EVERSIONS OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^n$

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Plan

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Introduction

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Part I : Geometry of Bounded Convex Domains

I.1 Preliminaries.

I.1.1 Basic notation.

The standard complex structure of $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$, given by the multiplication with $i = \sqrt{-1}$ is denoted by J_o. For any two elements $v, w \in \mathbb{C}^{n+1}$, $< v, w >= \sum_i = v^i \overline{w^i}$ is the standard Hermitian product and $|v| = \sqrt{< v, v >}$.

If (M, J) is a complex manifold endowed with the complex structure J, we denote by $T^{10}M$ and $T^{01}M$ the holomorphic and anti-holomorphic tangent spaces, i.e. the holomorphic submanifolds of $T^{\mathbb{C}}M$ generated by the subspaces

$$T^{10}_x M = \{\ v \in T^{\mathbb{C}}_x M\ :\ Jv = iv\ \} =$$

$$= \{\ v \in T^{\mathbb{C}}_x M\ :\ v = V - iJV \text{ for some } V \in T_x M\ \}\ ,$$

$$T^{01}_x M = \{\ v \in T^{\mathbb{C}}_x M\ :\ Jv = -iv\ \} =$$

$$= \{\ v \in T^{\mathbb{C}}_x M\ :\ v = V + iJV \text{ for some } V \in T_x M\ \}\ .$$

Similarly $T^{*10}(M)$ and $T^{*01}(M)$ are the holomorphic subbundle of $T^{*\mathbb{C}}M$ generated by the holomorphic and anti-holomorphic 1-forms, respectively. Notice that any 1-form $\theta \in T_x M \subset T^*M$ is naturally identified with an element in $T^{\mathbb{C}}_x M$ and hence admits a unique decomposition into $\theta = \theta^{10} + \theta^{01}$, with $\theta^{10} \in T^{*10}_x M$ and $\theta^{01} \in T^{*01}_x M$, respectively.

For any real valued function $\rho : \mathcal{U} \subset M \to \mathbb{R}$ defined on an open set $\mathcal{U} \subset M$, we denote by $d\rho$, $d^{10}\rho$ and $d^{01}\rho$ the differential of ρ and the component of $d\rho$ into $T^{*10}(M)$ and $T^{*01}(M)$, respectively, so that

$$d\rho_x = d^{10}\rho_x + d^{01}\rho_x$$

at any point. Using a set of complex coordinates $(z_0, \dots z_n)$ on $\mathcal{U}$, $d^{10}\rho$ and $d^{01}\rho$ assume the expressions

$$d^{10}\rho = \frac{\partial \rho}{\partial z^i} dz^i \qquad d^{01}\rho = \frac{\partial \rho}{\partial \overline{z^i}} d\overline{z^i}\ .$$

Let $\imath : S \subset M$ be the natural embedding of a real hypersurface $S \subset M$ of the complex manifold (M, J). Then we denote by $\mathcal{D} \subset TS$ the distribution defined by

$$\mathcal{D}_x = \{\ v \in T_x S\ : Jv \in T_x S\ \} = T_x S \cap J(T_x S)$$

where we identified T_xS with its image $\imath_*(T_xS) \subset T_{\imath(x)}M$. We also denote by $\mathcal{D}^{10} \subset T^{\mathbb{C}}S$ and $\mathcal{D}^{01} \subset T^{\mathbb{C}}S$ the so-called *holomorphic and anti-holomorphic distribution of* S, that is the distributions defined by

$$\mathcal{D}^{10}_x = T^{\mathbb{C}}_x S \cap T^{10}M \ , \qquad \mathcal{D}^{01}_x = T^{\mathbb{C}}_x S \cap T^{01}M \ .$$

Recall that each subspace $\mathcal{D}^{10}_x$ can be also naturally identified with the subspace

$$\mathcal{D}^{10}_x = \{ \ v \in \mathcal{D}^{\mathbb{C}}_x \ : \ v = V - iJV \text{ for some } V \in \mathcal{D}_x \ \} \ .$$

For any vector $V \in TM$ and $v \in T^{\mathbb{C}}M$, we denote by V^{10} and $\mathrm{Re}(v)$ the vectors in $T^{\mathbb{C}}M$ and TM, respectively, defined by

$$V^{10} = \frac{1}{2}(V - iJV) \ , \qquad \mathrm{Re}(v) = \frac{v + \bar{v}}{2} \ .$$

Similarly we define $V^{01} = \overline{V^{10}}$ and $\mathrm{Im}(v) = -J\mathrm{Re}(v) = -i\frac{v-\bar{v}}{2}$, so that

$$V = V^{10} + V^{01} \ , \quad v = \mathrm{Re}(v) + i\mathrm{Im}(v) \ , \qquad \mathrm{Re}(V^{10}) = \frac{V}{2} \ , \quad \mathrm{Re}(v)^{10} = \frac{v}{2} \ .$$

Using this notation, it is clear that for any real hypersurface $S \subset M$

$$\mathcal{D}^{10} = \{ \ v \in T^{\mathbb{C}}S \ : \ v = V^{10} \ \text{, for some } V \in \mathcal{D}\} \ , \qquad \mathcal{D} = \mathrm{Re}(\mathcal{D}^{10}) \ .$$

A (local) 1-form θ in T^*S will be called *(local) defining form for* $\mathcal{D}$ if $\mathrm{Ker}\theta_x = \mathcal{D}_x$ for any x on which θ is defined. Similarly, a (local) $\mathbb{C}$-valued 1-form Θ in $T^{\mathbb{C}}S \cap T^{10}M$ is called *(local) holomorphic defining form for* $\mathcal{D}^{10}$ if $\mathrm{Ker}\Theta_x \cap T^{10}M = \mathcal{D}^{10}_x$.

Note that if θ and θ' are two defining 1-forms then they are of the form $\theta' = \lambda \cdot \theta$ for some never vanishing smooth function λ. A similar results holds for two holomorphic defining 1-forms.

Let $S \subset M$ be a real hypersurface and let $\mathcal{U} \subset M$ an open subset with a non-empty intersection with S. A *defining function for* S *on* $\mathcal{U}$ is any smooth function $\rho : \mathcal{U} \to \mathbb{R}$ such that

$$S \cap \mathcal{U} = \{\rho = 0\} \qquad \text{and} \qquad d\rho_x \neq 0 \text{ for all } x \in S \ .$$

If ρ is a defining function for S, we call *defining 1-form for* $\mathcal{D} \subset TS$ *associated with* ρ the form θ^ρ defined by

$$\theta^\rho(V) = d\rho(JV) \qquad \text{for all } V \in T(S \cap \mathcal{U}) \ .$$

It is clear that $\mathrm{Ker}\theta^\rho = \mathcal{D} \cap (T(S \cap \mathcal{U}))$.

We call *holomorphic defining 1-form for* $\mathcal{D}^{10} \subset TS$ *associated with* ρ the $\mathbb{C}$-valued 1-form Θ^ρ defined by

$$\Theta^\rho = \theta^{\rho 10} = \theta^\rho + iJ\theta^\rho = -id^{10}\rho$$

It is clear that $\mathrm{Ker}\Theta^\rho = \mathcal{D}^{10} \cap T^{\mathbb{C}}(S \cap \mathcal{U})$ and that $\theta^\rho = 2\mathrm{Re}(\Theta^\rho)$.

In a system of complex coordinates ($z^i = x^i + iy^i$) we have that

$$\theta^\rho = \left.\frac{\partial\rho}{\partial x^i}\right|_S dy^i - \left.\frac{\partial\rho}{\partial y^i}\right|_S dx^i \ , \qquad \Theta^\rho = -i\left.\frac{\partial\rho}{\partial z^i}\right|_S dz^i \ .$$

Let θ be a defining 1-form for $\mathcal{D}$. Then the *Levi form of* $\mathcal{D}$ *associated with* θ is the J-invariant bilinear form in $S^2(\mathcal{D}^*)$ defined by

$$\mathcal{L}^\theta_x \ : \ \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R} \ , \qquad \mathcal{L}^\theta_x(X,Y) = d\theta_x(X,JY)$$

Notice that if θ and θ' are two defining 1-forms, then $\mathcal{L}^\theta_x \in \mathbb{R}^* \mathcal{L}^{\theta'}_x$ for any $x \in S$.

If Θ is a holomorphic defining 1-form corresponding to $\theta = \mathrm{Re}(\Theta)$, we call *Levi form of* $\mathcal{D}^\mathbb{C}$ *associated with* Θ the Hermitian form on $\mathcal{D}^\mathbb{C}$ defined by

$$\mathcal{L}^\Theta_x \ : \ \mathcal{D}^{10}_x \times \mathcal{D}^{10}_x \to \mathbb{C} \ ,$$

$$\mathcal{L}^\Theta_x(\frac{X-iJX}{2}, \frac{Y-iJY}{2}) = 2\mathcal{L}^\theta_x(\frac{X-iJX}{2}, \frac{Y+iJY}{2}) =$$
$$= -i2d\theta_x(\frac{X-iJX}{2}, \frac{Y+iJY}{2}) = \mathcal{L}^\theta_x(X,Y) + i\mathcal{L}^\theta_x(X,JY) \ .$$

Observe that for any $X \in \mathcal{D}_x$,

$$\mathcal{L}^\theta_x(X,X) = \mathcal{L}^\Theta_x(X^{10}, X^{10}) \ ,$$

and hence $\mathcal{L}^\theta$ and $\mathcal{L}^\Theta$ determine the same quadratic form on $\mathcal{D}$.

If θ^ρ and Θ^ρ are associated with some defining function ρ for S, the Levi forms $\mathcal{L}^\Theta$ can be evaluated as follows. Let $\frac{X-iJX}{2}, \frac{Y-iJY}{2} \in \mathcal{D}^{10}_x \subset T^{10}M$; with no loss of generality we may assume that they are the values at x of two commuting holomorphic vector fields defined on a neighborhood $\mathcal{U} \in M$. If we write $\frac{X-iJX}{2} = v^i\frac{\partial}{\partial z^i}$ and $\frac{Y-iJY}{2} = w^i\frac{\partial}{\partial z^i}$, then we have that

$$\mathcal{L}^\Theta(\frac{X-iJX}{2}, \frac{Y-iJY}{2}) = -i(X-iJX)\left[(J(Y+iJY))(\rho)\right] =$$
$$= -(X-iJX)([(Y+iJY)(\rho)] = -4\frac{\partial^2\rho}{\partial z^i\partial\overline{z^j}}v^i\overline{w^i}$$

The distribution $\mathcal{D}$ (and the hypersurface S) is called *Levi non-degenerate* if $\mathcal{L}^\theta$ (or $\mathcal{L}^\Theta$) is of maximal rank at all points. Note that the definition is independent on the choice of the defining form θ.

I.1.2 Extremal and critical discs of a bounded domain.

Δ_r denotes the disc of radius r in $\mathbb{C}$ and centered in the origin:

$$\Delta_r = \{z \in \mathbb{C} \ : \ |z| < r\} \ .$$

Δ denotes the disc of radius 1.

For any $\zeta = e^{i\phi} \in \partial\Delta$, we denote by n_ζ and t_ζ the unit normal vector and the unit tangent vector to $\partial\Delta$ at ζ, i.e.

$$n_\zeta = \cos\phi \left.\frac{\partial}{\partial x}\right|_\zeta + \sin\phi \left.\frac{\partial}{\partial y}\right|_\zeta , \qquad n^{10}_\zeta = \zeta \left.\frac{\partial}{\partial \zeta}\right|_\zeta , \tag{I.1.1}$$

$$t_\zeta = J n_\zeta , \qquad t^{10}_\zeta = i\zeta \left.\frac{\partial}{\partial \zeta}\right|_\zeta . \tag{I.1.2}$$

With $D \subset \mathbb{C}^{n+1}$, we will always denote a strongly pseudoconvex, bounded domain with smooth boundary. The boundary ∂D will be also denoted by $S = \partial D$. It is a smooth Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$.

For any holomorphic map $f : \bar\Delta \to \mathbb{C}^{n+1}$ we denote by

$$f'(\zeta) = (f'_0(\zeta), \dots, f'_n(\zeta))$$

the usual complex derivative of f. Note that under the natural identification $\mathbb{C}^{n+1} = T_{f(\zeta)}\mathbb{C}^{n+1}$, $f'(\zeta)$ is equal to

$$f'(\zeta) = f_*(\left.\frac{\partial}{\partial\zeta}\right|_\zeta) \in f_*(T^{10}\mathbb{C}) \subset T^{10}\mathbb{C}^{n+1} \tag{I.1.3}$$

More generally, for any holomorphic map $f : \bar\Delta \to (M, J)$ into a complex manifold (M, J) we will use the symbol $f'(\zeta)$ for the vector

$$f'(\zeta) = f_*(\left.\frac{\partial}{\partial\zeta}\right|_\zeta) \in f_*(T^{10}\mathbb{C}) \subset T^{10}_{f(\zeta)}M \tag{I.1.4}$$

We call *disc attached to* $S = \partial D$ any holomorphic embedding $f : \bar\Delta \longrightarrow \mathbb{C}^n$ such that $f(\partial\Delta) \subset S$.

We call *disc properly embedded in* D any holomorphic embedding $f : \bar\Delta \longrightarrow \mathbb{C}^n$ such that

a) $f(\Delta) \subset D$;
b) f is attached to $S = \partial D$;
c) $f'(\zeta) \notin \mathcal{D}^{10}_{f(\zeta)} \subset T^{\mathbb{C}}S$ (or equivalently $f_*(n_\zeta) \notin \mathcal{D}_{f(\zeta)}$).

The following Lemma shows that, in order to be properly embedded in a strongly pseudoconvex domain D, only a) and b) have to be satisfied.

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Lemma 1.1. *(see e.g.* [Le4]*) Let $f : \Delta \to \bar D$ be a disc which is contained in a strongly pseudo-convex domain D and it is attached to $S = \partial D$. Then f is properly embedded and $f_*(n_\zeta) \neq TS$ for any $\zeta \in \partial\Delta$.*

Proof. We have to show that, for a suitable choice of the defining function ρ of S, we have that $d\rho(f_*(n_\zeta)) > 0$ for any $\zeta \in \partial\Delta$. Since D is strongly pseudo-convex we may assume that ρ is strictly plurisubharmonic and smooth in a neighborhood of $\bar D$ and for all point $x \in D$, $\rho(x) < 0$. Then the function $u(\zeta) = \rho(f(\zeta))$ is subharmonic on $\bar{\Delta}$ and attains its maximum on all points of $\partial\Delta$. Then, by Hopf lemma, we have that $d\rho(f_*(n_\zeta)) > 0$ for any $\zeta \in \partial\Delta$. $\square$

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Let us denote by $T^{*10}(S)$ the pull-back via the natural immersion $\imath : S \to \mathbb{C}^{n+1}$ of the holomorphic bundle $\pi : T^{*10}\mathbb{C}^{n+1}$, so that the following diagram commutes

$$\begin{array}{ccc} T^{*10}S & \xrightarrow[\tilde{\imath}]{} & T^{*10}\mathbb{C}^{n+1} \\ \pi\downarrow & & \downarrow\pi \\ S & \xrightarrow{\imath} & \mathbb{C}^{n+1} \end{array}$$

Let also $E^{10}(S) \subset T^{*10}S$ be the subbundle constituted by all the holomorphic defining 1-forms Θ for the distribution $\mathcal{D}$.

Note that any properly embedded disc $f : \bar{\Delta} \to \bar{D}$ determines an embedding

$$\tilde{f} : \partial\Delta \to E^{10}(S)$$

$$\tilde{f}(e^{i\theta}) = \Theta^f_{f(e^{i\theta})} \tag{I.1.5}$$

where $\Theta^f_{f(e^{i\theta})}$ is the unique element in $\pi^{-1}(f(e^{i\theta})) \in E^{10}(S)$ such that

$$\Theta^f_{f(e^{i\phi})}(ie^{i\phi}f'(e^{i\phi})) = \Theta^f_{e^{i\phi}}(f_*(t^{10})) = 1 \tag{I.1.6}$$

Note that $\Theta^f_{f(e^{i\phi})}$ can be defined as the holomorphic part of the unique defining 1-form $\theta^f_{f(e^{i\phi})}$ such that $\theta^f_{f(e^{i\phi})}(f_*(t)) = 1$.

¿From definitions, if ρ is a defining function for S, one can immediately prove that for any $\zeta = e^{i\phi}$ we have that

$$\Theta^f(\zeta) = \frac{\zeta}{q(\zeta)} \left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)} dz^i \tag{I.1.7}$$

where

$$q(\zeta) = \zeta \left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)} \left.\frac{\partial f^i}{\partial\zeta}\right|_{\zeta} . \tag{I.1.8}$$

The definition of Θ^f is clearly independent on ρ; this implies that (I.1.7) gives always the same holomorphic 1-form, independently on the defining function ρ (see also [Le4]).

Definition 1.2. A disc $f : \Delta \to D$ is called *extremal disc* if for any holomorphic map $g : \Delta \to D$ such that

$$g(0) = f(0) , \qquad g'(0) \in \mathbb{C}f'(0) ,$$

then $|g'(0)| \leq |f'(0)|$.

A disc $f : \Delta \to D$ is called *critical (or stationary) disc* if

(1) it extends smoothly to $\bar{\Delta}$ to give a properly embedded disc in D;
(2) there exists an holomorphic embedding $\tilde{f} : \bar{\Delta} \longrightarrow T^{*10}\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ such that $\tilde{f}|_{\partial\Delta}$ coincides with the map defined in (I.1.5)

Note that if the embedding $\tilde{f}$ exists, then it is unique!!!.

In case f is critical, we call $\tilde{f}$ the *lift of f in $T^{*10}D$*. Note that in [Pa] $\tilde{f}$ is called *dual map of f (see* [Pa], *Def. 2.10).*

Theorem 1.3. [Le] *If $D \subset \mathbb{C}^n$ is strictly convex*[1], *then a disc $f : \Delta \to D$ is extremal if and only if it is critical.*

Definition 1.4. Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc and $\tilde{f}$ the corresponding lift in $T^{*10}D$. We call *inner extension of the CR structure along f* the subbundle

$$\pi : \tilde{\mathcal{D}} \subset f^{-1}(T\mathbb{C}^{n+1}) \to \bar{\Delta}$$

$$\tilde{\mathcal{D}} = \bigcup_{\zeta \in \bar{\Delta}} \tilde{\mathcal{D}}_\zeta \ , \qquad \tilde{\mathcal{D}}_\zeta = \{ \ V \in T_{f(\zeta)}\mathbb{C}^{n+1} \ : V^{10} \in \mathrm{Ker} \tilde{f}_{f(\zeta)} \ \} \subset T_{f(\zeta)}\mathbb{C}^{n+1} \ .$$

We also denote by $\tilde{\mathcal{D}}^{10}$ the corresponding holomorphic bundle

$$\pi : \tilde{\mathcal{D}}^{10} \subset f^{-1}(T^{10}\mathbb{C}^{n+1}) \to \bar{\Delta}$$

$$\tilde{\mathcal{D}}^{10} = \bigcup_{\zeta \in \bar{\Delta}} \tilde{\mathcal{D}}^{10}_\zeta \ , \qquad \tilde{\mathcal{D}}^{10}_\zeta = \{ \ v \in T^{10}_{f(\zeta)}\mathbb{C}^{n+1} \ : v \in \mathrm{Ker} \tilde{f}_{f(\zeta)} \ \} \subset T^{10}_{f(\zeta)}\mathbb{C}^{n+1} \ .$$

Note that, by definitions, $\tilde{\mathcal{D}}^{10}|_{\partial\Delta}$ coincides with the holomorphic distribution at the boundary points of the disc $f(\partial\Delta) \subset S = \partial D$.

Definition 1.5. Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc and $\tilde{f}$ the corresponding lift in $T^{*10}D$. We call *the Levi form of the embedded disc f* the 1-parameter family of Levi forms $\mathcal{L}^f_{e^{i\phi}}$ parametrized by the points $e^{i\phi} \in \partial\Delta$ defined by

$$\mathcal{L}^f_{e^{i\phi}} = \mathcal{L}^{\tilde{f}}_{f(e^{i\phi})} \in (T^{\mathbb{C}*} \otimes T^{\mathbb{C}*})(\mathbb{C}^{n+1})|_{f(\partial\Delta)} \tag{I.1.9}$$

If ρ a defining function for $S = \partial D$ on a neighborhood $\mathcal{U}$ of $f(\partial\Delta)$ and $\xi = (z^0, \dots, z^n) : \mathcal{U} \to \mathbb{C}^{n+1}$ is a system of complex coordinates on $\mathcal{U}$, we call *the holomorphic symmetric form of the embedded disc f in the coordinates ξ* the 1-parameter family of symmetric forms $\mathcal{B}^{f,\xi}_{e^{i\phi}}$ parametrized by the points $e^{i\phi} \in \partial\Delta$ defined by

$$B^{f,\xi}_{e^{i\phi}}(X, Y) = \frac{1}{q(e^{i\phi})} \ (X - iJX)(Y - iJY(\rho))|_{e^{i\phi}} \tag{I.1.10}$$

where $q(e^{i\phi})$ is defined in (I.1.8) and X and Y represent two vector fields in $\mathcal{D}$ which assume the prescribed value at $f(\zeta) \in S$.

In complex coordinates, $\mathcal{L}^f_{e^{i\phi}}$ and $\mathcal{B}^{f,\rho}_{e^{i\phi}}$ assume the following forms:

$$\mathcal{L}^f_{e^{i\phi}} = \left[\frac{\zeta}{q(\zeta)} \frac{\partial^2 \rho}{\partial z^i \partial \overline{z^j}}(f(\zeta)) \right] dz^i \otimes d\overline{z^j} \ , \tag{I.1.9'}$$

[1]see §I.1.3 for the precise definition of *strict convexity*

$$\mathcal{B}^{f,\rho}_{e^{i\phi}} = \left[\frac{\zeta}{q(\zeta)} \frac{\partial^2 \rho}{\partial z^i \partial z^j}(f(\zeta)) \right] dz^i \otimes dz^j \ . \qquad \text{(I.1.10')}$$

Notice that the Hermitian forms $\mathcal{L}^f_\zeta$, $\zeta \in \partial\Delta$, is a Levi form determined by 1-form $\tilde{f}$ and hence it is invariant under changes of complex coordinates in $\mathbb{C}^{n+1}$ (or equivalently, under biholomorphisms) and under multiplication of the defining function by a non-vanishing function.

