Complexification of group actions

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Abstract

In this paper we at first give a survey of some results in [15], and later we give a proof of a partial converse of the Theorem stated in the first part.

1 Introduction

The geometric invariant theory of Mumford and its interpretation in terms of moment maps of hamiltonian actions, is dealing essentialy with the holomorphic (algebraic) action of a reductive complex Lie group on a complex space see [12]. A reductive complex group is the complexification of a compact real Lie group see [8]. Suppose that a complex space X is endowed with an action of a compact Lie group by holomorphic tranformations. In order to study this in the framework of geometric invariant theory it is convenient to embed X as an open K invariant subset of a "good" complex space X^* with an holomorphic action of the reductive complexified group $K^{\mathbf{C}}$. By the results of Palais [13] and Heinzner-Iannuzzi [7], in a broad varaity of situations a canonical space X^* as above can be constructed as a non necessarly Hausdorff complex space. However Heinzner proved in [6] that if X is Stein, then X^* exists is Hausdorff and in fact it is a Stein space. Later Heinzer and Iannuzzi proved that if X is holomorphically convex then X^* exists is Hausdorff and holomorphically convex. Iannuzzi recently dealt in [9] with the case of the action of some non compact group on Taut manifolds. He proved that also in such cases X^* exists and it is Hausdorff. In the first part of this paper we give a survay of the results in [15] where the existence of an Hausdorff complexification is proved when the space X has some kind of weak pseudoconveity. In this case the same pseudoconvexity is inherited by the complexification X^*. In the second part of this article we consider the case $K = S^1$ and prove a partial converse of the main Theorem stated in the first part.

2 Preliminaries

2.1 Local actions

In this section we give a quick exposition of the Palais theory in [13] on local actions of Lie groups and on its adaptation to the complex analytic setting given in [7]. For more details we refer to the original papers.

Let X be a reduced complex space, let G be a connected Lie group,

A local action of G on X is given by a real analytic map $\Phi : \Omega \to X$ where Ω is an open neighborhood of the neutral section $e \times X$ in $G \times X$ with the following properties:

i) For all $x \in X$ the open subset $\Omega_x = \{g \in G : (g, x) \in \Omega\}$ is connected
ii) For fixed $g \in G$ the map $x \to \Psi(g, x) = gx$ is holomophic when defined
iii)For every $g, h \in G$ and $x \in X$ we have $ex = x$
and if
$(gh, x) \in \Omega,\ (h, x) \in \Omega$, and
$(g, hx) \in \Omega,$ then
$(g, hx) = (gh, x)$.

For a complex Lie group the local action is said to be holomorphic if Ψ is holomorphic.

To every local action of G we can associate a linear map ξ from the Lie algebra g of G to the algebra of holomorphic vector fields on X given by $v \to \frac{\partial exp(tv)x)}{\partial t}_{|t=0}$. This map is an homomorphism of Lie algebras. Conversely every such homomorphism induces a local action. If G is a complex Lie group, the local holomorphic G action corresponds to $\mathbf{C}$ linear homomorphisms. A product $N \times U$ where N is an open connected neighborhood of e in G and U is an open set of X is called an elementary slice pair if $N = N^{-1}$, $N^2 \times U \subseteq \Omega$ and $N^2 \times N^2 U \subseteq \Omega$. Given an elementary slice pair $N \times U$ and $h \in G$ we define as a slice chart the map $\Phi : Nh \times U \to G \times X$ defined as $\Phi(g, x) = (g, gh^{-1}x)$. The map Φ is then an homoeomorphism of $Nh \times U$ onto an open set of $G \times X$.

