THE DYNAMICS OF HOLOMORPHIC MAPS NEAR CURVES OF FIXED POINTS.

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ABSTRACT. Let M be a two dimensional complex manifold and $f : M \to M$ a holomorphic map. Let $S \subset M$ be a curve made of fixed points of f, *i.e.* $\mathrm{Fix}(f) = S$. We a study the dynamics near S in case f acts as the identity on the normal bundle of the regular part of S. Besides results of local nature, we prove that if S is a globally and locally irreducible compact curve such that $S \cdot S < 0$ then there exists a point $p \in S$ and a holomorphic f-invariant curve with p on the boundary which is attracted by p under the action of f. These results are achieved introducing and studying a family of local holomorphic foliations related to f near S.

INTRODUCTION

Let M be a two-dimensional complex manifold, $\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $f : M \to M$ be holomorphic and $p \in M$. A *parabolic curve* for f at p is the image of an injective analytic disc $\varphi : \Delta \to M$ such that φ is continuous up to the boundary of Δ, $p = \varphi(1)$, $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$ and for any $q \in \varphi(\Delta)$ it follows that $\lim_{n\to\infty} f^n(q) = p$. Moreover φ is said to be *tangent* to a direction $[v] \in \mathbb{CP}^1$ at p if $[\varphi(\zeta)] \to [v]$ for $\zeta \to 1$ (where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^2 - \{0\}$ onto $\mathbb{CP}^1$).

Theorem 0.1 (Écalle, Hakim, Abate). *If f has an isolated fixed point $p \in M$ and $df_p = Id$ then there exists at least one parabolic curve for f at p.*

This theorem is a complete generalization of the well known one-dimensional Leau-Fatou flower theorem.

The "flower theorem in two-dimensions" has quite an odd story. The first who (partially) proved it in the 1980's was J. Écalle [8] who, using his theory of formal series and resurgence, was able to produce small pieces of f-invariant curves attached to the point p in case f is "generic". In the middle of the nineties, several people felt a need for a complete analytic proof of such a theorem. After some preliminary results of T. Ueda [17] and B. Weickert [18], a major step in this direction has been done by M. Hakim [10] who proved the "flower theorem" for "generic maps" (not only in $\mathbb{C}^2$ but also in $\mathbb{C}^n$). Her idea was to look at $f - Id$ near p. That is to say, if some conditions on the first non-zero homogeneous polynomial in the expansion of $f - Id$ are satisfied, then one can apply some Banach spaces techniques to construct parabolic

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curves. After his results in [1], Abate understood that if Hakim's conditions at p were not satisfied then one could have tried to blow-up the point p to reach a more favorable situation on the exceptional divisor. This is exactly the same strategy exploited by C. Camacho and P. Sad [6] to show that any holomorphic foliation on a two dimensional complex manifold has a separatrix at a singular point. Indeed Abate's proof of the flower theorem follows the same lines of Camacho-Sad argument. Abate defines "singularities" and "reduced singularities" for a holomorphic map and proves a reduction theorem which, roughly speaking, says that after a finite number of blow ups one gets a holomorphic map with only "reduced singularities" on the exceptional divisor. Then he defines an index for holomorphic maps on curves of fixed points and proves an index theorem which allows to localize the characteristic classes of the curve near the singularities of the map. After that he has formally the same ingredients as in the Camacho-Sad theory, and he can argue in the same way to obtain a point where Hakim's theory applies.

Besides giving a complete analytic proof of the flower theorem, Abate's work made evident that, firstly, the dynamics near isolated fixed points can be well understood only once one understands the dynamics near curves of fixed points, secondly, some results on the older theory of holomorphic foliations can be properly translated to give new results in discrete dynamics. An evidence of this second claim is a recent work of the author and F. Tovena [4], in which it is proved a discrete dynamics analogous of a generalization of the Camacho-Sad index Theorem due to T. Suwa [16].

In a sense this similarity with holomorphic foliations is to be expected for one can always find a *formal* vector field along the curve of fixed points of a holomorphic map (provided this map is tangent to the identity on such a curve) in such a way that the map is the time one flow of this formal vector field, and then use the theory of formal foliations.

One aim of this paper is to provide a new approach which avoids the use of formal foliations but provides an actual link between holomorphic foliations and holomorphic maps. Given a holomorphic self-map f of M which fixes $p \in M$ and such that $df_p = Id$, we associate to f a family of holomorphic 1-forms $\Omega_{f,p}$, see (2.1), which is composed by forms whose flows are "first-order approximations" of f at p. That is to say, the normalizations of such forms all have the same linear part at p (up to scalar multiples). This allows to define a *(reduced) singularity* of f as a (reduced) singularity of the family $\Omega_{f,p}$. With this approach, the *Reduction Theorem 2.7* follows directly from the Seidenberg Reduction Theorem for holomorphic foliations. Then we turn our attention to the case f has a curve of fixed points, say $S \subset M$. In this case, if $p \in S$, we select from the family of 1-forms $\Omega_{f,p}$ those forms which might generate foliations which have S locally as a leaf (see (3.1)). It will turn out that the existence of such forms is dynamically very relevant. Indeed f is said *degenerate* on S if such forms do not exist, *non-degenerate* if they do. In case f is non-degenerate on S, starting from such forms, we define a flat connection for the normal bundle of (the regular part of) S, outside the singularities of f on S. In particular, in case f has no singularities on S and S is smooth then we have a vanishing theorem (see Theorem 3.7). If S is compact (no matters whether smooth or not) and f has singularities on S then the characteristic classes of S localize around the singularities of f on S producing "residual indices" (see Theorems 4.2, 5.2) and we recover the index theorems of [2] and [4]. All these results can also be regarded as topological obstructions for a curve to be the fixed points locus of a (non-degenerate) holomorphic self-map of M.

The second target of these notes is to study the dynamics near a curve $S \subset M$ of fixed points of f, in case f acts as the identity on the normal bundle N_S of (the regular part of) S. Let p be a smooth point of S. The differential df_p has two eigenvalues (counting multiplicity) at p. One must be 1. The other eigenvalue gives the action of f on $N_{S,p}$. If this eigenvalue has modulo $\neq 1$ then the *center stable/unstable manifold theorem* [19] provides a clear picture of the dynamics of f near S at p (see also [14]). Here we deal with the case where 1 is the only eigenvalue of df on S. This situation is the one we find blowing up a fixed point $q \in M$ where $df_q = Id$, and therefore seems to deserve a special care. After relating the previous work about indices and singularities to blow-ups, we give an algorithm for producing parabolic curves starting from a point $p \in S$ where S is (possibly singular) reduced at p, the fixed points set $\mathrm{Fix}(f) = S$ near p and the index is a number $\notin \mathbb{Q}^+ \cup \{0\}$. Our algorithm is a generalization (and "translation to discrete dynamics") of J. Cano's work [7], which seems to allow new results also in the holomorphic foliations case. This is the key to several results. For instance, we show that f is degenerate on a curve S (and identically acting on the normal bundle N_S) if and only if f has a parabolic curve at all but a discrete set of points of S (see Proposition 6.11). Finally we show that if S is compact, globally and locally irreducible and $S \cdot S < 0$ then there exists at least a point of S where f has at least one parabolic curve (see Theorem 6.13). From this we give a new proof of Theorem 0.1.

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1. PRELIMINARY RESULTS

Let M be a complex two dimensional manifold, S a (possibly singular) irreducible curve in M, $f : M \to M$ holomorphic such that $f|_S = Id_S$, $f \neq Id_M$. For $p \in S$ we denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions at p, and by $\mathcal{I}(S)_p \subset \mathcal{O}_p$ the ideal of germs vanishing on S.

If $U \subset M$ is a coordinate set and $\phi : U \to \mathbb{C}^2$ a local chart and $l \in \mathcal{O}_p$ a defining function for S at p then

$$\phi \circ f \circ \phi^{-1} = Id + (l \circ \phi^{-1})^T G, \tag{1.1}$$

for some germ $G = (G_1, G_2)$ of holomorphic self-map of $\mathbb{C}^2$ at $(0, 0)$, $G \not\equiv 0$ on $\phi(S)$ and $T \geq 1$. It is easy to see that T is independent of ϕ, l. We call $T_p(f, S) := T$ the *order of f on S at p*.

Remark 1.1. From now on we omit to write explicitly the local chart ϕ when not indispensable, *e.g.*, we write simply $f = Id + l^T G$ instead of (1.1). We denote by h' the gradient of $h \in \mathcal{O}_p$ in the given local chart and by $< H, G >$ the scalar product of two germs H, G of holomorphic self-maps of $\mathbb{C}^2$.

Note that if $H \in \mathcal{O}_p$ and f is as in (1.1), then

$$\frac{H \circ f - H}{l^T} \equiv < H', G > \mod \mathcal{I}(S)_p$$

Definition 1.2. We say that f is *non-degenerate* on S at p if for a defining function l of S at p

$$\frac{l \circ f - l}{l^T} \equiv 0 \mod \mathcal{I}(S)_p,$$

i.e., if $< l', G > \equiv 0$ on S near p.

Note that if f is non-degenerate on S at p for some defining function l then it is so for any defining function. For the proof of this and for a detailed discussion of non-degeneracy we refer the reader to [4]. Here we content ourselves to state the following result from [4]:

Proposition 1.3. *If the curve S is globally irreducible then f is degenerate at $p \in S$ if and only if f is degenerate at $q \in S$ for every $q \in S$.*

Now suppose that S is smooth. Then one may choose local coordinates $\{(x, y), U\}$ around p such that $p = (0, 0)$, $S \cap U = \{y = 0\}$ and, if we write $f = (f_1, f_2)$,

$$\begin{cases} f_1(x, y) & = x + y^\mu g(x, y) \\ f_2(x, y) & = b(x)y + y^\nu h(x, y) \end{cases} \tag{1.2}$$

for some holomorphic functions g, h such that $g(x, 0) \not\equiv 0$, $h(x, 0) \not\equiv 0$ and natural numbers $\mu \geq 1$, $\nu \geq 2$. Note that if $b(x) = 1$ then $T_p(f, S) = \min\{\mu, \nu\}$ and if $b(x) \neq 1$ then $T_p(f, S) = 1$. A straightforward computation shows

$$df_{(x,0)} = \begin{pmatrix} 1 & * \\ 0 & b(x) \end{pmatrix}$$

where "$*$" is certainly 0 on $U \cap S$ if $\mu \geq 2$.