The symmetric holomorphic forms $\mathcal{B}^{f,\xi}_{e^{i\phi}}$ is independent on the choice of the defining function ρ, in the sense that if ρ is replaced by some other defining function $\rho' = \lambda\rho$, then the value of $(X + iJX)(Y + iJY(\rho'))$ is the multiple by $\lambda_{f(\zeta)}$ of the value $(X + iJX)(Y + iJY(\rho))$; hence the normalization factor $\frac{1}{q(e^{i\phi})}$ makes $\mathcal{B}^{f,\xi}$ independent on this change of ρ. On the other hand,

$\mathcal{B}^{f,\xi}$ is not invariant under
change of coordinates (or equivalently, under biholomorphisms).

(see [Pa], Th.2.17 (3)).

We conclude this subsection with two crucial lemmata.

Lemma 1.6. *(see* [Le], [Le2], [Le3]*) Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc. Then there exists a field of holomorphic frames $(e_1, \ldots, e_n)_\zeta$, $\zeta \in \bar{\Delta}$, for the subspaces $\tilde{\mathcal{D}}^{10}_\zeta \subset T^{10}_{f(\zeta)}\mathbb{C}^{n+1}$ such that*

$$\mathcal{L}^f_{e^{i\phi}}(e_\alpha, e_\beta) = \delta_{\alpha\beta} \ ,$$

for any $e^{i\phi} \in \partial\Delta$.

It follows that, for any $\zeta \in \bar{\Delta}$, the set of holomorphic vectors

$$(f'(\zeta), e_1(\zeta), \ldots, e_n(\zeta))$$

constitutes a holomorphic basis for $T^{10}_{f(\zeta)}\mathbb{C}^{n+1}$. We will call $(f'(\zeta), e_1(\zeta), \ldots, e_n(\zeta))$ the *unitary frame field adapted to the disc f*. We will denote by $e_0(\zeta) = f'(\zeta)$ and we will use the symbol $e_i(\zeta)$ to denote any vector of the unitary basis; we will use the symbol $e_\alpha(\zeta)$, with *an index denoted by a greek letter*, to denote the vectors which span the subspace $\tilde{\mathcal{D}}^{10}_\zeta$.

Lemma 1.7. [Pa] *Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc. Then there exists a defining function ρ of D such that for any $\zeta \in \bar{\Delta}$*

$$\tilde{f}(\zeta) = \frac{1}{\bar{\zeta}} \frac{\partial \rho}{\partial z^i}(f(\zeta)) dz^i \ . \qquad \text{(I.1.11)}$$

Using this defining function, we have that for any $\zeta \in \partial\Delta$ the function q defined in (I.1.8) and the lift $\tilde{f}$ verify

$$q(\zeta) = 1 \ , \qquad \forall |\zeta| = 1 \qquad \text{(I.1.12)}$$

$$\tilde{f} = \frac{\partial^2 \rho}{\partial z^i \partial \overline{z^j}}(f(\zeta)) d\overline{z^j}(f'(\zeta)) dz^i \qquad \text{(I.1.13)}$$

$$\frac{\partial \tilde{f}}{\partial \zeta} = \frac{1}{\bar{\zeta}} \frac{\partial^2 \rho}{\partial z^i \partial z^j}(f(\zeta))dz^j(f'(\zeta))dz^i \tag{I.1.14}$$

Any defining function ρ which verifies the claim of Lemma 1.7 will be called *normalized defining function.*

I.1.3 The Kobayashi metric and the Jacobi fields of a bounded convex domain.

Recall that a complex Finsler manifold (M, J, F) is a complex manifold (M, J) endowed with a complex Finsler metric F, which is a continuous function

$$F \colon TM \longrightarrow \mathbb{R}^+$$

which is smooth on $TM \setminus \{\text{zero section}\}$ and verifies the following two properties:

a) $F(u) > 0$ for any $u \neq 0$;
b) $F(\lambda u) = |\lambda| F(u)$ for any $u \neq 0$ and $\lambda \in \mathbb{C}^*$.

The complex Finsler metric F is called *strongly pseudoconvex* if any Finsler pseudosphere at a point x

$$S_F(x) = \{\ v \in T_xM \ : \ F(v) = 1\}$$

is strongly pseudoconvex as real hypersurface of $T_xM \simeq \mathbb{C}^n$.

In the following, we will always assume that the Finsler metric is strongly pseudoconvex.

We are interested in a special class of complex Finsler manifold, namely in the so-called Lempert manifolds.

Definition 1.8. A complex manifold (M, J) is a *Lempert manifold* if

(1) the infinitesimal Kobayashi pseudo-distance k_M is a strongly pseudoconvex Finsler metric;
(2) for any non-vanishing complex vector $w \in T^{\mathbb{C}}_x M \subset T^{\mathbb{C}} M$, there exists a complex curve $\gamma_w : U \subset \mathbb{C} \to M$ from an open set $U \subset \mathbb{C}$ into M, such that $\gamma_w(0) = x$, $\gamma'_w(0) \in \mathbb{R}w$ and $\gamma(U)$ is a totally geodesic submanifold of M;
(3) the metric, which is induced by k_M on the totally geodesic complex curve $\gamma_w(U)$, is Kähler and with constant holomorphic curvature equal to -4;
(4) the (finite) Kobayashi distance d_M, determined by k_M, is complete and the exponential map $\exp : T_xM \to M$ is a diffeomorphism for any $x \in M$.

The interest for Lempert manifolds comes from the following theorem. For a given domain D with smooth boundary ∂D, we will say that it is *strictly convex domain* if at any point $y \in \partial D$, the tangent hyperplane H_y at y intersects ∂D just at the point y. We will say that D *is strongly convex* if the the normal curvatures of ∂D are positive at any point.

Theorem 1.9. *(*[Fa], [Le]*)*

(1) *For any Lempert manifold (M, J, F), the complex Finsler metric F coincides with its Kobayashi metric k_M;*
(2) *A smoothly bounded, strongly convex domain $D \subset \mathbb{C}^{n+1}$ with smooth boundary is a Lempert manifold.*

Note. In case $D \subset \mathbb{C}^{n+1}$ is smoothly bounded and strictly convex, Lempert proves that the Kobayashi metric is a complex Finsler metric with strictly convex indicatrix. In all this paper, we will not consider this case, but it could be quite interesting which of the arguments used here can be applied to the strictly convex case in place of the strongly convex case.

Let (M, J, F) be a Lempert manifold. Any totally geodesic holomorphic embedding $f : \Delta \subset \mathbb{C} \to M$ will be called *critical disc* (or *extremal disc* or *complex geodesic*). The following corollary follows quite directly from definitions.

Corollary 1.10. *Let (M, J, F) be a Lempert manifold. Then:*

(1) *for any $x \neq y \in M$, there exists a unique critical disc $f : \Delta \to M$ such that*

$$x = f(0)\ , \qquad y = f(t) \text{ for some } t \in (0, 1)\ ; \tag{I.1.15}$$

(2) *for any $x \in M$ and any $0 \neq v \in T_xM$ there exists a unique critical disc $f : \Delta \to M$ such that*

$$x = f(0)\ , \qquad v^{10} \in \mathbb{R}f'(0)\ ; \tag{I.1.16}$$

(3) *for any $x \in M$ and any $0 \neq v \in T_xM$, the unique complete geodesic γ such that $\gamma_0 = x$ and $\gamma'_0 = \frac{v}{F(v)}$ is*

$$\gamma_s = f(\tanh(\frac{s}{2})) = f(\frac{e^s - 1}{e^s + 1})\ , \tag{I.1.17}$$

where f is the unique critical disc which verifies (I.1.16) with $x = \gamma_0$ and $v = \gamma'_0$.

In the following for any pair $x \neq y \in M$ and any $0 \neq v \in T_xM$, we will denote by

$$f_{x,y} : \Delta \to M\ , \qquad\qquad f_{x,v} : \Delta \to M$$

the critical discs which verify (I.1.15) and (I.1.16), respectively.

Let $\gamma : [a, b] \to M$ be a geodesic of a Lempert manifold and let $V : (-\delta, \delta) \times [a, b] \to M$ be a smooth map such that $V(s, *) = \gamma^{(s)} : [a, b] \to M$ is a geodesic for any $s \in [-\delta, \delta]$ and with $\gamma^{(0)} = \gamma$. We call V a *1-parameter family of geodesics centered at γ*.

A vector field I on $\gamma([a, b])$ is called *Jacobi field for γ* if and only if it is of the form

$$I_{\gamma_t} = \frac{d}{ds}(V(*, t))\bigg|_{s=0}$$

for some 1-parameter family of geodesics centered at γ.

Let $f : \Delta \subset \mathbb{C} \to M$ be an extremal disc of a Lempert manifold and let $\mathcal{V} : (-\delta, \delta) \times \Delta \to M$ be a smooth map such that $\mathcal{V}(s, *) = f^{(s)} : \Delta \to M$ is an extremal disc for any $s \in [-\delta, \delta]$ and with $\gamma^{(0)} = \gamma$. We call $\mathcal{V}$ a *1-parameter family of extremal disc centered at* f.

A vector field $J : \Delta \to T^{10}M$ is called *holomorphic Jacobi field for the disc* f if and only if it is of the form

$$J_\zeta = \left[\frac{d}{ds}(\mathcal{V}(*, \zeta))\bigg|_{s=0} \right]^{10}$$

for some 1-parameter family of extremal discs centered at f.

Notice that if we consider the pull-back bundle $\pi : f^{-1}(T^{10}M) \to \Delta$ (which is a holomorphic vector bundle over Δ), then any holomorphic Jacobi field J is a section $J : \Delta \to f^{-1}(T^{10}M)$ of this bundle.

This section is indeed holomorphic. In fact, for any $\zeta\Delta$,

$$J'(\zeta) = \frac{\partial}{\partial s}\mathcal{V}'(\zeta, *)\bigg|_{s=0} \in T^{10}_{f(\zeta)}M$$

since any vector $\mathcal{V}'(\zeta, s)$ is a holomorphic vector, being $\mathcal{V}(*, s)$ a 1-parameter family of holomorphic maps.

If J is a holomorphic Jacobi field for the disc f and if $\hat{\gamma} : [a, b] \to \Delta$ is a geodesic for the disc Δ, endowed with the Poincarè metric, then the real vector field

$$I_{f(\hat{\gamma}_t)} = \mathrm{Re}\,(J \circ \hat{\gamma}) \tag{I.1.18}$$

is a Jacobi field at the points of the geodesic $\gamma = f \circ \hat{\gamma}$. Furthermore:

Lemma 1.11. *Let $\gamma : [a, b] \to M$ be a complete geodesic (parameterized by arc length) of a Lempert manifold and let $I : \gamma([a, b]) \to TM$ a Jacobi field for γ. Then there exists a unique extremal disc $f : \Delta \to M$ and a unique holomorphic Jacobi field $J : \Delta \to T^{10}M$ such that*

$$\gamma(t) = f(\tanh(t))\ , \qquad I_{\gamma_t} = \mathrm{Re}(J_{f(\tanh(t))})\ . \tag{I.1.19}$$

In particular, (I.1.18) and (I.1.19) establish a one-to-one correspondence between the Jacobi fields along a complete geodesic γ, parameterized by arc length, and the holomorphic Jacobi fields along the extremal disc f such that $\gamma(t) = f(\tanh(t))$.

Proof. Let f be the unique extremal disc such that $f(0) = \gamma_0$ and $f_*\left(\frac{\partial}{\partial x}\big|_0\right) \in \mathbb{R}\dot{\gamma}_0$. Since $f : \Delta \to D$ is a totally geodesic embedding, γ is a geodesic of the embedded disc $f(\Delta)$; since the induced metric on Δ coincides with the Poincarè metric, then $\hat{\gamma}_t = f^{-1}(\gamma_t)$ is the geodesic of Δ, , parameterized by arc length, passing through the origin and tangent to $\frac{\partial}{\partial x}\big|_0$, i.e.

$$\hat{\gamma}_t = \tanh(t)\ .$$

Consider a 1-parameter family of geodesics $\gamma^{(s)}$ centered at γ, which corresponds to the Jacobi field I. For each of such geodesic, consider an extremal disc constructed in the same way we did for the disc f associated with γ. Then the corresponding holomorphic Jacobi field verifies the claim. $\square$

Corollary 1.12. *A holomorphic Jacobi field J along an extremal disc $f : \Delta \to M$ of a Lempert manifold is uniquely determined by the values* $\mathrm{Re}(J(0))$ *and* $\mathrm{Re}(J'(0))$.

Moreover, the map

$$J \mapsto \mathrm{Re}(J'(0))$$

determines a one-to-one correspondence between

$$\mathcal{J}(f(0)) = \{\ J : \Delta \to T^{10}M\ \ : J \textit{ is a holomorphic Jacobi field for } f\ ,\ J(0) = 0\ \}$$

and

$$T_V(S_F(f(0))) \subset T_V(T_{f(0)}M)\ ,$$

where $S_F(f(0)) \subset T_{f(0)}M$ is the Finsler sphere at $f(0)$ and $V = \mathrm{Re}(f'(0))$.

Proof. By Lemma 1.11, the holomorphic Jacobi field J uniquely determined by the corresponding Jacobi field I along the complete geodesic $\gamma = f(\tanh(t))$. On the other hand, I_{γ_t} is a solution of a non-linear second order differential equation in the independent variable t, with smooth coefficients. In particular, I_{γ_t} is uniquely determined the values $I_{\gamma_0} = \mathrm{Re}(J(0))$ and $\left.\frac{dI_{\gamma_t}}{dt}\right|_{t=0} = \mathrm{Re}(J'(0))$.

For the second claim, notice that a holomorphic Jacobi field J along f is in $\mathcal{J}(f(0))$ if and only if the corresponding Jacobi field $I_{\gamma_t} = \mathrm{Re}(J(\tanh(t)))$ is equal to the first derivative at $s = 0$ of a 1-parameter family of geodesics of the form

$$\exp_{f(0)}(V(s))$$

where $V(s)$ is a curve in $S_F(f(0)) \subset T_{\gamma_0}M$ with $V(0) = \dot\gamma_0 = \mathrm{Re}(f'(0))$. For any such 1-parameter family of geodesics, one has that

$$\left.\exp_{f(0)\,*}\right|_0 (\dot V(0)) = \left.\frac{dI_{\gamma_t}}{dt}\right|_{t=0} = \mathrm{Re}(J'(0))$$

On the other hand, by a suitable modification to the Finsler case of the arguments used in Riemannian and Hermitian geometry, one can check that

$$\left.\exp_{f(0)\,*}\right|_0 = Id_{T_{f(0)}M}\ .$$

¿From this we get that

$$\mathrm{Re}(J'(0)) = \dot V(0)$$

as we needed to prove. □

I.2 Jacobi fields of bounded convex domains.

Let $D \subset \mathbb{C}^{n+1}$ be a strongly pseudoconvex bounded domain with smooth boundary which is also a Lempert domain if endowed with its Kobayashi metric. We will shortly call any such domain a *Lempert domain.*

Let $f : \Delta \to D$ be a critical disc. We want to determine for any value $V, W \in T_{f(0)}D$ the unique holomorphic Jacobi field J such that

$$\mathrm{Re}(J(0)) = V\ , \qquad \mathrm{Re}(J'(0)) = W\ .$$

First of all, observe that for any 1-parameter family $\mathcal{V}$ of extremal disc centered at f there exists a 1-parameter family $\tilde{V}$ of embedded discs in $T^{*10}\mathbb{C}^{n+1}$ defined by

$$\tilde{V}(s,\zeta) = \tilde{f}^{(s)}(\zeta)$$

where $\tilde{f}^{(s)}(\zeta)$ is the lift in $T^{*10}\mathbb{C}^{n+1}$ of the embedded disc $f^{(s)} = \mathcal{V}(s,*)$.

We call *lift in* $T^{10}(T^{*10}\mathbb{C}^{n+1})$ *of a holomorphic Jacobi field* J the holomorphic map

$$\tilde{J} : \Delta \to T^{10}(T^{*10}\mathbb{C}^{n+1})$$

$$\tilde{J}(\zeta) = \left[\frac{d}{ds}\tilde{V}(s,\zeta)\bigg|_{t=0}\right]^{10} .$$

It is clear that if $\pi : T^{*10}\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the standard projection map, then

$$\pi_*(\tilde{J}(\zeta)) = J(\zeta) ,$$

for any $\zeta \in \bar{\Delta}$.

Notice also that, since $T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}$ is a vector space, we have the natural identification

$$T^{10}_{\tilde{f}(\zeta)}(T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}) = T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}$$

and hence $\tilde{J}$ is naturally identified with a map

$$\tilde{J} : \bar{\Delta} \to T^{*10}\mathbb{C}^{n+1} , \qquad \zeta \mapsto \tilde{J}(\zeta) \in T^{*10}_{f(\zeta)}\mathbb{C}^{n+1} .$$

Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc and J a holomorphic Jacobi field. Let also ρ a normalized defining function (see §I.1.2, Lemma 1.7) and $f^{(s)}$ a 1-parameter family of extremal discs associated with J. Then, if we denote by $q_s : \bar{\Delta} \to \mathbb{C}$ the 1-parameter family of functions

$$q_s(\zeta) = \zeta \left.\frac{\partial\rho}{\partial z^i}\right|_{f^{(s)}(\zeta)} \left.\frac{\partial f^{(s)i}}{\partial \zeta}\right|_{\zeta}$$

we have that the values $\tilde{J}(\zeta)$ for $\zeta = e^{i\phi} \in \partial\Delta$ may be expressed in terms of ρ, q_s and J as follows:

$$\tilde{J}(\zeta) =$$

$$= \left[-\frac{\zeta}{q(\zeta)^2}\left.\frac{\partial q_s(\zeta)}{\partial s}\right|_0 \left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)} + \frac{\zeta\overline{J^j(\zeta)}}{q(\zeta)}\left.\frac{\partial^2\rho}{\partial z^i\partial\bar{z}^j}\right|_{f(\zeta)} + \frac{\zeta J^j(\zeta)}{q(\zeta)}\left.\frac{\partial^2\rho}{\partial z^i\partial z^j}\right|_{f(\zeta)}\right] dz^i =$$

$$= \left[-\left.\frac{\partial q_s(\zeta)}{\partial s}\right|_{s=0} \tilde{f}(\zeta) + \zeta\overline{J^j(\zeta)}\left.\frac{\partial^2\rho}{\partial z^i\partial\bar{z}^j}\right|_{f(\zeta)} + \zeta J^j(\zeta)\left.\frac{\partial^2\rho}{\partial z^i\partial z^j}\right|_{f(\zeta)}\right] dz^i$$

Consider now the following decomposition of the vectors $J(\zeta)$ as

$$J(\zeta) = J^{//}(\zeta) + J^{\perp}(\zeta) , \qquad J^{//}(\zeta) \in \mathbb{C}f'(\zeta) , \quad J^{//}(\zeta) \in \mathcal{D}^{10}_{\zeta} .$$

Such unique decomposition can be done by expressing $J(\zeta)$ in terms of an adapted unitary basis $(e_0 = f', e_1, \dots e_n)_\zeta$, as given in Lemma 1.6, and by taking as $J^\perp$ the projection into $\mathrm{span}\{e_1, \dots, e_n\}$.

Let us express $J^{//}$ as

$$J^{//}(\zeta) = \lambda^J(\zeta) f'(\zeta) \ . \tag{I.2.1}$$

Now, observe that by Lemma 1.7,

$$\overline{J^{//j}(\zeta)}\ \frac{\partial^2 \rho}{\partial z^i \partial \bar z^j}\bigg|_{f(\zeta)} dz^i = \overline{\lambda^J(\zeta)} \tilde f(\zeta) \ , \qquad J^{//j}(\zeta)\ \frac{\partial^2 \rho}{\partial z^i \partial z^j}\bigg|_{f(\zeta)} dz^i = \bar\zeta \lambda^J(\zeta)\ \frac{\partial \tilde f}{\partial \zeta}\bigg|_\zeta$$

and hence, recalling the definition of $\mathcal{L}^f$ and $\mathcal{B}^{f,\xi}$, we may write $\tilde J(\zeta)$, for any $\zeta \in \partial\Delta$ as

$$\begin{gathered}\tilde J(\zeta) = \\ = \left[- \frac{\partial q_s(\zeta)}{\partial s}\bigg|_{s=0} \tilde f(\zeta) + \zeta \overline{J^{//j}(\zeta)}\ \frac{\partial^2 \rho}{\partial z^i \partial \bar z^j}\bigg|_{f(\zeta)} + \frac{\zeta J^{//j}(\zeta)}{q(\zeta)}\ \frac{\partial^2 \rho}{\partial z^i \partial z^j}\bigg|_{f(\zeta)} \right] dz^i + \\ + \zeta \mathcal{L}^f_\zeta(*, J^\perp) + \zeta \mathcal{B}^{f,\xi}_\zeta(*, J^\perp) = \\ = \left[- \frac{\partial q_s(\zeta)}{\partial s}\bigg|_{s=0} \tilde f(\zeta) + \zeta \overline{\lambda^J(\zeta)} \right] \tilde f(\zeta) + \lambda^J(\zeta)\ \frac{\partial \tilde f}{\partial \zeta}\bigg|_\zeta + \zeta \mathcal{L}^f_\zeta(*, J^\perp) + \zeta \mathcal{B}^{f,\xi}_\zeta(*, J^\perp)\end{gathered} \tag{I.2.2}$$

Since $\tilde J : \bar\Delta \to T^{*10}\mathbb{C}^{n+1}$ is holomorphic, for any holomorphic section of $\tilde{\mathcal{D}}^{10}$ which vanishes at the origin

$$\tau_o = \zeta \cdot \tau : \bar\Delta \to \tilde{\mathcal{D}}^{10} \subset T^{10}\mathbb{C}^{n+1} \ ,$$

we have that

$$0 = \tilde J_0[\tau_o] = \frac{1}{2\pi} \int_0^{2\pi} \tilde J_{e^{i\phi}}[\tau_o] d\phi = \frac{1}{2\pi} \int_0^{2\pi} \tilde J_{e^{i\phi}}[\tau] e^{i\phi} d\phi \ . \tag{I.2.3}$$

Notice that, since the function

$$\lambda^J(\zeta)\ \frac{\partial \tilde f}{\partial \zeta}\bigg|_\zeta (\tau)$$

is holomorphic for any holomorphic function λ^J and any holomorphic section $\tau : \bar\Delta \to \tilde{\mathcal{D}}^{10}$ and since $\tilde f(\tau) \equiv 0$ for any such section τ, we have that (I.2.2) and (I.2.3) give to the following condition on the component $J^\perp$ of the holomorphic Jacobi field J:

$$0 = \frac{1}{2\pi} \int_0^{2\pi} e^{i2\phi} \left\{ \mathcal{L}^f_{e^{i\phi}}(\tau, J^\perp) + \mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, J^\perp) \right\} d\phi \tag{I.2.4}$$

for any holomorphic section $\tau : \bar\Delta \to \tilde{\mathcal{D}}^{10}$.

