to appear: Trans. Amer. Math. Soc.

FIXED POINTS OF COMMUTING HOLOMORPHIC MAPPINGS OTHER THAN THE WOLFF POINT.

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ABSTRACT. Let Δ be the unit disc of $\mathbb{C}$ and let $f, g \in \mathrm{Hol}(\Delta,\Delta)$ be such that $f \circ g = g \circ f$. For $A > 1$, let $\mathrm{Fix}_A(f) := \{p \in \partial\Delta | \lim_{r\to 1} f(rp) = p, \lim_{r\to 1} |f'(rp)| \leq A\}$. We study the behavior of g on $\mathrm{Fix}_A(f)$. In particular we prove that $g(\mathrm{Fix}_A(f)) \subseteq \mathrm{Fix}_A(f)$. As a consequence, besides conditions for $\mathrm{Fix}_A(f) \cap \mathrm{Fix}_A(g) \neq \emptyset$, we prove a conjecture of C. Cowen [Co2] in case f and g are univalent mappings.

1. INTRODUCTION

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $h : \Delta \to \mathbb{C}$ be holomorphic and $x \in \partial\Delta$. We say that h has non-tangential limit $L \in \mathbb{C}$ at x if $\lim_{k\to\infty} h(z_k) = L$ for any sequence $\{z_k\} \subset \Delta$ such that $z_k \to x$ non-tangentially. Shortly we write either

$$\text{K-}\lim_{z\to x} h(z) = L,$$

or $h(x) = L$. The use of this notation should not be confusing since we do not assume the maps to be continuous up to the boundary.

By the Wolff-Denjoy Theorem if $f \in \mathrm{Hol}(\Delta,\Delta)$, $f \neq Id$, there exists one distinguished point $\tau_f \in \overline{\Delta}$, the *Wolff point* of f, such that $f(\tau_f) = \tau_f$ and $|f'(\tau_f)| \leq 1$. Moreover if $\tau_f \in \partial\Delta$ then $0 < f'(\tau_f) \leq 1$.

If $g \in \mathrm{Hol}(\Delta,\Delta)$, $g \neq Id$, and $f \circ g = g \circ f$ then Behan [Be]—generalizing a previous result of Shields [Shi] for the case f, g continuous up to $\partial\Delta$—proved that f and g have the same Wolff point unless they are two hyperbolic automorphisms. A consequence of this is that whenever $f, g \in \mathrm{Hol}(\Delta,\Delta)$ commute than they must share a fixed point (in the sense of non-tangential limit if this point is on the boundary). This result has been generalized to domains in several variables by the author (see [Br1], [Br2]), proving that a generic characteristic of commuting holomorphic mappings is that of sharing a "fixed point" in the closure of the domain.

By the Schwarz Lemma f has at most one fixed point in Δ (provided $f \neq Id$). However f could have more than one fixed point (as non-tangential limit) at the boundary. Let us denote by $\mathrm{Fix}(f)$ the set of fixed points of f, *i.e.*

$$\mathrm{Fix}(f) := \{p \in \partial\Delta : f(p) = p\} \cup \{\tau_f\}.$$

1991 *Mathematics Subject Classification.* Primary 30D05; Secondary 30C80, 30E25, 47B33.

Key words and phrases. fixed points; Wolff point; commuting mappings.

† Partially supported by Progetto MURST di Rilevante Interesse Nazionale *Proprietà geometriche delle varietà reali e complesse* and GNSAGA.

A natural question posed by C. Cowen [Co2] in 1984 was whether g must have these points fixed, as well. Simple examples show that this is not the case:

Example 1.1. Let $f(z) = (z)^2$ and $g(z) = (z)^3$ (here $(z)^2 = z \cdot z$, $(z)^3 = z \cdot z \cdot z$). Then the point -1 is fixed for g but $f(-1) = 1$. Note however that $f(-1)$ is a fixed point for g.

The conclusion of the previous example is not surprising since, for g, f continuous up to $\partial\Delta$, it follows easily that $g(\mathrm{Fix}(f)) \subset \mathrm{Fix}(f)$. Cowen himself provided an even worse example in which there exists a point $p \in \mathrm{Fix}(f)$ (with $f'(p) < \infty$) such that $g(p)$ is the Wolff point of f (see *Example 4.12* in [Co2], see also Example 4.3).

Notwithstanding such examples, Cowen conjectured the following:

(1) if the Wolff point τ_f of f is on the boundary and $f'(\tau_f) < 1$ then for any $g \in \mathrm{Hol}(\Delta,\Delta)$ such that $f \circ g = g \circ f$ it follows $\mathrm{Fix}(f) = \mathrm{Fix}(g)$,
(2) if the Wolff point of f belongs to Δ, $f \notin \mathrm{Aut}(\Delta)$, and $f'(\tau_f) \neq 0$ then there exists $m \in \mathbb{N}$ such that for any $g \in \mathrm{Hol}(\Delta,\Delta)$, $g \notin \mathrm{Aut}(\Delta)$ and $f \circ g = g \circ f$, if we set $g^m := g^{m-1} \circ g$, $g^1 = g$, it follows $\mathrm{Fix}(f) = \mathrm{Fix}(g^m)$,
(3) if $f'(\tau_f) \neq 0$ then for any $g \in \mathrm{Hol}(\Delta,\Delta)$ such that $f \circ g = g \circ f$ and f, g have two fixed points in common then $\mathrm{Fix}(f) = \mathrm{Fix}(g)$.

Aside from a lack of counterexamples, some geometrical intuitions and analogies with the continuous semigroup case (see *Theorem 5.2 in* [Co1]), Cowen was able to support his idea with the following result, whose proof is based on the properties of linear fractional models (see [Co1]):

Proposition 1.2 (Cowen [Co2]). *Suppose $f, g \in Hol(\Delta,\Delta)$, $f \circ g = g \circ f$. Moreover suppose that f' is continuous on $\overline{\Delta}$, $\tau_f \in \partial\Delta$, $f'(\tau_f) < 1$, and there exists $p \in Fix(f)$, $p \neq \tau_f$ such that $f(\overline{\Delta} - \{\tau_f, p\}) \subset \Delta$. Then $g(p) = p$.*

After that we had not knowledge of any further step toward the (dis)proving of Cowen's conjecture(s). We became acquainted with some counterexamples to such conjectures constructed by means of Blaschke products due to Chalendar and Mortini [ChMo] only after a version of this work was under reviewing. The author thanks R. Tauraso for bringing the paper [ChMo] to his attention.

In this paper we restrict our attention to *repelling boundary fixed points*, giving some positive results in the direction of Cowen's conjectures. According to Poggi Corradini [PC2], borrowing a word from the discrete dynamics, we give the following definition.

Definition 1.3. Let $f \in \mathrm{Hol}(\Delta,\Delta)$ and $x \in \partial\Delta$. We say that x is a *boundary repelling fixed point*, shortly BRFP, for f if $f(x) = x$ and $1 < f'(x) < +\infty$. For $A > 1$ we let

$$\mathrm{Fix}_A(f) := \{x \in \partial\Delta : f(x) = x \text{ and } f'(x) \leq A\}.$$

Note that by the Julia-Wolff-Carathéodory Theorem (see Theorem 2.1) if $x \in \partial\Delta$ is a fixed point for $f \in \mathrm{Hol}(\Delta,\Delta)$ then $f'(x) \in (0, +\infty]$.

By definition the Wolff point τ_f of f belongs to $\mathrm{Fix}_A(f)$ only if $\tau_f \in \partial\Delta$.

Our main result is the following

Theorem 1.4. *Let $f \in Hol(\Delta,\Delta)$ not the identity. Let $g \in Hol(\Delta,\Delta)$ be non-constant and such that $f \circ g = g \circ f$. Let $A > 1$. Then $g(Fix_A(f)) \subseteq Fix_A(f)$,* i.e. *for any $p \in Fix_A(f)$ the map g has non-tangential limit $g(p) \in Fix_A(f)$.*

The proof is based on the existence of *backward iteration sequences* for f at each $p \in \mathrm{Fix}_A(f)$ due to Poggi Corradini (see Section 2), and some fine properties of "abstract backward iteration sequences" which we are going to prove.

