Plurisubharmonic functions on taut manifolds with symmetries

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Abstract

In this note we show that if a taut manifold X has a group of biholomorphisms which is not descrete, then there exists on X a non constant bounded continuous plurisubharmonic function. We also show that hyperbolc Stein manifolds with non decrete biholomorphisms groups carry non constant bounded plurisubharmonic function (which may all be discontinuous).

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1 Introduction

Loosly speaking hyperbolic manifolds tends to behave like bounded domains whereas on the other extreme we have manifolds like the total euclidean space $\mathbf{C}^n$. In this note we show that hyperbolic Stein manifolds X with a non desrete biholomorphisms groups carry non constant bounded plurisubharmonic functions. If X is not necessarly Stein but it is taut, then it carries continuous bounded non constant plurisubharmonic functions.Note however the the hypothesis on the authomorphism group is essential. In fact (as it is well known) the biholomorphism group of an hyperbolic compact manifold is finite. On the other hand we can choose a one dimentional torus T with a point p delated, such that $X = T \setminus \{p\}$ has a finite non trivial biholomorphisms group. On the other hand X is a Stein taut manifolds with no bounded plurisubharmonic functions. For the definition of hyperbolic and taut manifolds see [7].

2 Preliminaries

2.1 Local actions

In this section we give a quick exposition of the Palais theory in [8] on local actions of Lie groups and on its adaptation to the complex analytic setting given in [4]. For more details we refer to the original papers.

Let X be a complex manifold, let G be a connected Lie group,

A local action of G on X is given by a real analytic map $\Phi : \Omega \to X$ where Ω is an open neighborhood of the neutral section $e \times X$ in $G \times X$ with the following properties:

$i)$ For all $x \in X$ the open subset $\Omega_x = \{g \in G : (g,x) \in \Omega\}$ is connected
$ii)$ For fixed $g \in G$ the map $x \to \Psi(g,x) = gx$ is holomophic when defined
$iii)$For every $g, h \in G$ and $x \in X$ we have $ex = x$
and if
$(gh,x) \in \Omega,\ (h,x) \in \Omega,$ and
$(g,hx) \in \Omega,$ then
$(g,hx) = (gh,x).$

For a complex Lie group the local action is said to be holomorphic if Ψ is holomorphic.

To every local action of G we can associate a linear map ξ from the Lie algebra g of G to the algebra of holomorphic vector fields on X given by $v \to \frac{\partial exp(tv)x)}{\partial t}_{|t=0}$. This map is an homomorphism of Lie algebras. Conversely every such homomorphism induces a local action. If G is a complex Lie group, the local holomorphic G action corresponds to $\mathbf{C}$ linear homomorphisms. The "Palais leaves" of the local action are by definition the maximal integral submanifolds of the distribution $\xi(g)$ coming from the diagonal local action of G on $G \times X$ obtained from the left action of G on itself. Let us consider the space

$$X^* = G \times X \ / \ \text{ modulo leaves}$$

with the quotient topology. We call X^* the Palais globalization of the local action. Note that the leaves are invariant with respect to the diagonal local action. Moreover the right action of G on the first factor of $G \times X$ induces a global action of G on X^*. Let $\Pi : G \times X \to X^*$ the natural projection and let $j : X \to X^*$ given by $j(x) = \Pi(e,x)$. Then both Π and j are continuous open mappings, see [4]. Moreover if G is complex each leaf is a complex locally closed submanifold of $G \times X$ and the restriction to the leaf of the projection on the first factor is a locally biholomorphic map. The local action is said to be univalent if for each leaf the above local biholomorphic map is injective. In other words (since the right action of G on the first factor of $G \times X$ tranforms leaves into leaves) the action is univalent if and only if j is injective. For a univalent local action we can use the map j and its traslated maps by elements of G to define an holomorphic atlas on X^* which makes X^* a possibly non Hausodorff complex manifold. For notational convenience a non necessarly Hausdorff complex manifold will be called in this paper a quasi complex manifold. The map j becomes then an holomorphic open locally G equivariant embedding. Note that if Y is a quasi complex manifold with a global G action the map $(g,y) \to g^{-1}y$ induces a G equivariant biholomorphism of Y^* onto Y. We also have the following

Lemma 2.1 *The space X^* has the following universal property. If Y is a quasi complex manifold with an holomorphic G action and $f : X \to Y$ a locally G equivariant holomorphic map, then there exists a unique holomorphic G equivariant map $f^* : X^* \to Y$ such that $f^* \circ j = f$. Moreover $Gj(X) = X^*$.*

Proof.
see Theorem 2 of section 3 in [4]. ■

In the sequel we will identify X with $j(X) \subseteq X^*$.

Any quasi complex manifold with a G action satisfing the condition of the Lemma is called a universal globalization of X.

A local G action is univalent if and only if it admits some globalization, in such case it admits a universl globalization, i.e. X^*, in the category of quasi complex manifolds. There are however example of spaces with a local G action which admit a universal globalization in the category of Hausdorff complex manifolds which is different from X^*.