We may now prove the following.

Theorem 2.1. *Let $D \subset \mathbb{C}^{n+1}$ be a smoothly bounded, strongly convex domain and let $f : \bar{\Delta} \to \bar{D}$ be an extremal disc. Let $J = \lambda^J f' + J^\perp : \bar{\Delta} \to T^{10}\mathbb{C}$ be a holomorphic Jacobi field, of which $J^\perp : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ is the corresponding holomorphic section $\tilde{\mathcal{D}}^{10}$. Let also $(e_i)_\zeta$ a unitary frame field adapted to f (see Lemma 1.6) and, for any holomorphic map $H : \bar{\Delta} \to \tilde{D}^{10}$ of the form $H = H_\alpha e^\alpha$, with H_α holomorphic, let us denote by $\nabla H = H'_\alpha e^\alpha$.*

Then:

a) *for any pair of vector $v = v^\alpha e_\alpha(0)$ and $w = w^\alpha e_\alpha(0) \in \mathbb{C}^n$, let us denote by $j_{v,w} : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ the section of the form*
$$j_{v,w}(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta) \; ; \tag{I.2.5}$$
then there exists a unique holomorphic section $j^{(2)}_{v,w} : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ such that
$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\mathcal{L}^f_{e^{i\phi}}(\tau, j^{(2)}_{v,w}) + e^{i4\phi}\mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, j^{(2)}_{v,w})\right\} d\phi =$$
$$= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{e^{i2\phi}\mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, j_{v,w})\right\} d\phi \tag{I.2.6}$$
for any holomorphic section $\tau : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$;

b) *if $J^\perp(0) = v = v^\alpha e_\alpha(0)$ and $\nabla J^\perp\big|_0 = w = w^\alpha e_\alpha(0)$, then $J^\perp(\zeta)$ is equal to*
$$J^\perp(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta) + \zeta^2 j^{(2)}_{v,w}(\zeta) \; , \tag{I.2.7}$$
where $j^{(2)}_{v,w}(\zeta) : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ is the unique section which verifies (I.2.6) with $j_{v,w}(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta)$.

Proof. a) Observe that the left hand side of (I.2.6) can be rewritten as
$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\mathcal{L}^f_{e^{i\phi}}(e^{i2\phi}\tau, e^{i2\phi} j^{(2)}_{v,w}) + \mathcal{B}^{f,\xi}_{e^{i\phi}}(e^{i2\phi}\tau, e^{i2\phi} j^{(2)}_{v,w})\right\} d\phi =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} Hess(\rho)_{f(e^{i\phi})}(\mathrm{Re}(e^{i2\phi}\tau), \mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w})) d\phi \tag{I.2.8}$$
where
$$Hess(\rho)_{f(e^{i\phi})}(\mathrm{Re}(e^{i2\phi}\tau), \mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w}))$$
denotes the real Hessian of the defining function ρ evaluated at the point $f(e^{i\phi})$ along the vectors $\mathrm{Re}(e^{i2\phi}\tau)$, $\mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w}) \in T_{f(e^{i\phi})}\mathbb{C}^{n+1}$. By hypothesis of strict linear convexity, the Hessian $Hess(\rho)$ is positive definite at all tangent spaces $T_{f(e^{i\phi})}\partial D$. From this it follows that (I.2.8) is vanishing for any section τ if and only if the section $e^{i2\phi} j^{(2)}_{v,w}$ is identically equal to zero. Same conclusion holds for the section $j^{(2)}_{v,w}$ and this proves the uniqueness part. For the existence part, see [Pa].

b) It is clear that (I.2.4) holds for any holomorphic section τ if and only if
$$0 = \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{i2\phi}\left\{\mathcal{L}^f_{e^{i\phi}}(\tau, J^\perp) + \mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, J^\perp)\right\} d\phi\right\} \tag{I.2.8'}$$
for any holomorphic section τ. From the hypothesis, it follows that $J^\perp(\zeta)$ is of the form (I.2.7) for some holomorphic function $j^{(2)}_{v,w}$. Substituting (I.2.7) into (I.2.8) it follows that $j^{(2)}_{v,w}$ has to verify (I.2.6) for any τ and this concludes the proof. □

Theorem 2.2. *Let $D \subset \mathbb{C}^{n+1}$ be a smoothly bounded, strongly convex domain and let $f : \bar{\Delta} \to \bar{D}$ be an extremal disc. Let also $J = \lambda^J f' + J^\perp : \bar{\Delta} \to T^{10}\mathbb{C}$ be a holomorphic Jacobi field, of which $J^\perp : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ is the corresponding holomorphic section $\tilde{\mathcal{D}}^{10}$.*

Then if $\lambda^J(\zeta)$ is of the form

$$\lambda^J(\zeta) = \alpha + ib\zeta - \bar{\alpha}\zeta^2 \tag{I.2.9}$$

where $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}$.

Proof. Let ρ be a normalized defining form for ∂D (see Lemma 1.7) and let $f^{(s)}$ a 1-parameter family of critical discs, centered at f, associated with the Jacobi field J.

This implies that for any s we have that

$$\rho(f^{(s)}(\zeta))\Big|_{\partial\Delta} = 0 \quad \Rightarrow \quad d\rho(\mathrm{Re}(J))|_{\partial\Delta} = \mathrm{Re}\left[d^{10}\rho(J)\big|_{\partial\Delta}\right] \equiv 0 \ .$$

This can be rewritten as

$$\mathrm{Re}\left[\bar{\zeta}\tilde{f}(J)\Big|_{\partial\Delta}\right] = \mathrm{Re}\left[\bar{\zeta}\lambda^J(\zeta)\big|_{\partial\Delta}\right] \equiv 0 \tag{I.2.10}$$

By developing $\lambda^J(\zeta)$ into Fourier series, we obtain from (I.2.10) that $\lambda^J(\zeta)$ is exactly of the form (I.2.9). □

I.3 Geodesics, Busemann functions and horospheres in complex Finsler spaces.

In all this section, M will denote a complex Finsler manifold with complex Finsler metric F. We will also denote by $\mathfrak{D}$ the *distance function* of F, i.e. the continuos function $\mathfrak{D} : M \times M \to \mathbb{R}^+$ defined by

$$\mathfrak{D}(x, y) = \inf_{\gamma \in \mathcal{G}(x,y)} \int_a^b F(\dot{\gamma}_t) dt$$

where the infimum is taken over the family $\mathcal{G}(x, y)$ of all piecewise $\mathcal{C}^1$ curves $\gamma : [a, b] \subset \mathbb{R} \to M$ with $\gamma_a = x$, $\gamma_b = y$.

As it is usually meant, by *arc of geodesic* we always indicate a smooth curve $\gamma : [a, b] \subseteq \mathbb{R} \to M$, such that for any $t \in [a, b]$, there exists a $\varepsilon > 0$, so that for all $t' \in [t - \varepsilon, t + \varepsilon]$ the curve $\gamma|_{[t',t]}$ has length equal to $\mathfrak{D}(\gamma_t, \gamma_{t'})$. With the term *geodesic* we will always mean an arc of geodesic $\gamma : \mathbb{R} \to M$ defined over the whole real axis. For the main properties of geodesic and arc of geodesic in a Finsler space (or, more generally, in a metric space) see e.g. [Bu], [BC], [AP], [Sp].

In particular, we have to mention that any complex Finsler manifold (M, F), endowed with its distance function d_F is in the category of *G-spaces*, as defined by H. Busemann (see [Bu], §II.15). This fact is relevant to us mainly because it means that most of the theorems in [Bu], which hold in this class of metric space, are clearly true for any of our complex Finsler manifolds.

Now, we need to introduce a few other terms, which will be quite helpful in discussing the properties of geodesics of a complex Finsler manifold.

Given two distinct points $x, y \in M$, by *oriented segment between x and y* we mean an arc of geodesic $\gamma : [0, c] \to M$ such that $\gamma_0 = x$, $\gamma_c = y$ and $c = d_F(x, y)$. Also, by *oriented straight line*, we actually mean a geodesic $\gamma : \mathbb{R} \to M$ so that $d_F(\gamma_{t_1}, \gamma_{t_2}) = |t_1 - t_2|$ for any $t_1, t_2 \in \mathbb{R}$. Note that any oriented straight line is parameterized by arc length.

Two oriented straight lines, which differs by a shift of the parameter, will be said *equivalent*. Two non-equivalent oriented straight lines with the same image in M are said to be *with opposite orientation.*

With the expression *ray originating from the point x* we mean an arc of geodesic $\gamma : [0, +\infty) \to M$ such that $\gamma_0 = x$ and so that $\mathfrak{D}(\gamma_{t_1}, \gamma_{t_2}) = |t_1 - t_2|$ for any $t_1, t_2 \in [0, +\infty)$. Any ray or oriented straight line γ will be called *prolongation of an oriented segment* $\zeta : [0, c] \to M$ if ζ is simply the restriction of γ on the interval $[0, c]$, i.e. $\zeta = \gamma|_{[0,c]}$.

The manifold M is called *straight* if it is a complete straight space according to the definition in [Bu]), i.e. if: a) it is complete as a topological space; b) every oriented segment can be prolonged into a ray. Note however that the only straight spaces we are going to deal with are the complex Finsler manifolds M, which verify a) and the following condition b'), which clearly imply b): b') for any $x \in M$ the exponential map $\exp_x : \mathcal{U} \subset T_xM \to M$ is defined over the entire T_xM and it is a diffeomorphism from T_xM onto M, then M is straight (the definition of exponential map of a complex Finsler space is the exact analogue of the one for Riemannian manifolds).

Let $\gamma : [0, +\infty) \to M$ be a ray originating from $x = \gamma_0$. A simple argument shows that for any $y \in M$, the limit $\lim_{s\to\infty} \mathfrak{D}(y, \gamma_s) - s$ exists and it is finite. For this reason, the following function is well defined.

Definition 3.1.

i) Consider an oriented straight line $\gamma : \mathbb{R} \to M$. Then the function

$$B_\gamma : M \to \mathbb{R}\ , \qquad B_\gamma(y) = \lim_{s\to\infty} \mathfrak{D}(y, \gamma_s) - s \tag{I.3.1}$$

is called *Busemann function of the oriented straight line γ.*

ii) For any oriented straight line γ and any $c \in \mathbb{R}$, the level set

$$S(\gamma_\infty)|_c = \{\ y \in M\ :\ B_\gamma(y) = c\ \}$$

is called *horosphere of the limit point of γ passing through γ_c.*

Quite often the horospheres are called 'limit spheres' and this is because they verify the following crucial property (see [Bu], Th.22.14).

Theorem 3.2. *Let γ be an oriented straight line in M, $\{s_n\}$ a sequence in $\mathbb{R}$ tending to $+\infty$. Then for any $c \in \mathbb{R}$, if we denote by $\mathfrak{D}_n = \mathfrak{D}(\gamma_{s_n}, \gamma_c)$, we have that for any $s_n > c$, $S_{\mathfrak{D}_n}(\gamma_{s_n}) \subset \{\ x\ : B_\gamma(x) < 0\ \}$ and*

$$\lim_{n\to\infty} S_{\mathfrak{D}_n}(\gamma_{s_n}) = S(\gamma_\infty)|_c\ ,$$

i.e. $S(\gamma_\infty)|_c$ *coincides with the set of accumulation points of the sequences* $\{y_n\}$ *with points* $y_n \in S_{\mathfrak{D}_n}(\gamma_{s_n})$.

The following definition and the next proposition will be crucial for studying the horospheres in convex domains.

Definition 3.3. Let $\eta : [0, +\infty) \to M$ be a ray originating from a point $x = \eta_0$. We say that η is a *co-ray of the oriented straight line* γ if there exists a sequence $\{x_n\}$ of points of M and a sequence of real numbers s_n so that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} s_n = +\infty$ and the oriented segments η_n with endpoints x_n and γ_{s_n} converge uniformly on compacta to the ray η.

An oriented straight line σ is called *asymptotic to the oriented straight line* γ if the ray $\sigma|_{[0,+\infty)}$ is a co-ray of γ.

Proposition 3.4. *If* η *is a co-ray of the oriented straight line* γ *originating from a point* x *of* $S = S(\gamma_\infty)|_c$*, then*

$$\mathfrak{D}(\eta_t, x) = \mathfrak{D}(\eta_t, S) \tag{I.3.2}$$

for any $t \in [0, +\infty)$*. In particular, if we denote by* $\mathfrak{D}_t : M \to \mathbb{R}$ *the function* $\mathfrak{D}_t(y) = \mathfrak{D}(\eta_t, y)$ *and if the horosphere* S *is a* $\mathcal{C}^1$ *hypersurface, then*

$$T_x(S) = Ker\ d\mathfrak{D}_t|_x \tag{I.3.3}$$

then for any $t > 0$.

Proof. (I.3.2) is simply the content of claim (22.18) in [Bu]. Then (I.3.3) follows from the fact that x is a minimum point for the function $\mathfrak{D}_t|_S$ and that $\mathfrak{D}_t$ is a smooth function (this last property is a direct consequence of definition and the smoothness of the complex Finsler metric F). □

We conclude this section, recalling the following result which is a direct consequence of definition, remarks before (22.16) in [Bu] and (23.1) in [Bu].

Proposition 3.5. *Let* M *be a straight manifold. Then for any oriented straight line* γ *and any* $x \in M$*, there exists exactly one oriented straight line* σ *with* $\sigma_0 = x$ *and which is asymptotic to* γ.

I.4 Busemann functions and horospheres of strongly convex domains.

Let $D \subset \mathbb{C}^n$ be a smoothly bounded, strongly convex domain in $\mathbb{C}^n$. By Lempert's results (see Theorem 1.9 (2)), we know that the Kobayashi metric k_D makes D a Lempert domain and hence a straight complex Finsler manifold, as defined in §I.3. We are going to denote by $\mathfrak{D}$ the (finite) Kobayashi distance of D.

We want now to give an explicit expression for the Busemann functions of D and prove that any horosphere in D is a smooth hypersurface of D.

Lemma 4.1. *For any $x \in D$, let us denote by $\mathfrak{D}_x : D \to \mathbb{R}$ the distance function $\mathfrak{D}_x(y) = \mathfrak{D}(x,y)$. Then, for any $y \in D$, with $y \neq x$,*

$$d\mathfrak{D}_x|_y = 4\cosh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right) Re\left[\tilde{f}_{x,y}|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})}\right] , \tag{I.4.1}$$

where $f_{x,y}$ denotes the unique critical disc which verifies (I.1.15).

Proof. Consider a vector $v \in T_yD$ and let $\sigma : [-\varepsilon, \varepsilon] \to D$ be a smooth curve with $\sigma_0 = y$ and $\sigma'_0 = v$ and consider the 1-parameter family of critical discs

$$f_t : \Delta \to D , \qquad f_t = f_{x,\sigma_t} .$$

¿From the definition of the critical discs f_t and Corollary 1.10, it follows that $f_t\left(\tanh\left(\frac{\mathfrak{D}_x(\sigma_t)}{2}\right)\right) = \sigma_t$, for any t, and hence that

$$v^{10} = J|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})} + f'_0|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})} \cdot \frac{1}{2\cosh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right)} \cdot \left[\left.\frac{d\mathfrak{D}_x(\sigma_u)}{du}\right|_{u=0}\right]^{10} , \tag{I.4.2}$$

where we denoted by J the holomorphic Jacobi vector field for f_0 defined by

$$J = \left.\frac{\partial f_t}{\partial t}\right|_{t=0} . \tag{I.4.3}$$

Notice that, since $f_t(0) = x$ for any t,

$$J(0) = 0 \in T^{10}_x D . \tag{I.4.4}$$

Consider now the family of lifts $\tilde{f}_t : \Delta \to T^{*10}D$ and recall that by definition, $\tilde{f}_t(f'_t) \equiv 1$ for any t. Then (I.4.2) implies

$$\tilde{f}_{x,y}(v) = \tilde{f}_0(v^{10}) = \tilde{f}_0\left(J|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})}\right) + \frac{1}{2\cosh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right)} \cdot \left[\left.\frac{d\mathfrak{D}_x(\sigma_u)}{du}\right|_{u=0}\right]^{10} \tag{I.4.5}$$

(we used the fact that $v = v^{10} + v^{01}$ and that $\tilde{f}(v^{01}) = 0$).

Now, by Theorem 2.2, we can decompose any holomorphic Jacobi field J into a sum $J = \lambda f' + J^{\perp}$, where $J^{\perp}$ is a holomorphic Jacobi vector field such that $\tilde{f}_0(J^{\perp}) \equiv 0$ and λ is a function of the form

$$\lambda : \Delta \to D , \qquad \lambda(\zeta) = \alpha + ib\zeta - \bar{\alpha}\zeta^2$$

where $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}$. ¿From (I.4.4), $\alpha = 0$ and hence by (I.4.5) we get

$$\begin{aligned} d\mathfrak{D}_x(v) &= \left.\frac{d\mathfrak{D}_x(\sigma_u)}{du}\right|_{u=0} = 2\mathrm{Re}\left[\left.\frac{d\mathfrak{D}_x(\sigma_u)}{du}\right|_{u=0}\right]^{10} = \\ &= 4\cosh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right) \mathrm{Re}\left[\tilde{f}_{x,y}|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})}(v) - ib\tanh\left(\frac{\mathfrak{D}_x(y)}{2}\right)\right] = \\ &= 4\cosh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right) \mathrm{Re}\left[\tilde{f}_{x,y}|_{\tanh(\frac{\mathfrak{D}_x(y)}{2})}(v)\right] . \end{aligned}$$

From this, (I.4.1) follows immediately. □

The proof of the following corollary is quite an immediate consequence of definitions and Lemma 4.1.

Corollary 4.2. *Let $x \neq y \in D$. If $f_{x,y}$ denotes the unique critical disc which verifies (I.1.15), then:*

(1) $f_{y,x} = f_{x,y} \circ T_{x,y}$, *where $T_{x,y}$ is the biholomorphism $T_{x,y} : \Delta \to \Delta$ defined by*

$$T_{x,y}(\zeta) = -\frac{\zeta - \tanh\left(\frac{\mathfrak{D}_x(y)}{2}\right)}{1 - \tanh\left(\frac{\mathfrak{D}_x(y)}{2}\right)\zeta} \ ; \tag{I.4.6}$$

(2) $\tilde{f}_{y,x}|_\zeta = \frac{1}{T'_{x,y}(\zeta)} \cdot \tilde{f}_{x,y}|_{T_{x,y}(\zeta)} = -\frac{\left(1-\tanh\left(\frac{\mathfrak{D}_x(y)}{2}\right)\zeta\right)^2}{1-\tanh^2\left(\frac{\mathfrak{D}_x(y)}{2}\right)} \cdot \tilde{f}_{x,y}|_{T_{x,y}(\zeta)}$;

(3) *the differential $d\mathfrak{D}_x|_y$ is equal to*

$$d\mathfrak{D}_x|_y = -4Re\left[\tilde{f}_{y,x}|_0\right] \ ; \tag{I.4.1'}$$

(4) *let γ be the oriented straight line $\gamma_s = f_{y,x}\left(\tanh\left(\frac{s}{2}\right)\right)$; then the subspace*

$$Ker\ d\mathfrak{D}_{\gamma_s}|_y = Ker\ Re\left[\tilde{f}_{y,\gamma_s}|_0\right]$$

is independent on s.

Proof. (1) and (2) are trivial. (3) and (4) follow from (2) and Lemma 4.1. □

Theorem 4.3. *(e.g.* [Le],[CHL]*) For any $x \in D$ and any $y \in \partial D$, there exists a unique critical disc $f_{x,y} : \Delta \to D$, whose continuos extension $f_{x,y} : \bar{\Delta} \to \bar{D}$ is so that*

$$f_{x,y}(0) = x \ , \qquad f_{x,y}(1) = y \ . \tag{I.4.7}$$

Note that if $\gamma : \mathbb{R} \to D$ is an oriented straight line and f_{γ_0,γ'_0} is the critical disc so that (I.1.17) holds, then

$$\lim_{s\to\infty} \gamma_s = f_{\gamma_0,\gamma'_0}(1) \ . \tag{I.4.8}$$

For this reason, for any oriented straight line γ and for the corresponding ray $\gamma|_{[0,+\infty)}$, we call the point (I.4.7) *the point at infinity of* γ and we denote it by γ_∞.