After that we study the behavior of g on $\mathrm{Fix}_A(f)$. If this set if finite—which is always the case if, *e.g.*, $\tau_f \in \Delta$ or $\tau_f \in \partial\Delta$ and $f'(\tau_f) < 1$—then g has *repelling cycles* made of BRFP's for f. More explicitly:

Definition 1.5. Let $g \in \mathrm{Hol}(\Delta,\Delta)$, $m \geq 1$ and $p_0, \ldots, p_{m-1} \in \partial\Delta$, $p_j \neq p_l$ for $l \neq j$. We say that $\{p_0, \ldots, p_{m-1}\}$ is a *cycle* for g if $g(p_j) = p_{j+1}$ for $j = 0, \ldots m-2$ and $g(p_{m-1}) = p_0$. The number m is called the *length of the cycle.* If in addiction $1 < |g'(p_j)| < \infty$ for $j = 0, \ldots, m-1$, we say that $\{p_0, \ldots, p_{m-1}\}$ is a *repelling cycle* for g.

Then the result is

Theorem 1.6. *Let $f \in Hol(\Delta,\Delta)$, not the identity. Let $g \in Hol(\Delta,\Delta)$ be non-constant and such that $f \circ g = g \circ f$. Let $A > 1$ and suppose that $Fix_A(f)$ is non-empty and finite. Then one and only one of the following may happen:*

1. *either there exist $m \geq 1$, $p_0, \ldots, p_{m-1} \in Fix_A(f)$, $p_l \neq p_j$ for $l \neq j$, and $1 < A' \leq A$ such that $\{p_0, \ldots, p_{m-1}\}$ is a repelling cycle for g and $f'(p_j) = A'$ for $j = 0, \ldots, m-1$,*
2. *or there exists $m \geq 1$ such that for any $p \in Fix_A(f)$ it follows that $g^m(p) = \tau_f$, where $\tau_f \in \partial\Delta$ is the Wolff point of f. In this case $(g^m)'(p) = \infty$ for any $p \in Fix_A(f) - \{\tau_f\}$.*

Starting from Theorem 1.6, adding some extra hypothesis to f and g, we get conditions for the existence of common fixed points other than the Wolff point (see Section 4). We also apply our results to the study of composition operators, giving a criterion for (non-)compactness of commuting composition operators (see Proposition 4.7). Finally we prove Cowen's conjectures (1) and (2) and partially (3) in case f, g are univalent on Δ and f', g' are continuous up to $\partial\Delta$.

Let us indicate with $\mathrm{Hol}^1_u(\Delta,\Delta)$ the class of maps which are univalent on Δ and whose derivatives extend continuously up to the boundary, *i.e.*,

$$\mathrm{Hol}^1_u(\Delta,\Delta) := \{h \in \mathrm{Hol}(\Delta,\Delta) \mid h \text{ is univalent and } h' \text{ is continuous on } \overline{\Delta}\}.$$

Note that if $f \in \mathrm{Hol}^1_u(\Delta,\Delta)$ then $f'(a) \neq 0$ for any $a \in \overline{\Delta}$. Then:

Theorem 1.7. *Suppose $f, g \in Hol^1_u(\Delta,\Delta)$ and the Wolff point τ_f of f is such that $\tau_f \in \partial\Delta$ and $f'(\tau_f) < 1$. If $f \circ g = g \circ f$ then $Fix(f) = Fix(g)$.*

Theorem 1.8. *Suppose $f \in Hol^1_u(\Delta,\Delta)$ and the Wolff point τ_f of f is such that $\tau_f \in \Delta$. Then there exists $m \in \mathbb{N}$ such that for any $g \in Hol^1_u(\Delta,\Delta)$, $g \notin Aut(\Delta)$, with $f \circ g = g \circ f$ then $Fix(f) = Fix(g^m)$.*

Proposition 1.9. *Suppose $f, g \in Hol^1_u(\Delta,\Delta)$ and $f \circ g = g \circ f$. Suppose $\tau_f \in \Delta$ and $Fix(f) \cap Fix(g)$ contains two points. Then $Fix(f) = Fix(g)$.*

The author wants to warmly thank Pietro Poggi Corradini for many stimulating electronic discussions about topics related to this paper.

2. Preliminary results on backward iteration sequences.

We start recalling the fundamental Julia-Wolff-Carathéodory Theorem (for a proof see, *e.g.*, [Ab] or [Sha]).

Theorem 2.1. *Let $h \in$ Hol(Δ,Δ) and $x \in \partial\Delta$. Let*

$$\alpha_x(f) := \liminf_{z \to x} \frac{1-|f(z)|}{1-|z|}.$$

If $\alpha_f(x) < \infty$ then exists $y \in \partial\Delta$ such that $f(x) = y$ and $f'(x) = \overline{x}y\alpha_f(x)$. Moreover

$$f'(x) = \liminf_{z \to x} \frac{1-|f(z)|}{1-|z|} = K\text{-}\lim_{z \to x} \frac{y-f(z)}{x-z}. \tag{2.1}$$

For the reader convenience we also report here some straightforward consequences of *Theorem 4.1 and Lemma8.2* of [CoPo], which we are going to use in the follows.

Theorem 2.2 (Cowen-Pommerenke [CoPo])**.** *Let $f \in$ Hol(Δ,Δ) and let $\tau_f \in \overline{\Delta}$ be its Wolff point. Then*

1. *If either $\tau_f \in \Delta$ or $\tau_f \in \partial\Delta$ and $f'(\tau_f) < 1$ then for any $A > 1$ the set $Fix_A(f)$ is finite.*
2. *If $\tau_f \in \partial\Delta$ and $f'(\tau_f) = 1$ then for any $A > 1$ the set $Fix_A(f)$ might only accumulate at τ_f.*
3. *If f is univalent, $p \in \partial\Delta$ and $|f'(p)| < \infty$ then for any point $q \in \partial\Delta$, $q \neq p$ such that $f(q) = f(p)$ it follows $|f'(q)| = \infty$.*

Remark 2.3. If $\{w_n\} \subset \Delta$ converges to x radially, *i.e.* $\lim_{n\to\infty} \frac{1-|w_n|}{|1-w_n|} = 1$, and $f \in \mathrm{Hol}(\Delta,\Delta)$ is such that $f(x) = x$ then

$$\lim_{n \to 1} \frac{1-|f(w_n)|}{1-|w_n|} = f'(x). \tag{2.2}$$

Indeed $f'(x) \leq \liminf_{n\to 1} \frac{1-|f(w_n)|}{1-|w_n|}$ by the Theorem 2.1. Conversely

$$\frac{1-|f(w_n)|}{1-|w_n|} \leq \frac{|1-f(w_n)|}{|1-w_n|} \cdot \frac{|1-w_n|}{1-|w_n|},$$

and again by the Theorem 2.1 and the hypothesis on $\{w_n\}$ the right hand side of the above equation tends to $f'(x)$, giving the reverse inequality.

Let us indicate by $\omega(z,w)$ the *hyperbolic distance* on Δ (we refer the reader to, *e.g.*, [Ab] for the definition and the properties of ω). For $a \in \Delta$ the map $T_a(z) := \frac{a-z}{1-\overline{a}z}$ is an automorphism of Δ and $T_a(a) = 0$. Then for $z, w \in \Delta$

$$\omega(z,w) = \frac{1}{2}\log\frac{1+|T_z(w)|}{1-|T_z(w)|}.$$

By the Schwarz-Pick Lemma $\omega(f(z), f(w)) \leq \omega(z,w)$ for any $f \in \mathrm{Hol}(\Delta,\Delta)$, with equality at $z \neq w$ if and only if f is an automorphism of Δ.

Paraphrasing Poggi Corradini (see [PC2]) we define a class of sequences associated to a holomorphic self-map of Δ, which are going to be the main tool in the proof of our result.

Definition 2.4. Let $f \in \mathrm{Hol}(\Delta,\Delta)$. A sequence $\{w_n\} \subset \Delta$ is a *backward iteration sequence* if $f(w_{n+1}) = w_n$ and there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\omega(w_{n+1}, w_n) < C.$$

Remark 2.5. We want to stress out that in our definition a backward iteration sequence is a sequence whose steps are bounded in the hyperbolic metric (whereas in [PC2] this is not required).

Note that by *Lemma 2.1 in* [PC2] if $f \in \mathrm{Hol}(\Delta,\Delta)$ and $x \in \partial\Delta$ is a BRFP for f then a backward iteration sequence $\{w_n\}$ converging to x such that for any $n \in \mathbb{N}$

$$\omega(w_n, w_{n+1}) \leq \frac{1}{2}\log f'(x),$$

actually converges radially to x.

The existence of backward iteration sequences at boundary repelling fixed points is guaranteed by the following theorem.

