_G) = \frac{\langle v, n_\tau\rangle}{\langle \varphi_G'(1), n_\tau\rangle} \cdot \frac{\langle \varphi_H'(1), n_\tau\rangle}{\langle v, n_\tau\rangle} \cdot \beta_1(h_H) \cdot \frac{\langle \varphi_G'(1), n_\tau\rangle}{\langle \varphi_H'(1), n_\tau\rangle} = \beta_1(h_H).$$

□

Note that if $h \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point in Δ so does any $g \in \mathrm{Hol}(\Delta, \Delta)$ conjugated to h. Therefore if $f \in \mathrm{Hol}(D, D)$ and $G \in \mathcal{G}$, $G \cap D \neq \emptyset$, with abuse of terminology we say that *f_G has a fixed point in Δ* to mean that any $h \in f_G$ has a fixed point in Δ. As custom we denote by $Fix(f)$ the set of fixed points of f in D.

By a result of Vigué [17] the set $Fix(f)$ is a submanifold of D.

Proposition 6.2. *Let $f \in Hol(D, D)$ and let $G \in \mathcal{G}$ with $G \cap D \neq \emptyset$. Then*

(1) *If $\alpha_G(f) < 1$ and f_G has no fixed points in Δ then f has no fixed points in D, its Wolff point belongs to G and $\alpha(f) = \alpha_G(f)$.*
(2) *If $\alpha_G(f) = 1$ and f_G has no fixed points in Δ then either f has no fixed points in D, its Wolff point belongs to G and $\alpha(f) = 1$ or $dim Fix(f) \geq 1$ and $G \cap \overline{Fix(f)} \neq \emptyset$.*

In particular if f_G has no fixed points in Δ then $\alpha_G(f) = \alpha(f)$.

Proof. (1) Let $\varphi_G : \Delta \to D$ be a complex geodesic such that $\varphi_G(\overline{\Delta}) = G$ and $h := \tilde{p}_G \circ f \circ \varphi_G$ has Wolff point 1 (here, as usual, we let $\tilde{p}_G$ be the left-inverse of φ_G). Let $\tau := \varphi_G(1)$. By Lemma 6.1 the boundary dilatation coefficient of f at τ is independent of the base point. Hence

$$\beta_\tau(f) = \beta_1(h) = \alpha_G(f) < 1.$$

Suppose $Fix(f) \neq \emptyset$. If $\tau \in \overline{Fix(f)}$ there exists $H \in \mathcal{G}$ such that $\tau \in H$ and $H \subset \overline{Fix(f)}$ (see [17]). Therefore $f_H = id_\Delta$ and by Lemma 6.1 $\beta_\tau(f) = \beta_1(id) = 1$, contradiction. Hence $\tau \notin \overline{Fix(f)}$. By the very definition of horospheres (see [2]) there exists $R_0 > 0$ so that for any $0 < R < R_0$, $\overline{E(\tau, R)} \cap \overline{Fix(f)} = \emptyset$. By *Theorem 2.4.16 and Corollary 2.6.48 of* [1] (see also [3] and [4]) for any $0 < R < R_0$, $z \in E(\tau, R)$ and $m \in \mathbb{N}$ it holds $f^m(z) \in E(\tau, R)$. Therefore $\{f^m(z)\}$ accumulates at some $z_0 \in \overline{E(\tau, R)}$. By the fundamental theorem on iteration (see *Theorem 2.1.29 of* [1] or [16]) this implies that τ belongs to the closure of the *limit manifold* X of f. Now X is a complex submanifold of D such that $Fix(f) \subseteq X$, $f(X) = X$ and $f|_X$ is an automorphism of X. Again this implies $\beta_\tau(f) = 1$, contradiction. Therefore f has no fixed points in D and its iterates accumulate at τ, *i.e.* τ is the Wolff point of f.

As for (2), arguing as before and retaining the same notation, if f has fixed points in D then $\tau \in \overline{X}$. Thus if $z_0 \in Fix(f)$ then f is the identity on the complex geodesic joining z_0 to τ and therefore $\tau \in \overline{Fix(f)}$. □

Let us now examine the case of sequences of converging maps.