Proposition 4.4. *Let γ be an oriented straight line of D and $\eta : [0,+\infty) \to D$ a ray originating from the point $y = \eta_0$. Then η is a co-ray of γ if and only if $\eta_\infty = \gamma_\infty$.*

In particular, for any $y \in D$, there exists exactly one co-ray η of γ which originates from y.

Proof. For $x \neq y \in D$ or $x \in D$ and $y \in \partial D$, the symbol $f_{x,y}$ will always denote the unique critical disc which verifies (I.1.15) or (I.4.7).

We first want to show that the critical disc f_{y,γ_∞} is equal to the limit of the sequence of critical discs $\{f_{y,\gamma_n}\}$, for $n \to +\infty$.

In fact, by Lemma A.1 (see Appendix), the sequence f_{y,γ_n} converges uniformly on $\bar{\Delta}$ to a critical disc f_o such that $f_o(0) = \lim_{n\to+\infty} f_{y,\gamma_n}(0) = y$. On the other hand,

$$\lim_{n\to\infty} \mathfrak{D}(y,\gamma_n) \geq \lim_{n\to\infty} \mathfrak{D}(\gamma_0,\gamma_n) - \mathfrak{D}(y,\gamma_0) = +\infty$$

and hence, since $\lim_{n_k\to+\infty}\tanh\left(\frac{\mathfrak{D}(y,\gamma_{n_k})}{2}\right) = 1$, for some suitable subsequence $\{f_{y,\gamma_{n_k}}\}$, we get

$$f_{y,\gamma_\infty}(1) = \lim_{n_k\to+\infty}\gamma_{n_k} = \lim_{n_k\to+\infty} f_{y,\gamma_{n_k}}\left(\tanh\left(\frac{\mathfrak{D}(y,\gamma_n)}{2}\right)\right) = f_o(1)\ .$$

So $f_o = f_{y,\gamma_\infty}$ as we wanted to show. As a consequence, it follows that the ray $\eta_t = f_{y,\gamma_\infty}\left(\tanh\left(\frac{t}{2}\right)\right)$, which is the unique ray originating from y and with γ_∞ as point at infinity, is a co-ray of γ.

Conversely, assume that η is a co-ray of γ and let $\gamma^{(n)}$ be a sequence of rays with the following requirements:

i) $\lim_{n\to+\infty}\gamma_0^{(n)} = y = \eta_0$;
ii) the intersections $\gamma_{s_n} = \gamma^{(n)}([0,+\infty))\cap\gamma(\mathbb{R})$ tend to $\lim_{n\to\infty}\gamma_{s_n} = \gamma_\infty$;
iii) the sequence of the oriented segment which goes from y to γ_{s_n} converge uniformly on compacta to η (note that the rays $\gamma^{(n)}$ are the prolongation of such oriented segments).

Let us denote by $v_n = {\gamma^{(n)}}'_0$ the tangent vectors at the origin of the rays $\gamma^{(n)}$ and by $f_{\gamma_0^{(n)},v_n}$ the unique critical disc which verifies (I.1.16) for the point $\gamma_0^{(n)}$ and the vector v_n. Since $\lim_{n\to\infty}\gamma_0^{(n)} = y\in D$, by Corollary A2, there exists a subsequence $f_{\gamma^{(n_k)},v_{n_k}}$ which converges to a critical disc f_{y,v_o}. In particular, it follows that

$$\eta_t = f_{y,v_o}\left(\tanh\left(\frac{t}{2}\right)\right)\ .$$

Now, denotes by s'_n the real numbers such that $\gamma_{s_n} = \gamma^{(n)}([0,+\infty))\cap\gamma(\mathbb{R}) = \gamma^{(n)}_{s'_n}$ and notice that

$$\lim_{n\to\infty} s'_n = \lim_{n\to\infty}\mathfrak{D}(\gamma_0^{(n)},\gamma^{(n)}_{s'_n}) \geq \lim_{n\to\infty}(\mathfrak{D}(\gamma_0,\gamma_{s_n}) - \mathfrak{D}(\gamma_0^{(n)},\gamma_0)) =$$
$$= \lim_{n\to\infty} s_n - \mathfrak{D}(y,\gamma_0)) = +\infty\ .$$

Then we may write that

$$\eta_\infty = \lim_{t\to\infty} f_{y,v_o}\left(\tanh\left(\frac{t}{2}\right)\right) = \lim_{k\to\infty} f_{y,v_o}\left(\tanh\left(\frac{s'_{n_k}}{2}\right)\right)\ ;$$

on the other hand

$$\lim_{k\to\infty} f_{y,v_o}\left(\tanh\left(\frac{s'_{n_k}}{2}\right)\right) = \lim_{k\to\infty} f_{\gamma_0^{(n_k)},v_{n_k}}\left(\tanh\left(\frac{s'_{n_k}}{2}\right)\right) = \lim_{k\to\infty}\gamma^{(n_k)}_{s'_{n_k}} =$$
$$= \lim_{k\to\infty}\gamma_{s_{n_k}} = \gamma_\infty$$

and this concludes the proof. □

¿From Proposition 4.4, one can easily check the following properties of the class of oriented straight lines of D:

(1) the relation between two oriented straight lines, defined by

$$\gamma\simeq\eta \qquad \Leftrightarrow \qquad \gamma \text{ is asymptotic to } \eta$$

is an equivalence relation;
(2) the equivalence classes of asymptotic oriented straight lines are in one-to-one correspondence with the points of the boundary ∂D;
(3) for any $y \in \partial D$, let us call *horospherical distribution associated with* y the family $\mathcal{D}^{(y)}$ of tangent subspaces defined by

$$\mathcal{D}^{(y)}_x = \text{Ker Re}\left(\tilde{f}_{x,y}|_0\right) \subset T_x D$$

(as usual, $f_{x,y}$ denotes the unique critical disc which verifies (I.4.7)); then for any $y \in \partial D$, the associated distribution $\mathcal{D}^{(y)}$ is a distribution.

Remark 4.5. By Corollary 4.2, the horospherical distribution $\mathcal{D}^{(y)}$ can be also defined as the family of subspaces

$$\mathcal{D}^{(y)}_x = \text{Ker } d\mathfrak{D}_{\gamma_s}|_x \subset T_x D$$

where γ_s is any point of an oriented straight line γ passing through x and with $\gamma_\infty = y$.

Theorem 4.6. *Let $y \in \partial D$ and $\gamma : \mathbb{R} \to D$ be an oriented straight line with point at infinity $\gamma_\infty \in \partial D$. Then:*
i) *the horospherical distribution $\mathcal{D}^{(y)}$ is an integrable distribution;*
ii) *the maximal integral leaves of $\mathcal{D}^{(y)}$ coincide with the horospheres $S(\gamma_\infty)|_c$ centered at the limit point of γ;*
iii) *the Busemann function B_γ of the straight line γ is*

$$B_\gamma(x) = -\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{x,\gamma_\infty})|_1\right|\right) .$$

Proof. Since $\dim_{\mathbb{R}} \mathcal{D}^{(y)} = 2n-1$, (ii) implies (i). To show (ii), we need the following Lemma.

Lemma 4.7. *For any oriented straight line γ, the associated Busemann function B_γ is $\mathcal{C}^1$ and, for any $x \in D$ the differential $dB_\gamma|_x$ is equal to*

$$dB_\gamma|_x = -4Re(\tilde{f}_{x,\gamma_\infty}|_0) , \tag{I.4.9}$$

where f_{x,γ_∞} is the unique critical disc such that (I.4.7) holds, where we set $y = \gamma_\infty \in \partial D$.

Proof. Consider a point $x \in D$, a vector $v \in T_x D$ and a regular curve $\sigma : [-\varepsilon, +\varepsilon] \to D$ such that $\sigma_0 = x$ and $v = \dot{\sigma}_0$. We want to prove that

$$v(B_\gamma) = \left.\frac{dB_\gamma(\sigma_t)}{dt}\right|_{t=0} = -4\text{Re}(\tilde{f}_{x,\gamma_\infty}(v)) . \tag{I.4.10}$$

Note that, by Theorem A1 (see Appendix), this immediately implies that B_γ is $\mathcal{C}^1$ and that (I.4.9) holds.

By (I.4.1'), we know that for any $s \in \mathbb{R}$

$$\left.\frac{d\mathfrak{D}_{\gamma_s}(\sigma_t)}{dt}\right|_{t=0} = d\mathfrak{D}_{\gamma_s}|_x(v) = -4\mathrm{Re}\left[\tilde{f}_{x,\gamma_s}(v)\right] \ ,$$

and, by Corollary A2 in the Appendix (applied with $k > 0$), we may infer that

$$\lim_{s\to+\infty} \left.\frac{d\mathfrak{D}_{\gamma_s}(\sigma_t)}{dt}\right|_{t=0} = \lim_{s'\to 1} \left.\frac{d\mathfrak{D}_{\gamma_{\tanh(s'/2)}}(\sigma_t)}{dt}\right|_{t=0} = -4\mathrm{Re}\left[\tilde{f}_{x,\gamma_\infty}(v)\right] \ . \tag{I.4.11}$$

On the other hand, by definition,

$$B_\gamma(\sigma_t) = \lim_{s\to\infty} \left(\mathfrak{D}_{\gamma_s}(\sigma_t) - s\right) \ ,$$

and hence from (I.4.11) and a classical property of real analysis (see e.g. [PS], Th.1.6, Ch. III)

$$-4\mathrm{Re}\left[\tilde{f}_{x,\gamma_\infty}(v)\right] = \lim_{s\to\infty} \left.\frac{d\mathfrak{D}_{\gamma_s}(\sigma_t)}{dt}\right|_{t=0} = \left.\frac{dB_\gamma(\sigma_t)}{dt}\right|_{t=0} = v(B_\gamma) \qquad \square$$

Since the horospheres $S(\gamma_\infty)|_c$ are the level sets of the Busemann function B_γ, from Lemma 4.7 we obtain that they are integral leaves of the horospherical distribution associated with $y = \gamma_\infty$. Since they have dimension $2n-1$, it is quite simple to check that they are maximal integral leaves and this proves (ii).

To prove (iii), we first show that for any $x \in D$

$$d\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty} f'_{*,\gamma_\infty})|_1\right|\right)|_x = \frac{1}{2} dB_\gamma|_x = \ . \tag{I.4.12}$$

Consider a vector $v \in T_xD$ and a regular curve $\sigma : [-\varepsilon, +\varepsilon] \to D$ such that $\sigma_0 = x$ and $v = \dot{\sigma}_0$. Then, we have that

$$v\left(\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{*,\gamma_\infty})|_1\right|\right)\right) = \frac{d}{dt}\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{\sigma_t,\gamma_\infty})|_1\right|\right)\Big|_{t=0} =$$

$$= \mathrm{Re}\left[\frac{1}{\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{x,\gamma_\infty})|_1}\tilde{f}_{\gamma_0,\gamma_\infty}(J'(1))\right] \ , \tag{I.4.13}$$

where J is the holomorphic Jacobi vector field for f_{x,γ_∞} defined by

$$J = \left.\frac{\partial f_{\sigma_t,\gamma_\infty}}{\partial t}\right|_{t=0} \ . \tag{I.4.14}$$

We may now compute the value of $\tilde{f}_{\gamma_0,\gamma_\infty}(J'(1))$ as follows.

By Theorem 2.2, we can uniquely decompose J into a sum

$$J = (\alpha + ib\zeta - \bar{\alpha}\zeta^2)f' + J^\perp$$

where $J^{\perp}$ is a holomorphic Jacobi vector field such that $\tilde{f}_{x,\gamma_\infty}(J^{\perp}) \equiv 0$ and where $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}$. On the other hand, since for any t

$$f_{\sigma_t,\gamma_\infty}(1) = \gamma_\infty$$

we conclude from that $J(1) = 0$. In particular, we get that

$$J^{\perp}(1) = 0 \ , \qquad (\alpha + ib - \bar{\alpha}) = i(2\mathrm{Im}(\alpha) + b) = 0 \ . \tag{I.4.15}$$

Now, note that by definition of $J^{\perp}$ and (I.4.15)

$$0 = \frac{\partial}{\partial\theta}\left[\tilde{f}_{x,\gamma_\infty}(J^{\perp}(e^{i\theta}))\right]\Big|_{\theta=0} = \frac{\partial}{\partial\theta}\left[\tilde{f}_{x,\gamma_\infty}|_{e^{i\theta}}\right]\Big|_{\theta=0}(J^{\perp}(1)) + \tilde{f}_{x,\gamma_\infty}(J^{\perp\prime}(1)) =$$
$$= \tilde{f}_{x,\gamma_\infty}(J^{\perp\prime}(1)) \ .$$

On the other hand, by (I.1.7), it is clear that $\tilde{f}_{x,\gamma_\infty}|_1 = k \cdot \tilde{f}_{\gamma_0,\gamma_\infty}$ for some $k \neq 0$ and hence

$$\tilde{f}_{\gamma_0,\gamma_\infty}(J^{\perp\prime}(1)) = 0 \ . \tag{I.4.16}$$

¿From (I.4.15) and (I.4.16), we obtain

$$\tilde{f}_{\gamma_0,\gamma_\infty}(J'(1)) = \tilde{f}_{\gamma_0,\gamma_\infty}((ib - 2\bar{\alpha})f'_{x,\gamma_\infty}(1)) = -2\mathrm{Re}(\alpha)\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{x,\gamma_\infty}(1))$$

and hence

$$v\left(\log\left(\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{*,\gamma_\infty})|_1\right)\right) = -2\mathrm{Re}(\alpha) =$$
$$-2\mathrm{Re}[\tilde{f}_{x,\gamma_\infty}(J_0)] = -2\mathrm{Re}[\tilde{f}_{x,\gamma_\infty}(v)] = \frac{1}{2}dB_\gamma(v) \ .$$

This proves (I.4.12), which signifies, in different terms, that the level sets of the function

$$-\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{x,\gamma_\infty})|_1\right|\right)$$

coincide with the level sets of the Busemann function.

To conclude, it is sufficient to check that at all points of γ, which is transversal a to all level sets of B_γ, we have

$$-\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{x,\gamma_\infty})|_1\right|\right)\Big|_{x=\gamma_s} = -s = B_\gamma(\gamma_s) \ .$$

To prove this, recall that by Corollary 4.2

$$f_{\gamma_s,\gamma_\infty} = f_{\gamma_0,\gamma_\infty} \circ T_{\gamma_0,\gamma_s}$$

where $T_{x,y}$ is the function defined in (I.4.6). In particular

$$f'_{\gamma_s,\gamma_\infty}(1) = -\frac{1+\tanh(\frac{s}{2})}{1-\tanh(\frac{s}{2})}f_{\gamma_0,\gamma_\infty}(T_{\gamma_0,\gamma_s}(1)) = -e^s f_{\gamma_0,\gamma_\infty}(T_{\gamma_0,\gamma_s}(1))$$

and

$$-\log\left(\left|\tilde{f}_{\gamma_0,\gamma_\infty}(f'_{\gamma_s,\gamma_\infty})|_1\right|\right) = -\log(e^s) = -s$$

and this concludes the proof. $\square$

Part II: The Eversive map

II.1 Preliminaries.

II.1.1 CR structures of hypersurface type.

Let S be a smooth manifold of dimension $2n-1$. An *integrable CR structure of hypersurface type on* S is a complex subbundle $\mathbb{L} \subset T^{\mathbb{C}}M$, which verifies the following conditions:

a) $\mathbb{L}_x \cap \overline{\mathbb{L}}_x = \{0\}$ and $\dim_{\mathbb{C}} \mathbb{L}_x = n-1$ for any $x \in S$; b) $\mathbb{L}$ is involutive, that is $[L_1, L_2] \in \mathbb{L}$ at any point x and for any pair of sections L_i of $\mathbb{L}$.
Note that a CR structure is uniquely determined by the pair $(\mathcal{D}, J)$, where $\mathcal{D}$ is the regular distribution of codimension one in TS, given by $\mathcal{D}_x = \mathrm{Re}(\mathbb{L}_x \oplus \overline{\mathbb{L}}_x)$, while J is the family of complex structures on the subspaces $\mathcal{D}_x$ induced by the action of $\mathbb{C}$ on $\mathbb{L}$. Indeed, a pair $(\mathcal{D}, J)$, made of a regular distribution of codimension one plus a smooth family complex structure on the subspaces $\mathcal{D}_x$, is associated with an integrable CR structures of hypersurface type if and only if

$$J([JX, Y] + [X, JY]) \in \mathcal{D} \ , \tag{II.1.1}$$

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0 \ , \tag{II.1.2}$$

for any two fields X, Y in $\mathcal{D}$.

Note that if S is an embedded real hypersurface of $\mathbb{C}^{n+1}$, then holomorphic distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}S$ (see §I.1) is always an integrable CR structure of hypersurface type. Since it is well known that any real analytic integrable CR structure of hypersurface type can be *locally* realized as the holomorphic distribution of some real hypersurface in $\mathbb{C}^n$, it is fair to say that the concept of integrable CR structure of hypersurface type codifies the properties of the holomorphic distribution of a real hypersurface S, with independently on the properties of the embedding of S as a submanifold of $\mathbb{C}^n$.

In the following, we will always use the short expressions '*CR structure*' and '*CR manifold*' to to mean '*integrable CR structure of hypersurface type*' and '*manifold endowed with an integrable CR structure of hypersurface type*', respectively.

For any CR manifold $(S, \mathcal{D}, J)$, the *(local) defining 1-forms* θ and the *(local) holomorphic defining 1-forms* for the distribution $\mathcal{D}$ are defined exactly as we have done for the holomorphic distribution of a real hypersurfaces (see §I.1).

Moreover for any defining 1-form θ and any holomorphic defining 1-form $\Theta = \theta^{10}$, the associated Levi forms $\mathcal{L}^\theta$ and $\mathcal{L}^\Theta$ are defined as in §I.1, i.e.

$$\mathcal{L}^\theta_x \ : \ \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R} \ , \qquad \mathcal{L}^\theta_x(X, Y) = d\theta_x(X, JY)$$

$$\mathcal{L}^\Theta_x \ : \ \mathcal{D}^{10}_x \times \mathcal{D}^{10}_x \to \mathbb{C} \ , \qquad \mathcal{L}^\Theta_x\left(\frac{X - iJX}{2}, \frac{Y - iJY}{2}\right) = \mathcal{L}^\theta_x(X, Y) + i\mathcal{L}^\theta_x(X, JY) \ .$$

As usual, $(S, \mathcal{D}, J)$ is called *Levi non-degenerate* if $\mathcal{L}^\theta$ (or $\mathcal{L}^\Theta$) is of maximal rank at all points.

Notice that $(S, \mathcal{D}, J)$ is Levi non-degenerate if and only if $\mathcal{D}$ is a *contact distribution*, i.e. if and only if for any local defining 1-form θ for $\mathcal{D}$,

$$\mathrm{Ker}\ d\theta_x|_{\mathcal{D}_x} = \{\ X \in \mathcal{D}_x \ : d\theta_x(X, *) \equiv 0\ \} = \{0\} \tag{II.1.3}$$

for any $x \in S$.

II.1.2 Symplectifications and Hyperplane Bundles of Levi non-degenerate CR manifolds.

An essential object for the study a CR manifold $(S, \mathcal{D}, J)$ is the so called *symplectification E_S of* $(S, \mathcal{D})$, that is the subbundle of the cotangent bundle $\pi : T^*M \to M$, defined

$$E_S \overset{\text{def}}{=} \{ \ \theta \in T^*_x S \ , \ x \in S \ : \ \text{Ker}\ \theta = \mathcal{D}_{\pi(\theta)} \ \} \ . \tag{II.1.4}$$

E_S is a principal bundle over S, with structural group equal to $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Let us still denote by $\pi : E_S \to S$ the natural projection map induced by the standard projection $\pi : T^*S \to S$.

Then, the *tautological 1-form of* E_S is defined as

$$\omega|_\theta \overset{\text{def}}{=} \pi^*\theta \ , \tag{II.1.5}$$

for any $\theta \in E_S$. Notice that ω has a nowhere degenerate exterior differential $d\omega$, so that $(E_S, d\omega)$ is a symplectic manifold. Moreover, a (local) diffeomorphism $f\colon S \to S$ maps $\mathcal{D}$ into $\mathcal{D}$ if and only if the natural induced map $\tilde{f}\colon T^*S \to T^*S$ verifies: a) $\tilde{f}(E_S) \subset E_S$; b) $\tilde{f}^*(d\omega) = d\omega^0$ (i.e. it is a symplectomorphism); c) $\tilde{f}$ commutes with the action of $\mathbb{R}^*$.

In the following, for any symplectification $\pi : E_S \to S$ of a CR manifold S, we denote by V_E the vertical vector whose flow Φ^V_t coincides with the action of $\mathbb{R}^+$ on E, i.e. such that for any form $\theta \in E_S$

$$\Phi^{V_E}_t(\theta) = e^t \cdot V \ .$$

If E_S admits a global section, then it consists of two connected components, say E^+_S and E^-_S, each one with structural group $\mathbb{R}^+$. Each of these components is called *oriented symplectification of* S.

In all the following, we will always assume that E_S admits a global section (like e.g. the boundary $S = \partial D$ of a bounded domain in $\mathbb{C}^{n+1}$), that such a global section has been fixed and we will call the connected component which contains the image of this section *the symplectification of* S, without any further specification.

Now, we want to introduce an object, which will be crucial in all the following developments.