Let $N_S := TM|_S/TS$ be the *normal bundle* of S in M. In the local coordinates $(U, (x, y))$ the projection $[\frac{\partial}{\partial y}]$ of $\frac{\partial}{\partial y}$ under the natural map $TM|_S \to N_S$ is a base frame for N_S over $U \cap S$. Therefore the action of f on N_S over $U \cap S$ is given by

$$[\frac{\partial}{\partial y}] \mapsto [df(\frac{\partial}{\partial y})] = [b(x)\frac{\partial}{\partial y}].$$

Hence $b(x) \equiv 1$ if and only if the action of f over N_S is the identity.

Since N_S has rank one, if S is compact then the action of f on N_S is constant and hence $b(x) \equiv b(f)$ is a constant. Therefore $b(x) \equiv 1$ if and only if f acts on N_S as the identity. That is to say, if S is smooth and compact the spectrum of df_p is $\{1\}$ at some—and hence any—point $p \in S$ if and only if f acts as the identity on N_S.

We remark that if $b(x) \neq 1$ at $p \in S$ then $T_p(f, S) = 1$ and f is degenerate on S at p since $(y \circ f - y)/y = b(x) - 1 \not\equiv 0$ on $y = 0$.

Remark 1.4. Suppose f is given by (1.2) and acts on N_S as the identity (then $b(x) = 1$). Up to a linear change of coordinates we can always assume that either $T_p(f, S) = \mu = \nu$ if f is degenerate on S or $T_p(f, S) = \nu = \mu - 1$ if f is non-degenerate on S. In particular if f is degenerate on S then $\mu \geq 2$ and $df = Id$ along S.

2. SINGULARITIES OF A MAP ALONG A CURVE AND THE REDUCTION THEOREM

Let M be a two dimensional complex manifold, $p \in M$ and let $f : M \to M$ be holomorphic and such that $f(p) = p$, $df_p = Id$. For $w, z \in \mathcal{O}_p$ let

$$\omega^{w,z,p} := (w \circ f - w)dz - (z \circ f - z)dw.$$

Let us consider the family of germs of holomorphic foliations given by

$$\Omega_{f,p} := \{\omega^{w,z,p} = 0 : w, z \in \mathcal{O}_p, dw_p \wedge dz_p \neq 0\}. \tag{2.1}$$

Let $\hat{\omega}^{w,z,p}$ be the form $\omega^{w,z,p}$ with coefficients divided by their greatest common divisor in $\mathcal{O}_p$. Note that $\hat{\omega}^{w,z,p} = \omega^{w,z,p}$ if and only if p is a isolated fixed point of f.

Definition 2.1. We say that p is a *singularity of* f if $\hat{\omega}^{w,z,p}[p] = 0$ for all w, z.

Choose local coordinates (x, y) near p such that $p = (0, 0)$ and $f = Id + G$, with G a germ of holomorphic self-map of $\mathbb{C}^2$ at $(0, 0)$. Let $G = (G_1, G_2) = h(G_1^\circ, G_2^\circ)$ with h the greatest common divisor of G_1, G_2 and G_1° and G_2° coprime in $\mathcal{O}_p$. In [2] the *pure order* of f at p is defined as the minimum of the order of vanishing of G_j°, $j = 1, 2$, at p. In [2] a point is a singularity for f if the pure order is at least one. Note that a point is a singularity of f according to Definition 2.1 if and only if it is a singularity according to [2].

Singularities are the only relevant points for dynamics, indeed we have

Proposition 2.2 (Abate, [2]). *Let M be a two dimensional manifold, $S \subset M$ a smooth curve and $p \in S$. Let $f : M \to M$ be holomorphic and such that Fix$(f) = S$ near p. Assume f is non-degenerate on S. If p is not a singularity for f on S then p cannot be an attracting point for f; in particular there are no parabolic curves for f at p.*

Let $\varrho := [a_{11}x + a_{12}y + q_1(x, y)]dx - [a_{21}x + a_{22}y + q_2(x, y)]dy$ be a holomorphic one form, with $a_{ij} \in \mathbb{C}$ and $q_j(x, y)$ of order at least two in $(0, 0)$. By definition the *linear part of* ϱ is given by the linear transformation $(x, y) \mapsto (a_{21}x + a_{22}y, a_{11}x + a_{12}y)$ and its eigenvalues are called the *eigenvalues of* ϱ at $(0, 0)$.

Suppose $p \in M$ is a singularity for f. Let us denote by $\hat{\omega}_0^{w,z,p}$ the linear part of $\hat{\omega}^{w,z,p}$ at p and by $\lambda_1^{w,z,p}$, $\lambda_2^{w,z,p}$ its eigenvalues at p.

Lemma 2.3. *If $\tilde{w}, \tilde{z} \in \mathcal{O}_p$ and $d\tilde{w}_p \wedge d\tilde{z}_p \neq 0$ then there exists $u \in \mathcal{O}_p^*$ such that*

$$\hat{\omega}_0^{\tilde{w},\tilde{z},p} = u\hat{\omega}_0^{w,z,p}.$$

In particular if both the eigenvalues are different from zero, then the ratio $\lambda_1^{w,z,p}/\lambda_2^{w,z,p}$ is independent of w, z and if $\lambda_j^{w_0,z_0,p} = 0$ ($j = 1$ or 2) for some w_0, z_0 then it is so for all w, z.

Proof. We are going to prove the statement in local coordinates $\{U,(x,y)\}$ such that $p=(0,0)$. Write $f = Id + hG$ with $h \in \mathcal{O}_{(0,0)}$ of order ≥ 0 at $(0,0)$ and $G = (G_1, G_2)$ a germ of holomorphic self-map of $\mathbb{C}^2$ at $(0,0)$ with G_1, G_2 relatively prime in $\mathcal{O}_{(0,0)}$. Note that $h=0$ is the fixed points set of f at $(0,0)$, thus its order at $(0,0)$ is 0 if and only if f has an isolated fixed point at p. Let $G_0(x,y)$ be the linear part of G at $(0,0)$. Then, indicating with L_2 the terms of order ≥ 2 at $(0,0)$ and taking into account that $G(0,0)=(0,0)$, we have

$$
\begin{aligned}
&\frac{(w\circ f - w)}{h}dz - \frac{(z\circ f - z)}{h}dw \\
&\quad = (z_x < w', G > - w_x < z', G >)dx + (z_y < w', G > - w_y < z', G >)dy + L_2 \\
&\quad = [z_x(0,0) < w'(0,0), G_0(x,y) > - w_x(0,0) < z'(0,0), G_0(x,y) >]dx \\
&\quad + [z_y(0,0) < w'(0,0), G_0(x,y) > - w_y(0,0) < z'(0,0), G_0(x,y) >]dy + L_2 \\
&\quad =< [0, (z_x w_y - w_x z_y)(0,0)], G_0(x,y) > dx \\
&\quad - < [(z_x w_y - w_x z_y)(0,0), 0], G_0(x,y) > dy + L_2 \\
&\quad = (z_x w_y - w_x z_y)(0,0)[< (0,1), G_0(x,y) > dx - < (1,0), G_0(x,y) > dy] + L_2.
\end{aligned}
$$

Since $0 \neq dz_p \wedge dw_p = (z_x w_y - w_x z_y)(0,0) dx \wedge dy$, the statement follows. □

We are now in the position to give the following:

Definition 2.4. Let $p \in M$ be a singularity for f. We say that p is a *reduced singularity* for f if p is a reduced singularity for $\omega^{w,z,p}$ for some—and hence any—w, z. That is to say

($\star_1$) either the eigenvalues $\lambda_1^{w,z,p}, \lambda_1^{w,z,p} \neq 0$ and $\lambda_1^{w,z,p}/\lambda_2^{w,z,p} \notin \mathbb{Q}^+$ or
($\star_2$) $\lambda_1^{w,z,p} \neq 0$ and $\lambda_2^{w,z,p} = 0$.

To summarize we defined the (reduced) singularities of f to be the (reduced) singularities of *all* the family $\Omega_{f,p}$.

Reduced singularities of type ($\star_1$) are very important in both the continuous and discrete dynamics. Indeed in the continuous case a reduced ($\star_1$) singularity implies existence of separatrices; in the discrete case we have:

Theorem 2.5 (Abate, Hakim)**.** *Let M be a two dimensional complex manifold, $f: M \to M$ holomorphic, $S \subset M$ a curve such that $f|_S = Id|_S$. Let $p \in S$ be a smooth point of S and suppose f is non-degenerate on S at p. If Fix$(f) = S$ near p and p is a reduced ($\star_1$) singularity of f then there exists at least one parabolic curve for f at p not contained in S.*

For a proof of this result see [2], [10], where a more precise statement about the actual number of parabolic curves is given.

Let $p \in M$. A *blow-up* or *quadratic transformation* of p is a two dimensional complex manifold $\tilde{M}$ together with a proper holomorphic map $\pi : \tilde{M} \to M$ such that $D := \pi^{-1}(p)$, called the *exceptional divisor*, is a projective complex line and $\pi : \tilde{M} - D \to M - \{p\}$ is biholomorphic (see, *e.g.*, [11]). Suppose that $p \in M$ is a singularity for f. By definition p is a singularity for $\omega^{w,z,p}$ for all $\omega^{w,z,p} \in \Omega_{f,p}$. The 1-forms $\pi^*(\omega^{w,z,p})$ are identically zero on

D. However one may normalize them dividing the coefficients of the forms by their greatest common divisor in order to obtain 1-forms $\tilde{\omega}^{w,z,p}$ with only isolated singularities on D. Once fixed $w, z \in \mathcal{O}_p$, the well known theorem of Seidenberg (see, *e.g.* [5]) assures that after a finite number of blow-ups one obtains a complex two dimensional manifold, still denoted by $\tilde{M}$, together with a proper holomorphic map, still denoted by $\pi : \tilde{M} \to M$ such that

(1) $D := \pi^{-1}(p) = \cup_{\alpha=1}^N D_\alpha$, with D_α complex projective lines intersecting transversally each other and no three of them intersecting at one point,
(2) $\pi : \tilde{M} - D \to M - \{p\}$ is biholomorphic,
(3) The 1-form $\tilde{\omega}^{w,z,p}$ has only reduced singularities on D.

