DUAL MAPS AND KOBAYASHI DISTANCE OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^{n+1}$

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Abstract. We describe the Kobayashi distance and Busemann functions of bounded strictly convex domains in terms of boundary values of critical discs, their dervatives and their dual maps.

1. Introduction.

In the fundamenthal paper [Le], Lempert studied the Kobayashi metric on a bounded stricly convex domain with smooth boundary. He described the extremal holomorphic discs of this domains. In [Sp. T.] we introduced a bridge between the C-R geometry of the boundary of the domain and the Kobayashi geometry of the interior. We called this bridge the "eversive map". In the present paper we continue the descripction of the relation of Kobayashi and C-R geometry, by describing Kobayashi Busemann functions and orospheres in termis of boundary data coming from the extremal discs.

2. Preliminaries.

2.1 Basic notation.

The standard complex structure of $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$, given by the multiplication with $i = \sqrt{-1}$, is denoted by J. For any two elements $v, w \in \mathbb{C}^{n+1}$, we denote by $< v, w >= \sum_i v^i \overline{w^i}$ the standard Hermitian product and $|v| = \sqrt{< v, v >}$. The vectors $(\epsilon_1, \dots, \epsilon_k)$ will always denote the elements of the standard basis of $\mathbb{R}^k$. By Δ we denote the unit disc $\Delta = \{z \in \mathbb{C} \ : \ |z| < 1\} \subset \mathbb{C}$. We always consider Δ endowed with the Poincaré metric $ds^2_\Delta = \frac{d\zeta d\bar{\zeta}}{(1-|\zeta|^2)^2}$, with constant Gaussian curvature -4.

Let $D \subset \mathbb{C}^{n+1}$ be a domain. $T^{10}D$ and $T^{01}D$ denote the holomorphic and anti-holomorphic tangent spaces, respectively, i.e.

$$T^{10}_x D = \{ \ v \in T^{\mathbb{C}}_x M \ : \ v = V - iJV \text{ for some } V \in T_x D \ \} \ , \quad T^{01}_x D = \overline{T^{10}_x D} \ .$$

Similarly $T^{*10}(D)$ and $T^{*01}(D)$ are the subbundles of $T^{*\mathbb{C}}D$ generated by the holomorphic and anti-holomorphic 1-forms, respectively. Accordingly, if $v \in T_x D \subset$

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$T^{\mathbb{C}}_x D$ (resp. $\theta \in T^*_x D \subset T^{*\mathbb{C}} D$) we denote by $v = v^{10} + v^{01}$ (resp. $\theta = \theta^{10} + \theta^{01}$) the canonical decomposition into holomorphic and anti-holomorphic part.

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface in $\mathbb{C}^{n+1}$ and assume that $\rho : \mathcal{U} \subset \mathbb{C}^{n+1} \to \mathbb{R}$ is a *local defining function for* S, i.e. a function with $d\rho_x \neq 0$ for any $x \in S \cap \mathcal{U}$ and such that $S \cap \mathcal{U} = \{ \ x \in \mathcal{U} \ : \ \rho(x) = 0 \ \}$. By $\mathcal{D} \subset TS$ and $\mathcal{D}^{10} \subset T^{\mathbb{C}} S$ we denote the *real holomorphic distribution* and the *complex holomorphic distribution*, respectively, that are the distributions defined by

$$\mathcal{D}_x = \{ \ v \in T_x S \ : Jv \in T_x S \ \} = T_x S \cap J(T_x S) \ ,$$

$$\mathcal{D}^{10}_x = \{ \ v \in T^{10}_x S \ : \mathrm{Re}(v) \in \mathcal{D}_x \ \} = \mathcal{D}^{\mathbb{C}}_x \cap T^{10}_x \mathbb{C}^{n+1} \ .$$

Clearly, for any $x \in \mathcal{U}$

$$\mathcal{D}_x = \mathrm{Ker}(d\rho_x \circ J) \ , \qquad \mathcal{D}^{10}_x = \mathrm{Ker}\left(i d^{10} \rho_x \right) \cap T^{10}_x \mathbb{C}^{n+1} \ .$$

It is also clear that the complex structure J of $\mathbb{C}^{n+1}$ induces on each subspace $\mathcal{D}_x$ a complex structure, which we may denote by J_x. The pair $(\mathcal{D}, J)$ formed by $\mathcal{D}$ and the smooth family of complex structures $J = \{ J_x : \mathcal{D}_x \to \mathcal{D}_x \ , \ x \in S \}$ is an (integrable) CR structure of codimension one. (for definition and main property of CR structures, see e.g. [Bo]). In case $M = S$ is a smooth real hypersurface in $\mathbb{C}^{n+1}$, the CR structure we described above is called *induced CR structure of* S.