Definition 1.1. Let $(S, \mathcal{D}, J)$ be an orientable, Levi non-degenerate CR manifold. We call *hyperplane bundle of the symplectification* E_S the subbundle $H(E_S) \subset TE_S$ of the tangent space of E_S defined by

$$H(E_S) = \{ v \in T_\theta E_S \ , \ \theta \in E_S \ : \ \omega_\theta(v) = 1 \} \ . \tag{II.1.6}$$

Denote by $\hat{\pi} : H(E_S) \to E_S$, where $\hat{\pi}$ denotes the projection induced by the standard projection $\hat{\pi} : TE_S \to E_S$. Then it is clear that a standard fiber $H_\theta = \hat{\pi}^{-1}(\theta)$ is a hyperplane in $T_\theta E_S$ and $\dim_{\mathbb{R}} H_\theta = 2n+1$. Moreover, if we denote by

$\hat{\pi}' : \pi \circ \hat{\pi} : H(E_S) \to S$, we have that a standard fiber $H_x = \hat{\pi}'^{-1}(x)$ is diffeomorphic to $H_\theta \times \mathbb{R}^+ \simeq \mathbb{R}^{2n+2}$.

In case $S = \partial D$ is the smooth boundary of a strongly pseudo-convex bounded domain $D \subset \mathbb{C}^{n+1}$ and if $\rho : \mathcal{U} \to \mathbb{R}$ is a (local) defining function for S (see §S I.1), then there is a natural trivialization of a symplectification E_S

$$T^\rho : E_S \to S \times \mathbb{R}^+$$

defined by

$$\theta = \lambda \theta^\rho_x \xrightarrow{T^\rho} (x, |\lambda|) \in S \times \mathbb{R}^+ \tag{II.1.7}$$

for any $\theta \in E_S(x)$.

Using this trivialization, the tautological 1-form ω into the 1-form ω^ρ on $S \times \mathbb{R}^+$ defined as

$$\omega^\rho_{(x,\lambda)} = |\lambda| \pi^*(\theta^\rho) \tag{II.1.8}$$

Moreover, the map

$$T^\rho_* : H(E_S) \longrightarrow T(S^\rho) \times T\mathbb{R}^+$$

maps $H(E_S)$ diffeomorphically onto the subbundle $\widetilde{H(E_S)} = T^\rho_*(H(E_S)) \subset T(S) \times T\mathbb{R}^+$, with fibers equal to

$$\widetilde{H(E_S)}_{(x,\lambda)} = \{\ (X, a)\ :\ X \in T_x S\ ,\ a \in \mathbb{R} = T_\lambda \mathbb{R}^+\ ,\ \lambda \cdot \theta^\rho(X) = 1\ \}\ . \tag{II.1.9}$$

Using the map T^ρ, it is quite simple to check the following Lemma, which gives some technical information on the symplectification E useful for the following developments.

Lemma 1.2. *Let $\pi : E_S \to S$ be an oriented symplectification of a Levi non-degenerate CR manifold $(S, \mathcal{D}, J)$ and let ω be the tautological 1-form. Then:*

(1) *$d\omega$ is a symplectic 2-form on E_S;*
(2) *for any two vectors $0 \neq v, f \in T_\theta E_S$ such that $\pi_*(v) = 0$ and $\omega_\theta(f) \neq 0$,*

$$d\omega_\theta(f, v) \neq 0\ ;$$

(3) *let $V_\theta \subset T_\theta E_S$ be the vertical subspace w.r.t. π and denote by $V_\theta^{\perp_{d\omega}} = \{\ X \in T_\theta E_S\ :\ d\omega_\theta(X, V_\theta) = 0\ \}$; then*

$$\tilde{\mathcal{D}}_\theta = (Ker\ \omega_\theta) \cap V_\theta^{\perp_{d\omega}}$$

projects isomorphically onto $\mathcal{D}_{\pi(\theta)}$ via π_*.

II.1.3 The Chern-Moser bundle of a Levi non-degenerate CR manifold.

We now want to introduce, for any Levi non-degenerate CR manifold $(S, \mathcal{D}, J)$ the bundle $\pi_P : P \to S$ of adapted frames bundle as introduced in [CM].

First of all, observe that if $\hat{\theta}\colon U \subset S \to E_S$ is a local section of the symplectification E_S of the Levi non-degenerate CR manifold S, then it is always possible to find other $2n$ forms $\hat{\theta}^i$ on U which verify

$$d\hat{\theta} = \sum_{i=1}^{p} \hat{\theta}^i \wedge \hat{\theta}^{i+n} - \sum_{j=p+1}^{n} \hat{\theta}^j \wedge \hat{\theta}^{j+n} \quad , \quad J^* \hat{\theta}^i|_{\mathcal{D}} = \hat{\theta}^{i+n}|_{\mathcal{D}} \qquad 1 \le i \le n \tag{II.1.10}$$

(see [CM]).

So, if for any $\hat{\theta}_x \in E_s$, we denote by $\omega^i_{\hat{\theta}_x}$, $1 \le i \le 2n$, the 1-forms

$$\omega^i_{\hat{\theta}_x} \stackrel{\text{def}}{=} (\pi)^* \hat{\theta}^i_x$$

we obtain that

$$d\omega|_{\hat{\theta}_x} = \sum_{i=1}^{p} \omega^i_{\hat{\theta}_x} \wedge \omega^{i+n}_{\hat{\theta}_x} - \sum_{i=p+1}^{n} \omega^i_{\hat{\theta}_x} \wedge \omega^{i+n}_{\hat{\theta}_x} + \omega_{\hat{\theta}_x} \wedge \omega^{2n+1}_{\hat{\theta}_x}$$

In particular, since any $\theta \in E_S$ can be assumed to be equal to $\theta = \hat{\theta}_{\pi(\theta)}$ for some local section $\hat{\theta} : U \subset S \to E_S$, this shows that any $\theta \in E_S$, there exist $2n$ 1-forms $\omega^i \in T^*_\theta E_S$ so that

$$d\omega|_\theta = \sum_{i=1}^{p} \omega^i \wedge \omega^{i+n} - \sum_{i=p+1}^{n} \omega^i \wedge \omega^{i+n} + \omega_\theta \wedge \omega^{2n+1} \tag{II.1.11}$$

If we denote by ω^α and $\omega^{\overline{\alpha}}$ the *holomorphic* and *antiholomorphic* complex valued forms $\omega^i + i\omega^{i+n}$ and $\omega^i - i\omega^{i+n}$, $\quad i = 1, \ldots, n$ in $T^{\mathbb{C}}_\theta E_S$, (II.1.11) can be more conveniently written as

$$d\omega|_\theta = \frac{1}{2i} \left(\sum_{\alpha=1}^{p} \omega^\alpha \wedge \omega^{\overline{\alpha}} - \sum_{\beta=p+1}^{n} \omega^\beta \wedge \omega^{\overline{\beta}} \right) + \omega_\theta \wedge \omega^{2n+1} \quad . \tag{II.1.11'}$$

Now, from the previous remark, we see that the following definition is meaningful. However, before giving the definition, we want to recall that a *linear frame* u at a point x of a manifold M of dimension m is an $\mathbb{R}$-linear isomorphism $u\colon \mathbb{R}^m \to T_x M$ and that it can be always identified with the corresponding basis $(f_1, \ldots f_m)$ in $T_x M$ where

$$f_i = u(\epsilon_i) \in T_x M \ ,$$

and $(\epsilon_1, \ldots, \epsilon_m)$ is the standard basis of $\mathbb{R}^m$. For any linear frame u on $T_x M$, the point $x = \pi(u)$ is called *base point of* u.

Definition 1.3. Let $(S, \mathcal{D}, J)$ be a CR manifold with a Levi non-degenerate CR structure $(\mathcal{D}, J)$ and let $\pi : E_S \to S$ the associated symplectification.

The *Chern-Moser bundle* of S is the family $P_{CM}(S)$ of all linear frames

$$\{f_0, \ldots, f_{2n+1}\} \subset T_\theta E_S$$

at the tangent spaces of E_S, which verify the following:

a) $\omega_\theta(f_0) = 1$;
b) $\pi_*(f_{2n+1}) = 0$;
c) for any integer $1 \leq i \leq n$, $\omega_\theta(f_i) = \omega_\theta(f_{n+i}) = 0$, $d\omega_\theta(f_{2n+1}, f_i,) = \omega_\theta(f_{2n+1}, f_{n+i}) = 0$ and $J\pi_*(f_i) = \pi_*(f_{n+i})$ (the last condition is meaningful, because, by Lemma 1.2, the previous conditions imply that $\pi_*(f_i)$ and $\pi_*(f_{i+n})$);
d) the vectors $e_k = \pi_*(f_k) - iJ\pi_*(f_k)$, $k = 1, \ldots, n$, constitute a holomorphic unitary basis w.r.t. $\mathcal{L}^\Theta_x$ at $x = \pi(\theta)$ (see definition of $\mathcal{L}^\Theta_x$ in §II.1.1) and $d\omega_\theta(f_0, f_{2n+1}) = 1$.

Notice that the conditions a), b), c) and d) for a frame $\{f_0, \ldots, f_{2n+1}\}$ are equivalent to the following conditions on the corresponding dual coframe $\{\omega^0, \ldots, \omega^{2n+1}\} \subset T^*_\theta E_S$:

a') $\omega^0 = \omega_\theta$;
b') there exists 1-forms θ^i in $T^*_{\pi_*(\theta)}M$ such that $(\theta, \theta^1, \ldots, \theta^{2n})$ constitutes a coframe for $T^*_{\pi_*(\theta)}M$ such that $\omega^i = \pi^*\theta^i$ for any $i = 1, \ldots, 2n$;
c') for any integer $1 \leq i \leq n$, the 1-forms θ^i verify $\theta^{i+n} = J^*\theta^i$;
d') $d\omega_\theta$ can be written in terms of the 1-forms ω^i as in (II.1.11).

By the previous remarks, we know that at least one coframe which verifies a'), b'), c') and d') exists. Actually, if we denote by $\pi' : P_{CM}(S) \to E_S$ the standard projection map which associates to any frame its corresponding base point, $P_{CM}(S)$ has a bundle structure over S given by the projection

$$\pi_{CM} : P_{CM}(S) \to S \ , \pi_{CM} = \pi \circ \pi'(\{f_0, \ldots, f_{2n+1}\}) \ .$$

The Chern-Moser bundle $P_{CM}(S)$, considered as a bundle over E_S by means of the projection $\pi' : P_{CM}(S) \to E_S$, is a reduction the linear frame bundle $L(E_S)$, with structural group G, given by all the matrices $A \in GL(2n+1, \mathbb{C})$ of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ A^\alpha_0 & A^\alpha_\beta & 0 & 0 \\ A^{\overline{\alpha}}_0 & 0 & A^{\overline{\alpha}}_{\overline{\beta}} & 0 \\ A^{2n+1}_0 & ig_{\rho\overline{\sigma}}A^\rho_\beta A^{\overline{\sigma}}_0 & -ig_{\rho\overline{\sigma}}A^{\overline{\sigma}}_{\overline{\beta}}A^\rho_0 & 1 \end{pmatrix} , \tag{II.1.12}$$

where $A^{2n+1}_0 \in \mathbb{R}$, $A^{\overline{\alpha}}_0 = \overline{A^\alpha_0}$, $A^{\overline{\alpha}}_{\overline{\beta}} = \overline{A^\alpha_\beta}$, $A^\alpha_\rho A^{\overline{\beta}}_{\overline{\sigma}} g_{\alpha\overline{\beta}} = g_{\rho\overline{\sigma}}$ and $g_{\alpha,\overline{\beta}}$ are the components of the standard pseudohermitian inner product on $\mathbb{C}^n$, of signature $(p, n-p)$.

In particular, G contains the subgroup of matrices

$$U_{p,n-p} \simeq \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A^\alpha_\beta & 0 & 0 \\ 0 & 0 & A^{\overline{\alpha}}_{\overline{\beta}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} ,$$

and it can be checked that there is a natural identification between $P_{CM}(S)/U_{p,n-p}$ and $H(E_S)$.

Note that by construction *a (local) diffeomorphism $h\colon E_S \to E_S$ is induced by a CR transformation of S if and only if it induces an automorphism of $P_{CM}(S)$.*

II.1.4 The unitary frame bundle of a complex Finsler manifold.

Let (M, J, F) be a complex Finsler manifold of complex dimension $n+1$, with complex structure J and complex Finsler metric $F : TM \to \mathbb{R}^+$.

We call *Finsler sphere bundle* of M the subbundle $S(M) \subset TM$ determined by all Finsler spheres

$$S(M) = \bigcup_{x\in M} S_F(x) = \bigcup_{x\in M} \{\ v \in T_xM \ :\ F(v) = 1\ \}\ .$$

We also denote by $\mathbb{P}TM$ the projectivized tangent bundle

$$\mathbb{P}TM = \bigcup_{x\in M} (T_xM \setminus \{0\})/\mathbb{C}^*\ .$$

It is clear that $S(M)$ has a natural structure of S^1-bundle over $\mathbb{P}TM$.

In the next definition we introduce the concept of unitary frames of a complex Finsler manifold. In all formulas, for any frame $u = \{f_0, \ldots, f_{2n-1}\}$ we use the symbols $f_1, \ldots, f_{2n-1}$ also to denote the vectors in $T_{f_0}(T_xM)$ which correspond to the vectors of u via the natural identification map

$$\imath_{f_0} : T_xM \to T_{f_0}(T_xM)\ , \qquad \imath_{f_0}(v) = \dot\alpha|_{t=0}\ , \quad \text{where } \alpha_t = f_0 + tv\ .$$

Moreover, we recall that a *complex linear frame* at a point x is a $\mathbb{R}$-linear isomorphism $u\colon \mathbb{R}^{2n+2} \to T_xM$, so that $u(iv) = J \circ u(v)$ for any $v \in \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$.

If u is a complex frame, we denote by u^{10} the corresponding holomorphic basis

$$u^{10} = \{e_\alpha = u(\varepsilon_\alpha) = \frac{1}{2}(f_{2\alpha} - \sqrt{-1}f_{2\alpha+1})\}\ ,$$

where $\varepsilon_\alpha = \epsilon_{2\alpha} - i\epsilon_{2\alpha+1}$ are the elements of the standard holomorphic basis of $\mathbb{C}^{n+1}$.

We denote by $\pi : L^{\mathbb{C}}(M) \to M$ the bundle of all complex linear frames of M and we recall that $L^{\mathbb{C}}(M)$ is a principal bundle with $GL_{n+1}(\mathbb{C})$ as structure group.

Definition 1.4. A complex linear frame $u = \{f_0, \ldots, f_{2n+1}\} \subset T_xM$ of a complex Finsler manifold (M, J, F) is called *unitary* if

a) $f_0 \in S_F(x)$ and $f_1 = Jf_0$;
b) the vectors $f_2, \ldots, f_{2n+1}$ span the subspace $\mathcal{D}_{f_0}$ of $T_{f_0}S_F(x)$ of the CR distribution $\mathcal{D}$ of $S_F(x) \subset T_xM$, and $f_{2i+1} = Jf_{2i}$ for any $i = 1, \ldots, n-1$;
c) the holomorphic vectors $e_1 = f_2 - if_3, \ldots, e_n = f_{2n} - if_{2n+1}$ constitute a unitary basis for $\mathcal{D}_{f_0}$ with respect to the Levi form $\mathcal{L}^{\Theta}_{f_0}$ associated with the holomorphic defining 1-form for $\mathcal{D}_{f_0}$ determined by the defining function of the hypersurface $S_F(x) \subset T_xM$ given by

$$\rho(v) = F^2(v) - 1\ .$$

The subbundle $U_F(M) \subset L^{\mathbb{C}}(M)$ of all unitary frames is called *unitary frame bundle of* $S(M)$.

Notice that, by definitions (see §I.1.1), the Levi form $\mathcal{L}^{\Theta}_{f_0}$ is equal to

$$\mathcal{L}^{\Theta}_{f_0} : \mathcal{D}^{10}_{f_0} \times \mathcal{D}^{10}_{f_0} \to \mathbb{C}$$

$$\mathcal{L}^{\Theta}_{f_0}(\frac{X - iJX}{2}, \frac{Y - iJY}{2}) = d\theta_{f_0}(X, Y) + id\theta_{f_0}(X, JY)$$

where $\theta_{f_0} = J^* d\rho_{f_0}|_{T_{f_0} S_F(x)}$. Note also that θ_{f_0} is also the unique 1-form in $T^*_{f_0} S_F(x)$, which vanishes on the subspace $\mathcal{D}_{f_0}$ and such that

$$\theta_{f_0}(Jf_0) = d\rho_{f_0}(f_0) = 2F(f_0)(f_0 F)|_{f_0} = 2 ,$$

where we used the fact that $F(\lambda v) = |\lambda| F(v)$ for any $v \in T_x M$ and any $\lambda \in \mathbb{C}^*$.

If we denote by $\pi : L^{\mathbb{C}}(M) \to M$ the usual projection map, then the same projection map $\pi : U_F(M) \to M$ makes $U_F(M)$ subbundle of $L^{\mathbb{C}}(M)$ on which U_n acts freely and fiber preserving. Moreover, since each Finsler sphere $S_F(x) \subset T_x M$ is invariant under the action of group

$$T^1 = \{ \exp(tJ) \ t \in \mathbb{R} \}$$

then, from definition, it can be checked that also the group T^1 acts $U_F(M)$ freely and fiber preserving.

Actually, there is a natural identification between $U_F(M)/U_{n-1}$ and $S(M)$, namely, if $\pi' : U_F(M) \to U_F(M)/U_n$ is the standard projection, then

$$\pi'((f_0, Jf_0, \dots, f_{2n-1})) \in U_F(M)/U_n \ \mapsto \ f_0 = S_F(x) ,$$

where $x = \pi((f_0, Jf_0, \dots, f_{2n+1}))$.

Similarly there is a natural identification between $U_F(M)/(U_n \times T^1)$ and $\mathbb{P}TM = S(M)/T^1$.

II.2 The eversive map of bounded convex domains.

In all this section, we will assume that D is a smoothly bounded, strongly convex domain $D \subset \mathbb{C}^{n+1}$.

Let us denote by F the Kobayashi metric of D and let $S(D)$ the corresponding Finsler sphere bundle. On the other hand, ∂D is a Levi non-degenerate CR manifold and we may consider the hyperplane bundle $H(E_{\partial D})$ of an oriented symplectification $E_{\partial D}$ of ∂D.

The goal of this section is to introduce a natural diffeomorphism $\mathcal{E}_D : S(D) \to H(E_{\partial D})$, which is invariant under biholomorphism between domains.

Lemma 2.1. *For any $x \in D$ and $v \in T_x D$, let $f : \bar{\Delta} \to \bar{D}$ be the continuos extension to $\bar{\Delta}$ of the unique critical disc such that (I.1.16) holds, and let $\tilde{f}_{x,v} : \bar{\Delta} \to T^{*10}\mathbb{C}^{n+1}$ be the continuos extension of its lift in $T^{10*}D$. Let also ρ be a defining function for ∂D (i.e. such that $D = \{ z \in \mathcal{U} \subset \mathbb{C}^{n+1} : \rho(z) < 0 \}$ for some neighborhood $\mathcal{U}$ of D) and assume that $E_{\partial D}$ is the oriented symplectification of ∂D, which contains the forms θ^ρ as defined in §I.1.1.*

Assume that $\alpha_{x,v} : \mathbb{R} \to (T^*\partial D)^{\mathbb{C}}$ is the curve

$$\alpha_{x,v}(t) = [-i\tilde{f}_{x,v}|_{e^{it}}]\Big|_{T\partial D} . \tag{II.2.1}$$

Then $\alpha_{x,v}$ is indeed a curve in $\alpha_{x,v} : \mathbb{R} \to T^*\partial D$ i.e.

$$\alpha_{x,v}(t) = [-i\tilde{f}_{x,v}|_{e^{it}}]\Big|_{T\partial D} = \Big(Re[-i\tilde{f}_{x,v}|_{e^{it}}]\Big)\Big|_{T\partial D} .$$

Furthermore, $\alpha_{x,v}(t)$ takes value in $E_{\partial D}$ and the tangent vector $\alpha'_{x,v}|_{t=0}$ is in $H(E_{\partial D})$.

Proof. First of all, notice that, by definitions, $\left[-i\tilde{f}_{x,v}|_{e^{it}}\right]$ vanishes on any vector of the distribution $\mathcal{D} \subset T\partial D$, underlying the holomorphic distribution $\mathcal{D}^{10}$, and that (recall (I.1.2) and (I.1.3))

$$\left[-i\tilde{f}_{x,v}|_{e^{is}}\right](f_{x,v*}(t_{e^{is}})) = \frac{1}{2}[\tilde{f}_{x,v}|_{e^{is}}](f'_{x,v}|_{e^{is}}) + \frac{1}{2}\overline{[\tilde{f}_{x,v}|_{e^{is}}](f'_{x,v}|_{e^{is}})} = 1 .$$

Since $T_{e^{is}}\partial D = \mathbb{R}f_{x,v\,*}(t_{e^{is}}) \oplus \mathcal{D}_{e^{is}}$, it is clear that $[-i\tilde{f}_{x,v}|_{e^{it}}]\Big|_{T\partial D}$ takes value in $\mathbb{R}$.

After this observation the claim that $\alpha_{x,v}(t)$ is a curve in $E_{\partial D}$ is immediate from definitions. Now, we only have to observe that, by definitions,

$$\omega(\frac{d}{ds}\alpha_{x,v}\Big|_{s=0}) = \alpha_{x,v}|_0(\frac{d}{ds}\pi \circ \alpha_{x,v}(s)\Big|_{s=0}) =$$

$$= \frac{1}{2}[-i\tilde{f}_{x,v}|_{e^{is}}](f_{x,v}|_{e^{is}*}(t_1)) + \frac{1}{2}\overline{[-i\tilde{f}_{x,v}|_{e^{is}}](f_{x,v}|_{e^{is}*}(t_1))} =$$

$$= \frac{1}{2}[-i\tilde{f}_{x,v}|_{e^{is}}](f_{x,v}|_{e^{is}*}(t_1^{10})) + \frac{1}{2}\overline{[-i\tilde{f}_{x,v}|_{e^{is}}](f_{x,v}|_{e^{is}*}(t_1))} =$$

$$= \frac{1}{2}[\tilde{f}_{x,v}|_{e^{is}}](f'_{x,v}|_1) + \frac{1}{2}\overline{[\tilde{f}_{x,v}|_{e^{is}}](f'_{x,v}|_1)} = 1 . \qquad \square$$

Indeed, from now on, we will assume that $E_{\partial D}$ *is the oriented symplectification of ∂D, which contains the 1-forms θ^ρ, determined by a defining function ρ defined on a neighborhood $\mathcal{U}$ of D, which is also a defining function for D* (i.e. such that $D = \{ z \in \mathcal{U} \subset \mathbb{C}^{n+1} : \rho(z) < 0 \}$). With this assumption, Lemma 2.1 is always assumed to be true.