A subset V is a neighborhood of (h_0, y_0) for the leaf topology in $G \times X$ if and only if there exists a slice chart Φ such that $(h_0, y_0) \in \Phi(Nh \times x_0) \subseteq V$ for some $x_0 \in U$. A connected component of $G \times X$ in the leaf topology is called a leaf of the local action. If X is smooth then the leaves are the maximal integral submanifolds of the distribution $\xi(g)$ coming from the diagonal local action of G on $G \times X$ obtained from the left action of G on itself. Let us consider the space

$$X^* = G \times X \ / \ \text{ modulo leaves}$$

with the quotient topology. We call X^* the Palais globalization of the local action. Note that the leaves are invariant with respect to the diagonal local action. Moreover the right action of G on the first factor of $G \times X$ induces a global action of G on X^*. Let $\Pi : G \times X \to X^*$ the natural projection and let $j : X \to X^*$ given by $j(x) = \Pi(e, x)$. Then both Π and j are continuous open mappings, see [7]. Moreover if G is complex each leaf is a complex locally closed submanifold of $G \times X$ and the restriction to the leaf of the projection on the first factor is a locally biholomorphic map. The local action is said to be univalent if for each leaf the above local biholomorphic map is injective. In other words (since the right action of G on the first factor of $G \times X$ tranforms leaves into leaves) the action is univalent if and only if j is injective. For a univalent local action we can use the slice charts to define an holomorphic atlas on X^* which makes X^* a possibly non Hausodorff complex space. For notational convenience a non necessarly Hausdorff complex space will be called in this paper a quasi complex space. The map j becomes then an holomorphic open locally G equivariant embedding. Note that if Y is a quasi complex space with a global G

action the map $(g, y) \to g^{-1}y$ induces a G equivariant biholomorphism of Y^* onto Y. We also have the following

Lemma 2.1 *The space X^* has the following universal property. If Y is a quasi complex space with an holomorphic G action and $f : X \to Y$ a locally G equivariant holomorphic map, then there exists a unique holomorphic G equivariant map $f^* : X^* \to Y$ such that $f^* \circ j = f$. Moreover $Gj(X) = X^*$.*

Proof.
see Theorem 2 of section 3 in [7]. ■

Any quasi complex space with a G action satisfing the condition of the Lemma is called a universal globalization of X. Clearly a universal globalization if it exists it is unique. A local G action is univalent if and only if it admits some globalization, in such case it admits a universl globalization, i.e. X^*, in the category of quasi complex spaces. There are however example of spaces with a local G action which admit a universal globalization in the category of Hausdorff complex spaces which is different from X^*.

We now consider the case where there exists a global G action on the complex space X by holomorphic tranformation. Assume moreover that the Lie algebra of $G^{\mathbf{C}}$ is the complexification of the Lie algebra of G. This is the case for example if G is compact of simply connected. The global action of G induces a local holomorphc action of the complexified group $G^{\mathbf{C}}$.

Lemma 2.2 *If there exists a global action on X of the group $(\mathbf{R}, +)$, then the local $(\mathbf{C}, +)$ action is univalent. Similarly if there exists a global action of S^1 then the local action of $\mathbf{C}^*$ is univalent.*

Proof. In the case of $(\mathbf{R}, +)$ let F be an $(\mathbf{R}, +)$ equivariant connected Riemann domain over $\mathbf{C}$, then the action of $(\mathbf{R}, +)$ on F is free and proper. The quotient $Fmod\ \mathbf{R}$ is then a connected non compact one dimensional real manifold and the induced map from $Fmod\ \mathbf{R}$ into $\mathbf{R}$ is a local diffeomorphism, hence it is injective. Therefore by $(\mathbf{R}, +)$ equivariance and freeness the original map is also injective. A similar proof works for the case of S^1. ■

Definition 2.3 *Let G be a connected Lie group with Lie algebra g. The group G is said to by polar if the map from $G \times g$ into $G^{\mathbf{C}}$ defined as $(g, v) \to gexp(iv)$ is a diffeomorphism.*

For example $(\mathbf{R}^n, +)$ is polar and every compact connected Lie group is polar.

Definition 2.4 *Let G be a real connected polar Lie group with Lie algebra g. Let $G^{\mathbf{C}}$ be the complexification of G. Let Y be a quasi complex space with a $G^{\mathbf{C}}$ action such that X is a G invariant open subset of Y and $G^{\mathbf{C}}X = Y$. We say that X is orbit connected in Y if for every $x \in X$ the set*

$$\Omega_x = \{g \in G^{\mathbf{C}} : gx \in X\}$$

is connected. We say that X is orbit convex in Y if for every $x \in X$ and $v \in g$ such that $exp(iv)x \in X$ one has $exp(itv)x \in X$ for every t in the interval $[0, 1]$.