Proposition 6.3. *Let* $\{f_k\} \subset Hol(D, D)$ *be such that* $f_k \to f \in \overline{Hol(D, D)}$*. Suppose that for any* k *there exists* $G_k \in \mathcal{G}$ *with* $G \cap D \neq \emptyset$ *such that* $(f_k)_{G_k} = id_{G_k}$*. If* $G_k \to G \in \mathcal{G}$ *then one and only one of the following cases is possible:*

(1) $G \in \partial D$ *and the map* f *is the constant* $z \mapsto G$*.*
(2) $G \in \partial D$*,* $f \in Hol(D, D)$ *has no fixed points and* G *is its Wolff point.*
(3) $G \in \partial D$*,* $f \in Hol(D, D)$ *has fixed points in* D*, dim*$Fix(f) \geq 1$ *and* $G \in \overline{Fix(f)}$*.*
(4) $G \in D$*,* $f \in Hol(D, D)$ *has fixed points in* D*, dim*$Fix(f) \geq 1$ *and* $G \cap D \subset Fix(f)$*.*

Proof. First suppose that $G \in \partial D$. Then there exists a sequence $\{z_k\}$ such that $z_k \in G_k$ for any k and $z_k \to G$. If f has no fixed points in D then the result follows from Theorem 3.10. If f has a fixed point $z_0 \in D$ then this is exactly *Théorème 4.3 of* [18]. However let us give another proof based on our method. Let $\varphi : \Delta \to D$ be a complex geodesic such that $\varphi(0) = z_0$, $\varphi(1) = 1$ and let $\tilde{p}$ be the associated left-inverse. If we prove that $\beta_G(f) \leq 1$ then by Lemma 6.1 it follows that $\tilde{p} \circ f \circ \varphi$ fixes 0 and has boundary dilatation coefficient at 1 less than or equal to 1, which is impossible by the Wolff Lemma [20] unless f fixes $\varphi(\Delta)$. Suppose that (up to subsequences)

$$\frac{z_k - G}{\|z_k - G\|} \to v, \tag{6.1}$$

for some $v \notin T^{\mathbb{C}}_G \partial D$. For any k let $\varphi_k : \Delta \to D$ be the complex geodesic such that $\varphi_k(0) = z_k$ and $\varphi_k(1) = G$. Let $H_k := \varphi_k(\overline{\Delta})$. By Theorem 5.10 and equation (6.1) $H_k \to H \in \mathcal{G}$ where $H \cap D \neq \emptyset$. Therefore we can reparameterize the φ_k's in such a way that $\varphi_k \to \eta$ for some complex geodesic η with left inverse $\tilde{q}$. Hence $\tilde{p}_k \circ f_k \circ \varphi_k$ converges to $\tilde{q} \circ f \circ \eta$. Moreover $\tilde{p}_k \circ f_k \circ \varphi_k$ has fixed point ζ_k and $\zeta_k \to 1$. By Remark 1.8 $\tilde{q} \circ f \circ \eta$ has Wolff point 1 or it is the identity. In both cases $\beta_1(\tilde{q} \circ f \circ \eta) \leq 1$ and Lemma 6.1 gives $\beta_G(f) \leq 1$. We are led to show that it is actually possible to find such a $\{z_k\}$ for which equation (6.1) holds. By *Theorem 2 of* [9] for any k large G_k is almost parallel to some direction in $T^{\mathbb{C}}_G \partial D$ at any point of $G_k \cap D$. Since G_k is transversal to ∂D, if $z_k \in G_k$ is such that $d(z_k, \partial G_k)$ is the maximum among all $z \in G$ then such $\{z_k\}$ realizes equation (6.1) up to subsequences.

Suppose now that $G \cap D \neq \emptyset$. Then $id = (f_k)_{G_k} \to f_G$ implying that $f_G = id$ and—arguing as in Remark 3.8—$f(z) = z$ for any $z \in G \cap D$, as claimed. □

Let $f \in \mathrm{Hol}(D, D)$. Let us denote by $\overline{Fix}(f)$ the set of fixed points in D, *i.e.* $Fix(f)$, together with any point $x \in \partial D$ such that f has non-tangential limit x at x and $\beta_x(f) \leq 1$.

As a corollary of Proposition 6.3 we have the following Theorem (see also *Theorem p. 1700 in* [10]):

Theorem 6.4. *Let* $\{f_k\} \subset Hol(D, D)$ *be such that* $f_k \to f$*. Then* $\limsup \overline{Fix}(f_k) \subseteq \overline{Fix}(f)$*.*

Proof. Consider the family $\Gamma := \{G_\nu\}$ such that $G_\nu \in \mathcal{G}$ and $G_\nu \subset \overline{Fix}(f_k)$ for some k. By [17] for any $z, w \in Fix(f_k)$, $z \neq w$, there exists $G_k \in \mathcal{G}$ with $G_k \cap D \neq \emptyset$ such that $G_k \subset \overline{Fix}(f_k)$ and $z, w \in G_k$. Therefore the cluster set of Γ coincides with $\limsup \overline{Fix}(f_k)$. Let G be in the cluster set of Γ. Then there exists a sequence $G_m \in \mathcal{G}$ such that $G_m \to G$ and either $G_m \cap D \subset \mathrm{Fix}(f_m)$ or G_m is the Wolff point of f_m (these are the only possibilities by Proposition 6.2). Hence $G \subset \overline{Fix}(f)$ by Proposition 6.3 or Theorem 3.10. $\square$

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