Using Lemma 2.1, we may consider the following definition.

Definition 2.2. Let D be a smoothly bounded, strongly convex domain $D \subset \mathbb{C}^{n+1}$, endowed with the complex Finsler metric F given by the Kobayashi metric of D. Let $S(D)$ and $H(E_{\partial D})$ be the Finsler sphere bundle and the hyperplane bundle of D and $E_{\partial D}$, respectively.

We call *eversive map* the map

$$\mathcal{E}_D : S(D) \to H(E_{\partial D})$$

defined as follows: for any $v \in T_x D$, $x \in D$, let

$$\mathcal{E}_D(v) = \alpha'_{x,v}|_0 \in H(E_{\partial D})$$

where $\alpha_{x,v}$ is the curve defined by (II.2.1).

Theorem 2.3. *Let D is a smoothly bounded, strongly convex domain $D \subset \mathbb{C}^{n+1}$, endowed with the complex Finsler metric F given by the Kobayashi metric of D. Let also $U_F(D)$ and $P_{CM}(\partial D)$ the unitary frame bundle of (D, F) and the Chern-Moser bundle of ∂D, respectively.*

Then there exists a U_n-equivariant diffeomorphism

$$\tilde{\mathcal{E}}_D : U_F(D) \longrightarrow P_{CM}(\partial D)$$

such that the following diagram commutes

$$\begin{array}{ccc} U_F(D) & \xrightarrow{\tilde{\mathcal{E}}_D} & P_{CM}(\partial D) \\ \hat{\pi}\downarrow & & \downarrow\hat{\pi} \\ S(D) = U_F(D)/U_n & \xrightarrow{\mathcal{E}_D} & H(E_{\partial D}) = P_{CM}(\partial D)/U_n \end{array} \tag{II.2.2}$$

In particular, the eversive map $\mathcal{E}_D$ is a diffeomorphism.

Theorem 2.3 is a direct consequence of Theorem 4.1 and Proposition 3.5, which we will discuss in the next sections.

II.3 Geodesic deformation of a Finsler sphere bundle.

In all this section, we will assume that (M, J, F) is a Lempert manifold. Let us consider the following definition.

Definition 3.1. Let (M, J, F) be a Lempert manifold. Then, for any $t \in]0, \infty[$ let $(S_t(M), \rho_t)$ be the pair defined as:

a) $S_t(M) \subset M \times M$ is the sphere bundle $\pi : S_t(M) \to M$, where each fiber $\pi^{-1}(x) = S_t(x)$ is the sphere centered at the point x, i.e.

$$S_t(x) = \exp_x(tS_F(x)) = \{y \in M \ : \ y = \exp_x(tv) \ , \ F(v) = 1 \ \} \ ;$$

b) ρ_t is the smooth real function on $M \times M$ defined by

$$\rho_t(x, y) = F^2(\frac{\exp^{-1}(y)}{t}) - 1 \ .$$

We call $(S_t(M), \rho_t)$ *the equipped sphere bundle of radius* t of M. The family $\{(S_tM, \rho_t)\}$ is called *the geodesic deformation of the Finsler sphere bundle* $S(M)$ *of* (M, J, F).

Here is some additional notation, which we will need in the following. For a sphere $S_t(x) \subset M$ of radius t, we denote by $(\mathcal{D}_{x,t}, J_{x,t})$ the CR structure on $S_t(x)$ induced by the complex structure J of M. In particular, we recall that for any $p \in S_t(x)$, the subspace $\mathcal{D}_{x,t}|_p$ is defined as the maximal J-invariant subspace of $T_pS_t(x)$, i.e.

$$\mathcal{D}_{x,t}|_p = \{ \ v \in T_pS_t(x) \ : Jv \in T_pS_t(x) \ \} \ .$$

The symbol $J_{x,t}$ denotes the family of complex structures induced by J on the subspaces $\mathcal{D}_{x,t}|_p$.

We also define as $\mathcal{L}_{x,t}$ the family of Levi forms on the holomorphic subspaces of $\mathcal{D}^{\mathbb{C}}_{x,t}$, defined using the defining function for $S_t(x)$ given by $\rho_{x,t} \overset{\text{def}}{=} \rho_t|_{\pi^{-1}(x)}$.

Proposition 3.2. *Let* (SM, ρ) *the H-sphere bundle of the Lempert manifold* (M, J, F) *(see Def 3.2 [Sp]). Let also* (S_tM, ρ_t) *the geodesic H-sphere bundles of radius* t*, and let* ϕ_t *be the diffeomorphism*

$$\phi_t : SM \longrightarrow S_tM$$

$$\phi_t(x, y) = (x, \exp_x(ty)) \ .$$

Then, for any Finsler sphere $S_F(x) \subset SM$*, the restriction of* ϕ_t *on* $S_F(x)$ *induces a diffeomorphism between* $S_F(x)$ *and* $S_t(x)$*, which maps the* J*-invariant distribution* $\mathcal{D}_x$ *of* $S_F(x)$ *onto the distribution* $\mathcal{D}_{x,t}$*, i.e. it is a contactomorphism between* $(S_F(x), \mathcal{D}_x)$ *and* $(S_t(x), \mathcal{D}_{x,t})$*.*

Moreover, if $(\mathcal{D}_x, J_x^{(t)})$ *denotes the CR structure on the Finsler sphere* $S_F(x)$ *induced from* $S_t(x)$ *by the diffeomorphism* ϕ_t*, i.e. given by the distribution* $\mathcal{D}_x = (\phi_t)_*^{-1}(\mathcal{D}_{x,t})$ *and with complex structure*

$$J_x^{(t)} = (\phi_t)_*^{-1} \circ J_{x,t} \circ \phi_{t*} \ ; \tag{II.3.1}$$

then:

a) *for any* $v \in S_F(x)$*,* $\lim_{t\to 0} J_x^{(t)}|_{\mathcal{D}_x|_v} = J_x|_{\mathcal{D}_x|_p}$ *where* J_x *is the the complex structure of the natural CR structure of* $S_F(x) \subset T_xM$*;*
b) *for any* $v \in S_F(x)$*, the pull-back* $\phi_t^*\mathcal{L}_{x,t}|_{\phi_t(x,v)}$ *of the Levi form of the sphere* $S_t(x)$*, given by the functions* $\rho_t(x, y) = F^2(\frac{\exp^{-1}(y)}{t}) - 1$*, is equal to*

$$\phi_t^*\mathcal{L}_{x,t}|_v(\frac{X - iJX}{2}, \frac{Y - iJY}{2}) = d\theta_v(X, Y) + id\theta_v(X, J^{(t)}Y)$$

where $\theta_v = J^*d\rho_v|_{T_vS_F(x)}$*,* $\rho = F^2 - 1$ *and* J *is the complex structure of* T_xM*;*
c) *the collection of CR structures* $(\mathcal{D}_x, J_x^{(t)})$ *on the Finsler spheres* S_x *depends smoothly on the pairs* (x, t)*, for any* $(x, t) \in M \times [0, +\infty)$*.*

Proof. For the first claim, we just have to show that, for any vector $y \in S_F(x) \subset T_xM$ and any vector $w \in T_y(S_F(x)) \subset T_y(T_xM) \simeq T_x$, such that $Jw \in T_y(S_F(x))$, the following is true:

$$J \exp_{x\,*}(w) \in T_{exp_x(ty)} S_{x,t}\ .$$

But this is exactly the claim (2) of Lemma AII.4 (see Appendix II).

To prove (a), it suffices to show that for any $v \in \mathcal{D}_x|_p$

$$\lim_{t\to 0} J^{(t)}_x(v) = J_x(v)\ .$$

By definitions,

$$J^{(t)}_x(v) = (\phi_t)^{-1}_* \circ J_{x,t} \circ \phi_{t*}(v)$$

and the conclusion follows from the following facts: if $v \in T_y S_F(x) \subset T(T_xM)$, then

$$\lim_{t\to 0} \phi_{t*}(v) = \lim_{t\to 0} \exp_{x\,*}|_{ty}(v) = Id_{T(T_xM)}(v) = v\ ; \tag{II.3.2}$$

on the other hand, by definition of $J^{(t)}$,

$$J(\phi_{t*}(v)) = \phi_{t*}(J^{(t)}(v))\ ,$$

where J is the complex structure of $T_{exp_x(ty)}M \simeq T(T_{exp_x(ty)}M)$. Therefore,

$$\lim_{t\to 0} J^{(t)}_x(v) = \lim_{t\to 0} \phi_{t*}(J^{(t)}_x(v)) = J \lim_{t\to 0} \phi_{t*}(v) = J_x(v)\ . \tag{II.3.3}$$

To prove (b), notice that, by definitions (see §I.1.1)

$$\phi_t^* \mathcal{L}_{x,t}|_v(\frac{X - iJX}{2}, \frac{Y - iJY}{2}) =$$

$$= d\theta^{(t)}_{\phi_t(v)}(\phi_{t*}(X), \phi_{t*}(Y)) + i d\theta^{(t)}_v(\phi_{t*}(X), J\phi_{t*}(Y)) =$$

$$= d(\phi_t^*\theta^{(t)})_v(X, Y) + i d(\phi_t^*\theta^{(t)})_v(X, J^{(t)}Y)$$

where $\theta^{(t)}_y = J^* d\rho_t|_{T_y S_t(x)}$, for any $y \in S_t(x)$. On the other hand, for any $X \in T_v S_F(x)$,

$$(\phi_t^*\theta^{(t)})_v(X) = \theta^{(t)}(\phi_{t*}(X)) = d\rho_t(\phi_{t*}(JX)) = d\left(F^2(JX)\right) = \theta_v(X)\ .$$

(c) follows immediately from definitions and the smoothness of the map ϕ_t as a map from $TM \times [0, +\infty)$ into $M \times M$. $\square$

Definition 3.3. Let (M, J, F) be a Lempert manifold. Then, for any $t \in]0, \infty[$ let $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$ be the triple where:

a) $SM \subset TM$ is the sphere bundle where each fiber $\pi^{-1}(x) = S_F(x) \subset T_xM$ is the Finsler sphere of unit length vectors in T_xM;
b) $(\mathcal{D}, J^{(t)})$ is the family of CR structures on the Finsler spheres $S_F(x) \subset SM$, where $\mathcal{D}_x$ is the distribution of the maximal J-invariant subspaces of $TS_F(x)$ and $J^{(t)}_x$ is the family of complex structures on the subspaces $\mathcal{D}_x|_p \subset \mathcal{D}_x$, which are induced by the CR structure of the the geodesic sphere $S_t(x)$ of center x and radius t, by means of the diffeomorphism $\phi^t(v) = \exp_x(tv)$ (see Prop. 3.2 and in particular (II.3.1));
c) $\mathcal{L}^{(t)}$ is the family of Levi forms on the holomorphic tangent subspaces of the CR structures $(\mathcal{D}_x, J^{(t)}_x)$, which are the pull-backs via the diffeomorphism $\phi^t(v) = \exp_x(tv)$ of the Levi forms $\mathcal{L}^{(t)}_x$ of the geodesic spheres $S_t(x)$, associated to the defining functions $\rho^{(t)}_x$ (see remarks after Def. 3.1).

We call $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$ *the exponential pull-back of the equipped sphere bundle of radius t*. The family of all exponential pull-backs is called *geodesic deformation of the CR structure of $S(M)$*.

Finally, we say that the geodesic deformation of the CR structure of $S(M)$ is *complete* if there exists a triple $(SM, (\mathcal{D}, J^{(\infty)}), \mathcal{L}^{(\infty)})$ with the following properties:

a) $SM \subset TM$ is the Finsler sphere bundle of M;
b) $(\mathcal{D}, J^{(\infty)})$ is a family of CR structures on the Finsler spheres $S_F(x) \subset SM$, where $\mathcal{D}_x$ is the distribution of the maximal J-invariant subspaces of $TS_F(x)$ and $J^{(\infty)}_x$ is a smooth family of complex structures on the subspaces $\mathcal{D}_x|_p \subset \mathcal{D}_x$ with the following property: for any $v \in \mathcal{D}_x|_p$

$$J^{(\infty)}(v) = \lim_{t\to\infty} J^{(t)}(v) \ ;$$

c) $\mathcal{L}^{(\infty)}$ is a family of Levi forms on the holomorphic tangent subspaces of the CR structures $(\mathcal{D}_x, J^{(\infty)}_x)$, with the following property: for any $v \in \mathcal{D}_x|_p$

$$\mathcal{L}^{(\infty)}(v - iJ^{(\infty)}(v), v + iJ^{(\infty)}(v)) = \lim_{t\to\infty} \mathcal{L}^{(t)}(v - iJ^{(t)}(v), v + iJ^{(t)}(v)) \ ;$$

d) the collection of CR structures $(\mathcal{D}_x, J^{(\operatorname{arctanh}(s))}_x)$ on the Finsler spheres S_x depends smoothly on the pairs (x, s), for any $(x, s) \in M \times [0, 1]$.

Note that any strongly convex, smoothly bounded domain $D \in \mathbb{C}^{n+1}$ has a complete geodesic deformation. In fact,

Proposition 3.4. *Let $D \subset \mathbb{C}^{n+1}$ be a strongly convex, smoothly bounded domain endowed with the complex Finsler metric F equal to its Kobayashi infinitesimal metric.*

Then the geodesic deformation $\{(S(D), (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})\}$ of the CR structure of $S(D) \subset TD$ is complete if one consider the triple $(SM, (\mathcal{D}, J^{(\infty)}), \mathcal{L}^{(\infty)})$, where

i) *the CR structures $(\mathcal{D}, J^{(\infty)})$ are given by the pull-backs on the Finsler spheres $S_x \subset T_xD$ of the CR structure and of the boundary ∂D, by means of the map $\Phi_x : S_x \to \partial D$ defined as $\Phi_x(v) = f_{x,v}(1)$;*

ii) *for any* $(x, v) \in S(M)$, *the Levi form* $\mathcal{L}^{(\infty)}|_{x,v}$ *is the pull-back via the map* $\Phi_x : S_x \to \partial D$ *of the Levi form* $\mathcal{L}_{\Phi_x(v)}$ $Phi_x(v) \in \partial D$ *defined by*

$$\mathcal{L}^{x,\infty}_{\Phi_x(v)}(\frac{X - iJX}{2}, \frac{Y - iJY}{2}) = d\theta^{(x,\infty)}_{\Phi_x(v)}(X, Y) + id\theta^{(x,\infty)}_{\Phi_x(v)}(X, J\,Y)$$

where $\theta^{(x,\infty)}$ *is the smooth field of contact forms on* ∂D *defined by*

$$\theta^{(x,\infty)}_y = -i2\tilde{f}_{x,y}|_{T_y\partial D}$$

and $f_{x,y}$ *is the unique critical disc which verifies (I.4.7).*

Proof. Observe that the map ϕ_t given in Proposition 3.2 verifies

$$\lim_{t\to\infty} \phi_t(x, y) = \Phi_x(y)$$

for any $y \in S_F(x)$. This implies that, for any x,

$$\lim_{t\to\infty} J^{(t)}|_{\mathcal{D}_x} = \lim_{t\to\infty} (\phi_t)^{-1}_* \circ J \circ \phi_{t*}|_{\mathcal{D}_x} = (\Phi_x)^{-1}_* \circ J \circ \Phi_{x*} = J^{(\infty)}\ .$$

On the other hand, the 1-form on $S_F(x)$ given by

$$\Phi^*_x(\theta^{(x,\infty)}_{\Phi_x(v)})$$

is such that, at any $v \in S_F(x)$, vanishes on $\mathcal{D}_x|_v$ and

$$\Phi^*_x(\theta^{(x,\infty)}_{\Phi_x(v)})|_v(Jv) = -i2\tilde{f}_{x,\Phi_x(v)}(\Phi_{x*}(Jv)) = 2\tilde{f}_{x,\Phi_x(v)}(f'_{x,\Phi_x(v)}|_1) = 2$$

Hence, by the observations after Definition 1.4, the 1-form $\Phi^*_x(\theta^{(x,\infty)}_{\Phi_x(v)})$ coincides with $\theta_v = J^* d\rho_v$, where $\rho = F^2 - 1$ and, by Proposition 3.2 (b),

$$\Phi^*_x(\mathcal{L}^{x,\infty}_{\Phi_x(v)})(\frac{X - iJX}{2}, \frac{Y - iJY}{2}) = d\theta_v(X, Y) + id\theta_v(X, J^{(\infty)}Y) =$$

$$= \lim_{t\to\infty} \Big[d\theta_v(X, Y) + id\theta_v(X, J^{(t)}Y)\Big] = \lim_{t\to\infty} \mathcal{L}^{(t)}(\frac{X - iJX}{2}, \frac{Y - iJY}{2})\ . \qquad \square$$

Our goal is to prove that the geodesic deformation of the CR structure of $S(M)$ provides a deformation of the unitary frame bundle $U_F(M)$ as well. For proving this, we need to introduce the concept of *unitary frames* for the exponential pull-back of an equipped sphere bundle.

Definition 3.4. Let (M, J, F) be Lempert manifold and $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$ the exponential pull-back of an equipped sphere bundle of radius t.

We say that a linear frame $u = \{f_0, \ldots, f_{2n+1}\}$ at $x = \pi(u) \in M$ is *unitary for the pull-backed sphere bundle* $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$ if

a) $f_0 \in S_F(x)$ and $f_1 = Jf_0$;
b) the vectors $f_2, \ldots, f_{2n+1}$ span the maximal J-invariant subspace $\mathcal{D}_{f_0}$ of $T_{f_0}S_F(x)$ and $f_{2j+1} = J^{(t)}f_{2j}$ for any $j = 1, \ldots, n+1$;
c) the vectors $e_1 = f_2 - \sqrt{-1}J^{(t)}f_2, \ldots, e_n = f_{2n} - \sqrt{-1}J^{(t)}f_{2n+1)}$ constitute a unitary basis for $\mathcal{D}_{f_0}$ with respect to the Levi form $\mathcal{L}^{(t)}_x|_{f_0}$ given by $\mathcal{L}^{(t)}$;

The bundle $U_{(t)}(M)$ of all unitary frames of $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$ is called *unitary frame bundle associated with spheres of radius t*. In case the geodesic deformation of M is complete, then we denote by $U_{(\infty)}(M)$ the bundle of unitary frames for $(SM, (\mathcal{D}, J^{(\infty)}), \mathcal{L}^{(\infty)})$, where in this case the unitary frames have been defined in the same way of the unitary frames for $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$.

Proposition 3.5. *Let (M, J, F) be Lempert manifold and for any $t \in (0, +\infty)$ let $U_{(t)}(M)$ be the unitary frame bundle associated with spheres of radius t. Let us also assume that for $t = 0$, then $U_{(0)}(M) = U_F(M)$ and, in case the geodesic deformation of M is complete, that $U_{(\infty)}(M)$ is the unitary frame bundle of $(SM, (\mathcal{D}, J^{(t)}), \mathcal{L}^{(t)})$.*

Then, for any $t_1, t_2 \in [0, +\infty)$, there exists a U_n-bundle isomorphism between $U_{(t_1)}(M)$ and $U_{(t_2)}(M)$.

In case the geodesic deformation of M is complete, then $U_{(\infty)}(M)$ is a U_n-bundle equivalent to $U_{(t)}(M)$ for any $t \in [0, +\infty]$.

Proof. Let $t_1, t_2 \in [0, +\infty)$ and let $P = \bigcup_{t \in [t_1, t_2]} U_{(t)}(M)$. By construction and Prop. 3.2, P has a structure of principal bundle over $S_F(M) \times [t_1, t_2]$ with structure group U_n. Then by some standard facts on principal bundles (see e.g. Cor. 2.6.2 in [Mo]), it follows that $P_{M \times t_1}$ is equivalent to $P_{M \times t_2}$ as U_n-bundle.

The second claim can be checked in a similar way. $\square$

In the next section, we will prove that the unitary frame bundle $U_{(\infty)}(M)$ of a bounded convex domain $M = D$ can be identified with the Chern-Moser bundle $P_{CM}(S)$ of the CR manifold $S = \partial D$ diffeomorphic to a sphere and that $P_{CM}(S)/U_n = H(E_S)$. This will prove Theorem 2.3 by means of Proposition 3.5.

II.4 The eversive map between the unitary and the Chern-Moser bundle of a bounded convex domain.

In this section (M, J, F) will always denote a smoothly bounded, sctrictly convex domain $M = D \subset \mathbb{C}^{n+1}$ endowed with the infinitesimal Kobayashi's metric $F = k_D$.

We recall that, by Proposition I.4.4, the relation between oriented straight lines given by

$$\gamma \simeq \gamma' \qquad \Leftrightarrow \qquad \gamma|_{[0,+\infty)} \text{ is a co-ray of } \gamma'$$

is an equivalence relation and that the equivalence classes of oriented straight lines are in one-to-one correspondence with the points of ∂M. Therefore, in all the following we will identify the set ∂M with the set of all equivalence classes of oriented straight lines.

Theorem 4.1. *Let $M = D \subset \mathbb{C}^{n+1}$ be a smoothly bounded, strongly convex domain endowed with the Finsler metric F given by the infinitesimal Kobayashi metric of D. Let also $S = \partial M$ endowed with the CR structure induced by $\mathbb{C}^{n+1}$.*

Then there exists a U_n-equivariant diffeomorphism

$$\Phi : U_{(\infty)}(M) \to P_{CM}(\partial M)$$

such that the induced map

$$\phi : U_{(\infty)}(M)/U_n = S(D) \to P_{CM}(\partial M)/U_n$$

coincides with the eversive map given in Definition 2.2.