Lemma 2.5 *Let G be a real connected polar Lie group with Lie algebra g. Let $G^{\mathbf{C}}$ be the complexification of G. Let Y be a quasi complex space with a $G^{\mathbf{C}}$ action such that X is a G invariant open subset of Y and $G^{\mathbf{C}}X = Y$. Then Y is $G^{\mathbf{C}}$ equivariantly biholomorphic to the Palais complexification X^* of X if and only if X is orbit connected in Y.*

Proof. See [1]. ■

Lemma 2.6 *Let K be a compact connected Lie group, let F be a connected complex manifold with a K action. Let $\pi : F \to K^{\mathbf{C}}$ be a K equivariant Riemann domain projection, (where K acts on $K^{\mathbf{C}}$ by left multiplication). Then π is injective if and only if F is holomorphically separeted.*

Proof.
See the proof of Proposition 3 of [7]. ■

Corollary 2.7 *Let X be a holomorphically separeted complex space, let K be a compact Lie group acting on X by biholomorphisms, then the local action of $K^{\mathbf{C}}$ is univalent.*

2.2 P-pseudoconvex spaces

Definition 2.8 *As in [2] and [11] we say that a complex space X is P-pseudoconvex If it is convex with respect to the family of plurisubharmonic functions, i.e. if for every compact set H the hull $\tilde{H}$ of H with respect to the family of plurisubharmonic functions is relatively compact. The complex space X is said to be weakly pseudoconvex if it carries a plurisubharmonic exaustion function, and for $0 \leq k \leq \infty$ it is C^k weakly pseudoconvex if it carries a C^k plurisubharmonic exaustion function.*

Note that X is C^0 weakly pseudoconvex if and only if it is convex with respect to the family of continuous plurisubharmonic functions, see [14].

Examples 2.9 *Every Stein space is C^∞ weakly pseudoconvex, more generally by using the Remmert reduction one can see that every holomorphically convex space is C^∞ weakly pseudoconvex.*

Example 2.10 *Let D be a relativly compact domain in a complex manifold X and assume that for a given $k \geq 2$ the domain D has a C^k defining function on X which is plurisubharmonic in a neighborhood of ∂D, then D is C^0 weakly pseudoconvex. In [5] Grauert constructs an example af a D as above which is not holomorphically convex.*

Example 2.11 *Let X be a compact complex manifold and L an holomorphic line bundle on X. Let $k \geq 2$ and h be a C^k hermitian metric on L with non negative curvature. Let D^* be the set of vectors in the total space of L^* having norm smaller then 1 with respect to the dual metric h^*. Then D^* is C^0 weakly pseudoconvex. Note that, as observed by Grauert, given an holomorphic function f on D^*, the degree h homogeneus component of the Taylor expansion of f along the fibers gives an holomorphic section of L^h.*

Example 2.12 *Let X and L as above, assume that $c_1(L)$ belongs to the closure of the cone of (Chern classes of) effective line bundles in $H^2(X, \mathbf{R})$. Then L has a semipostive singular metric, see [3]. Then the domain D^* defined as above is weakly pseudoconvex. Suppose moreover that the domain D^* admits some non constant holomorphic function, hence for some k the bundle L^k admits some non trivial holomorphic section s. We can consider on L the singular metric $h = (\frac{1}{\sum_j |\theta(s_j)|^2})^{1/k}$, where θ is a local trivialization, and $s_1, \dots s_N$ is a base of $H^0(X, L^k)$. So D^* is C^0 weakly pseudoconvex.*

3 Main Theorem

In [15] we prove the Theorem below.