If $\pi : \tilde{M} \to M$ is a quadratic transformation of $p \in M$, the map f induces a holomorphic map $\tilde{f} : \tilde{M} \to \tilde{M}$ such that $\pi \circ f = \tilde{f} \circ \pi$ and $\tilde{f}$ acts on the exceptional divisor D as df_p, if df_p is invertible (see [1]). In particular if p is a singularity for f then $df_p = Id$ and $\tilde{f}|_D = Id|_D$. The map $\tilde{f}$ has isolated singularities on D. In [2] Abate proves directly an analogous of Seidenberg's reduction theorem for the map f (see *Theorem 2.3 in* [2]). Here we give another version of such a theorem, with a simpler proof based on Seidenberg's theorem. Before that, we need another definition:

Definition 2.6. Let $p \in M$ be such that $f(p) = p$ and $df_p = Id$. Let $\pi : \tilde{M} \to M$ be the blow-up at p such that $D := \pi^{-1}(p)$ is a complex projective line. We say that p is *dicritical* for f if $\tilde{f} : \tilde{M} \to \tilde{M}$ is degenerate on D.

The point p is dicritical for f if and only if it is dicritical for any 1-form $\omega^{w,z,p}$. Indeed $\tilde{f}$ is degenerate on D if and only if D is not invariant for $\pi^*(\omega^{w,z,p})$ for any w, z, which is the definition of *dicritical point* for foliations (see *e.g.* [5]).

Theorem 2.7 (Reduction Theorem)**.** *Let M be a two dimensional complex manifold. Let $f : M \to M$ be holomorphic. Let $p \in M$ be a singularity of f. Then there exists a two dimensional complex manifold $\tilde{M}$, a proper holomorphic map $\pi : \tilde{M} \to M$ and a holomorphic map $\tilde{f} : \tilde{M} \to \tilde{M}$ such that*

(1) *$D := \pi^{-1}(p) = \cup_{\alpha=1}^N D_\alpha$, where D_α are complex projective lines which intersect transversally each other and no three of them intersect at one point.*
(2) *$\pi : \tilde{M} - D \to M - \{p\}$ is biholomorphic.*
(3) *$\pi \circ \tilde{f} = f \circ \pi$.*
(4) *$\tilde{f}|_D = Id|_D$.*
(5) *$\tilde{f}$ has only isolated reduced or dicritical singularities on D.*

Proof. Let p be a non-dicritical singularity for f. Let $\{U, (x, y)\}$ be local coordinates around p so that $p = (0, 0)$ and write $f = (f_1, f_2)$ as

$$\begin{cases} f_1(x,y) = & x + l(x,y)g(x,y) \\ f_2(x,y) = & y + l(x,y)h(x,y) \end{cases} \tag{2.2}$$

where g, h are coprime and l does not divide g and h in $\mathcal{O}_p$. Note that we may assume $l \equiv 1$ if and only if $\mathrm{Fix}(f) = \{p\}$ near p. Moreover, since p is a singularity for f then $g(0,0) = 0$ and $h(0,0) = 0$. Let $\mu(g) \geq 1$ (*respect.* $\mu(h) \geq 1$) be the order of vanishing of g (*respect.* of h) at $(0,0)$. Write the homogeneous polynomials expansion of g and h as follows

$$g(x,y) = g_{\mu(g)}(x,y) + g_{\mu(g)+1}(x,y) + g^{\bullet}(x,y),$$
$$h(x,y) = h_{\mu(h)}(x,y) + h_{\mu(h)+1}(x,y) + h^{\bullet}(x,y).$$

Note that $g_{\mu(g)} \not\equiv 0$, $h_{\mu(h)} \not\equiv 0$. Let $\mu(l) \geq 0$ be the order of vanishing of l at $(0,0)$. Clearly $\mu(l) + \mu(g) \geq 2$, $\mu(l) + \mu(h) \geq 2$. Let $\pi : \tilde{M} \to M$ be the blow-up at p. Let (u,v) be local coordinates on $\tilde{M}$ such that $\pi(u,v) = (u, uv)$ and $D := \pi^{-1}(0,0) = \{u = 0\}$. Write $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ and let $l(u,uv) = u^{\mu(l)}\tilde{l}$, with $\tilde{l}(0,v) \not\equiv 0$. Then

$$\begin{aligned}
\tilde{f}_1(u,v) =& u + l(u,uv)g(u,uv) = u + \tilde{l}u^{\mu(l)+\mu(g)}[g_{\mu(g)}(1,v) + O(|u|)],\\
\tilde{f}_2(u,v) =& \frac{uv + l(u,uv)h(u,uv)}{u + l(u,uv)g(u,uv)} = v + \tilde{l}u^{\mu(l)-1}[u^{\mu(h)}h_{\mu(h)}(1,v)\\
&+ u^{\mu(h)+1}h_{\mu(h)+1}(1,v) - u^{\mu(g)}vg_{\mu(g)}(1,v) - u^{\mu(g)+1}vg_{\mu(g)+1}(1,v)\\
&- u^{\mu(l)+\mu(g)+\mu(h)-1}\tilde{l}h_{\mu(h)}(1,v)g_{\mu(g)}(1,v) + O(|u|^{\min(\mu(g),\mu(h))+2})]
\end{aligned}$$

where, as usual, $O(|u|^m)$ stands for terms of order at least m in u. From this expression follows that *p is dicritical for f if and only if $\mu(g) = \mu(h)$ and $yg_{\mu(g)}(x,y) = xh_{\mu(h)}(x,y)$.*

Let $\tilde{p} := (0,v)$. Choose $w = u$, $z = v$.

First suppose $\mu(h) > \mu(g)$. Then $\mu(h) + \mu(l) \geq 3$ and

$$\begin{aligned}
\hat{\omega}^{u,v,\tilde{p}} =& [ug_{\mu(g)}(1,v) + O(|u|^2)]dv\\
&- [-vg_{\mu(g)}(1,v) - vug_{\mu(g)+1}(1,v) + u^{\mu(h)-\mu(g)}h_{\mu(h)}(1,v) + O(|u|^2)]du
\end{aligned}$$

On the other hand, a straightforward calculation shows that

$$\pi^*(\omega^{x,y,p}) = l(u,uv)\{ug(u,uv)dv - [h(u,uv) - vg(u,uv)]du\},$$

that is

$$\begin{aligned}
\widehat{\pi^*(\omega^{x,y,p})} =& [ug_{\mu(g)}(1,v) + O(|u|^2)]dv\\
&- [-vg_{\mu(g)}(1,v) - vug_{\mu(g)+1}(1,v) + u^{\mu(h)-\mu(g)}h_{\mu(h)}(1,v) + O(|u|^2)]du. \qquad (2.3)
\end{aligned}$$

Therefore $\tilde{p}$ is a singularity for $\tilde{f}$ if and only if it is a singularity for $\pi^*(\omega^{x,y,p})$ and the part of lowest degree of $\omega^{u,v,\tilde{p}}$ is equal to the part of lowest degree of $\pi^*(\omega^{x,y,p})$ at $\tilde{p}$.

If $\mu(h) < \mu(g)$ then a similar reasoning leads to the same conclusion.

Suppose finally $\mu(g) = \mu(h)$. This implies that $\mu(l) + \mu(g) \geq 2$. As observed before, since p is non-dicritical for f then $h_{\mu(h)}(1, v) - v g_{\mu(g)}(1, v) \neq 0$ and therefore

$$\begin{aligned}\hat{\omega}^{u,v,\tilde{p}} &= [u g_{\mu(g)}(1, v) + O(|u|^2)]dv \\ &\quad - [h_{\mu(h)}(1, v) - v g_{\mu(g)}(1, v) + u(h_{\mu(h)+1}(1, v) - v g_{\mu(g)+1}(1, v)) \\ &\qquad + u^{\mu(l)+\mu(g)-1}\tilde{l} h_{\mu(h)}(1, v) g_{\mu(g)}(1, v) + O(|u|^2)]du\end{aligned}$$

If $\mu(l) + \mu(g) > 2$ we can argue as before to find the same conclusion. So we are left to analyze the case $\mu(l) + \mu(g) = 2$. The singularity for $\tilde{f}$ on the exceptional divisor D are given by $(0, v_0)$ where v_0 is such that

$$h_{\mu(h)}(1, v_0) - v_0 g_{\mu(g)}(1, v_0) = 0.$$

By (2.3), these are exactly the singularities of $\pi^*(\omega^{x,y,p})$. But, unless $h_{\mu(h)}(1, v_0) = 0$, the linear part of $\widehat{\pi^*(\omega^{x,y,p})}$ at $(0, v_0)$ is different to the linear part of $\hat{\omega}^{u,v,\tilde{p}}$ at $(0, v_0)$. However a straightforward calculation shows that the eigenvalues of $\hat{\omega}^{u,v,\tilde{p}}$ at $(0, v_0)$ are given by $g_{\mu(g)}(1, v_0)$ and $\frac{\partial}{\partial v}(-h_{\mu(h)}(1, v) + v g_{\mu(g)}(1, v))_{|v=v_0}$, which are exactly the eigenvalues of $\widehat{\pi^*(\omega^{x,y,p})}$ at $(0, v_0)$.

Summing up, we have shown that the singularities of $\omega^{u,v,\tilde{p}}$ on the chart (u, v) of D are exactly the singularities of $\pi^*(\omega^{x,y,p})$ on such a chart and also that the part of lowest degree or, when this degree is one, the eigenvalues of $\omega^{u,v,\tilde{p}}$ are equal to the part of lowest degree or to the eigenvalues of $\pi^*(\omega^{x,y,p})$ at such singularities. The same holds for the other chart of D and hence the result follows from the Seidenberg reduction theorem applied to the 1-form $\omega^{x,y,p}$. □

Remark 2.8. Let f be given by (2.2). In the proof of Theorem 2.7 we saw that the point p is dicritical for f if and only if $\mu(h) = \mu(g)$ and $y g_{\mu(g)}(x, y) \equiv x h_{\mu(h)}(x, y)$. Moreover it follows easily that if p is dicritical for f then $\tilde{f}$ acts as the identity on the normal bundle of the exceptional divisor D.