Proof. Let $(x, v, u = \{f_0, \ldots, f_{2n+1}\}) \in U_{(\infty)}(M)$ and consider the element in $(y, \theta, u' = \{f'_0, \ldots, f'_{2n+1}\}) \in P_{CM}(S)$, with $y \in S = \partial M$, $\theta \in \pi^{-1}(y) \subset E_S$ and $u' = \{f'_0, \ldots, f'_{2n+1}\} \subset T_\theta E_S$, defined as follows.

For any vector $w \in S_F(x) \subset T_x M$ in some Finsler sphere, let us consider the element $\psi(x; w) \in E_{\partial D}$ defined as

$$\psi(x; w) = -i\tilde{f}_{x,w}|_1 \tag{5.1}$$

where $f_{x,w}$ is the unique critical disc so that (I.1.17) holds true w.r.t. the oriented straight line $\gamma_{x,v}$ originating from x and tangent to v. Recall that

$$\pi(\psi(x; w)) = \lim_{t\to\infty} \exp_x(tw) \ .$$

The fact that $\psi(x; w) \in E_{\partial D}$ is proved in Lemma 2.1.

Now, if $(x, v, u = \{f_0, \ldots, f_{2n+1}\}) \in U_{(\infty)}(M)$, we set

$$y = \pi(\psi(x; v)) = \lim_{t\to\infty} \exp_x(tw) \ , \quad \theta = \psi_x(w) \ ,$$

$$f'_0 = \psi(x; \cdot)_*(f_1) \ , \quad f'_1 = \psi(x; \cdot)_*(f_2) \ , \quad \ldots, f'_{2n} = \psi(x; \cdot)_*(f_{2n}) \ . \tag{5.2}$$

Finally, we set f'_{2n+1} as the unique vector in $T_{\psi(x;v)}E$, such that

$$\pi_*(f'_{2n+1}) = 0 \ , \qquad d\omega_{\psi(x;v)}(\psi(x; \cdot)_*(f_0), f'_{2n+1}) = 1 \ . \tag{5.3}$$

Notice that, if T denotes the vertical vector field of $E_{\partial D}$ whose 1-parameter flow coincides with the 1-parameter group of transformations

$$\theta \in E_{\partial D} \mapsto e^t\theta \in E_{\partial D}$$

then (5.3) is equivalent to

$$f'_{2n+1} = \frac{1}{d\omega_{\psi(x;v)}(\psi(x; \cdot)_*(f_0), T)} \cdot T|_{\psi_x(v)} \tag{5.4}$$

We claim that $(y, \theta, u' = \{f'_0, \ldots, f'_{2n+1}\}) \in P_{CM}(S)$. In fact, f'_0 verifies a) of Definition 1.3, by Lemma 2.1. Moreover $f'_1, \ldots, f'_{2n}, f'_{2n+1}$ verify b), they are annihilated by ω_θ (because their projection onto ∂D is in $\lim_{t\to\infty} \exp_{x\,*}|_{tv}(\mathcal{D}_v) \subset \mathcal{D}_y$) and they verify d) by definition of frames in $U_{(\infty)}(M)$.

It remains to check that for any $i = 1, \ldots, 2n$,

$$d\omega_\theta(f'_0, f'_i) = 0 \ , \tag{5.5}$$

so that also c) of Definition 1.3 is verified.

Observe that, if we cosider a smooth section $\{g_0, \ldots, g_{2n+1}\}$ of frames which belong to $U_{(\infty)}(M)$ on a neighborhood of $(x, v) \in S(M)$, so that

$$\{g_0, \ldots, g_{2n+1}\}|_{(x,v)} = \{f_0, \ldots, f_{2n+1}\} ,$$

then

$$d\omega_\theta(f_0', f_i') = d\psi(x; \cdot)^*\omega(f_1, f_{i-1}) = \left(\mathcal{L}_{g_1}\psi(x; \cdot)^*\omega\right)(f_{i-1}) ,$$

since

$$\left(\mathcal{L}_{g_1}\psi(x; \cdot)^*\omega\right) = d\left(\psi(x; \cdot)^*\omega(g_1)\right) + \imath_{g_1} d\left(\psi(x; \cdot)^*\omega\right) = \imath_{g_1} d\left(\psi(x; \cdot)^*\omega\right)$$

because $(\psi(x; \cdot)^*\omega(g_1)) \equiv 1$ by definition of ψ and of Lemma 2.1.

Now observe that the flow of the vector field g_1 (which coincides at any point $(y; w) \in S(M)$ with the vector $Jw \in T_y S_F(y)$) coincides with the fundamental vector field associated to the axction of S^1 on $S(M)$ defined by

$$S^1 \times S(M) \to S(M) , \qquad (e^{it}; (x, v)) \mapsto (x, e^{it}v) .$$

Now, for any $y \in D$, this action is a 1-parameter family of CR transformations of the spheres $S_F(y) \subset T_y M$ and therefore, it is also a 1-parameter family of contact transformations of $(S_F(y), \mathcal{D}_y)$. This implies that for any $i \geq 2$,

$$\left(\mathcal{L}_{g_1}\psi(x; \cdot)^*\omega\right)(f_{i-1}) = \left(\mathcal{L}_{g_1}\psi(x; \cdot)^*\omega(g_{i-1})\right)_{(x,v)} - \left(\psi(x; \cdot)^*\omega([g_1, g_{i-1}])\right)_{(x,v)} =$$

$$= \left(\psi(x; \cdot)^*\omega([g_1, g_{i-1}])\right)_{(x,v)} = 0$$

where we used the fact that $\psi(x; \cdot)^*\omega(g_{i-1})_{(x,v)} \equiv 0$ and that $[g_1, g_{i-1}]_{(x,v)}$ is in $\mathcal{D}_x$, for the previos remarks and the fact that $g_{i-1}|_{(y,w)} \in \mathcal{D}_y$ for any (y, w).

This concludes the proof of (5.5) and of the fact that $(y, \theta, u' = \{f_0', \ldots, f_{2n+1}'\}) \in P_{CM}(S)$.

We define the map Φ as the map which sends any frame

$$(x, v, u = \{f_0, \ldots, f_{2n+1}\}) \in U_{(\infty)}(M)$$

into the corresponding frame

$$(y, \theta, u' = \{f_0', \ldots, f_{2n+1}'\}) \in P_{CM}(S) .$$

It is clear that Φ is smooth since the map $\psi : S(D) \to \partial D$ is smooth as it is proved in ??????.

The fact that Φ is U_n-equivariant can be checked using just the definition of the action of U_n on $U_{(\infty)}(M)$ and the definition of Φ. $\square$

II.5. Relation between the Chern-Moser connection and the non-linear Hermitian connection of a bounded convex domain.

We recall here the definition of non-linear connection and of a Cartan connection on a bundle of frames of manifold.

Definition 5.1. Let $P \subset L(N)$ be a subbundle of the bundle of linear frames on a real manifold N. Then:

(1) a (non-linear) connection on $P \subset L(N)$ is a distribution $\mathcal{H}$ of dimension equal to $\dim N$, which is complementary to the vertical distribution; in case there exists an action of a Lie group H on P which preserves the bundle structure, a (non-linear) connection $\mathcal{H}$ is called *nice* if it is H-invariant;

(2) in case P s a principal H-bundle and there exists an injective homomorphism $\imath : H \to G$ from H into a Lie group G of dimension $\dim G = \dim H + \dim N$, a Cartan G-connection on P is an H-invariant smooth family of linear isomorphisms

$$\omega : T_u P \to \mathfrak{g} = Lie(G) \ , \qquad u \in P \ ,$$

which maps the vertical subspaces $\mathcal{V}_u \subset T_u P$ isomorphically into $\mathfrak{h} = Lie(\imath(H)) \subset Lie(G)$.

(3) let $\mathcal{G} \subset Diff(M)$ be a group of diffeomorphisms of M which preserve the subbundle P; then a non-linear connection $\mathcal{H}$ or a Cartan connection ω on P is called *canonical with respect to* $\mathcal{G}$ if and only if any element of $\mathcal{G}$ acts on P in such a way that it transforms the connection into itself.

Consider a Lempert manifold (M, J, F) and let $\pi : P = U_{(t)}(M) \to M$ be one of its unitary frame bundles associated with the spheres of radius t. Let also $\mathcal{H}^{(t)}$ be a nice non-linear connection on $U_{(t)}(M)$ (i.e. U_n-invariant), which is canonical w.r.t. the group of biholomorphisms of M. Then we may define an absolute parallelism on $U_{(t)}(M)$ associated with $\mathcal{H}^{(t)}$ as follows.

First of all, recall that any vertical subspace $\mathcal{V}_u \subset T_u U_{(t)}(M)$ is linearly isomorphic to $\mathfrak{u}_n + \mathbb{R}^{2n+1} = \mathfrak{u}_n + \mathbb{C}^n + \mathbb{R}$. Let fix a basis $\{E_k\}$ for $\mathfrak{u}_n$, take the canonical basis $\{\epsilon_j\}$ of $\mathbb{R}^{2n} = \mathbb{C}^n$ and let $\{t\}$ be the standard basis for $\mathbb{R}$. Then consider the corresponding *generalized fundamental vector fields on* $U_{(t)}(M)$, namely $\tilde{E}_k$, $\tilde{\epsilon}_k$ and $\tilde{t}$, as they have been defined in Def. 4.5 in [Sp]. We recall that they are all vertical vector fields and that they span $\mathcal{V}_u \subset T_u U_{(t)}(M)$ at any point $u \in U_{(t)}(M)$.

Finally, for any $u = \{f_0, \dots, f_{2n+1}\} \in U_{(t)}(M)$, consider the vectors $\hat{f}_i \in \mathcal{H}^{(t)}$ which projects onto the vectors $f_i \in T_{\pi(u)}M$. The set of vector fields

$$\{\hat{f}_i, \tilde{t}, \tilde{\epsilon}_k, \tilde{E}_k\}$$

constitutes an *absolute parallelism, which is preserved by any biholomorphism of* M. We will call it *absolute parallelism on* $U_{(t)}(M)$ *determined by the non-linear connection* $\mathcal{H}^{(t)}$.

Consider now a smoothly bounded, strictly convex domain $M = D \subset \mathbb{C}^{n+1}$ endowed with its infinitesimal Kobayashi metric $F = k_D$. Let $S = \partial D$ and $P_{CM}(S)$ the Chern-Moser bundle of S. Let also $\omega_u : T_u P_{CM}(S) \to \mathfrak{su}_{n,1}$ the Chern-Moser Cartan connection introduced in [CM]. This family of isomorphisms ω_u define an absolute parallelism on $P_{CM}(S)$ as follows. Consider a basis $\{t, epsilon_i, E_k, f_0, f_i\}$ for $\mathfrak{su}_{n,1}$ with the following properties: if we consider the standard graded decomposition of $\mathfrak{su}_{n,1}$

$$\mathfrak{su}_{n,1} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

we assume that t is a basis for $\mathfrak{g}_{-2}$, $\{\epsilon_i\}$ is a basis for $\mathfrak{g}_{-1}$, $\{E_k\}$ a basis for the semisimple part of $\mathfrak{g}_0$ (which is $\mathfrak{u}_n$), $\{f_0\}$ is a basis for the center of $\mathfrak{g}_0$ and $\{f_i\}$ is a basis for $\mathfrak{g}_1 + \mathfrak{g}_2$. Then let $\{\tilde{t}, \tilde{\epsilon}_i, \tilde{E}_k, \hat{f}_0, \hat{f}_i\}$ be the vector fields of $P_{CM}(S)$ which verify at any point u

$$\omega_u(\tilde{t}) = t\ , \qquad \omega_u(\tilde{\epsilon}_i) = \epsilon_i\ , \qquad \text{etc.}$$

Also in this case, we may say that $\{\tilde{t}, \tilde{\epsilon}_i, \tilde{E}_k, \hat{f}_0, \hat{f}_i\}$ is *an absolute parallelism, which is preserved by any CR transformations of S*. We will call it *absolute parallelism on $P_{CM}(S)$ determined by the Cartan connection ω*. Observe that there is some arbitrarity in the definition of such absolute parallelism, namely it depends on the choice of the basis for $\mathfrak{su}_{n,1}$.

In the following statement, by *smooth family of non-linear connections $H^{(t)}$*, we mean that there exists a manifold B which contains as embedded submanifolds al bundles $U_{(t)}(M)$ and that the distributions $\mathcal{H}^{(t)}$ vary smoothly on t as vector subspaces in the tangent bundle TB.

Theorem 5.2. *Let $M = D \subset \mathbb{C}^{n+1}$ be a smoothly bounded, strongly convex domain endowed with the Finsler metric F given by the infinitesimal Kobayashi metric of D. Let also $S = \partial M$ endowed with the CR structure induced by $\mathbb{C}^{n+1}$.*

Then there exists a smooth 1-parameter family of nice non-linear connections $\mathcal{H}^{(t)}$ on the unitary bundle $U_{(t)}(M)$, with the following properties:

(1) *for any t, the connection $\mathcal{H}^{(t)}$ is canonical w.r.t. the group of biholomorphisms of M;*
(2) *for any t, the self-parallel curves w.r.t. the covariant derivation determined by $\mathcal{H}^{(t)}$ coincide with the geodesics of (M, J, F);*
(3) *the non-linear connection $\mathcal{H}^{(\infty)}$ gives an absolute parallelism on $U_{(\infty)}(M)$ which is mapped into an absolute parallelism associated with the Cartan connection of $P_{CM}(S)$, by the the U_n-equivariant diffeomorphism*

$$\Phi : U_{(\infty)}(M) \to P_{CM}(\partial M)$$

defined by Theorem 4.1.

Remark. The relevance of this result (if it is correct) is that we can immediately write down the structure equations of the parallelism determined by $\mathcal{H}^{(\infty)}$, because they coincide with the structure equations of the parallelism of the Cartan connection of $P_{CM}(S)$. Then, using this structure equations, it is also possible to write down the equations for the Jacobi fields and check that they verify the same equations for *any* strictly convex domain, in particular the same equations of the Jacobi fields of the unit ball. From this it should follow that the Busemann curvature of any strictly convex domain is negative.

The construction of the connections $\mathcal{H}^{(t)}$ is as follows.

For any $x \in M$, let us call *n-dimensional complex affine frame at x* any triple

$$(y, w, u = \{f_2, \dots, f_{2n+1}\})\ ,$$

where $y \in M$, w is a complex n-dimensional hyperplane in $T_x M$ and $\{f_2, \ldots, f_{2n+1}\}$ is a complex linear frame of $w \subset T_y M$. We will denote by $A_n^{\mathbb{C}}(M)$ the collection of all n-dimensional complex affine frame.

For any complex affine frame (y, u) at x, we will call x *the base point of the complex affine frame* (y, w, u) and the map which associates the base point to any complex affine frame will be denoted by $\pi : A^{\mathbb{C}}(M) \to M$. Also, pair (y, w) may be considered as an element of the manifold of n-dimensional hyperplanes in tangent space of M, which is biholomorphic to $\mathbb{P}TM = TM/\mathbb{C}^*$ and which can be identified with $M \times Gr_n(\mathbb{C}^{n+1})$. The map which associates the element $(x, y, w) \in M \times M \times Gr_n(\mathbb{C}^{n+1})$ to an affine frame (y, w, u) at a point x, will be denoted by $\hat{\pi} : A_n^{\mathbb{C}}(M) \to M \times M \times Gr_n(\mathbb{C}^{n+1})$.

It is clear that $A_n^{\mathbb{C}}(M)$ is a $GL_n(\mathbb{C})$-bundle over $M \times M \times Gr_n(\mathbb{C}^{n+1})$ and hence it has a natural structure of complex manifold which makes the projection $\hat{\pi}$ a biholomorphism, with fibers biholomorphic to $GL_n(\mathbb{C})$ (see e.g. the remarks in Example 2.5, Ch.IX, Vol. II in [KN]). We denote by $\hat{J}$ the complex structure of $A_n^{\mathbb{C}}(M)$.

Furthermore, since $M \subset \mathbb{C}^{n+1}$, we have a natural embedding of $A_n^{\mathbb{C}}(M)$ into $A_n^{\mathbb{C}}(\mathbb{C}^{n+1})$, which is holomorphic.

At this point, we remark that for any $t \in (0, \infty]$, the bundle $U_{(t)}(M)$ has a natural immersion $\imath_t : U_{(t)}(M) \to A_n^{\mathbb{C}}(M)$ whose image $\tilde{U}_{(t)}(M) = \imath_t(U_{(t)}(M))$ is a subbundle of $A_n^{\mathbb{C}}(M)$ with structural group $U_n \subset GL_n(\mathbb{C})$ over the base $\hat{\pi} \circ \imath_t(U_{(t)}(M)) \subset M \times M \times Gr_n(\mathbb{C})$.

In order to define $\imath_t$, we first need to recall that, by construction, if $(x; u) \in U_{(t)}(M)$, with u linear frame at $T_x M$, then

$$u' = \{\exp_{x\,*}|_{tf_0}(f_0), \ldots, (\exp_{x\,*}|_{tf_{2n+1}}(f_{2n+1})\}$$

is a complex linear frame at $y = \exp_x(tf_0)$. So we may set

$$\imath_t : U_{(t)}(M) \to A_n^{\mathbb{C}}(M) \subset A_n^{\mathbb{C}}(\mathbb{C}^{n+1}) \ ,$$

$$\begin{aligned} \imath_t(x; u) = \big(x; \big(&\exp_x(tf_0), \operatorname{span}_{\mathbb{C}}\{\exp_{x\,*}|_{tf_0}(f_2), \ldots, (\exp_{x\,*}|_{tf_{2n+1}}(f_{2n+1})\}, \\ &u' = \{\exp_{x\,*}|_{tf_0}(f_2), \ldots, (\exp_{x\,*}|_{tf_{2n+1}}(f_{2n+1})\}\big)\big) \ . \end{aligned} \tag{II.4.2}$$

When $t = \infty$, for any (x, u), we set $\imath_\infty(x, u)$ equal to $\lim_{t\to\infty} \imath_\infty(x, u)$.

Now, for any $t \in [0, +\infty)$, we denote by $\mathcal{H}^{(t)}$ the distribution on $U_{(t)}(M)$ defined as follows:

$$X \in \mathcal{H}_u^{(t)} \subset T_u U_{(t)}(M) \qquad \Leftrightarrow \qquad \hat{J}(\imath_{t*}(X)) \in \imath_{t*}(T_u U_{(t)}(M)) \ . \tag{II.4.3}$$

In other words, $\mathcal{H}^{(t)}$ coincides with the distribution underlying the holomorphic distribution of the real submanifold $\tilde{U}_{(t)}(M) \subset A_n^{\mathbb{C}}(M)$.

Lemma 4.3. *For any $t \in (0, \infty]$ the distribution $\mathcal{H}^{(t)}$ verifies the following two properties at any $u \in U_{(t)}(M)$:*

1. $\dim_R \mathcal{H}_u^{(t)} = 2n + 2$*;*
2. $\mathcal{H}_u^{(t)} \cap \mathcal{V}_u^{(t)} = \{0\}$.

In particular, it determines a non-linear connection on $U_{(t)}(M)$ (according to Def. 3.7 in [Sp]*), which is U_n-equivariant.*

Proof. For any $x \in M$ let us denote by $\mathbb{L}_x$ the fiber of $A_n^{\mathbb{C}}(M)$ (or of $A_n^{\mathbb{C}}(\mathbb{C}^{n+1})$) over x under the projection map $(x; (y, w, u = \{f_2, \dots, f_{2n+2}\}) \mapsto x$.

Denote also by $\mathbb{V}_x = \mathbb{L}_x \cap \tilde{U}_{(t)}(M)$ the fiber over x of $\tilde{U}_{(t)}(M) = \imath_t(U_{(t)}(M))$. We claim that $\mathbb{V}_x$ is a totally real submanifold of $\mathbb{L}_x$. Notice that from this, it would follow (2) immediately. The proof of the claim is an immediate modification of the proof of Lemma 3.10 in [Sp], once we observe that the spheres $S_t(x)$ in M are all convex (For Stefano: *Actually, the proof requires that the spheres are strictly pseudoconvex; can we guarantee it?*).

To prove (1), it remains to observe that

$$\begin{gathered}\dim_{\mathbb{R}} A_n^{\mathbb{C}}(M) = 2(n+1) + (2(n+1) + 2n) + 2n^2 = 2(n+1) + 2(n+1)^2 \ , \\ \dim U_{(t)}(M) = 2(n+1) + (2n+1) + n^2 = \dim A_n^{\mathbb{C}}(M) - (n+1)^2 \ .\end{gathered}$$

¿From this it follows that

$$\begin{aligned}\dim A^{\mathbb{C}}(M) \geq \dim T_u U_{(t)}(M) + \dim \hat{J}_F T_u U_{(t)}(M) - \dim \mathcal{H}_u^{(t)} = \\ = 2 \dim A_n^{\mathbb{C}}(M) - 2(n+1)^2 - \dim \mathcal{H}_u^{(t)} \ ,\end{aligned}$$

i.e. that

$$\dim \mathcal{H}_u^{(t)} \geq \dim A_F^{\mathbb{C}}(M) - 2(n+1)^2 = 2(n+1)$$

and this proves (1).

The last claim follows from the fact that the action of $GL_n(\mathbb{C})$ (and hence of U_n) on $A_n^{\mathbb{C}}(M)$ is holomorphic and hence the subspaces $\mathcal{H}^{(t)}$ are U_n-equivariant by construction. □

Lemma 4.4. *For any $t \in [0, \infty]$, let us denote by $\mathcal{V}^{U_n}$ the distribution which is vertical w.r.t. the projection $\hat{\pi} : U_{(\infty)}(M) \to U_{(\infty)}(M)/U_n$. Then if X and Y are two vector fields of M such that $[X, Y] = 0$ and if $\tilde{X}, \tilde{Y}$ are the unique two vector fields in $\mathcal{H}^{(\infty)}$, which project onto X and Y, respectively, then*

$$[\tilde{X}, \tilde{Y}]_{(x,u)} \in \mathcal{V}^{U_n}_{(x,u)} \ .$$

In particular, the distribution $\mathcal{H}^{(\infty)} \oplus \mathcal{V}^{U_n}$ is integrable.