Theorem 3.1 *Let X be a P-pseudoconvex space, let K be a compact connected Lie group acting on X by biholomorphisms. Assume that the local action of $K^{\mathbf{C}}$ is univalent. Than the Palais globalization X^* of X is a Hausdorff complex space. Moreover X is orbit convex in X^*, and the holomorphic functions on X can be approximated by holomorphic functions on X^* in the compact-open topology. If X is weakly pseudoconvex or C^0 weakly pseudoconvex then so is X^*. Moreover if X is smooth and C^k weakly pseudoconvex with $k \geq 2$, then so is X^*.*

It is however not clear when a local $K^{\mathbf{C}}$ action on P-pseudoconvex space is univalent. We know that this is the case if X is holomorphically separeted, but holomorphically separeted P-pseudoconvex spaces are very closed to being Stein, hence in this case there is not much new. In the case of general P-pseudoconvex spaces we proved in [15] the following partial results.

Proposition 3.2 *Let K be a compact connected Lie group. Let $K^{\mathbf{C}}$ be the complexification of K. Let Z be a complex space with a K action such that the local $K^{\mathbf{C}}$ action is univalent. Let X be a P-pseudoconvex space with a K action, assume that there exists a locally injective holomorphic K equivariant map $f : X \to Z$. Then the local $K^{\mathbf{C}}$ action on X is univalent.*

Proof.

Let Y be the (non necessarly Hausdorff) Palais complexification of Z. Apply Proposition 4.1 of [15] to the pair (X, Y).

■

On the other hand as observed in [15] we have

Proposition 3.3 *Let Y be a P-pseudoconvex space, let K be a compact connected Lie group so that $K^{\mathbf{C}}$ is acting localy in a univalent way on Y. Let X be a complex space with an action of K. Let $\phi : X \to Y$ be a proper holomorphic K equivariant map. Then X is P-pseudoconvex, and the local action of $K^{\mathbf{C}}$ on X is univalent.*

The only conclusive result about univalence is obtained in [15] in the commutative case, i.e.

Proposition 3.4 *Let X be a P-pseudoconvex space, let T be a torus acting on X by biholomorphisms. Then the local action of $T^{\mathbf{C}}$ is univalent.*

4 Partial converse

We now desire to understand at what extend the main Theorem stated above can be reversed. We consider here the case $K = S^1$. Recall that in this case the local $\mathbf{C}^*$ is univalent, hence the Palais leaves are closed complex analytic submanifolds of complex dimension 1 in $\mathbf{C}^* \times X$. In this section we are goig to prove the following

Theorem 4.1 *Let X be a complex manifold with an holomorphic action of S^1. Consider the diagonal action of S^1 on $Y = \mathbf{C}^* \times X$. Let X^* be the Palais globalization of X, with respect to the local action of $\mathbf{C}^*$. (Such globalization exists since the local action of $\mathbf{C}^*$ coming from a global action of S^1 is allways univalent). Then X^* is Hausodrff if and only if there exists a continuous function ϕ on Y such that ϕ is an exaustion and for every F Palais leaf in $\mathbf{C}^* \times X$ the restriction of ϕ to F is subharmonic.*

Remark 4.2 *Observe that for a general compact group K the analogous of the above Theorem, as it stands, can not be true. In fact, since each Palais leaf is a Riemann domain over the Stein manifold $K^{\mathbf{C}}$; the existence of a function ϕ as in the Theorem would immediatly imply that each leaf is Stein. If $dim K > 1$ this is not necessarly the case even for manifolds with Hausdorff globalizations.*

Let $\Pi : \mathbf{C}^* \times X \to X^*$ the canonical projection, let $G : \mathbf{C}^* \times X \to C^* \times X^*$ given by $G(z, x) = (z, \Pi(z, x))$. Then G is holomorphic and by univalence of the local action of $\mathbf{C}^*$ it is injective. Let $\Omega = G(\mathbf{C}^* \times X)$. Then $\Omega = \{(z, p) \in \mathbf{C}^* \times X^* : zp \in X\}$. In fact if (z, p) is in Ω then $p = \Pi(z, x)$ and $zp = \Pi(z/z, x) = \Pi(1, x) \in X$. Viceversa if $zp \in X$ and $p = \Pi(w, x)$ then $zp = \Pi(w/z, x) = \Pi(1, y)$ with $y \in X$. Therefore $\Pi(w, x) = \Pi(z, y)$. So $p = \Pi(z, y)$ and $(z, p) = G(z, y)$. From above we also see that $G^{-1}(z, p) = (z, zp)$. Hence G is a biholomorphism of $\mathbf{C}^* \times X$ with $\Omega \subseteq \mathbf{C}^* \times X^*$. Note that $\mathbf{C}^*\Omega = \mathbf{C}^* \times X$ and Ω is orbit convex in $\mathbf{C}^* \times X$. By the Lemma 2.5 the Palais globalization Ω is $\mathbf{C}^* \times X$.