3. CONNECTIONS ON SMOOTH CURVES AND THE VANISHING THEOREM

Let M be a two dimensional complex manifold, $S \subset M$ a curve. Let $f : M \to M$ be holomorphic and such that $f_{|S} = Id_{|S}$. Let Σ_S be the set of singular points of S. Let $U \subset M$ be a coordinate set with coordinates functions (x, y). Let l be a defining function of S on U, *i.e.*, $S \cap U = \{(x, y) : l(x, y) = 0\}$ with $dl_p \neq 0$ for any $p \in U \cap (S - \Sigma_S)$. Up to shrink U, we can assume that $dl_p \neq 0$ for any $p \in U - \Sigma_S$. Let $\tau \in \mathcal{O}(U)$ be such that $d\tau_q \neq 0$ for any $q \in U$ and $d\tau_p \wedge dl_p \neq 0$ for any $p \in (U - \Sigma_S) \cap S$. We call such a τ a *transverse* to S.

In the rest of the paper, when not differently stated, we use to name with the letter l any defining function of S and with the letter τ any transverse to S.

Let $T := T_p(f, S)$ be the order of f on S at p for some $p \in U$. Using the coherence of the ideals sheaf of S it is easy to see that $T_q(f, S) = T$ for any $q \in U$ (see the proof of *Lemma 2 in*

[4]). Therefore the following is a well defined holomorphic one form on U:

$$\omega^{l,\tau} := \frac{\tau \circ f - \tau}{l^T} dl - \frac{l \circ f - l}{l^T} d\tau. \tag{3.1}$$

The map f is non-degenerate on $S \cap U$ if and only if S is a leaf for the family of holomorphic foliations on U given by $\{\omega^{l,\tau} = 0\}$ when varying l among the defining functions of S and τ among the transverses to S. Indeed f is non-degenerate on S if and only if there exists $\tilde{h} \in \mathcal{O}(U)$ such that

$$l \circ f - l = l^{T+1} \tilde{h}.$$

Note that, even if f is non-degenerate on S, the foliations $\{\omega^{l,\tau} = 0\}$ on U really depend on l and τ; in particular, apart from $S \cap U$, they generally do not share other leaves.

For the rest of this section suppose $S \subset M$ is smooth. If $p \in S$ is a singularity for f then, in particular, $\omega^{l,\tau}[p] = 0$. However the converse is easily seen to be true: if $p \in S$ is such that $\omega^{l,\tau}[p] = 0$ for any τ, l then p is a singularity for f.

Let f be non-degenerate on S at p and suppose $p \in S$ is *not* a singularity for f. Let (x, y) be local coordinates such that $p = (0, 0)$ and let $f = (f_1, f_2)$ be given by

$$\begin{cases} f_1(x,y) &= x + l^T g(x,y), \\ f_2(x,y) &= y + l^T h(x,y). \end{cases} \tag{3.2}$$

The map f is, by definition, non-degenerate on S at p if and only if $l_x g + l_y h \equiv 0$ on S. Let τ be a transverse to S at p. If it were

$$\frac{\tau \circ f - \tau}{l^T}[p] = (g\tau_x + h\tau_y)[p] = 0$$

then it would be $(\tau_x l_y - \tau_y l_x)[p] = 0$, against $d\tau_p \wedge dl_p \neq 0$. Therefore *if f is non-degenerate on S at p then $p \in S$ is a singularity for f if and only if there exists one—and hence any—transverse τ to S such that*

$$\frac{\tau \circ f - \tau}{l^T}[p] = 0.$$

Let L be the line bundle associated to the divisor S. That is to say, if U_j, U_k are coordinate sets such that $U_j \cap U_k \neq \emptyset$ and S is given by $\{l_j = 0\}$ on U_j and by $\{l_k = 0\}$ on U_k, then $l_{jk} := \frac{l_j}{l_k} \in \mathcal{O}^*(U_j \cap U_k)$ are cocycle functions defining L. Then, since $l_{jk} dl_k = dl_j$ on $U_j \cap U_k \cap S$, it follows that $\{dl_j\}$ is a vector bundle homomorphism between $TM_{|S}$ and $L_{|S}$ whose kernel is TS. Therefore $L_{|S} \simeq N_S$, the normal bundle of S.

Fix a chart U with local coordinates (x, y), suppose $S = \{(x,y) \in U : l(x,y) = 0\}$, and let $\tau \in \mathcal{O}(U)$ be a transverse to $S \cap U$. We have the following exact sequence:

$$0 \longrightarrow TS \longrightarrow TM|_S \xrightarrow{dl} N_S \longrightarrow 0. \tag{3.3}$$

Remark 3.1. The morphism dl allows to define a natural holomorphic frame for N_S on $U \cap S$; that is to say, if v is a holomorphic section of $TM|_S$ such that $dl(v) \equiv 1$ then $dl(v)$ can be thought of as a holomorphic frame for N_S on U, which in the sequel we will always denote by E.

Suppose that f has no singularities on $S \cap U$ and $p \in S \cap U$. Let us define the following operator on $TS_p \times N_S|_U$ with value in N_S:

$$\begin{aligned} &\theta^{l,\tau} : T_pS \times N_S|_U \to N_{S,p} \\ &\theta^{l,\tau} : (X, s) \mapsto \theta^{l,\tau}_X(s) := dl_p([\tilde{X}, \tilde{s}]|_S), \end{aligned} \tag{3.4}$$

where $\tilde{s}$ is a section of TM near $S \cap U$ such that $dl(\tilde{s}|_S) = s$ and $\tilde{X}$ is a section of TM on U such that $\tilde{X}(p) = X$ and $\omega^{l,\tau}(\tilde{X}) = 0$ on U.

Lemma 3.2. *For any $X \in T_pS$ and $s = v \cdot E$ section of N_S over $S \cap U$ it follows*

$$\theta^{l,\tau}_X(s) = \left\{ X \cdot v - v \frac{l \circ f - l}{l(\tau \circ f - \tau)}[p] d\tau_p(X) \right\} E. \tag{3.5}$$

In particular $\theta^{l,\tau}_X(s)$ as defined in (3.4) *depends only on X and s and not on $\tilde{s}$, $\tilde{X}$ chosen to define it.*

Proof. Let $\tilde{s} = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ be such that $dl(\tilde{s}|_S) = v$. On U a basis for TS is given by $l_y\frac{\partial}{\partial x} - l_x\frac{\partial}{\partial y}$, therefore $X = \lambda(l_y\frac{\partial}{\partial x} - l_x\frac{\partial}{\partial y})[p]$ for some $\lambda \in \mathbb{C}$. For $T = T_p(f, S)$ the order of f on S at p, let us set

$$\begin{aligned} \tilde{g} =& \frac{\tau \circ f - \tau}{l^T}, \\ \tilde{h} =& \frac{l \circ f - l}{l^{T+1}}. \end{aligned}$$

Since $\omega^{l,\tau} = (-\tilde{h} l \tau_x + \tilde{g} l_x)dx + (-\tilde{h} l \tau_y + \tilde{g} l_y)dy$ it follows that

$$\tilde{X} = wk \left\{ (-\tilde{h} l \tau_y + \tilde{g} l_y)\tfrac{\partial}{\partial x} + (-\tilde{g} l_x + \tilde{h} l \tau_x)\tfrac{\partial}{\partial y} \right\}$$

where $k \in \mathcal{O}^*(U)$ and $w(p) = \frac{\lambda}{k(p)\tilde{g}(p)}$ (note that $\tilde{g} \neq 0$ on S by the previous discussion). Thus

$$\begin{aligned} [\tilde{X}, \tilde{s}] = \{ &wk(-\tilde{h} l \tau_y + \tilde{g} l_y)A_x + wk(-\tilde{g} l_x + \tilde{h} l \tau_x)A_y \\ &- A\tfrac{\partial}{\partial x}(wk(-\tilde{h} l \tau_y + \tilde{g} l_y)) - B\tfrac{\partial}{\partial y}(wk(-\tilde{h} l \tau_y + \tilde{g} l_y))\}\tfrac{\partial}{\partial x} \\ &+ \{wk(-\tilde{h} l \tau_y + \tilde{g} l_y)B_x + wk(-\tilde{g} l_x + \tilde{h} l \tau_x)B_y \\ &- A\tfrac{\partial}{\partial x}(wk(-\tilde{g} l_x + \tilde{h} l \tau_x)) - B\tfrac{\partial}{\partial y}(wk(-\tilde{g} l_x + \tilde{h} l \tau_x))\}\tfrac{\partial}{\partial y} \end{aligned}$$

Therefore, after a straightforward calculation we find

$$\begin{aligned}
dl_p([\tilde{X},\tilde{s}]|_{l=0}) &= wkv\tilde{h}(l_x\tau_y - \tau_x l_y)[p] + wk\tilde{g}(-Bl_x l_{yy} + Bl_y l_{xy} \\
&\quad - Al_x l_{xy} + Al_y l_{xx} + l_x l_y A_x - l_x^2 A_y - l_y l_x B_y + l_y^2 B_x)[p] \\
&= v\lambda\frac{\tilde{h}}{\tilde{g}}(l_x\tau_y - \tau_x l_y)[p] - \lambda(l_y\tfrac{\partial}{\partial x} - l_x\tfrac{\partial}{\partial y})[p](dl_p(\tilde{s})) \\
&= -v\frac{\tilde{h}}{\tilde{g}}[p]d\tau_p(X) + X\cdot v,
\end{aligned}$$

as wanted. □

Now we want to see how $\theta^{l,\tau}$ varies when varying τ and l.