Proof. It suffices to show that

$$\hat{\pi}_* \circ \imath_{\infty *}([X, Y]) = [(\hat{\pi} \circ \imath_\infty)_*(X), J_o(\hat{\pi} \circ \imath_\infty)_*(Y)] = 0$$

where we used the symbol $\hat{\pi}$ also for the natural map

$$\hat{\pi} : \imath_\infty(U_{(\infty)}(M)) \to \imath_\infty(U_{(\infty)}(M))/U_n \subset M \times (\mathbb{C}^{n+1} \times Gr_n(\mathbb{C})) \ .$$

Since the distribution $\mathcal{H}^{(\infty)} = \imath_\infty|_*(\mathcal{H}^{(\infty)})$ is $\hat{J}$-invariant, and since the map $\hat{\pi}$ is holomorphic, it follows that $\hat{\pi} \circ \imath_{\infty *}(X)$ belongs to the holomorphic distribution of $\hat{\pi} \circ \imath_\infty(U_{(\infty)}(M)) \subset M \times (\mathbb{C}^{n+1} \times Gr_n(\mathbb{C}))$. However, by Webster's theorem in [We] and the fact that ∂M is strictly convex, we have that the holomorphic distribution $\hat{\pi} \circ \imath_\infty(U_{(\infty)}(M))$ projects isomorphically at any point onto the tangent spaces of M. From this and the hypothesis of $[X, Y] = 0$, we may conclude that $\hat{\pi}_* \circ \imath_{\infty *}([X, Y])$ (to be written better). □

(To be continued)

Appendix I

Theorem A1. *Let $D \subset \mathbb{C}^{n+1}$ be a bounded, strictly convex domain with smooth boundary and let*

$$Crit_{k,\alpha}(D) = \{\ f\ \ : \bar{\Delta} \to \bar{D}\ ,\ f \text{ is a } \mathcal{C}^\infty \text{ extension of a critical disc } f : \Delta \to D\ \}$$

endowed with the topology induced by the Banach space $\mathcal{C}^{k,\alpha}(\bar{\Delta}; \mathbb{C}^{n+1})$, $k \geq 1$ and $0 \geq \alpha < 1$.

Then $Crit_{k,\alpha}$ is a submanifold of $\mathcal{C}^{k,\alpha}(\bar{\Delta}; \mathbb{C}^{n+1})$ and the map

$$\Psi : S(D) \subset TD \longrightarrow Crit(D)$$

$$\Psi(x, v) = f_{x,v}$$

where $f_{x,v}$ is the unique critical disc such that $f_{x,v}(0) = 0$ and $v \in \mathbb{R}Re(f'_{x,v}(0))$, is a diffeomorphism.

Proof. Stefano will do it. □

Corollary A2. *Assume the hypothesis of Theorem A1. If $\{f_n\}$ is a sequence in $Crit_{k,\alpha}(D)$, then at least one of the following two possibilites may occur:*

(1) *there exists a subsequence $\{f_{n_k}\}$ and an element $f \in Crit_{k,\alpha}(D)$, so that f_{n_k} converges to f in $Crit_{k,\alpha}(D)$;*
(2) *there exists a subsequence $\{f_{n_k}\}$ which converges uniformily on compacta of Δ to a constant map $f(\zeta) \equiv y \in \partial D$.*

Proof. Since D is bounded, by Ascoli-Arzela theorem, there exists a subsequence $\{f_{n_k}\}$ converging uniformily on compacta to a holomorphic map $f : \Delta \to \bar{D}$. If $f(\Delta) \subset \partial D$, then case (2) occurs. In case there exists some ζ_o such that $x_o = f(\zeta_o) \in D$, we may consider a $\sigma \in Aut(\Delta)$ such that $(f \circ \sigma)(0) \in D$. Hence, if we denote by $\hat{f}_{n_k} = f_{n_k} \circ \sigma$, we have that the sequence

$$\{\ (\hat{f}_{n_k}(0), \frac{\mathrm{Re}(f'_{n_k}(0))}{|f'_{n_k}(0)|})\ \} \subset S(D) \subset TD$$

admits a converging subsequence to an element $(x_o, v) \in S_{x_o} \subset T_{x_o}D$. So, by Theorem A1, we have that $\hat{f}_{n_k}$ converges in $\mathrm{Crit}_{k,\alpha}$ to $(f \circ \sigma)$ and case (1) occurs. □

Lemma A3. *Let $\sigma : [-\varepsilon, \varepsilon] \to D$ and $\tau : [1-\varepsilon, 1] \to \bar{D}$ be two regular curves such that $\sigma_0 = x \in D$ and $\tau_1 = y \in \partial D$. For any $(r, s) \in [-\varepsilon, \varepsilon] \times [1-\varepsilon, 1]$, denote by*

$$F_{r,s} : \bar{\Delta} \to \bar{D}$$

the continuous extension to $\bar{\Delta}$ of the unique critical disc such that $F_{r,s}(0) = \sigma_r$ and $F_{r,s}(t) = \tau_s$, for some $t \in (0, 1]$. Let also denote by $\tilde{F}_{r,s}$ the corresponding two-parameter family of lifts in $T^{*10}D$.

Then for any $r_o \in [-\varepsilon, \varepsilon]$ the 1-parameter families of critical discs $V_s = F_{r_o,s}$ and of lifts $\tilde{V}_s = \tilde{F}_{r_o,s}$ converge uniformly on $[1-\varepsilon, 1] \times \bar{\Delta}$ to the critical disc $V_1 = F_{r_o,1}$ and to the lift $\tilde{V}_1 = \tilde{F}_{r_o,1}$, respectively.

Proof. Ask Stefano.!!! □

Appendix II

In all the following, (M, J, F) is a complex Finsler manifold. All notation is the same adopted in [Sp] and [Sp1] and we refer the reader to those papers for it.

Lemma AII.1. *Let (M, J, F) be geodetically torsion free and of constant holomorphic sectional curvature equal to c. Let also $\dot{v}_\gamma$ be a local vector field on $\mathbb{P}TM$ which extends the vector $\dot{v}_\gamma$ at the points of a geodesic γ as defined in §5 in* [Sp1]. *Let also denote by $\dot{\gamma}$ the projection $\pi'_*(\dot{v}_\gamma)$ (which is clearly an extension of $\dot{\gamma}$ to a vector field defined on a neighborhood of the image of γ). We also assume that γ is parametrized by arc length. Then for any field $\hat{Y}$ on $\mathbb{P}TM$, if we denote by $\mathcal{X}_\gamma$, $\mathcal{Y}$ and $\mathcal{X}_{J\dot{\gamma}}$ three vector fields on $U_F(M)$ which project onto $\dot{v}_\gamma$, $\hat{Y}$ and $J\dot{\gamma}$, respectively, we have that at any point γ_t:*

a)

$$T_{\hat{Y}\dot{v}_\gamma}(v_\gamma) = u_\gamma\left(\sum_{\lambda=1}^{n-1}(T^\lambda_{\mu 0}\theta^\mu(\mathcal{Y}))\theta^0(\mathcal{X}_\gamma)e_\lambda + \sum_{\lambda=1}^{n-1}(T^{\bar{\lambda}}_{\bar{\mu}\bar{0}}\theta^{\bar{\mu}}(\mathcal{Y}))\theta^{\bar{0}}(\mathcal{X}_\gamma)e_{\bar{\lambda}}\right)\ ; \quad \text{(AII.1)}$$

in particular

$$g_{v_\gamma}(T_{\hat{Y}\dot{v}_\gamma}(v_\gamma), \dot{\gamma}) \equiv 0\ , \qquad g_{v_\gamma}(T_{\hat{Y}\dot{v}_\gamma}(v_\gamma), J\dot{\gamma}) \equiv 0\ ; \quad \text{(AII.2)}$$

b)

$$g_{v_\gamma}(R_{\dot{v}_\gamma\hat{Y}}\dot{\gamma}(v_\gamma), \dot{\gamma})|_{\gamma_t} \equiv 0 \quad \text{(AII.3)}$$

and

$$g_{v_\gamma}(R_{\dot{v}_\gamma\hat{Y}}\dot{\gamma}(v_\gamma), J\dot{\gamma})|_{\gamma_t} \equiv 2cg_{v_\gamma}(\pi'_*(\hat{Y}), \dot{\gamma})\ . \quad \text{(AII.4)}$$

Proof. a) Since γ is a geodesic, for any $u_{\gamma_t} \in \hat{\pi}^{-1}(v_{\gamma(t)})$ we have that for any $1 \leq \lambda \leq n-1$

$$\theta^0_{u_{\gamma_t}}(\mathcal{X}_\gamma) = \dot{\gamma}^{10}_t\ , \qquad \theta^{\bar{0}}_{u_{\gamma_t}}(\mathcal{X}_\gamma) = \dot{\gamma}^{01}_t\ , \qquad \theta^\lambda_{u_{\gamma_t}}(\mathcal{X}_\gamma) = \theta^{\bar{\lambda}}_{u_{\gamma_t}}(\mathcal{X}_\gamma) = 0\ ,$$

$$\varpi^\lambda_0|_{u_{\gamma_t}}(\mathcal{X}_\gamma) = \varpi^0_\lambda|_{u_{\gamma_t}}(\mathcal{X}_\gamma) = 0\ .$$

(see [Sp]; see also [Sp1] (5.6)). Therefore

$$u^{-1}(T_{\hat{Y}\dot{v}_\gamma}(v_{\gamma(t)})) = \Theta(\mathcal{Y}, \mathcal{X}_\gamma) + \Sigma(\mathcal{Y}, \mathcal{X}_\gamma) =$$

$$= \sum_\lambda T^\lambda_{\mu 0}\theta^\mu(\mathcal{Y})\theta^0(\mathcal{X}_\gamma)e_\lambda + \sum_{\bar{\lambda}} T^{\bar{\lambda}}_{\bar{\mu}\bar{0}}\theta^{\bar{\mu}}(\mathcal{Y})\theta^{\bar{0}}(\mathcal{X}_\gamma)e_{\bar{\lambda}} \quad \text{(AII.5)}$$

by the definition of Σ and of geodetically torsion free manifolds. ¿From (AII.5), the conclusion follows immediately.

b) Since γ is parametrized by arc length, $2\theta^0(\mathcal{X}_\gamma)\theta^{\bar{0}}(\mathcal{X}_\gamma) \equiv 1$, and hence using (4.4) in [Sp1], we have that

$$g_{v_\gamma}(R_{\dot{v}_\gamma\hat{Y}}\dot{\gamma}(v_\gamma), \dot{\gamma})|_{\gamma_t} = 2\mathrm{Re}\left[\Omega^0_0(\mathcal{X}_\gamma, \mathcal{Y})\theta^0(\mathcal{X}_\gamma)\theta^{\bar{0}}(\mathcal{X}_\gamma)\right]_{\gamma_t} = \mathrm{Re}\left[\Omega^0_0(\mathcal{X}_\gamma, \mathcal{Y})\right]_{\gamma_t} =$$

$$= \left[\Omega^0_0(\mathcal{X}_\gamma, \mathcal{Y}) + \Omega^{\bar{0}}_{\bar{0}}(\mathcal{X}_\gamma, \mathcal{Y})\right]_{\gamma_t} \equiv 0 ,$$

where the last step follows from the identity (5.40) in [Sp].

Similarly

$$g_{v_\gamma}(R_{\dot{v}_\gamma \hat{Y}}\dot{\gamma}(v_\gamma), J\dot{\gamma})|_{\gamma_t} = 2\mathrm{Re}\left[\Omega^0_0(\mathcal{X}_\gamma, \mathcal{Y})\theta^0(\mathcal{X}_\gamma)\theta^{\bar{0}}(\mathcal{X}_{J\dot{\gamma}})\right]_{\gamma_t} =$$

$$= -\sqrt{-1}\Omega^0_0(\mathcal{X}_\gamma, \mathcal{Y})|_{\gamma_t} + \sqrt{-1}\Omega^{\bar{0}}_{\bar{0}}(\mathcal{X}_\gamma, \mathcal{Y})|_{\gamma_t} = 2cg_{v_\gamma}(\pi'_*(\hat{Y}), \dot{\gamma}) ,$$

where the last identity follows from the results on the components of the curvature 2-form Ω^0_0 of an E-manifold as determined in Theorem 6.9 (1). □

Lemma AII.2. *Let (M, J, F) be an E-manifold and $\gamma = [-a, a] \to M$ be a geodesic through the point $p_o = \gamma_0$. Then the vector field along γ defined by*

$$I_\gamma|_{\gamma_t} = tJ \cdot \dot{\gamma}_t \tag{AII.6}$$

is a Jacobi vector field.

Proof. Let $p_o = \gamma_0$ and let $v = \dot{\gamma}_0 \in T_xM$. Since M is an E-manifold, then there exists a unique complex geodesic $f : \Delta \to M$, which passes through p_o and it is tangent to v. Since $f(\Delta)$ is a totally geodesic submanifold, γ is included in $f(\Delta)$ and it can be expressed as

$$\gamma_t = f(t) \ , \ t \in [-a, a] \in \mathbb{R}$$

Since the curves $e^{is} \cdot t$ are geodesics in Δ, the map

$$V : (-\delta, \delta) \times [-a, a] \to M$$

$$V(s, *) = f(e^{is} \cdot t)$$

is a 1-parameter family of geodesics of M centered at γ. In particular the vector field

$$I_\gamma = f_*(tJ_o \cdot \frac{\partial}{\partial t}) = J \cdot f_*(t\frac{\partial}{\partial t}) = tJ \cdot \dot{\gamma}_t$$

is a Jacobi vector field. □

Lemma AII.3. *Let M be an E-manifold of constant sectional curvature equal to c. For any $v \in \mathbb{P}_t xM$, let g_v the scalar product on T_xM defined by (1.8). Then if I is a Jacobi vector fields along the geodesic γ, then the following properties hold:*

(a) $\frac{d^2}{dt^2}g_{v_\gamma(t)}(I, \dot{\gamma}_t) = 0$ *and* $\frac{d^2}{dt^2}g_{v_\gamma(t)}(I, J\dot{\gamma}_t) - 2cg_{v_\gamma(t)}(I, \dot{\gamma}_t) = 0$;

(b) *if there exists a point t_o so that $g_{v_\gamma(t_o)}(I, \dot{\gamma}_{t_o}) = g_{v_\gamma(t_o)}(D_{\dot{v}_\gamma}I|_{v_\gamma}, \dot{\gamma}_{t_o}) = g_{v_\gamma(t_o)}(D_{\dot{v}_\gamma}I|_{v_\gamma}, J\dot{\gamma}_{t_o}) = 0$, then I is orthogonal to the subspace spanned by $\dot{\gamma}$ and $J\dot{\gamma}$ at all points γ_t.*

Proof. (a) From Prop. 3.4 and Th. 5.1 in [Sp1], we have that

$$\frac{d^2}{dt^2}g_{v_\gamma(t)}(I, \dot{\gamma}_t) = g_{v_\gamma(t)}(D_{\dot{v}_\gamma}\left(D_{\dot{v}_\gamma}I|_{v_\gamma}\right)|_{v_\gamma}, \dot{\gamma}_t) ,$$

$$\frac{d^2}{dt^2} g_{v_\gamma(t)}(I, J\dot{\gamma}_t) = g_{v_\gamma(t)}(D_{\dot{v}_\gamma}\left(D_{\dot{v}_\gamma} I|_{v_\gamma}\right)|_{v_\gamma}, J\dot{\gamma}_t) \ .$$

Then the conclusion follows easily by considering the equation (6.1) in [Sp1], verified by I and from Lemma AII.1 and Th.5.1 in [Sp1] again.

(b) Notice that

$$g_{v_\gamma(t_o)}(D_{\dot{v}_\gamma} I|_{v_\gamma}, \dot{\gamma}_{t_o}) = \left.\frac{d}{dt} g_{v_\gamma(t)}(I, J\dot{\gamma}_t)\right|_{t_o} \ .$$

Hence, from (a), it follows that I is orthogonal to $\dot{\gamma}$ at all points. Then using the second equation of (a), it follows also that I is orthogonal to $J\dot{\gamma}$ as well at all points. $\square$

In the following, we will consider the 'exponential map' exp which is defined exactly as in the classical Riemannian or Hermitian case. In fact, for any point $x \in M$ and any vector $v \in T_xM$ we set

$$\exp_x(v) = \gamma_1$$

where γ is the unique geodesic such that $\gamma_0 = x$, $\dot{\gamma}_0 = v$. Note that we have also that

$$\gamma_t = \exp_x(tv)$$

for any t belonging to the interval where γ is defined.

Lemma AII.4. *Let γ be a geodesic with $\dot{\gamma}_0 = kv \in T_xM$, where $F(v) = 1$. Let also $w \in T_v(S_x) \subset T_v(T_xM) \simeq T_xM$, where S_x is the Finsler sphere of unit length vectors at x. Then*

1. *if we identify w with a vector tangent to the sphere $k \cdot S_x$ of radius k, then $\exp_{x\,*}(w)$ is tangent to the geodesic sphere $S_x^{(k)}$ of radius k and it is equal to the value of I_w at γ_1, where I_w is the unique Jacobi field along γ such that $I_w(0) = 0$ and $D_{\dot{v}_\gamma} I_w|_{v_\gamma(0)} = w$;*
2. *if w and Jw are both tangent to $T_v(S_x)$, then $\exp_{x\,*}(w)$ and $J\exp_{x\,*}(w)$ are both tangent to the geodesic sphere $S_x^{(k)}$ (but in general $J\exp_{x\,*}(w) \neq \exp_{x\,*}(Jw)$!!!)).*

Proof. (1) Consider the 1-parameter family of geodesics defined by

$$V(s,t) = \exp_x(tv + sw) \ , \tag{AII.7}$$

which is clearly centered at $\gamma_t = \exp(tv)$.

Then, by definitions, $\exp_{x\,*}(w)$ is equal to the value at γ_1 of the Jacobi field I_w associated with the family (AII.7). Note also that, since $w \in T_v(S_x)$, the initial speed of the geodesics $\gamma_t^{(s)} = \exp_x(tv + sw)$ verifies

$$F(\dot{\gamma}_0^{(s)}) = F(\dot{\gamma}_0) + O(s)$$

and hence

$$dist(x, \dot{\gamma}_1^{(s)}) = dist(x, \gamma_0) + 0(s) = k + 0(s)$$

This implies that $I_w|_{\gamma_1} = \frac{d}{ds}(\gamma_1^{(s)})|_{s=0}$ is tangent to the geodesic sphere of radius k.

¿From definitions, it is also clear that $I_w(0) = 0$. It remains to show that $D_{\dot v_\gamma} I_w|_{v_\gamma}(0) = w$. Let $\mathcal{I}$ and $\mathcal{X}_\gamma$ be two vector fields on $U_F(M)$ which project onto I and $\dot v_\gamma$, respectively, and a frame $u \in \hat\pi^{-1}(v_\gamma(0))$ and let us compute $D_{\dot v_\gamma} I_w|_{v_\gamma(0)}$ using Def. 3.2 in citeSp1: we have

$$D_{\dot v_\gamma} I_w|_{v_\gamma(0)} = u(\mathcal{X}_\gamma(\theta(\mathcal{I})) + \omega_u(\mathcal{X}_\gamma)\cdot\theta(\mathcal{I})) = u(\mathcal{X}_\gamma(\theta(\mathcal{I}))\ , \tag{AII.8}$$

because $\theta(\mathcal{I}) = u(I_w(0)) = 0$. However, it simple to realize that (AII.8) is equal to

$$u(\mathcal{X}_\gamma(\theta(\mathcal{I})) = \frac{\partial}{\partial}\left(\frac{\partial}{\partial s}(V(s,t))\right)\Bigg|_{t=0,s=0} = \frac{d}{ds}\left(\dot\gamma_0^{(s)}\right)\Bigg|_{s=0} = \frac{d}{ds}\left(kv+sw\right)\Bigg|_{s=0} = w$$

and this proves (1).

(2) From (1) and Lemma AII.3 (b), for any $w \in T_v(k\cdot S_x)$, the vector $w' = \exp_{x\,*}(w)$ is tangent to the geodesic sphere $S_x^{(k)}$ and verifies

$$g_{v_\gamma(1)}(w', \dot\gamma_1) = 0$$

where g_v is the Hermitian inner product defined in (1.8). Since $\exp_x$ gives a diffeomorphism with the sphere $k\cdot S_x$ of radius k and the geodesic sphere $S_x^{(k)}$, we conclude that

$$T_{\gamma_1} S_x^{(k)} = \{\ w \in T_{\gamma_1}M\ \ : g_{v_\gamma(1)}(w, \dot\gamma_1) = 0\ \}\ .$$

In particular, this implies that the maximal J-invariant subspace $\mathcal{D}_{\gamma_1}^{(k)} \subset T_{\gamma_1} S_x^{(k)}$ is equal to

$$\mathcal{D}_{\gamma_1}^{(k)} = \{\ w \in T_{\gamma_1}M\ \ : g_{v_\gamma(1)}(w, \dot\gamma_1) = g_{v_\gamma(1)}(w, J\dot\gamma_1) = 0\ \}\ .$$

So the claim follows, if we can check that if w and Jw are both $T_v(k\cdot S_x)$, then the corresponding vectors $\exp_{x\,*}(w)$ and $\exp_{x\,*}(Jw)$ are orthogonal to both $\dot\gamma_1$ and $J\dot\gamma_1$. But this follows immediately from (1) and Lemma AII.3 (b). $\square$

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