Theorem 2.6 (Poggi Corradini [PC2])**.** *Let $f \in Hol(\Delta,\Delta)$. Suppose $x \in \partial\Delta$ is a BRFP for f. Then there exists a backward iteration sequence $\{w_n\}$ for f converging to x and such that for any $n \in \mathbb{N}$*

$$\omega(w_n, w_{n+1}) \leq \frac{1}{2}\log f'(x).$$

Before going ahead we need an insight in abstract backward iteration sequence.

Lemma 2.7. *Let $f \in Hol(\Delta,\Delta)$. Let $\{w_n\}$ be a backward iteration sequence for f. Then one and only one of the following cases is possible:*

1. *The map f has Wolff point $\tau_f \in \Delta$ and $w_n = \tau_f$ for any $n \in \mathbb{N}$.*
2. *There exists a unique point $x \in \partial\Delta$ such that $\lim_{n\to\infty} w_n = x$. Moreover the point x is a BRFP for f.*
3. *The map f has Wolff point $\tau_f \in \partial\Delta$ and the sequence $\{w_n\}$ is converging tangentially to τ_f.*

Proof. Suppose first that $\{w_n\}$ has an accumulation point $p \in \Delta$. Let $\tau_f \in \overline{\Delta}$ be the Wolff point of f. We claim that there exists a subsequence of $\{w_n\}$ which accumulates at τ_f. If $p = \tau_f$ this is tautological. Suppose then that $p \neq \tau_f$. Let

$$\mathcal{K} := \{f^m(w_n)\}_{m\leq n} = \{w_{n-m}\}_{m\leq n},$$

and

$$d := \inf_{q\in\overline{\mathcal{K}}}\{|q - \tau_f|\}.$$

Clearly τ_f is in the cluster set of $\{w_n\}$ if and only if $d = 0$. Suppose $d > 0$. Since $p \in \Delta$ and τ_f is the Wolff point of f then there exists $t \in \mathbb{N}$ such that $|f^t(p) - \tau_f| < d$. But if $w_{n_k} \to p$ then $f^t(w_{n_k}) \to f^t(p)$ and therefore $f^t(p) \in \overline{\mathcal{K}}$, against the definition of d. Therefore there exists a subsequence $\{w_l\}$ of $\{w_n\}$ which converges to τ_f. First suppose $\tau_f \in \partial\Delta$. Clearly in this case f cannot be an automorphism of Δ. Therefore, if we let E_0 be the horocycle (*i.e.* a open euclidean disc contained in Δ and tangent to τ_f at $\partial\Delta$) such that $p \in \partial E_0$, let E_1 be the horocycle such that $f(p) \in \partial E_1$ and let E_2 be the horocycle such that $f^2(p) \in \partial E_2$, then by the Wolff Lemma (see, *e.g.*, [Ab] or [Sha]) it follows

$$\overline{E_2} \subset E_1 \subset \overline{E_1} \subset E_0.$$

Hence there exist a neighborhood U of p and a neighborhood V of $f^2(p)$ such that $\overline{U} \cap \overline{E_1} = \emptyset$ and $V \subset E_1$. Since $w_{n_k} \to p$ then $w_{n_k-2} \to f^2(p)$. Therefore there exist $l_0 < l_k$ such that $w_{l_0} \in U$ and $w_{l_k} \in V$. By the Wolff Lemma $f(E_1) \subset E_1$.

But then $w_{l_0} = f^{l_k - l_0}(w_{l_k}) \in E_1$, contradiction. Hence $\tau_f \in \Delta$. Therefore for any fixed $r \in \mathbb{N}$

$$\omega(\tau_f, w_r) = \omega(f(\tau_f), f(w_{r+1})) \leq \omega(\tau_f, w_{r+1}) \leq \ldots \leq \omega(\tau_f, w_m),$$

for any $m > r$. Since $w_l \to \tau_f$ this is possible if and only if $w_n = \tau_f$ for any n.

Suppose now that the cluster set of $\{w_n\}$ is contained in $\partial\Delta$. Using the definition of ω it is easy to see (or see, *e.g.*, [Ab]) that

$$\frac{1}{2}\log f'(x) = \liminf_{z \to x}[\omega(0, z) - \omega(0, f(z))]. \tag{2.3}$$

Suppose that $\{w_{n_k}\}$ is converging to $x \in \partial\Delta$. Then

$$[\omega(0, w_{n_k}) - \omega(0, f(w_{n_k}))] \leq \omega(w_{n_k}, f(w_{n_k})) = \omega(w_{n_k}, w_{n_k - 1}) < C < \infty. \tag{2.4}$$

Therefore by Theorem 2.1 f has non-tangential limit at x, which is given by $y = \lim_{k\to\infty} f(w_{n_k})$ (this last assertion actually follows from a "metric version" of the Julia Lemma we stated in more general domains than the disc, see [Br2]) . Now

$$\omega(w_{n_k}, f(w_{n_k})) = \omega(w_{n_k}, w_{n_k - 1}) < C, \tag{2.5}$$

and therefore $\limsup_{k\to\infty} \omega(w_{n_k}, f(w_{n_k})) < C$ which is possible only if $x = y$. Hence any point in the cluster set of $\{w_n\}$ is either a BRFP for f or the Wolff point of f. Moreover if x is in the cluster set of $\{w_n\}$ then $f'(x) \leq A$ where $A := \exp(2C)$, by (2.4). Suppose that x is an isolated point in the cluster set of $\{w_n\}$. We claim that actually $\lim_{n\to\infty} w_n = x$. If not, there exists $\epsilon > 0$ and a sub-sequence $\{w_{n_l}\}$ such that if we set $U_\epsilon := \{z \in \Delta \ : |z - x| < \epsilon\}$ then $\{w_{n_l}\} \subset U_\epsilon$ but $w_{n_l+1} \notin \overline{U_\epsilon}$ for any n_l. But this contradicts (2.5). Hence either $\{w_n\}$ is converging to a unique point or the cluster set is a dense subset of an arc in $\partial\Delta$. In the last case there exist a point $x \in \mathrm{Fix}_A(f)$ and a sequence $\{x_n\} \subset \mathrm{Fix}_A(f)$ such that x is not the Wolff point of f and $x_n \to x$. This is impossible by Theorem 2.2.

Now suppose $\tau_f \in \partial\Delta$ is the limit of $\{w_n\}$. If f is an automorphism then clearly f must be a parabolic automorphism, $w_n = f^{-n}(w_0)$ for some $w_0 \in \Delta$ and $w_n \to \tau_f$ tangentially. Suppose f is not an automorphism. Transfer to the right half plane $H := \{w \in \mathbb{C} | \mathrm{Re} w > 0\}$ by means of the Cayley transform $C : \Delta \to H$ given by $C(z) := \frac{\tau_f + z}{\tau_f - z}$. Let $F := C \circ f \circ C^{-1}$ and $z_n := C \circ w_n$. Then $\{z_n\}$ is a backward iteration sequence for F and $z_n \to \infty$, the Wolff point of F. By the (right half plane) Wolff lemma $\mathrm{Re}(z_n) = \mathrm{Re}(f(z_{n+1})) > \mathrm{Re}(z_{n+1})$ and hence $\{\mathrm{Re}(z_n)\}$ is a strictly decreasing sequence and thus has a limit $r \geq 0$. Hence $\{z_n\} \subset \{w \in \mathbb{C} | \mathrm{Re}(w) > r\}$ and $\lim_{n\to\infty} \mathrm{Re}(z_n) = r$. Interpreting this in the unit disc we see that $w_n \to \tau_f$ tangentially either to $\partial E(1, \frac{1}{r})$ if $r > 0$ or to $\partial\Delta$ if $r = 0$. In both cases $w_n \to \tau_f$ tangentially. □

As the following examples show, a backward iteration sequence can converge to the Wolff point:

Example 2.8. If $f \in \mathrm{Hol}(\Delta,\Delta)$ is a parabolic automorphism then $\{f^{-n}(0)\}$ is a backward iteration sequence for f converging to its Wolff point.

Example 2.9. Let $\mathbb{H} := \{w \in \mathbb{C} : \Re e w > 0\}$. Let $V := \{x + iy \in \mathbb{H} : 1 \leq x \leq 2, y \leq 0\}$. Let $\sigma : \Delta \to \mathbb{H} - V$ be a Riemann mapping. Let $f(z) := \sigma^{-1}(\sigma(z) + i)$. Then $f \in \mathrm{Hol}(\Delta,\Delta)$ has Wolff point $\tau_f = \lim_{r\to+\infty} \sigma^{-1}(1 + ri) \in \partial\Delta$, f is not an automorphism and $w_n := \sigma^{-1}(3 - ni)$ is easily seen to be a backward iteration sequence converging to the Wolff point of f. Note however that $f'(\tau_f) = 1$.