Lemma 3.3. *Let $\tilde{\tau} \in \mathcal{O}(U)$ be such that $d\tilde{\tau}\wedge dl \neq 0$ on U. Let $\tilde{l} = ul$ for $u \in \mathcal{O}^*(U)$. Then for any $X \in T_pS$*

$$\frac{l\circ f - l}{l(\tau\circ f - \tau)}[p]d\tau_p(X) - \frac{\tilde{l}\circ f - \tilde{l}}{\tilde{l}(\tilde{\tau}\circ f - \tilde{\tau})}[p]d\tilde{\tau}_p(X) = -\{d(\log u)\cdot X\}s. \tag{3.6}$$

Proof. Write $f = (f_1, f_2)$ as in (3.2). Let $T = T_p(f,S)$ be the order of f on S at p. Let $\tilde{h} = \frac{l\circ f - l}{l^{T+1}}$. From (3.5) it follows that, if $X = \lambda(l_y\frac{\partial}{\partial x} - l_x\frac{\partial}{\partial y}) \in TS|_U$,

$$\begin{aligned}
&\left\{\frac{l\circ f - l}{l(\tau\circ f - \tau)}|_{l=0}d\tau - \frac{l\circ f - l}{l(\tilde{\tau}\circ f - \tilde{\tau})}|_{l=0}d\tilde{\tau}\right\}\cdot X \\
&= \tilde{h}|_{l=0}\frac{\tilde{\tau}_x\tau_y - \tau_x\tilde{\tau}_y}{(\tau_x g + \tau_y h)(\tilde{\tau}_x g + \tilde{\tau}_y h)}(-hdx + gdy)\cdot X \\
&= -\lambda\tilde{h}|_{l=0}\frac{\tilde{\tau}_x\tau_y - \tau_x\tilde{\tau}_y}{(\tau_x g + \tau_y h)(\tilde{\tau}_x g + \tilde{\tau}_y h)}(l_y h + l_x g) = 0,
\end{aligned}$$

since $l_y h + l_x g = 0$ on S by definition of non-degeneracy. Now we may assume $\tau = x$. Then

$$\begin{aligned}
&\left\{\frac{l\circ f - l}{l^{T+1}g}|_{l=0}dx - \frac{ul\circ f - ul}{ul^{T+1}g}|_{l=0}dx\right\}\cdot X \\
&\quad = \left\{\frac{\tilde{h}}{g}|_{l=0}dx - l\frac{u\circ f - u}{l^{T+1}ug}|_{l=0}dx - (u\circ f)\frac{l\circ f - l}{ul^{T+1}g}|_{l=0}dx\right\}\cdot X \\
&\quad = \left\{\frac{\tilde{h}}{g}|_{l=0}dx - \frac{u_x g + u_y h}{ug}|_{l=0}dx - \frac{\tilde{h}}{g}|_{l=0}dx\right\}\cdot X = -\left\{\frac{u_x}{u} + \frac{u_y}{u}\frac{h}{g}\right\}dx\cdot X.
\end{aligned}$$

Since f is non-degenerate on S, by definition, it follows that $\frac{h}{g} = -\frac{l_x}{l_y}$ on $\{l = 0\}$ and therefore

$$-\left\{\frac{u_x}{u} + \frac{u_y}{u}\frac{h}{g}\right\} dx \cdot X = \left\{\frac{-u_x l_y + u_y l_x}{u l_y}\right\} dx \cdot \lambda(l_y \tfrac{\partial}{\partial x} - l_x \tfrac{\partial}{\partial y})$$
$$= \lambda \frac{-u_x l_y + u_y l_x}{u} = -d(\log u) \cdot X,$$

as wanted. □

Remark 3.4. Let χ be a (local) biholomorphism on U with values in U. Let $V \subseteq U$ a open set such that $f(\chi^{-1}(V)) \subseteq U$. Let $\tilde{\tau} = \tau \circ \chi^{-1}$, $\tilde{l} = l \circ \chi^{-1}$ and $\tilde{f} = \chi \circ f \circ \chi^{-1}_{|V}$. Then, since

$$\frac{\tilde{l} \circ \tilde{f} - \tilde{l}}{\tilde{l}(\tilde{\tau} \circ \tilde{f} - \tilde{\tau})} d\tilde{\tau} = \chi^* \left(\frac{l \circ f - l}{l(\tau \circ f - \tau)} d\tau\right)$$

it follows that $\theta^{\tilde{l},\tilde{\tau}}$ is the expression of $\theta^{l,\tau}$ in the local coordinates $(\chi_1(x, y), \chi_2(x, y))$.

Now suppose f has no singularities on S. Let $\{U_j\}$ be a covering of S made of coordinate sets with local coordinates (x_j, y_j). For any j choose a defining function l_j such that $S \cap U_j = \{l_j = 0\}$. By (3.6) we may define on each U_j the operator

$$\theta^j := \theta^{l_j, \tau_j},$$

where τ_j is any transverse to S on U_j. Note that by (3.5) the operators θ^j are partial connections on $N_S|_{U_j}$. We want to show that they indeed glue together to give a partial connection for all of N_S. This is the case if and only if the 1-forms

$$\eta^j := \left\{-\frac{l_j \circ f_j - l_j}{l_j(\tau_j \circ f_j - \tau_j)}|_{l_j=0}\right\} d\tau_j \tag{3.7}$$

defined on the U_j's are such that

$$\eta^j = \eta^k + \frac{du_{kj}}{u_{kj}}, \tag{3.8}$$

whenever $U_k \cap U_j \neq \emptyset$ and $\{u_{jk}\}$ is the system of cocycles defining N_S relative to $\{U_j\}$, and this follows at once from Remark 3.4 and (3.6). Thus we have proved:

Proposition 3.5. *Suppose M is a two dimensional complex manifold, $S \subset M$ a smooth curve and $f : M \to M$ holomorphic such that $f|_S = Id_S$ and f is non-degenerate on S. Suppose f has no singularities on S. Then there exists a partial connection (in the sense of Baum-Bott* [3]*) θ on N_S defined locally by* (3.5)*.*

Let $(TS)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = TS \oplus \overline{TS}$. Since N_S is a holomorphic vector bundle on S we may endow $\overline{TS}$ with the partial connection

$$\overline{\partial} : N_S \to \Omega^{(0,1)}(N_S).$$

On N_S we define the operator

$$\nabla : N_S \to \Omega(N_S),$$

given on $(TS)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\nabla := \theta \oplus \overline{\partial}.$$

Proposition 3.6. *Suppose that f has no singularities on S. Then ∇ defines a connection for N_S. If U is a coordinate open set and (x, y) are local coordinates such that $S \cap U = \{y = 0\}$ then the connection 1-form with respect to the frame $[\frac{\partial}{\partial y}]$ of N_S on U is given by*

$$\left\{ -\frac{y \circ f - y}{y(x \circ f - x)}|_{y=0} \right\} dx. \tag{3.9}$$

Proof. It follows from (3.7) and the previous proposition that θ is a partial connection for N_S on TS. Therefore ∇ is a connection for N_S. Moreover

$$\nabla_{\frac{\partial}{\partial x}}([\frac{\partial}{\partial y}]) = \theta_{\frac{\partial}{\partial x}}([\frac{\partial}{\partial y}]),$$

and

$$\nabla_{\frac{\partial}{\partial \overline{x}}}([\frac{\partial}{\partial y}]) = 0.$$

From this and (3.5) the statement follows. □

Hence we have

Theorem 3.7 (Vanishing Theorem)**.** *Let M be a two dimensional complex manifold, $S \subset M$ a smooth curve, $f : M \to M$ holomorphic such that $f|_S = Id_S$ and f is non-degenerate on S. If f has no singularities on S then there exists a connection ∇ (called the* basic connection*) on N_S such that its curvature $K \equiv 0$. In particular the first Chern class $c_1(N_S) = 0$.*

Proof. By Proposition 3.6 the operator $\nabla = \theta \oplus \overline{\partial}$ is a connection for N_S whose 1-form η is (locally) given by (3.9) in the natural frame E. Since $K = d\eta - \eta \wedge \eta$ it follows that $K \equiv 0$. Also $c_1(N_S) = -\frac{1}{2\pi i}[c_1(\nabla)] = 0$. □

Remark 3.8. If S is non-compact then $c_1(L) = 0$ for any line bundle L on S (this follows at once from the long exact sequence associated to the exponential sequence). However if S is compact then this is not generally true, and the result is not obvious.

4. Residual index theorem in the smooth case

We use the notations of the previous section. Suppose S is compact, connected and smooth and f is non-degenerate on S. Then f has only a finite number of singularities on S. Let $\Sigma := \{p_1, \ldots, p_r\}$ be such singularities and $V := S - \Sigma$. For $\alpha = 1, \ldots, r$ let $\{U_\alpha\}$ be a coordinate set of M such that $T_\alpha := U_\alpha \cap S$ is non-empty and simply connected and $\{p_\alpha\} = \Sigma \cap T_\alpha$. Let ∇ be a basic connection for N_V as defined in the previous section. Let W_α be a simply connected open set in S such that $\overline{W_\alpha} \subset T_\alpha$. On each T_α let ∇_α be a connection for $N_S|_{T_\alpha}$. Let ψ be a C^∞ function on S such that ψ has support in $\cup_\alpha T_\alpha$ and $\psi|_{W_\alpha} \equiv 1$ for $\alpha = 1, \ldots, r$.

Let $\nabla_1 := \psi \sum \nabla_\alpha + (1-\psi)\nabla$. Then ∇_1 is a connection for N_S and, if K_1 is its curvature, it follows that

$$\frac{-1}{2\pi i}[K_1] = c_1(N_S) \in H^2(S, \mathbb{C}).$$

Note that $K_1 = K$ on $S - \cup_\alpha T_\alpha$, where K is the curvature of ∇. Thus by Theorem 3.7, K_1 has compact support contained in $\cup T_\alpha$.

In particular by the Poincaré duality it follows

$$H^2(S, \mathbb{C}) \simeq H_0(S, \mathbb{C}) \simeq \mathbb{C},$$

where the first isomorphism is given by integration on S. Therefore the first Chern number of N_S, which equals the self-intersection number of S denoted by $S \cdot S$, is given by

$$S \cdot S = \frac{-1}{2\pi i}\int_S K_1 = \frac{-1}{2\pi i}\sum_\alpha \int_{T_\alpha} K_1. \tag{4.1}$$

Let us give the following definition:

Definition 4.1. The *residual index* of f on S at $p \in S$ is given by

$$\text{Ind}(f, S, p) = \frac{1}{2\pi i}\int_{V_p} K_1,$$

where V_p is a simply connected open set in S containing p such that $V_p \cap \cup_\alpha T_\alpha = \emptyset$ if $p \notin \Sigma$ and $V_p = T_\alpha$ if $p = p_\alpha \in \Sigma$.