Therefore the above theorem is equivalent to the following

Theorem 4.3 *(second version)*

The Palais globalization of Ω with respect to the local $\mathbf{C}^*$ action is Hausodorff if and only if there exists a continuous exaustion of Ω which is subharmonic in the $\mathbf{C}^*$ direction.

Let $\mathcal{F}$ be the family of continuous functions in Ω which are subharmonic along the $\mathbf{C}^*$ direction. Let $\mathcal{F}_{S^1}$ be the family of S^1 functions in $\mathcal{F}$. We say that Ω is $\mathcal{F}$ convex if for every compact set K in Ω the hull of K with respect to $\mathcal{F}$, i.e. the set $\hat{K} = \{p \in \Omega : \phi(p) \leq Max_{q \in K} \phi(q) \text{ for every } \phi \in \mathcal{F}\}$ is compact.

Lemma 4.4 *The space Ω admits a continuous exaustion which is subharmonic along the $\mathbf{C}^*$ directions if and only if it is $\mathcal{F}$ convex.*

Proof.

The Lemma is proved in [14] when $\mathcal{F}$ is the family of all plurisubharmonic continuous functions, however the same proof works in our case. ■

Therefore the above theorem is equivalent to the following

Theorem 4.5 *(third version)*
The Palais globalization of Ω with respect to the local $\mathbf{C}^*$ action is Hausodorff if and only if Ω is $\mathcal{F}$ convex.

This is the Theorem which we will prove.

Lemma 4.6 *Let $K \subset \Omega$ be a compact S^1 invariant subset, then the hull $\hat{K}$ of K with respect to $\mathcal{F}$ coincides with the hull K^* of K with respect to $\mathcal{F}_{S^1}$. In particular $\hat{K}$ is also S^1 invariant.*

Proof. Clearly $\hat{K} \subseteq K^*$. Viceversa if $p \in K^*$ and $\phi \in \mathcal{F}$, define $\phi^*(x) = max_{z \in S^1} \phi(zx)$. Now $\{\phi^* > \alpha\} = S^1\{\phi > \alpha\}$. Moreover let $x_n \to x$ and $\phi^*(x_n) \to l = \limsup_{y \to x} \phi^*(y)$. Let $z_n \in S^1$ such that $\phi^*(x_n) = \phi(z_n x_n)$. Then up to subsequences $z_n \to z_0 \in S^1$ and $l = \phi(z_0 x) \leq \phi^*(x)$. Therefore ϕ^* is continuous. Hence $\phi^* \in \mathcal{F}_{S^1}$ and $\phi \leq \phi^*$, moreover since K is S^1 invariant $max_K \phi = max_K \phi^*$. Therefore $\phi(p) \leq \phi^*(p) \leq max_K \phi^* = max_K \phi$. i.e. $p \in \hat{K}$. ■

Lemma 4.7 *Let X be a Stein manifold with an holomorphic S^1 action, let X^* be its (Stein) Palais complexification, and let ϕ be a plurisubharmonic S^1 invariant function of X. Then for every $x \in X$ the function $\Phi(t) = \phi(e^t x)$ is convex in t.*

Proof.
The result is known in the case where ϕ is smooth and strictly plurisubharmonic, see [6]. In the general case, since X is Stein we know from [4] that there is a sequence ϕ_n of smooth strictly plurisubharmonic functions on X pointwise decreasing to ϕ. Integration on the S^1 orbits and using Dini theorem, we find a sequence of smooth stricly plurisubharmonic S^1 invariant functions pointwise decreasing to ϕ. The result follows. ■