It is not fortuitous that in the previous examples $f'(\tau_f) = 1$. Indeed the following result holds.

Proposition 2.10. *Let $f \in Hol(\Delta,\Delta)$ and let $\tau_f \in \partial\Delta$ be its Wolff point. If $f'(\tau_f) < 1$ then there are no backward iteration sequences for f converging to τ_f.*

The previous proposition was first proved by the author for *univalent mappings* in a preliminary version of this paper. After that, prof. Poggi Corradini pursued a deep study of backward iteration sequences (see [PC3]) finding a completely different proof of such a result working for general mappings. Here we present our original proof, referring the reader to [PC3] for the general case.

Proof of Proposition 2.10 for f univalent. Suppose $\{w_n\}$ is a backward iteration sequence converging to τ_f. Let $\lambda := f'(\tau_f)$. By *Theorem 3.2 in* [Co1] and subsequent remarks there exists a holomorphic map $\sigma : \Delta \to \mathbb{H} := \{w \in \mathbb{C} : \Re ew > 0\}$ such that σ is univalent on Δ and $\sigma \circ f(z) = \lambda\sigma(z)$. Let $\Phi(w) := \lambda w$ and $z_n := \sigma(w_n)$. Then it is easy to see that Φ is a hyperbolic automorphism of $\mathbb{H}$ and $\{z_n\}$ is a backward iteration sequence for Φ. Indeed

$$\Phi(z_n) = \Phi(\sigma(w_n)) = \sigma(f(w_n)) = \sigma(w_{n-1}) = z_{n-1}.$$

Therefore $z_n := \Phi^{-n}(z_0)$ (from this it follows easily that $\{z_n\}$ has bounded hyperbolic steps, but we do not need this here). Note that $\lim_{n\to\infty} z_n = \infty$ and $\lim_{n\to\infty} \Phi^n(z_0) = 0$. Now

$$\begin{aligned} \lim_{n\to\infty} \sigma^{-1}(z_n) &= \lim_{n\to\infty} w_n = \tau_f, \\ \lim_{n\to\infty} \sigma^{-1}(\Phi^n(z_0)) &= \lim_{n\to\infty} f^n(w_0) = \tau_f. \end{aligned} \tag{2.6}$$

Let $\gamma : (0,1) \to \mathbb{H}$ be a continuous curve joining 0 to ∞ and passing through z_n and $\Phi^n(z_0)$ for any n. By *Proposition 2.14 in* [Po] then $\sigma^{-1}(\gamma(t))$ has distinct limits for $t \to 0$ and $t \to 1$. But this contradicts (2.6). $\square$

Theorem 2.11 (Poggi Corradini [PC2])**.** *Let $f \in Hol(\Delta,\Delta)$. Suppose 1 is a BRFP for f, let $a := (f'(1)-1)/(f'(1)+1)$ and $\Phi(z) := (z-a)/(1-az)$. Then there exists $\sigma \in Hol(\Delta,\Delta)$ such that*

1. $K\text{-}\lim_{z\to 1} \sigma(z) = 1$,
2. $\sigma \circ \Phi(z) = f \circ \sigma(z)$ *for any* $z \in \Delta$,
3. σ *is* isogonal *at* 1, i.e. $K\text{-}\lim_{z\to 1} \arg \frac{1-\sigma(z)}{1-z} = 0$.

Moreover σ is unique up to precomposition with a Möbius transformation Ψ such that $\Psi \circ \Phi = \Phi \circ \Psi$.

We denote by (f, Φ, σ) the triple of functions as in Theorem 2.11, and call it a *pre-model for f*.

Remark 2.12. Let (f, Φ, σ) be a pre-model for f. If we set $a_k := \Phi^{-k}(r)$ for $r \in (-1,1)$ then $\{a_k\}$ is a backward iteration sequence for Φ converging to 1. Indeed a simple computation shows that Φ is a hyperbolic automorphism of Δ with Wolff point -1 and repelling fixed point 1—hence Φ^{-1} has Wolff point 1. Moreover since the automorphism $T_r(z) := (z-r)/(1-rz)$ is commuting with Φ, we have

$$\begin{aligned}\omega(a_k, a_{k+1}) &= \omega(\Phi^{-k}(r), \Phi^{-(k+1)}(r)) = \omega(r, \Phi^{-1}(r)) \\ &= \omega(T_r(r), T_r \circ \Phi^{-1}(r)) = \omega(T_r(r), \Phi^{-1} \circ T_r(r)) \\ &= \omega(0, \Phi^{-1}(0)) = \omega(0, a) = \frac{1}{2} \log f'(1),\end{aligned} \tag{2.7}$$

and therefore the steps are bounded. Let $z_k := \sigma(a_k)$. Then

$$f(z_k) = f \circ \sigma(a_k) = \sigma \circ \Phi(a_k) = \sigma(a_{k-1}) = z_{k-1},$$

and

$$\omega(z_k, z_{k+1}) = \omega(\sigma(a_k), \sigma(a_{k+1})) \leq \omega(a_k, a_{k+1}) = \frac{1}{2} \log f'(1) < \infty. \tag{2.8}$$

Therefore $\{z_k\}$ is a backward iteration sequence for f converging to 1.

What is really interesting for us is a converse of Remark 2.12:

Proposition 2.13. *Let $f \in Hol(\Delta,\Delta)$ be such that 1 is a BRFP for f, and let (f, Φ, σ) be a pre-model for f given by Theorem 2.11. Then for any backward iteration sequence $\{w_k\}$ for f converging to 1 such that for any $k \in \mathbb{N}$*

$$\omega(w_k, w_{k+1}) \leq \frac{1}{2} \log f'(1), \tag{2.9}$$

there exists a backward iteration sequence $\{z_k\}$ for Φ converging to 1 such that $\sigma(z_k) = w_k$ for any $k \in \mathbb{N}$. Moreover there exists $r \in (-1, 1)$ such that $z_k = \Phi^{-k}(r)$.

Proof. The proof of Theorem 2.11 (see [PC2]) is based on the existence of a backward iteration sequence for f—say $\{p_k\}$—such that $\omega(p_k, p_{k+1}) \leq 1/2 \log f'(1)$ for any $k \in \mathbb{N}$. Once one has such a sequence then σ turns out to be a limit of the sequence $\{f^n \circ T_n\}$ where T_n is an automorphism of Δ such that $T_n(0) = p_n$ for any n and $\sigma(a_k) = p_k$ for $a_k = \Phi^{-k}(0)$. Therefore given a backward iteration sequence $\{w_n\}$ satisfying (2.9), arguing as in the proof of Theorem 2.11, we can find another pre-model $(f, \Phi, \tilde{\sigma})$ for f at 1 such that $\tilde{\sigma}(a_k) = w_k$. By the uniqueness statement in Theorem 2.11 it follows that there exists $\Psi \in \mathrm{Aut}(\Delta)$ commuting with Φ such that $\tilde{\sigma} = \sigma \circ \Psi$. Let $z_k := \Psi(a_k) = \Phi^{-k}(\Psi(0))$. The sequence $\{z_k\}$ is a backward iteration sequence for Φ converging to 1. Indeed $z_k = \Psi(a_k) \to 1$ for Φ^{-1} is a hyperbolic automorphism with Wolff point 1. Moreover

$$\Phi(z_k) = \Phi \circ \Psi(a_k) = \Psi \circ \Phi(a_k) = \Psi(a_{k-1}) = z_{k-1},$$

and by (2.7)

$$\omega(z_k, z_{k+1}) = \omega(\Psi(a_k), \Psi(a_{k+1}) = \omega(a_k, a_{k+1})) = \frac{1}{2} \log f'(1) < \infty.$$