Note that $\text{Ind}(f, S, p) = 0$ for any $p \in S$ which is not a singularity for f.

The index theorem (due to Abate, who first proved it by other methods, see [2]) follows now directly from (4.1):

Theorem 4.2 (Index Theorem in the smooth case)**.** *Let M be a complex two dimensional manifold, $S \subset M$ a smooth compact connected curve and $f : M \to M$ holomorphic such that $f|_S = Id_S$ and f is non degenerate on S. Then*

$$\sum_{p \in S} \text{Ind}(f, S, p) = S \cdot S.$$

Now we are going to show that $\text{Ind}(f, S, p)$, for p a singularity of f, is in fact independent of the ∇_1 chosen to define it. To do so, we compute $\text{Ind}(f, S, p)$ explicitly.

Let $p \in S$ be a singularity for f, $\{(x, y), U\}$ local coordinates for M such that $p = (0, 0)$ and $S \cap U = \{y = 0\}$. Let $T := T_p(f, s)$ be the order of f on S at p. Suppose $f = (f_1, f_2)$ is given by

$$\begin{cases} f_1(x, y) = & x + y^T g(x, y) \\ f_2(x, y) = & y + y^{T+1} h(x, y) \end{cases} \tag{4.2}$$

where $g(x, 0) \not\equiv 0$. Suppose V, W are simply connected open sets with smooth boundary in S such that $p \in V \subset \overline{V} \subset W \subset \overline{W} \subset U \cap S$ and $\psi \equiv 1$ on V and ψ has compact support in W. Let η_1 be the connection 1-form of ∇_1 on U, η the connection 1-form of ∇ on $U - \{p\}$ and

η_α the connection 1-form of ∇_α (where ∇_α is a connection for N_S on W). Then, in the natural frame $[\frac{\partial}{\partial y}]$, by (3.9) we have

$$\eta_1 = \psi\eta_\alpha + (1-\psi)\eta = \psi\eta_\alpha - (1-\psi)\frac{h(x,0)}{g(x,0)}dx.$$

Since $K_1 = d\eta_1 - \eta_1 \wedge \eta_1 = d\eta_1$ then Stokes' theorem implies (recall that $\psi|_{\partial W} \equiv 0$)

$$\begin{aligned}\mathrm{Ind}(f,S,p) = \frac{-1}{2\pi i}\int_W K_1 = \frac{-1}{2\pi i}\int_W d\eta_1 = \frac{-1}{2\pi i}\int_{\partial W}\eta_1 \\ = \frac{-1}{2\pi i}\int_{\partial W} -\frac{h(x,0)}{g(x,0)}dx = \mathrm{Res}(\frac{h(x,0)}{g(x,0)}dx; x=0).\end{aligned}$$

Thus in the case S is smooth we have

$$\mathrm{Ind}(f,S,p) = \mathrm{Res}(\frac{h(x,0)}{g(x,0)}dx; x=0),$$

which is exactly the index defined in [2].

5. RESIDUAL INDEX THEOREM IN THE NON-SMOOTH CASE

In this section we let S have some isolated singularity. Then the setting is the following: M is a two dimensional complex manifold, $S \subset M$ is a compact (globally) irreducible connected compact curve and $f : M \to M$ holomorphic such that $f_{|S} = Id_S$ and f is non-degenerate on S.

Let $\Sigma := \Sigma_f \cup \Sigma_S$, where Σ_f is the set of singularities of f on S and Σ_S is the set of singularities of S. Let $\{U_j, (x_j, y_j)\}$ be local coordinates such that $S \cap U_j = \{l_j = 0\}$. Let L be the line bundle associated to the divisor S, whose cocycles are given by $l_j/l_k \in \mathcal{O}^*(U_j \cap U_k)$. Let V be a open neighborhood of Σ, such that $V = \cup V_\alpha$ and the V_α's are pairwise disjoint, simply connected and so that each one contains only one point of Σ, say p_α. Moreover, if $T := V \cap S$ and $T_\alpha := V_\alpha \cap S$ we require that ∂T_α is a smooth regular curve for any α. We are going to define a C^∞ connection $\tilde\nabla$ for L such that $\tilde\nabla$ restricted to $S - V$ coincides with the basic connection for N_{S-V} defined by the l_j's. To do this, let W be a tubular neighborhood of $S - \Sigma$ and $\rho : W \to S - \Sigma$ be the projection. Since $\rho^*(L|_{S-\Sigma})$ equals L once restricted to $S - \Sigma$, then $\rho^*(L|_{S-\Sigma})$ is C^∞ isomorphic to L over W. On $L|_W$ we put the pull back connection $\tilde\nabla_0 := \rho^*(\nabla_0)$ where ∇_0 is the basic connection for $L|_{S-\Sigma}$ defined as in section 3. On each V_α let $\tilde\nabla_\alpha$ be any connection for $L|_{V_\alpha}$. For any α let $\tilde V_\alpha \subset V_\alpha$ be a simply connected open set such that $p_\alpha \in \tilde V_\alpha$. Let ψ be a C^∞ function on M such that $\psi \equiv 1$ on $\tilde V_\alpha$ for any α and $\mathrm{supp}(\psi) \subset V$. Finally let $\tilde\nabla := \psi \sum \tilde\nabla_\alpha + (1-\psi)\tilde\nabla_0$. Then $\tilde\nabla$ is a connection for L over a neighborhood of S. Therefore, if $\tilde K$ is the curvature of $\tilde\nabla$, we have

$$S \cdot S = \int_{[S]} c_1(L) = \frac{-1}{2\pi i}\int_{[S]} \tilde K \tag{5.1}$$

where $[S] \in H_2(M, \mathbb{C})$ is the homology class of S. Since $\tilde{K} = K = 0$ on $S - T$, where K is the curvature of ∇_0 on $S - \Sigma$, then it follows that

$$\frac{-1}{2\pi i} \int_{[S]} \tilde{K} = \frac{1}{2\pi i} \sum_{\alpha} \int_{T_\alpha} \tilde{K}. \tag{5.2}$$

We are now in the position to define the residual index at singular points of S:

Definition 5.1. The *residual index* of f on S at $p_\alpha \in \Sigma_S$ is given by

$$\mathrm{Ind}(f, S, p_\alpha) = \frac{-1}{2\pi i} \int_{T_\alpha} \tilde{K}.$$

From (5.1) and (5.2) it follows the index theorem in the non-smooth case, already proved in [4] with a different technique:

Theorem 5.2 (Index Theorem in the non-smooth case)**.** *Let M be a two dimensional complex manifold, $S \subset M$ a globally irreducible compact connected curve (possibly with singularities) and $f : M \to M$ holomorphic such that $f|_S = Id_S$ and f is non degenerate on S. Then*

$$\sum_{p \in S} \mathit{Ind}(f, S, p) = S \cdot S.$$

Now we want to calculate the index in the case $p \in \Sigma_S$. Therefore we have to evaluate

$$\frac{-1}{2\pi i} \int_{T_\alpha} \tilde{K}.$$

We may suppose that on V_α the curve S is given by $l = 0$. By (3.5) the 1-form of the connection $\tilde{\nabla}$ on $S \cap V_\alpha$ with respect to the natural frame E associated to l is given by

$$\tilde{\eta} = \psi \eta_\alpha + (1 - \psi) \left\{ -\frac{l \circ f - l}{l(\tau \circ f - \tau)} |_{l=0} \right\} d\tau,$$

where τ is any transverse to S outside p and η_α is the connection 1-form of ∇_α (where ∇_α is a connection for L on V_α). Then $\tilde{K} = d\tilde{\eta} - \tilde{\eta} \wedge \tilde{\eta} = d\tilde{\eta}$ on T_α and by Stokes' theorem we have

$$\mathrm{Ind}(f, S, p) = \frac{-1}{2\pi i} \int_{T_\alpha} \tilde{K} = \frac{-1}{2\pi i} \int_{T_\alpha} d\tilde{\eta} = \frac{-1}{2\pi i} \int_{\partial T_\alpha} \tilde{\eta} = \frac{1}{2\pi i} \int_{\partial T_\alpha} \left\{ \frac{l \circ f - l}{l(\tau \circ f - \tau)} \right\} d\tau,$$

as in [4].

Remark 5.3. Note that if ∂T_α is not connected, the integral is a sum of integrals each of which gives the index of f at p of the irreducible component of S intersecting such component of ∂T_α.

6. DYNAMICS NEAR CURVES OF FIXED POINTS

In his paper, Abate [2] proves that if f is a (germ of) holomorphic diffeomorphism of $\mathbb{C}^2$ with an isolated fixed point $p \in \mathbb{C}^2$ and such that $df_p = Id$ then there exists at least one parabolic curve for f at p. In this section we are going to investigate what happens in the case $f : M \to M$

has a curve of fixed points and f acts as the identity on the normal bundle of the regular part of S.

In what follows we need these results about indices and blow-ups (see [2] and [4]):

Lemma 6.1. *Let M be a two dimensional complex manifold. Let $f : M \to M$ be holomorphic and $S \subset M$ a curve such that $f|_S = Id_S$ and f is non-degenerate on S. Let $p \in S$ be such that S is irreducible at p. $\pi : \tilde{M} \to M$ be the blow-up of M at p, and $\tilde{f} : \tilde{M} \to \tilde{M}$ be the map induced by f. Let $D := \pi^{-1}(p)$ and let $\tilde{S} := \overline{\pi^{-1}(S - \{p\})}$ be the strict transform of S. Then*

(1) *The map $\tilde{f}$ is non-degenerate on $\tilde{S}$.*
(2) *Let $\{\tilde{p}\} = D \cap \tilde{S}$. Then $\mathrm{Ind}(\tilde{f}, \tilde{S}, \tilde{p}) = \mathrm{Ind}(f, S, p) - m^2$, where $m \geq 1$ is the multiplicity of S at p (in particular $m = 1$ if and only if S is smooth at p).*

Now we may start our study of the dynamics. First, we need some calculations for the indices:

Lemma 6.2. *Let M be a two dimensional complex manifold, $S \subset M$ a curve. Let $f : M \to M$ be holomorphic such that $f|_S = Id_S$ and f is non-degenerate on each component of S.*

(1) *If $p \in S$ is a smooth point of S dicritical for f then $\mathrm{Ind}(f, S, p) = 1$.*
(2) *If $p \in S$ belongs to two smooth irreducible branches S_1, S_2 of S intersecting transversally at p and p is a non-dicritical reduced singularity $(\star_2)$ for f on S then either $\mathrm{Ind}(f, S_1, p) = 0$ or $\mathrm{Ind}(f, S_2, p) = 0$.*
(3) *If $p \in S$ belongs to two smooth irreducible branches of S intersecting transversally at p and p is a non-dicritical reduced singularity $(\star_1)$ for f on S, then*
$$\mathrm{Ind}(f, S_1, p) \cdot \mathrm{Ind}(f, S_2, p) = 1.$$
(4) *If $p \in S$ is a smooth point of S which is a non-dicritical reduced singularity $(\star_2)$ of f on S*
    - *either $\mathrm{Ind}(f, S, p) = 0$,*
    - *or, after one blow-up, the induced map $\tilde{f}$ has a reduced singularity $(\star_2)$ at the intersection of the strict transform of S with the exceptional divisor and a reduced singularity $(\star_1)$ on a smooth point of the exceptional divisor.*

The statement (2), (3) and (4) of Lemma 6.2 are in [2].