Proof of the theorem.
Assume that Ω is $\mathcal{F}$ convex, and by contraddiction that $\Omega^* = \mathbf{C}^* \times X^*$ is not Hausdorff. By acting with $\mathbf{C}^*$ we can then assume that there are two distinct points, $x_1 \in \Omega$ and $x_2 \in \Omega^*$ which do not have disjoint open neighborhoods. Let

$$\Delta_j = \{t \in \mathbf{R} : e^t x_j \in \Omega\} \text{ for j=1,2.}$$

Note that $\Delta_1 \cap \Delta_2 = \emptyset$ otherwise if we had $t_* \in \Delta_1 \cup \Delta_2$ both x_1 and x_2 would belong to $e^{-t_*}\Omega$ which is an Hausdorff open subset of Ω^*, and this is against the assumptions. Since Ω is orbit connected in Ω^* we have $\Delta_1 =]\alpha, \beta[$, and $\Delta_2 =]\gamma, \delta[$, with $\alpha < 0$, $\beta > 0$, and $\alpha < \beta \leq \gamma < \delta$. Since x_1 and x_2 have no disjoint neighborhoods, there exists a sequence p_n converging to x_1 and x_2 in the same time. Note that $0 \in \Delta_1$ and choose $t_0 \in \Delta_2$. Now for n large enough $p_n \in \Omega$ and $e^{t_0} p_n \in \Omega$ By orbit connectedness we therefore have $e^{\beta} p_n \in \Omega$. Let $K \subseteq \Omega$ be the S^1 invariant compact subset defined by

$$K = S^1 \left(\cup_n p_n \ \cup x_1 \ \cup_n e^{t_0} p_n \ \cup e^{t_0} x_2 \right).$$

Let $\phi \in \mathcal{F}_{S^1}$, and let M be the maximum of ϕ on K. For fixed $n \in N$ define $\Psi(t) = \phi(e^t p_n)$, then by Lemma 4.7 applied to the Stein manifold $(\mathbf{C}^* \times \{p_n\}) \cap \Omega$ the function $\Phi(t)$ is convex. Moreover $\Phi(0) \leq M$, $\Phi(t_0) \leq M$ hence $\Phi(\beta) \leq M$. So by Lemma 4.6 $e^\beta p_n \in \hat{K}$. Since by assumption $\hat{K}$ is compact, we can assume, up to subsequence, that $e^\beta p_n$ converges to $p_* \in \Omega$. Therefore for small positive $e^{\beta-\varepsilon} p_n$ converges to $e^{-\varepsilon} p_*$ in Ω. On the other hand, by definition of Δ_1 we have that $e^{\beta-\varepsilon} p_n$ converges to $e^{\beta-\varepsilon} x_1$ in Ω, therefore $e^\beta x_1 = p_* \in \Omega$. For positive small ε we then have $e^{\beta+\varepsilon} x_1 \in \Omega$, and this is against the definition of Δ_1. It follows that if Ω is $\mathcal{F}$ convex then X^* must be Hausdorff.

Assume viceversa that X^* is Hausdorff and let $p_1 : \Omega \to \mathbf{C}^*$ and $p_2 : \Omega \to X^*$ be the natural projections. Let K be a compact subset of Ω and let $\hat{K}$ be its $\mathcal{F}$ hull. We need to show that $\hat{K}$ is compact. Since every compact set is contained in its S^1 orbit, we can assume K and $\hat{K}$ to be S^1 invariant. We first prove that $p_2(K) = p_2(\hat{K})$. In fact let $x \in X^* \setminus p_2(K)$, then, since X^* is Hausdorff it is a complex manifold, therefore there exists a continuous function $f : X^* \to \mathbf{R}$ such that $f_{|p_2(K)} \equiv 0$ and $f(x) = 1$. Since $f \circ p_2 \in \mathcal{F}$ we conclude that x does not belong to $p_2(\hat{K})$.