Also

$$\sigma(z_k) = \tilde{\sigma} \circ \Psi^{-1}(\Psi(a_k)) = \tilde{\sigma}(a_k) = w_k,$$

as wanted. $\square$

Remark 2.14. Equations (2.3), (2.8) and Remark 2.3 imply that if $x \in \partial\Delta$ is a BRFP for f and $\{w_n\}$ is a backward iteration sequence for f converging to $x \in \partial\Delta$ with $\omega(w_n, w_{n+1}) \leq 1/2 \log f'(x)$ for any $n \in \mathbb{N}$ then

$$\frac{1}{2} \log f'(x) = \lim_{n\to\infty} [\omega(0, w_n) - \omega(0, f(w_n))] \leq \lim_{n\to\infty} \omega(w_n, w_{n+1}) \leq \frac{1}{2} \log f'(x),$$

and therefore

$$\frac{1}{2}\log f'(x) = \lim_{n\to\infty} \omega(w_n, w_{n+1}).$$

Corollary 2.15. *Let $f \in Hol(\Delta,\Delta)$ and let $x \in \partial\Delta$ be a BRFP for f. If $\{w_k\}$ and $\{z_k\}$ are two backward iteration sequences for f converging to 1 such that for any $k \in \mathbb{N}$*

$$\omega(w_k, w_{k+1}) \leq \frac{1}{2}\log f'(x)$$

and

$$\omega(z_k, z_{k+1}) \leq \frac{1}{2}\log f'(x),$$

then there exists $C > 0$ such that for any $k \in \mathbb{N}$

$$\omega(w_k, z_k) \leq C.$$

Proof. Up to conjugation we can suppose that $x = 1$. Let (f, Φ, σ) be a pre-model for f as given by Theorem 2.11. By Proposition 2.13 there exist two backward iteration sequences for Φ—say $\{a_k\}$ and $\{b_k\}$—converging to 1 and such that $\sigma(a_k) = z_k$ and $\sigma(b_k) = w_k$. Therefore

$$\omega(z_k, w_k) = \omega(\sigma(a_k), \sigma(b_k)) \leq \omega(a_k, b_k). \tag{2.10}$$

Now $\Phi(a_k) = a_{k-1}$ implies that $a_k = \Phi^{-1}(a_{k-1})$. Therefore

$$\omega(a_k, b_k) = \omega(\Phi^{-k}(a_0), \Phi^{-k}(b_0)) = \omega(a_0, b_0). \tag{2.11}$$

Putting together equations (2.10) and (2.11) we get the result. □

3. Boundary repelling fixed points.

In this section we prove Theorem 1.4 and, as a corollary, Theorem 1.6. The main idea is that if $x \in \partial\Delta$ is a BRFP for f and $\{w_n\}$ is a backward iteration sequence converging to x then $\{g(w_n)\}$ is a backward iteration sequence for f, which must converge to either a BRFP for f or to the Wolff point of f. Then using the Lindelöf theorem applied to a suitable curve we show that g has non-tangential limit to such a point. The results of the previous section allow then to evaluate f' to such a point and to get the result.

Proof of Theorem 1.4. Up to conjugation we can suppose 1 is a BRFP for f. Let $\{w_n\}$ be a backward iteration sequence for f converging to 1 (the existence is assured by Theorem 2.6) such that $\omega(w_n, w_{n+1}) \leq 1/2 \log f'(1)$. Let $\tau_f \in \overline{\Delta}$ be the Wolff point of f. If $\tau_f \in \Delta$ we can choose w_0 in such a way that $g(w_0) \neq \tau_f$. Indeed let (f, Φ, σ) be a pre-model for f given by Theorem 2.11. Since σ is non-constant the image $\sigma((0, 1/2))$ is a non-trivial curve in Δ. The map g being non-constant, there exists $w_0 \in \sigma((0, 1/2))$ such that $g(w_0) \neq \tau_f$. Let $z_0 \in (0, 1/2)$ be such that $\sigma(z_0) = w_0$. The sequence

$$w_n := \sigma(\Phi^{-n}(z_0))$$

is easily seen to be a backward iteration sequence for f converging to 1—just arguing as in the proof of Proposition 2.13—and (as in (2.7))

$$\begin{aligned}\omega(w_n, w_{n+1}) &= \omega(\sigma(\Phi^{-n}(z_0)), \sigma(\Phi^{-(n+1)}(z_0))) \\ &\leq \omega(\Phi^{-n}(z_0), \Phi^{-(n+1)}(z_0)) = \frac{1}{2}\log f'(1).\end{aligned}$$

Moreover $g(w_0) \neq \tau_f$ by construction. Let

$$w_n^1 := g(w_n).$$

We claim that $\{w_n^1\}$ is a backward iteration sequence for f with steps bounded by $1/2 \log f'(1)$. Indeed

$$f(w_n^1) = f \circ g(w_n) = g \circ f(w_n) = g(w_{n-1}) = w_{n-1}^1,$$

and

$$\omega(w_n^1, w_{n+1}^1) = \omega(g(w_n), g(w_{n+1})) \leq \omega(w_n, w_{n+1}) \leq \frac{1}{2} \log f'(1).$$

By Lemma 2.7 $\{w_n^1\}$ either converges to a point $p \in \partial\Delta$ which is a BRFP for f (or the Wolff point τ_f) or $w_n^1 = \tau_f \in \Delta$ for any $n \in \mathbb{N}$. But $w_0^1 = g(w_0) \neq \tau_f$ and therefore the last possibility cannot occur. The sequence $\{w_n^1\}$ is a backward iteration sequence converging to $p \in \partial\Delta$. Moreover by (2.3)

$$\begin{aligned} \frac{1}{2} \log f'(p) &\leq \lim_{n\to\infty} [\omega(0, w_n^1) - \omega(0, f(w_n^1))] \leq \lim_{n\to\infty} \omega(w_n^1, f(w_n^1)) \\ &= \lim_{n\to\infty} \omega(g(w_n), g(f(w_n))) \leq \lim_{n\to\infty} \omega(w_n, f(w_n)) \leq \frac{1}{2} \log f'(1). \end{aligned} \tag{3.1}$$

Therefore $f'(p) \leq f'(1)$. We have to show that g has non-tangential limit p at 1. Let (f, Φ, σ) be a pre-model for f at 1 given by Theorem 2.11, such that $\sigma(\Phi^{-n}(0)) = w_n$ (this is possible by Proposition 2.13). We are going to show that

$$\lim_{t\to 1} g(\sigma(t)) = p.$$

Once we have this the Lindelöf Theorem (see, *e.g.*, [Ab]) implies that g has non-tangential limit p at 1 (recall that $\lim_{t\to 1} \sigma(t) = 1$). Now

$$[0,1] = \cup_{k\in\mathbb{N}} \Phi^{-k}([0,a]). \tag{3.2}$$

Let $\{t_m\} \subset (0,1)$ be such that $g(\sigma(t_m)) \to q \in \overline{\Delta}$. By (3.2) for any m there exist $s_m \in [0,a)$ and $k_m \in \mathbb{N}$ such that $t_m = \Phi^{-k_m}(s_m)$. Therefore

$$\begin{aligned} \omega(g(\sigma(t_m)), g(w_{k_m})) &\leq \omega(\sigma(t_m), w_{k_m}) = \omega(\sigma(\Phi^{-k_m}(s_m)), \sigma(\Phi^{-k_m}(0))) \\ &\leq \omega(\Phi^{-k_m}(s_m), \Phi^{-k_m}(0)) = \omega(s_m, 0) \leq \omega(a, 0) < \infty. \end{aligned}$$

Hence, since $\{g(w_n)\}$ is converging to p,

$$q = \lim_{m\to\infty} g(\sigma(t_m)) = \lim_{m\to\infty} g(w_{m_k}) = p.$$

This means that $\lim_{t\to 1} g(\sigma(t)) = p$ as wanted. □

Remark 3.1. Suppose f, g are as in the hypothesis of Theorem 1.4. If there are no backward iteration sequences converging to the Wolff point τ_f of f then the proof of Theorem 1.6 shows that $g(\mathrm{Fix}_A(f) - \{\tau_f\}) \subset \mathrm{Fix}_A(f) - \{\tau_f\}$.

This is always the case if either $\tau_f \in \Delta$ or $\tau_f \in \partial\Delta$ and $f'(\tau_f) < 1$, by Proposition 2.10.

Remark 3.2. Let f, g be as in the hypothesis of Theorem 1.4 and let $\tau_f \in \partial\Delta$ be the Wolff point of f. Note that if $p \in \partial\Delta$ is a BRFP for f and $g(p) = \tau_f$ then $g'(p) = \infty$. Indeed if $\{w_n\}$ is a backward iteration sequence for f given by Theorem 2.6 converging to p, then $w_n \to p$ radially, by Remark 2.5. If $|g'(p)| < \infty$ then g would be isogonal at p and hence $\{g(w_n)\}$ would be a backward iteration sequence for f converging to τ_f non-tangentially, against Lemma 2.7.