Proof of Lemma 6.2.(1). By Remark 1.4 we may choose local coordinates (x, y) around p in such a way that $p = (0, 0)$, $S = \{y = 0\}$ and f is given by (4.2), where $T \geq 1$ and $g(x, 0) \not\equiv 0$, $h(x, 0) \not\equiv 0$. Since p is dicritical it follows from Remark 2.8 that $g_{\mu(g)}(x, y) = x h_{\mu(h)}(x, y)$, where $g_{\mu(g)}$ and $h_{\mu(h)}$ are the first non-zero terms in the homogeneous expansion of g and h respectively, and $\mu(g) = \mu(h) + 1$. Then, setting $\Gamma = (e^{2\pi i \theta}, 0)$ for $\theta \in (0, 1)$, $g(x, y) =$

$g_{\mu(g)}(x,y) + g^\bullet(x,y)$ and $h(x,y) = h_{\mu(h)}(x,y) + h^\bullet(x,y)$ we have

$$2\pi i \ \mathrm{Ind}(f,S,p) = \int_\Gamma \frac{y \circ f - y}{y(x\circ f - x)} dx = \int_\Gamma \frac{h_{\mu(h)}(x,y) + h^\bullet(x,y)}{g_{\mu(g)}(x,y) + g^\bullet(x,y)} dx$$
$$= \int_\Gamma \frac{h_{\mu(h)}(x,0)}{g_{\mu(g)}(x,0) + g^\bullet(x,0)} dx + \int_\Gamma \frac{h^\bullet(x,0)}{g_{\mu(g)}(x,0) + g^\bullet(x,0)} dx.$$

Now the second integral is the residue at $x = 0$ of the ratio of two holomorphic functions with numerator of order $\geq \mu(h)+1$ at 0 and denominator of order $\mu(g) = \mu(h)+1$ at 0, therefore the ratio is holomorphic at $x=0$ and the integral is zero. As for the first integral

$$\int_\Gamma \frac{h_{\mu(h)}(x,0)}{g_{\mu(g)}(x,0) + g^\bullet(x,0)} dx = \int_\Gamma \frac{h_{\mu(h)}(x,0)}{h_{\mu(h)}(x,0)x + g^\bullet(x,0)} dx$$
$$= \int_\Gamma \frac{1}{x + \frac{g^\bullet(x,0)}{h_{\mu(h)}(x,0)}} dx = 2\pi i,$$

since $g^\bullet(x,0)/h_{\mu(h)}(x,0)$ has order $\geq \mu(g)+1-\mu(h) = \mu(g)+1-\mu(g)+1 = 2$ at $x=0$. Therefore $\mathrm{Ind}(f,S,p) = 1$ as wanted. □

Now we are going to somehow mimic the work of Cano [7] for the continuous dynamics in order to find out an algorithm for producing parabolic curves. However, contrarily to the continuous case, we have to worry about dicritical points and degenerate curves, since it is by no means obvious that there must exist infinitely many parabolic curves in such situations. This is actually true as we show later, but the proof is based on the algorithm itself.

Definition 6.3. Let M be a two dimensional complex manifold, $f: M \to M$ holomorphic. Let $\mathrm{Fix}(f) = S$, for S a curve in M.

- We say that a point $p \in S$ is *of type* (C_1) if S is nonsingular at p, the map f is non-degenerate on S at p and
$$\mathrm{Ind}(f,S,p) \notin \mathbb{Q}^+ \cup \{0\}.$$
- We say that a point $p \in S$ is *of type* (C_2) if S has two nonsingular branches S_0, S_1 at p, intersecting transversally at p and there exists a real number $r > 0$ such that
$$\mathrm{Ind}(f,S_0,p) \notin \mathbb{Q}_{(\geq -1/r)} = \{a \in \mathbb{Q} : a \geq -1/r\}$$
$$\mathrm{Ind}(f,S_1,p) \in \mathbb{Q}_{(\leq -r)} = \{a \in \mathbb{Q} : a \leq -r\}.$$

Remark 6.4. If $q \in S$ is of type (C_2) then by Lemma 6.2.(2) and (3) the point q cannot be a reduced singularity for f on the branches of S at q. Note also that by Lemma 6.2 a point of type (C_1) or (C_2) cannot be dicritical.

We start with the following lemma whose proof goes exactly as in [7] (we only note that blowing up a (C_1) or (C_2) point produces a divisor on which the blow up map is non-degenerate by the previous note):

Lemma 6.5. *Let M be a two dimensional complex manifold, $f : M \to M$ be holomorphic and such that Fix$(f) = S$, for a curve S. Let $q \in S$ be a point of type $(C_1), (C_2)$. Let $\pi : \tilde{M} \to M$ be the blow-up at q and $E := \pi^{-1}(S)$ the total transform of S. Let $\tilde{f}$ be the holomorphic map induced by f on $\tilde{M}$. Then there exists a point $\tilde{q} \in E$ of type (C_1) or (C_2).*

Definition 6.6. Let S be a curve in a complex two dimensional manifold M. Let $f : M \to M$ be holomorphic and assume Fix$(f) = S$. Let $p \in S$. We say that p is an *appropriate singularity* for f if after a finite number of blow ups there exists a point of type (C_1) or (C_2) on the large transform of S.

The importance of appropriate singularities comes from the following result:

Proposition 6.7. *Let M be a two dimensional complex manifold, $S \subset M$ a curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$. Let $p \in S$ be an appropriate singularity for f. Then there exists at least one parabolic curve for f at p.*

Proof. By definition, after a finite number of blow ups we obtain a (C_1) or (C_2) point. By Lemma 6.5 and by Theorem 2.7 (and, if this is the case, by the theorem of resolution of singularities for curves, see [11]) after a finite number of blow-ups at (C_1) or (C_2) points we find either a nonsingular point of the total transform or a corner which is a reduced singularity for the induced map of type, respectively, (C_1) or (C_2). However by Remark 6.4 it cannot be of type (C_2). Thus there must be a reduced singularity of type (C_1) which is a smooth point of the total transform, say q. By Lemma 6.2.(4) we may assume (up to blow up once more if necessary) that the point q is a reduced singularity of type $(\star_1)$. Thus Theorem 2.5 applies producing (at least) a parabolic curve not contained in the total transform of S, which projects down to a parabolic curve for f at p. □

By definition a nonsingular point $p \in S$ of a curve S such that Fix$(f) = S$ near p, f is non-degenerate on S and Ind$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ is an appropriate singularity for f. However there are some more interesting examples, as the following results show:

Proposition 6.8. *Let M be a two dimensional complex manifold, $S \subset M$ be a (possibly singular and non-compact) curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$.*

(1) *If f acts on the normal bundle of the regular part of S as the identity and f is degenerate on S then every $p \in S$, except at most a discrete subset of S, is an appropriate singularity for f.*
(2) *Let $p \in S$ be such that S_p is a generalized irreducible cusp,* i.e., *there exist local coordinates (x, y) such that $p = (0, 0)$, $S = \{y^m = x^n\}$, $m < n$ and S is irreducible at $(0, 0)$. If f is non-degenerate on S and Ind$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then p is an appropriate singularity.*
(3) *Let $p \in S$ be such that S_p is an irreducible singularity for which at each step of the desingularization process the exceptional divisor is not in the tangent cone of the singularity. If f is non-degenerate on S and Ind$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then p is an appropriate singularity.*

Proof. 1. Let

$$\mathcal{S}_f := \{p \in S : \frac{l \circ f - l}{l^{T_p(f,S)}}[p] = 0, \forall\, l \in \mathcal{O}_p : (l)_p = \mathcal{I}(S)_p\}.$$

Note that $\mathcal{S}_f$ is a discrete set which contains the set of singularities of f on S. Let $\mathcal{S}_S$ be the (discrete) set of singular points of S. Let $\mathcal{S} := \mathcal{S}_f \cup \mathcal{S}_S$. Let $p \in S \setminus \mathcal{S}$. We can choose local coordinates (x, y) around p in such a way that $p = (0, 0)$, $S = \{y = 0\}$ and $f = (f_1, f_2)$ is given by

$$\begin{cases} f_1(x,y) & = x + y^T g(x,y) \\ f_2(x,y) & = y + y^T h(x,y) \end{cases}$$

where $T \geq 2$ (by Remark 1.4), $g(0,0) \neq 0$ and $h(0,0) \neq 0$. Write $g(x,y) = a_0 + g_1(x,y)$ where $a_0 \neq 0$, $g_1(0,0) = 0$ and $h(x,y) = b_0 + h_1(0,0)$ with $b_0 \neq 0$ and $h_1(0,0) = 0$. Now let $\pi : \tilde{M} \to M$ be the blow-up of M at p and $\tilde{f}$ the map induced by f. If (u, v) are local coordinates around $\pi^{-1}(p)$ such that $\pi(u, v) = (u, uv)$, then the exceptional divisor $D := \pi^{-1}(p)$ is given by $\{u = 0\}$ and the strict transform $\tilde{S}$ of S is given by $\{v = 0\}$. The point $\tilde{p} = (u = 0, v = 0)$ is the only intersection between $\tilde{S}$ and D. Writing $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ we have

$$\begin{cases} \tilde{f}_1(u,v) & = u + u^T v^T a_0 + v^T o(|u|^T) \\ \tilde{f}_2(u,v) & = v + u^{T-1} v^T [b_0 - v a_0] + v^T o(|u^{T-1}|) \end{cases}$$

Therefore the order of $\tilde{f}$ on D at $(0,0)$ is $T-1$, and $\tilde{f}$ is non-degenerate on D. Now we calculate the index of $\tilde{f}$ at $(0,0)$ on D. Let Γ be the cycle given by $u = 0$, $v = e^{2\pi i\theta}$ for $\theta \in [0,1]$. Then

$$2\pi i \ \mathrm{Ind}(f, D, (0,0)) = \int_\Gamma \frac{u \circ \tilde{f} - u}{u(v \circ \tilde{f} - v)} dv = \int_\Gamma \frac{a_0 + O(u)}{b_0 - v a_0 + O(u)} dv = 0.$$

Therefore $\mathrm{Ind}(\tilde{f}, D, (0,0)) = 0$. Since $D \cdot D = -1$ then there exists $q \in D - \tilde{S}$ such that $\Re e \mathrm{Ind}(\tilde{f}, D, q) < 0$, and then a (C_1) point as wanted.