We now consider the case where there exists a global $(\mathbf{R}, +)$ action on the complex manifold X by holomorphic tranformations. This induces a local holomorphc $\mathbf{C}$ action. See [6].

Lemma 2.2 *If there exists a global action on X of the group $(\mathbf{R}, +)$, then the local $(\mathbf{C}, +)$ action is univalent.*

Proof. Let F be an $(\mathbf{R}, +)$ equivariant connected Riemann domain over $\mathbf{C}$, then the action of $(\mathbf{R}, +)$ on F is free and proper. The quotient $Fmod\, \mathbf{R}$ is then a connected non compact one dimensional real manifold and the induced map from $Fmod\, \mathbf{R}$ into $\mathbf{R}$ is a local diffeomorphism, hence it is injective. Therefore by $(\mathbf{R}, +)$ equivariance and freeness the original map is also injective. Any Palais leaf is an F as above. ■

Definition 2.3 *Let Y be a quasi complex manifold with an holomorphic $(\mathbf{C}, +)$ we say that an $\mathbf{R}$ invariant open subset X of Y is orbit connected in Y if for every $y \in Y$ the set $\{t \in \mathbf{R} : ity \in X\}$ is connected.*

Lemma 2.4 *If X is a complex manifold with an holomorphic $\mathbf{R}$ action, then X is orbit connected in its Palais complexification X^*.*

Proof.
see [1]. ■

Let X be a complex manifold with an holomorphic action of $(\mathbf{R}, +)$ and let X^* be its Palais complexification. For $x \in X^*$ let

$$\alpha(x) = inf\{t \in \mathbf{R} : itx \in X\} \text{ and let } \beta(x) = -sup\{t \in \mathbf{R} : itx \in X\}.$$

Then α and β are $\mathbf{R}$ invariant upper semicontinuous funtions on X^* with value in $[-\infty, +\infty[$. On the other hand since X is orbit connected in X^* it folllows that

$$X = \{x \in X^* : \alpha(x) < 0, \quad \beta(x) < 0\}.$$

Moreover

$$\begin{aligned} &\alpha(isx) = \alpha(x) - s \\ &\beta(isx) = \beta(x) + s \\ &\text{for } s \in \mathbf{R}\,. \end{aligned} \tag{1}$$

3 Stein and taut manifolds with an R action

It is proved in [2] that if X is Stein then α and β are plurisubharmonic, (either of them could be constantly $-\infty$).

This proves the following

Proposition 3.1 *If X is an hyperbolic Stein manifold with non descrete biholomorphisms group, then there exists on X a non constant bounded plurisubharmonic function.*

Proof.

Since the connected component of the identity of the Lie group of biholomorphisms of X is not the trivial group, there exists a non trivial holomorphic action of $(\mathbf{R}, +)$ on X. On the other hand the maps α and β relative to this action can not be both constantly $-\infty$ otherwise we would have $X = X^*$, and an hyperbolic manifold can not have not trivial holomorphic $\mathbf{C}$ actions. Assume then β is not constantly $-\infty$. By equation (1) the function β is negative and non constant on X so $exp(\beta)$ is a non constant bounded plurisubharmonic function on X. We conclude similarly if α is not constantly $-\infty$. ■

Assume now that X is taut with non discete biholomorphisms group, and let us consider a non trivial holomorphic $(\mathbf{R}, +)$ action on X. We are going to show that α and β are continuous and plurisubharmonic even when X is not Stein. This claim will prove as above the

Proposition 3.2 *If X is a taut manifold with non descrete biholomorphisms group, then there exists on X a continuous non constant bounded plurisubharmonic function.*

Proof of the claim

Consider the map $F : \mathbf{C} \times X \to \mathbf{C} \times X^*$ defined by $F(z, x) = (z, \Pi(z, x))$, where $\Pi : \mathbf{C} \times X \to X^*$ is the natural projection. Then F is holomorphic and injective. Moreover the set $\Omega = F(\mathbf{C} \times X)$ is given by $\Omega = \{(z, p) : zp \in X\}$. So Ω is open and $F^{-1} : \Omega \to \mathbf{C} \times X$ is the map $F^{-1}(z, p) = (z, zp)$. Therefore F is a holomorphic embedding and the diagonal action of $\mathbf{R}$ on $\mathbf{C} \times X$ trasformes via F in the left multiplication on the first factor in $\mathbf{C} \times X^*$. Moreover the leaves of the Palais foliation transform via F in the fibers of the restriction to Ω of the projection P_2 on the second factor. Note that the Palais complexification of Ω is $\mathbf{C} \times X^*$. If α, β are the functions associated as above to the pair (X, X^*) then the maps α_Ω and β_Ω associated to the pair $(\Omega, \mathbf{C} \times X^*)$ are given by $\alpha_\Omega(z, x) = \alpha(x) - Imz$ and $\beta_\Omega(z, x) = \beta(x) + Imz$. Let U be any open subset of X^* which is biholomorphic to a ball in some $\mathbf{C}^N$. Assume $\beta < n$ on U. Let $\Omega_n = (\{Imz > -n\} \times U) \bigcap \Omega$. So

$$\Omega_n = (\{Imz > -n\} \times U) \bigcap F(\{Imz > -n\} \times X).$$

Then Ω_n is a taut open set in $\mathbf{C}^{N+1}$, hence it is Stein, see [7]. The Palais complexification of Ω_n is $\mathbf{C} \times U$ moreover we have $\alpha_{\Omega_n} = max(-n, \alpha(x)) - Imz$ and $\beta_{\Omega_n} = \beta(x) + Imz$. So that $max(-n, \alpha(x))$ and $\beta(x)$ are psurisubharmonic on U.