Theorem 1.6 will follow from the next result:

Lemma 3.3. *Let $f \in Hol(\Delta,\Delta)$, not the identity. Let $g \in Hol(\Delta,\Delta)$, non constant, be such that $f \circ g = g \circ f$. If there exists $l \in \mathbb{N}$ and $p \in \partial\Delta$ such that p is a BRFP of f and $g^l(p) = p$ then there exists a repelling cycle $\{p_0, \dots, p_{m-1}\}$ for g such that $m \leq l$, $p_0 = p$, the points p_j's for $j = 0, \dots, m-1$ are BRFP's for f and $f'(p_j) = f'(p)$ for $j = 0, \dots, m-1$.*

Proof. Let $p_0 := p$. By Theorem 1.4 the map g has non-tangential limit $p_1 \in \partial\Delta$ at p_0, for some BRFP p_1 of f and $f'(p_1) \leq f'(p_0)$ (the point p_1 cannot be the Wolff point of f for otherwise it would be the Wolff point of g as well and $g^l(p_1) = p_1$ for any $l \in \mathbb{N}$, or f, g would be hyperbolic automorphisms and still this would result contradictory). By induction define $p_j := g(p_{j-1})$ for $j = 1, \dots, l$ (note that $p_l = p_0$). Let $m \leq l$ be the smallest positive integer such that $g^m(p_0) = p_0$. Therefore $\{p_0, \dots, p_{m-1}\}$ are (different) BRFP's for f such that $g(p_j) = p_{j+1}$ for $j = 0, \dots, m-1$ (here $p_m = p_0$). By Theorem 1.4 we get $f'(p_j) \geq f'(p_{j+1})$ for $j = 0, \dots, m-1$, therefore $f'(p_j) = f'(p_0)$ for $j = 1, \dots, m-1$ as claimed. We want to show now that p_0 is a BRFP for g^m. Let $\{w_n\}$ be a backward iteration sequence for f at p_0 such that $\omega(w_n, w_{n+1}) \leq 1/2 \log f'(p_0)$. Then arguing as in the proof of Theorem 1.4 it follows that $\{g^m(w_n)\}$ is a backward iteration sequence for f converging to p_0 (since g^m has non-tangential limit p_0 at p_0 and, by Remark 2.3, $w_n \to p_0$ non-tangentially) and

$$\omega(g^m(w_n), g^m(w_{n+1})) \leq \omega(w_n, w_{n+1}) \leq \frac{1}{2} \log f'(p_0).$$

Then by (2.3) and Corollary 2.15

$$\frac{1}{2} \log (g^m)'(p_0) \leq \lim_{n\to\infty} [\omega(0, w_n) - \omega(0, g^m(w_n))] \leq \lim_{n\to\infty} \omega(w_n, g^m(w_n)) \leq C.$$

Therefore g^m has non-tangential limit p_0 at p_0 and $(g^m)'(p_0) < \infty$, *i.e.*, p_0 is a BRFP for g^m. By Theorem 2.1 this implies that

$$\lim_{r\to 1} \frac{1 - |g^m(rp_0)|}{1-r} = (g^m)'(p_0) < \infty.$$

Therefore there exists $K > 0$ such that for any $r \in (0,1)$

$$\begin{aligned} K > \frac{1 - |g^m(rp_0)|}{1-r} &= \frac{1 - |g^m(rp_0)|}{1 - |g(rp_0)|} \frac{1 - |g(rp_0)|}{1-r} \\ &= \frac{1 - |g^{m-1}(g(rp_0))|}{1 - |g(rp_0)|} \frac{1 - |g(rp_0)|}{1-r}. \end{aligned}$$

Since by the Schwarz Lemma the two factors in the right hand side of the above equation are bounded away from zero (see, *e.g., Lemma 1.2.4 of* [Ab]) then Theorem 2.1 implies that

$$|g'(p_0)| \leq \liminf_{r\to 1} \frac{1 - |g(rp_0)|}{1-r} < \infty.$$

By the same token and since $g(p_0) = p_1$, $|(g^{m-1})'(p_1)| < \infty$. Repeating the above argument we find that $|g'(p_j)| < \infty$ for $j = 1, \dots, m-1$ as wanted. □

Now we come to the proof of Theorem 1.6:

Proof of Theorem 1.6. By Theorem 1.4 if $q_0 \in \mathrm{Fix}_A(f)$ then g has non-tangential limit $q_1 := g(q_0)$ at q_0 and $q_1 \in \mathrm{Fix}_A(f)$. Let $q_2 = g(q_1) = g^2(q_0)$. Then $q_2 \in \mathrm{Fix}_A(f)$. Proceeding in this way, since $\mathrm{Fix}_A(f)$ is finite, we eventually either reach the Wolff point of f or find $s < t \in \mathbb{N}$, such that $q_t = q_s$.

In the last case we can choose t in such a way that $q_{s+j} \neq q_s$ for $j = 1, \ldots, t-s-1$. Let $m := t - s$ and $p_j := q_{s+j}$ for $j = 0, \ldots, m-1$. Therefore g is a permutation of $\{p_0, \ldots, p_{m-1}\}$ and the assertion comes from Lemma 3.3.

If every $p \in \mathrm{Fix}_A(f)$ is such that $g^{m_p}(p) = \tau_f$ for some $m_p \in \mathbb{N}$, since $\mathrm{Fix}_A(f)$ is finite, there exists $m \in \mathbb{N}$ such that $g^m(p) = \tau_f$ for any $p \in \mathrm{Fix}_A(f)$. Also $(g^m)'(p) = \infty$ for any $p \in \mathrm{Fix}_A(f)$ by Remark 3.2. □

4. Examples, consequences and applications

We start with the following:

Proposition 4.1. *Let $f, g \in Hol(\Delta,\Delta)$ with g non-constant and $f \circ g = g \circ f$. If the Wolff point τ_f of f is contained in Δ and there exists $A > 1$ such that $Fix_A(f)$ contains only one point, say p, then p is a BRFP for g.*

Proof. By Theorem 1.6 there exists a repelling cycle $\{p_0, \ldots, p_{m-1}\}$ for g made of elements of $\mathrm{Fix}_A(f)$. The only possibility is that $p_j \equiv p$ for $j = 0, \ldots, m-1$, *i.e.*, $m = 1$, and therefore p is a BRFP for g. □

More generally, with the same proof (see also Remarks 3.1 and 3.2):

Proposition 4.2. *Let $f, g \in Hol(\Delta,\Delta)$ with g non-constant and $f \circ g = g \circ f$. Let $\tau_f \in \overline{\Delta}$ be the Wolff point of f. Suppose that $Fix_A(f) - \{\tau_f\}$ contains only one point, say p for some $A > 1$. If either there are no backward iteration sequences for f converging to its Wolff point τ_f or $g'(p) < \infty$ then p is a BRFP for g.*

The following example shows that the condition on the non-existence of backward iteration sequences converging to the Wolff point of f cannot be deleted (see also *Example 4.12 in* [Co2]):

Example 4.3. Let f and σ be as in Example 2.9. Let $g(z) := \sigma^{-1}(\sigma(z)+2)$. Then $f \circ g = g \circ f$ and $g(\tau_f) = \tau_f$. However f fixes the point $x := \lim_{r \to -\infty} \sigma^{-1}(1/2 + ri)$ and $f'(x) < \infty$, but $g(x) = \tau_f$. Note that $g'(x) = \infty$.

Another consequence is the following

Proposition 4.4. *Let $f \in Hol(\Delta,\Delta)$ not the identity be such that its Wolff point τ_f belongs either to Δ or to $\partial\Delta$ and $f'(\tau_f) < 1$. Let $g \in Hol(\Delta,\Delta)$ be such that $f \circ g = g \circ f$ and g is non-constant. Suppose there exists $A > 1$ such that $Fix_A(f)$ is non-empty. Then there exists $m \geq 1$ such that f and g^m have (at least) m common BRFP's.*

Proof. By Theorem 2.2 $\mathrm{Fix}_A(f)$ has a finite number of points. Therefore we can apply Theorem 1.6 to get a repelling cycle $\{p_0, \ldots, p_{m-1}\}$ for g made of BRFP's for f (see Remark 3.1). Then g^m fixes each point of such a cycle. □

The positive integer m in Proposition 4.4 could be > 1 as the following example shows:

Example 4.5. Let $p_j := e^{2j\pi i/3}$ for $j = 0, 1, 2$. For $j = 0, 1, 2$ let $E_j \subset \Delta$ be the open disc of radius $1/2$ tangent to $\partial\Delta$ at p_j. Let $G := \cup_{j=0}^{3} E_j$. The region G is simply connected. Let $f : \Delta \to G$ be the Riemann mapping which fixes p_0, p_1, p_2 (see *Theorem 6 p.46* in [Go]). Let $g(z) := e^{2\pi i/3} z$. Clearly $f \circ g = g \circ f$, the map g has a repelling cycle made of $\{p_0, p_1, p_2\}$ and $g^3 = Id$.