2. If p is a dicritical point or f then we blow it up and find that the blow up map $\tilde{f}$ is degenerate on the exceptional divisor and acting as the identity on its normal bundle. Then p is an appropriate singularity by point 1.

Suppose p is not dicritical and $m > 2$ (otherwise p is (C_1)). We assume that $S = \{(x,y) : y^m - x^n = 0\}$ near $p = (0,0)$. Blowing up the point p, in the coordinates $x = u, y = uv$ it follows that the strict transform $\tilde{S}$ of S is given by $\{v^m - u^{n-m} = 0\}$ and the exceptional divisor $D = \{u = 0\}$. Thus $\tilde{p} = \tilde{S} \cap D$ is $\tilde{p} = (0,0)$. By Lemma 6.1 it follows $\mathrm{Ind}(\tilde{f}, \tilde{S}, \tilde{p}) \notin \mathbb{Q}_{(\geq -m^2)}$, where $\tilde{f}$ is the blow up of f. If there are no (C_1) points on the exceptional divisor $D \setminus \{\tilde{p}\}$ then by Theorem 4.2 it follows that $\mathrm{Ind}(\tilde{f}, D, \tilde{p}) \in \mathbb{Q}_{(\leq -1)}$ (and in particular $\tilde{p}$ is not dicritical). Since S is irreducible then $m \neq n - m$. Suppose first $n - m > m$. Blow up the point $\tilde{p}$ and denote by S_0 the strict transform of $\tilde{S}$, by D_0 the strict transform of D and by D_1 the new exceptional divisor. Also denote by f_1 the blow up of $\tilde{f}$. It follows that if $D_0 \cap D_1 = \{q_1\}$

and $S_0 \cap D_1 = \{q_0\}$ then $q_0 \neq q_1$. By Lemma 6.1 we have $\mathrm{Ind}(f_1, S_0, q_0) \notin \mathbb{Q}_{(\geq -2m^2)}$ and $\mathrm{Ind}(f_1, D_0, q_1) \in \mathbb{Q}_{(\leq -2)}$. Thus, if there are no (C_1) points on $D_1 \setminus \{q_0, q_1\}$ and q_1 is not (C_2) then again by Theorem 4.2 we have $\mathrm{Ind}(f_1, D_1, q_0) \in \mathbb{Q}_{(\leq -1/2)}$. Note that S_0 is given by $\{y^m - x^{n-2m} = 0\}$. Again, if $n - 2m > m$ we blow up the point and argue as before. We continue this way until finding $k \in \mathbb{N}$ such that $n - km < m$. At this point, if there are no (C_1) or (C_2) points on the large transform, arguing as before, denoting by q the intersection between the exceptional divisor D and the strict transform $\tilde{S}$ of S, and by $\tilde{f}$ the blow up of f, we have $\mathrm{Ind}(\tilde{f}, \tilde{S}, q) \notin \mathbb{Q}_{(\geq -km^2)}$ and $\mathrm{Ind}(\tilde{f}, D, q) \in \mathbb{Q}_{(\leq -1/k)}$. Now, blowing up at q we obtain a triple intersection at one point, say q_0. Arguing as before, if there are no (C_1) points on the exceptional divisor D_1 and f_0 denote the blow up of $\tilde{f}$, D_0 the strict transform of D, S_0 the strict transform of $\tilde{S}$, we have $\mathrm{Ind}(f_0, S_0, q_0) \notin \mathbb{Q}_{(\geq -km^2-(n-km)^2)}$, $\mathrm{Ind}(f_0, D_0, q_0) \in \mathbb{Q}_{(\leq -(k+1)/k)}$ and $\mathrm{Ind}(f_0, D_1, q_0) \in \mathbb{Q}_{(\leq -1)}$. From this point on, each time we blow up the singular point of the strict transform of S we obtain a triple intersection until the strict transform is nonsingular and intersecting the other two lines transversally. Also, if at each step there are no $(C_1), (C_2)$ points, we end up with a triple S_0, D_0, D_1 of nonsingular curves intersecting transversally at one point q and such that $\mathrm{Ind}(f, S_0, q) \notin \mathbb{Q}_{(\geq -M)}$, $\mathrm{Ind}(f, D_0, q) \in \mathbb{Q}_{(\leq -(1+1/r))}$ and $\mathrm{Ind}(f, D_1, q) \in \mathbb{Q}_{(\leq -1)}$, with $r > 0$ and $M > 2r + 1$ (this can be seen by an induction argument). Then, blowing up once more and arguing in the usual way, if there are no (C_1) points, we certainly obtain a (C_2) point.

3. The proof is similar to the one in part 2., for at each blow up the hypothesis on the resolution process implies that no three branches of the total transform intersect at a same point. We leave the details to the reader. □

It is an open question to say whether all the locally irreducible singularities which are the non-degenerate fixed points set of a holomorphic map are appropriate if the index of the map is not a positive rational number nor zero.

In particular by Propositions 6.7 and 6.8 we have

Theorem 6.9. *Let M be a two dimensional complex manifold, $S \subset M$ be a (possibly singular and non-compact) curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$. Suppose that f acts on the normal bundle of the regular part of S as the identity and that f is degenerate on S. Then for every $p \in S$, except at most a discrete subset of S, there exists at least one parabolic curve for f at p.*

Remark 6.10. If S is compact then f has parabolic curves at every point of S *but at most a finite set of points*.

Note that also the converse is true:

Proposition 6.11. *Let M be a two dimensional complex manifold, let $S \subset M$ be a (possibly non compact and singular) curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$. Suppose that f acts on the normal bundle of the regular part of S as the identity. If there exists a non-discrete subset $A \subset S$ such that f has parabolic curves at every $p \in A$ then f is degenerate on S.*

Proof. If f were non-degenerate on S then the union of the singularities of f on S and the singular points of S would form a discrete set B. By Proposition 2.2 then $A \subseteq B$, which contradicts the hypothesis. $\square$

Another application is to dicritical points (cfr. *Theorem 3.1.(ii) in* [2], where part of the following result is achieved by direct methods):

Proposition 6.12. *Let M be a two dimensional complex manifold, $p \in M$. Let $f : M \to M$ be holomorphic and such that $f(p) = p$, $df_p = Id$. Then p is dicritical for f if and only if for every direction but a finite number, there exists a parabolic curve for f at p tangent to such a direction at p.*

Proof. Let $\pi : \tilde{M} \to M$ be the blow-up of M at p, $D := \pi^{-1}(p)$ be the exceptional divisor and $\tilde{f} : \tilde{M} \to \tilde{M}$ be the holomorphic blow-up of f. Suppose p is dicritical for f. Then $\tilde{f}$ is degenerate on D and acts on the normal bundle of D as the identity (see Remark 2.8). Thus by Theorem 6.9 every point but a finite number of D has parabolic curves for $\tilde{f}$ which blow down to parabolic curves for f tangent to all directions but a finite number. Viceversa if each direction but a finite number is a tangent for a parabolic curve for f at p then $\tilde{f}$ has parabolic curves at every point of D but a finite number. If $\tilde{f}$ does not act as the identity on the normal bundle of D then it is non-degenerate by definition. If $\tilde{f}$ acts as the identity on the normal bundle of D then it must be degenerate on D by Proposition 6.11 and hence p is dicritical. $\square$

We also have the following global result:

Theorem 6.13. *Let M be a two dimensional complex manifold, $S \subset M$ a compact, globally and locally irreducible curve with $S \cdot S < 0$. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$ and f is non-degenerate on S. Then there exists a point $p \in S$ such that f has at least one parabolic curve at p.*

Proof. By Theorem 5.2 there exists $p \in S$ such that $\Re e\mathrm{Ind}(f, S, p) < 0$. If such a point is nonsingular for S then it is (C_1) and we are done. Same if it is a singularity satisfying one of the hypothesis of Proposition 6.8. In general however one can argue following the same lines of [15] (see also *p.40 in* [5]) for the holomorphic foliations case to prove the existence of an appropriate singularity for f on S and then apply Proposition 6.7. $\square$

Note that Theorem 6.13 and Theorem 6.9 imply that: *if M is a two dimensional complex manifold, $f : M \to M$ holomorphic, $S \subset M$ a compact, globally and locally irreducible curve such that $S \cdot S < 0$, Fix$(f) = S$ and f acts as the identity on the normal bundle of the regular part of S then there exists at least one point $p \in S$ such that f has parabolic curves at p.*

Also we recover *Abate's flowers theorem,* [2]:

Corollary 6.14 (Abate)**.** *Let f be a (germ of) biholomorphism of $\mathbb{C}^2$ such that 0 is a isolated fixed point for f and $df_0 = Id$. Then there exists a parabolic curve for f at 0.*

Proof. If 0 is dicritical then apply Proposition 6.12. If 0 is non-dicritical then blowing up 0 we find a holomorphic map $\tilde{f}$ which has the exceptional divisor D as fixed points set and is non-degenerate on it. Since $D \cdot D = -1$ then we can apply Theorem 6.13. $\square$

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