Define now the function $\delta_1 : p_2(K) \to \mathbf{R}$ by $\delta_1(x) = max_{(w,x)\in K} |w|$. We claim that δ_1 is upper semicontinuus. In fact let x_n converging to to x in $p_2(K)$ and $\delta_1(x_n)$ converging to $l = maxlim_{y\to x}\delta_1(y)$. There exist $w_n \in \mathbf{C}^*$ such that $(w_n, x_n) \in K$ and $|w_n| = \delta_1(x_n)$. Up to subsequence we can assume that w_n converges to w_* so $|w_*| = l$. Since $(w_*, x) \in K$ we have $\delta_1(x) \geq |w_*| = l$. So δ_1 is upper semiconinuous. Since X^* is a manifold there exists a sequence of continuus functions on $p_2(K)$ poinwise decreasing to δ_1 see for example [10]. By Uryson Lemma this functions can be continuously extended to all of X^*. Let f_n be such extensions. Let $g_n(w, x)$ be the restriction to $\tilde{\Omega}$ of the functions $|w| - f_n(x)$. Then $g_n \in \mathcal{F}$ for every $n \in \mathbf{N}$. If $(w, x) \in K$ then $g_n(w, x) = |w| - f_n(x) \leq |w| - \delta_1(x) \leq 0$. Therefore if $(w, x) \in \hat{K}$ we have $|w| - f_n(x) \leq 0$. Now taking the limit as n goes to infinity and recalling that $p_2(K) = p_2(\hat{K})$ we find that for $(w, x) \in \hat{K}$ $|w| - \delta_1(x) \leq 0$.

Applying a similar argument to the function $\delta_2 : p_2(K) \to \mathbf{R}$ defined by $\delta_2(x) = max_{(w,x)\in K} |1/w|$ we find that for $(w, x) \in \hat{K}$ $|1/w| - \delta_2(x) \leq 0$. In other words if $(w, x) \in \hat{K}$ then $x \in p_2(K)$ and we have

$$min_{(u,x)\in K} |u| \leq |w| \leq max_{(u,x)\in K} |u| \tag{1}$$

Viceversa let $(w, x) \in \Omega$ such that $x \in p_2(K)$ and

$$min_{(u,x)\in K} |u| \leq |w| \leq max_{(u,x)\in K} |u|.$$

Let $u_0 \in \mathbf{C}^*$ such that $(u_0, x) \in K$ and $|u_0| = min_{(u,x)\in K} |u|$. Let $u_1 \in \mathbf{C}^*$ such that $(u_1, x) \in K$ and $|u_1| = max_{(u,x)\in K} |u|$. By orbit connecteness $\{u \in \mathbf{C}^* : |u_0| \leq |u| \leq |u_1|\} \times \{x\} \subseteq \Omega$. Let $H = S^1(u_0, x) \cup S^1(u_1, x)$. By S^1 invariance of K we have $H \subseteq K$. Let $\phi \in \mathcal{F}$ then by the maximum principle $\phi((w, x)) \leq max_H \phi \leq max_K \phi$, i.e. $(w, x) \in \hat{K}$. Since the functions in $\mathcal{F}$ are continuous, $\hat{K}$ is closed in Ω. Let $(w_n x_n) \in \hat{K}$ Since $x_n \in p_2(K)$ and by the inequality (1)

$$0 < min_{p_1(K)} |u| \leq |w_n| \leq max_{p_1(K)} |u|$$

we can up to subsequence assume that x_n converges to $x \in p_2(K)$ in X^* and w_n converges to $w_0 \in \mathbf{C}^*$. Let $u_n \in \mathbf{C}^*$ such that $(u_n, x_n) \in K$ and $|u_n| = \delta_1(x_n)$,

Similarly let $v_n \in \mathbf{C}^*$ such that $(v_n, x_n) \in K$ and $|1/v_n| = \delta_2(x_n)$. We can assume that $u_n \to u_0 \in \mathbf{C}^*$ and $v_n \to v_0 \in \mathbf{C}^*$ with $(u_0, x) \in K$ and $(v_0, x) \in K$. By the inequality (1) we have

$$0 < |v_n| \leq |w_n| \leq |u_n| \tag{2}$$

and in the limit

$$0 < |v_0| \leq |w_0| \leq |u_0| \tag{3}$$

By orbit connectness (w_0, x) is in Ω, and (w_n, x_n) converges to (w_0, x_0) in Ω. So $\hat{K}$ is compact.

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