In a sense the previous example shows that Cowen's conjecture *(2)* would not hold if g is allowed to be an automorphism of Δ.

Next we show that Theorem 1.6 does not hold if $\mathrm{Fix}_A(f)$ is infinite (and this may happen only if f has Wolff point τ_f on the boundary and $f'(\tau_f) = 1$).

Example 4.6. Let $\mathbb{H} := \{w \in \mathbb{C} : \Re e w > 0\}$. Let

$$\begin{aligned} V := &\{x + iy \in \mathbb{H} : (x - 1/2)^2 + y^2 < 1/4\} \\ &\cup \{x + iy \in \mathbb{H} : x > 1/2, -1/2 < y < 1/2\}. \end{aligned}$$

Let $\sigma : \{x + iy \in \mathbb{H} : -1/2 < y < 1/2\} \to V$ be a Riemann mapping such that $\sigma(0) = 0$. Extend σ by reflection to all of $\mathbb{H}$. Let $G := \cup_{k\in\mathbb{Z}}(V + ik)$. Therefore $\sigma : \mathbb{H} \to G$ is univalent, $\sigma(ki) = ki$ and $\sigma'(ki) = \sigma'(0) < \infty$ for any $k \in \mathbb{Z}$. Let $C : \Delta \to \mathbb{H}$ be a Riemann mapping such that $C(1) = \infty$ and set $x_k := C^{-1}(ki)$. Let $f(z) := C^{-1}(\sigma(C(z)))$ and $g(z) := C^{-1}(C(z)+i)$. By construction $f\circ g = g\circ f$. Moreover $f(x_k) = x_k$, $f'(x_k) = f'(x_0) = A < \infty$ for any $k \in \mathbb{Z}$. Also g, f have Wolff point at 1 and $f'(1) = g'(1) = 1$. Since $g(x_k) = x_{k+1}$ it follows that there is no $m \in \mathbb{N}$ such that $g^m(p) = \tau_f = 1$ for any $p \in \mathrm{Fix}_A(f)$ and g has no (finite) repelling cycle made of points of $\mathrm{Fix}_A(f)$.

An application of the previous results is about compactness of commuting composition operators. If $f \in \mathrm{Hol}(\Delta,\Delta)$ and $H^2(\Delta)$ is the Hardy space of square summable holomorphic functions, then f induces an operator $C_f : H^2(\Delta) \to H^2(\Delta)$ given by $C_f(G) := G \circ f$ for any $G \in H^2(\Delta)$ (see, *e.g.*, [Sha]). Given $g \in \mathrm{Hol}(\Delta,\Delta)$ such that $f \circ g = g \circ f$ it is clear that $C_f \circ C_g = C_g \circ C_f$. A natural question is whether C_f compact implies C_g compact. The answer is generally negative, just consider $f(z) = 1/2z$ and $g(z) = z$ and apply the *Compactness Criterion p.43 in* [Sha]. However we have the following:

Proposition 4.7. *Let $f, g \in Hol(\Delta,\Delta)$ be such that $f \circ g = g \circ f$. Suppose that $Fix_A(f)$ is non-empty for some $A > 1$. Then C_f and C_g are not compact.*

Proof. If f has Wolff point $\tau_f \in \partial\Delta$ then by Behan's Theorem (see [Be]) g has Wolff point $\tau_g \in \partial\Delta$. Since $f'(\tau_f) \le 1$, $g'(\tau_g) \le 1$ then Theorem 2.1 and the Compactness Criterion (see *Theorem p.43 in* [Sha]) imply that C_f, C_g are not compact. Suppose $\tau_f \in \Delta$. By Theorem 2.2 $\mathrm{Fix}_A(f)$ is finite. Hence by Theorem 1.6 there exists a point $p \in \mathrm{Fix}_A(f) \subset \partial\Delta$ such that $|f'(p)| < \infty$ and $|g'(p)| < \infty$. By Theorem 2.1 and *Theorem p. 43 in* [Sha] this implies that C_f and C_g are not compact. □

5. The case of univalent functions

In this section we restrict ourselves to the case f and g are two univalent holomorphic self-maps of Δ. If $f \circ g = g \circ f$ and f is univalent it is not true in general that g is also univalent. However, whenever $f'(\tau_f) < 1$, then g is univalent by *Corollary 4.9 in* [Co2]. The main point is to understand how a repelling cycle could be for univalent maps. For this we first need to introduce some notations. Let $\{p_0, \ldots, p_{m-1}\}$ be a cycle for $g \in \mathrm{Hol}(\Delta,\Delta)$. We give p_0 the number 0. Then moving counterclockwise on $\partial\Delta$ starting from p_0 we label with 1 the very next point p_{j_1} of the cycle. We proceed counterclockwise to the next point p_{j_2} of the cycle and number it with 2, and so on. After introducing this labelling we have a way of seeing g as a permutation of $\{0, \ldots, m - 1\}$. For $l, k \in \{0, \ldots, m - 1\}$ we write $g : l \mapsto k$ if $g(p_{j_l}) = p_{j_k}$.

Lemma 5.1. *Let $g \in Hol(\Delta,\Delta)$ be univalent. Let $\{p_0, \ldots, p_{m-1}\}$ be a cycle (not necessarily a repelling cycle) for g with length $m > 1$. Then there exists $k \in \{1, \ldots, m-1\}$ such that g acts on the cycle $\{p_0, \ldots, p_{m-1}\}$ as the permutation*

$$j \mapsto j + k \mod (m)$$

for $j \in \{0, \ldots, m-1\}$.

Lemma 5.1 says that a univalent map has only few ways of acting on its cycles. Moreover it is not difficult to see that the k in the Lemma cannot be a divisor of m. Taking iterates it is possible to construct examples with $k > 1$. We wish to thank C. de Fabritiis for pointing out this fact.

Proof of Lemma 5.1. If $m = 2, 3$ the assertion is clear. Suppose $m > 3$ and that the conclusion of the lemma is false. Up to relabelling the cycle we can suppose that $g : 0 \mapsto k$ and $g : 1 \mapsto k'$ with $k' \neq k + 1$. First suppose $k = 1$. Let V be the arc on $\partial\Delta$ with end points p_1 and p_2 and which does not contain p_0. By hypothesis p_1 is associated to 1 and p_2 to some $l > 2$. Therefore there exists some $p_{j_s} \in V$ with $j_s \neq 1, 2$. Let $s, r \in \{3, \ldots, m-1\}$ be respectively the maximum and the minimum integer such that $p_s, p_r \in V$ (possibly $p_s = p_r$). Let γ be the segment contained in Δ and joining p_0 to p_1 and let σ be the segment in Δ which joins p_{r-1} to p_s. By construction $\gamma \cap \sigma = \emptyset$. Now $g(\gamma)$ is a Jordan curve which joins p_1 to p_2 and whose closure disconnects V from $\partial\Delta - V$. But $g(\sigma)$ is a Jordan curve which connects $p_r \in V$ to $p_{s+1} \notin V$ and therefore $g(\gamma)$ must intersect $g(\sigma)$ somewhere in Δ, against the fact that g is univalent. Suppose now $k > 1$. We are in the situation $g : 0 \mapsto k$ but g does not map 1 to $k + 1$. Let $l \in \{0, \ldots, m-1\}$ be such that $g : 1 \mapsto k - 1$, $g : 2 \mapsto k - 2$, ..., $g : l \mapsto k - l$ and $g : l + 1 \not\mapsto k - l - 1$. Certainly $l < k$ because otherwise if $k < m - 1$ then g would map k to 0, *i.e.*, the cycle would contain $k < m - 1$ elements or, if $k = m - 1$ then $g : 0 \mapsto k - 1$ and $g : k - 1 \mapsto 0$. Then $l < k$. Note that g cannot map $l + 1$ to $k - l + 1$ since it was $k - (l - 1)$ to be mapped there. We can relabel the cycle in such a way that 0 corresponds to p_{j_l}. In this case $g : 0 \mapsto k' = k - l$, $g : 1 \mapsto k_1$ with $k_1 \neq k' - 1$ and $k_1 \neq k' + 1$. If $k' = 1$ we can proceed as before. If $k' > 1$ we argue as follows. Let q_0 be the point of the cycle in position 0 and q_{j_1} the one in position 1. Let γ be the segment in Δ joining q_0 to q_{j_1} and V the arc on $\partial\Delta$ with extremes $q_1 := g(q_0)$ and $q_{j_1+1} := g(q_{j_1})$ and which does not contain q_0. The arc V minus the extremes contains more than one point of the cycle. If $q_2 := g(q_1) \in V$ let $r \in \{3, \ldots, m-1\}$ be the maximum such that $q_r \in V - \{q_1, q_{j_1+1}\}$. Then the segment σ in Δ between q_1 and q_r is such that $\gamma \cap \sigma = \emptyset$. But $g(\gamma) \cap g(\sigma) \neq \emptyset$ for $g(\gamma)$ has extremes q_1, q_{j_1+1} and $g(\sigma)$ has extremes $q_2 \in V$ and $q_{r+1} \notin V$. This contradicts the univalency of g. If $q_2 \notin V$ then we choose $r \in \{3, \ldots, m-1\}$ to be the minimum such that $q_r \in V - \{q_1, q_{j_1+1}\}$. The segment $\sigma \subset \Delta$ which joins p_{r-1} and p_1 is such that $\sigma \cap \gamma = \emptyset$ but $g(\sigma) \cap g(\gamma) \neq \emptyset$. Again a contradiction. □