Letting n go to infinity we see that α and β are plurisubharmonic on U. Let us prove continuity.

By formula (1) it is suffcient to prove the continuity of α and β on X. It is even sufficient to prove the following claim :

If $x \in X$ and x_n converges to x in such a way that $\alpha(x_n)$ converges to l_1 and $\beta(x_n)$ converges to l_2 then $l_1 = \alpha(x)$ and $l_2 = \beta(x)$.

Note that by upper semicontinuity we allways have $-\infty \leq l_1 \leq \alpha(x) < 0$ and $-\infty \leq l_2 \leq \beta(x) < 0$.

Let $l_1(n) = l_1 + 1/n$ for finite l_1 and $l_1(n) = -n$ for $l_1 = -\infty$, and let us define $l_2(n)$ similarly. For n large enough we have $l_1 < l_1(n) < 0 < -l_2(n) < -l_2$. For fixed large n the sequence of holomorphic functions $f_k(z) = zx_k$ maps for large k, the domain $\{l_1(n) < Imz < -l_2(n)\}$ into X.

By assurption $f_k(0)$ converges to x in X. Since X is taut we can, up to subsequences, assume that f_k converges uniformly on the compact sets of X to an holomorphic map g with values in X. On the other hand it is proved in [5] that for X taut the space X^* is Hausdorff, hence $g(z) = zx$.

We have therefore proved that if $l_1(n) < Imz < -l_2(n)$ then $zx \in X$, i. e. that $\alpha(x) \leq l_1(n)$ and $\beta(x) \leq l_2(n)$. Letting n go to infinity and using upper semicontinuity of α and β we find $\alpha(x) = l_1$ and $\beta(x) = l_2$. So that the claim is proved.

Let us now consider the function $\alpha + \beta$ which for X Stein or taut is a negative $\mathbf{C}$ invariant plurisubharmonic function on X^*. If $\alpha + \beta \equiv -\infty$ then, since α and β are plurisubharmonic, then either α or β ara constantly $-\infty$. Assume for example $\alpha = -\infty$, then X is defined in X^* by the condition $\beta < 0$. In this case X^* is Hausdorff, in fact $\{\beta < n\} = inX$ is Hausdorff for every positive integer n, and so is X^*. A more interesting case is when $\alpha + \beta$ is a finite negative constant.

Proposition 3.3 *Assume that X is a Stein or taut with an holomorphic action of $(\mathbf{R}, +)$ such that $\alpha+\beta$ is a negative constant $-c$. Assume moreover that $H^1(X, \mathbf{R}) = 0$. Then there exists a complex manifold S such that X is $\mathbf{R}$ equivariantly biholomorphic to the product $\{-c < Imz < 0\} \times S$ with the action of $(\mathbf{R}, +)$ by translations on the first factor.*

Proof.

Since $\beta = -c - \alpha$ then α and β are continuous, actually they are pluriharmonic functions. Let us first show that X^* is Hausdorff. Let x_1 and x_2 be two points with no disjoint neighborhoods; by acting with $\mathbf{C}$ we can assume that $x_1 \in X$, so $\alpha(x_1) < 0$ and $\beta(x_1) < 0$. Since α and β are continuous $\alpha(x_2) < 0$ and $\beta(x_2) < 0$. So $x_2 \in X$ and since X is Hausdorff $x_1 = x_2$. Since $H^1(X, \mathbf{R}) = 0$ there exists an holomorphic function $f : X \to \mathbf{C}$ such that $Imf = \beta$. Fix $x_0 \in X$, by formula (1) the holomorphic function $f(zx_0) - f(x_0) - z$ on the strip $\{\alpha(x_0) < Imz < -\beta(x_0)\}$ is real and vanishes at $z = 0$. Therefore f is locally $\mathbf{C}$ equivariant. It follows that f can be extended to an holomorphic $\mathbf{C}$ equivariant map, which we keep calling f, from X^* into $\mathbf{C}$. Since it is $\mathbf{C}$ equivariant, then f is onto and with no critical points. Let $S = f^{-1}(0)$. The map $(z, x) \to zx$ from $\mathbf{C} \times S$ into X^* is a $\mathbf{C}$ equivariant biholomorhism with inverse $x \to (f(x), -f(x)x)$. Now X is defined inside X^* by the inequalities $\alpha < 0$ and $\beta < 0$ i.e. by the inequalities $-c < Imf < 0$. The proof is complete. ■

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