Proposition 5.2. *Let $g \in Hol(\Delta,\Delta)$ be univalent and let $\{p_0, \ldots, p_{m-1}\}$ be a cycle for g. Let τ_g be the Wolff point of g. If $m > 1$ then $Fix(g) = \{\tau_g\}$ and $\tau_g \in \Delta$.*

Proof. Suppose first $m > 2$. By Lemma 5.1 the map g can only act on the cycle as the permutation $j \mapsto j + k \mod (m)$, for some $k \in \{0, \ldots, m-1\}$ fixed. Suppose $p \in \partial\Delta$ is such that $g(p) = p$. Up to relabelling we can suppose that p_0 is the closest point of the cycle to p moving from p on $\partial\Delta$ counterclockwise. Let γ be the segment in Δ joining p to p_0 and σ the segment between p_{j_1}—the point in position 1—and p_{m-1}. Note that p_{m-1} is in position $-k$, *i.e.* in position k moving clockwise from p_0. By construction $\gamma\cap\sigma=\emptyset$. The closure of the Jordan curve $g(\gamma)$ connects p to p_1. Let V be the arc on $\partial\Delta$ with end points p and p_1 which contains p_0. The closure of the curve $g(\sigma)$ joins p_0 with p_{j_1+1}. Since $p_0\in V$ (and it is not an extreme of V) and $p_{j_1+1}\notin V$ it follows that $g(\gamma)\cap g(\sigma)\neq\emptyset$, against the fact that g is univalent. Therefore g cannot have any fixed point on $\partial\Delta$. This implies that the only fixed point of g is its Wolff point τ_g and it must belong to Δ.

We are left to consider the case $m=2$. Assume by contradiction that $\tau_g\in\partial\Delta$. Let γ_j be the segment between τ_g and p_j, $j=0,1$. Then $g(\gamma_j)$ is a curve joining τ_g to p_j. Since g is isogonal at τ_g this implies that $g(\gamma_0)\cap g(\gamma_1)\cap\Delta\neq\emptyset$, which is impossible for g is univalent. Therefore $\tau_g\in\Delta$. Suppose $p\in\partial\Delta$ is such that $g(p)=p$. Let σ be the segment between τ_g and p oriented by declaring $p-\tau_g$ a positive direction. Let γ be a smooth oriented curve in Δ such that $\tau_g\in\gamma$, $\gamma\cap\sigma=\{\tau_g\}$. We also require that p_0,p_1 are the end points of γ and γ is non-tangential to $\partial\Delta$ at p_0,p_1. Since g preserves the orientation at τ_g (for $g'(\tau_g)\neq 0$) then the curves $g(\gamma)$ and $g(\sigma)$ must intersect at τ_g with the same orientation as $\gamma\cap\sigma$. But this clearly implies that $g(\gamma)$ intersects $g(\sigma)$ at some point $q\in\Delta-\{\tau_g\}$, contradicting the fact that g is univalent. □

This is enough to prove Cowen's conjectures for univalent mappings, C^1 up to the boundary.

Proof of Theorem 1.7. If $\mathrm{Fix}(f)=\{\tau_f\}$ or f—and hence g—is a hyperbolic automorphism of Δ then the result follows from the Behan Theorem (see [Be]).

Suppose $\mathrm{Fix}(f)$ contains more than one point and f is not an automorphism. Then $\tau_f=\tau_g$ and there exists $A>1$ such that $\mathrm{Fix}_A(f)$ is non-empty. Let

$$A:=\max_{p\in\partial\Delta}|f'(p)|. \tag{5.1}$$

Certainly $1<A<\infty$. By Theorem 2.2 the set $\mathrm{Fix}_A(f)$ is finite and $\mathrm{Fix}_A(f)=\mathrm{Fix}(f)$. By Proposition 5.2, g may have only repelling cycles of length 1. Therefore if we take any $p\in\mathrm{Fix}_A(f)$ and argue as in the proof of Lemma 3.3 we eventually find a finite sequence $p,g(p),\ldots,g^m(p)$ such that $g^m(p)$ is a fixed point of g (note that if $p\neq\tau_f$ then $g^m(p)\neq\tau_f$ by Remark 3.1). We claim that $m=1$, *i.e.*, $g(p)=p$. Suppose not. Let $q_0:=g^{m-1}(p)$ and $q_1:=g^m(p)$. Then $g(q_0)=g(q_1)=q_1$ and $|g'(q_0)|<\infty$, $|g'(q_1)|<\infty$. But this contradicts the point (3) of Theorem 2.2. □

Proof of theorem 1.8. Suppose $\mathrm{Fix}(f)$ contains more than one point (otherwise the result is trivial). Define A as in equation 5.1. Then $\mathrm{Fix}_A(f)=\mathrm{Fix}(f)-\{\tau_f\}$ is a non-empty finite set, by Theorem 2.2. We partition $\mathrm{Fix}_A(f)$ giving the following equivalence relation: for $p,q\in\mathrm{Fix}_A(f)$ set $p\sim q$ if $f'(p)=f'(q)$. The number $N\in\mathbb{N}$ of equivalence classes is finite and each class contains only a finite number of elements, say $a_j\in\mathbb{N}$ for $j=1,\ldots,N$. Let

$$m:=\prod_{j=1}^N a_j!=\prod_{j=1}^N(a_j-1)(a_j-2)\cdots 1.$$

We claim that if g is as in the hypothesis then $\mathrm{Fix}(g^m)=\mathrm{Fix}(f)$. Clearly $\tau_f=\tau_g$. Let $p\in\mathrm{Fix}_A(f)$. Arguing as in the proof of Lemma 3.3 we can find a repelling cycle $\{g^l(p),\ldots,g^{l+m_p}(p)\}$ for g of length $m_p+1\geq 1$ made of BRFP's for f such that $f'(g^{l+j}(p))=f'(g^l(p))$ for any $j=0,\ldots,m_p$. If $l>0$ let $q_0:=g^{l-1}(p)$ and

$q_1 = g^{l+m_p}(p)$. Then $q_0 \neq q_1$ and $g(q_0) = g(q_1)$, which contradicts point (3) of Theorem 2.2 since $|g'(q_0)| < \infty$, $|g'(q_1)| < \infty$. Therefore $l = 0$, *i.e.*, each BRFP for f is contained in a (unique) repelling cycle for g, and such a cycle is made of BRFP's where f' attains the same value. Hence the length of each cycle divides m and so g^m fixes every BRFP for f, as wanted. □

And finally:

Proof of Proposition 1.9. In case $\tau_f \in \Delta$, then $\tau_g = \tau_f$ and there exists $p \in \partial\Delta$ BRFP for f and g. Therefore by Proposition 5.2 the map g may have only repelling cycle of length 1. The reasoning now goes exactly as in the proof of Theorem 1.7. □

Remark 5.3. The previous proofs show that actually Theorems 1.7, 1.8 and Proposition 1.9 hold for a bigger class than $\mathrm{Hol}^1_u(\Delta,\Delta)$. Namely they hold in the class of univalent maps on Δ whose derivatives are *bounded* on $\overline{\Delta}$. We thank M. Landucci for making this remark to us.

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