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface. We call *(local) defining 1-form* any 1-form θ (defined on a subset $\mathcal{U} \subset S$) such that $\mathrm{Ker}\, \theta_x = \mathcal{D}_x$ at any point $x \in \mathcal{U}$. The holomorphic part $\Theta = \theta^{10}$ will be called *(local) holomorphic defining 1-form*. If ρ is a defining function for S, then we call $\theta \overset{\text{def}}{=} -\frac{1}{2}(d\rho_x \circ J)$ the defining 1-form associated with ρ.

For a given defining 1-form θ, we denote by $\mathcal{L}^\theta_x$ the bilinear form on $\mathcal{D}_x$

$$\mathcal{L}^\theta_x \ : \ \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R} \ , \qquad \mathcal{L}^\theta_x(X, Y) = \frac{1}{2} d\theta_x(X, JY) = \mathrm{Re}\left(-i d\theta(X^{10}, \overline{Y^{10}}) \right) \ .$$

Notice that, in case θ and $\theta' = \lambda(x)\theta$ are two distinct defining 1-forms, then $\mathcal{L}^{\theta'}_x = \lambda(x) \cdot \mathcal{L}^\theta_x$ for any point x. This shows that the value $\mathcal{L}^\theta_x$ is, up to multiplication by a scalar factor, independent on θ. If $(z^1, \ldots, z^n)$ is a system of complex coordinates centered at x and if we consider a (local) defining function ρ with associated defining 1-form $\theta_x = -\frac{1}{2} d\rho_x \circ J$, then for any $X^{10} = v^i \frac{\partial}{\partial z^i}$, $Y^{10} = w^i \frac{\partial}{\partial z^i}$, we have

$$\mathcal{L}^\theta_x(X, Y) = \mathrm{Re}\left(\frac{\partial^2 \rho}{\partial z^i \partial \overline{z^j}} v^i \overline{w^j} \right) \ .$$

In other words, $\mathcal{L}^\theta_x$ is (up to a scalar multiple) the classical *Levi form of* S *at* x and it is an Hermitian form. In the following, we will say that $\mathcal{L}^\theta_x$ *is the Levi form rescaled by the defining form* θ.

Recall that if $S = \partial D$ is the boundary of a strictly convex domain, then $\mathcal{L}^\theta_x$ is positive definite. It follows that any defining form θ is a *contact form* and $\mathcal{D}$ is a contact distribution.

Let $D \subset \mathbb{C}^{n+1}$ be a bounded domain with a smooth boundary $S = \partial D$. We call *disc attached to* $S = \partial D$ any holomorphic embedding $f : \bar{\Delta} \longrightarrow \mathbb{C}^n$ such that $f(\partial\Delta) \subset S$. We call *disc properly embedded in* D any holomorphic embedding $f : \bar{\Delta} \longrightarrow \mathbb{C}^n$ such that

a) $f(\Delta) \subset D$;
b) f is attached to $S = \partial D$;
c) $f'(\zeta) \notin \mathcal{D}^{10}_{f(\zeta)} \subset T^{\mathbb{C}}S$ for any $\zeta \in \partial\Delta$.

Throughout the following, D is always assumed to be a bounded, strictly linearly convex domain in $\mathbb{C}^{n+1}$ with smooth boundary.

2.2 Extremal and critical discs of a bounded domain.

Let $S = \partial D$ and $(\mathcal{D}, J)$ its induced CR structure. Denote by $\pi : T^{*10}(S) \to S$ the pull-back on S of the bundle $\pi : T^{*10}\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ via the natural immersion map $\imath : S \to \mathbb{C}^{n+1}$. Let also $E^{10}(S) \subset T^{*10}S$ be the subbundle of all holomorphic defining 1-forms Θ for $\mathcal{D}$. Then for any properly embedded disc $f : \bar{\Delta} \to \bar{D}$, we consider the map

$$\tilde{f} : \partial\Delta \to E^{10}(S) \ , \qquad \tilde{f}(\zeta) = i\zeta\Theta^f|_{f(\zeta)} \tag{2.2}$$

where we denote by $\Theta^f|_{f(\zeta)}$ the unique element in $\pi^{-1}(f(\zeta)) \subset E^{10}(S)$ such that

$$\Theta^f|_{f(\zeta)}\Big(\left.\frac{\partial f(e^{i\phi}\cdot\zeta)}{\partial\phi}\right|_{\phi=0}\Big) = \Theta^f|_{f(\zeta)}(i\zeta f'(\zeta)) = 1 \ .$$

If ρ is a defining function for $S = \partial D$, we have that, for any $\zeta \in \partial\Delta$,

$$\tilde{f}(\zeta) = i\zeta\ \Theta^f\big|_\zeta = \frac{\zeta}{q(\zeta)}\left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)} dz^i \ , \qquad \text{where } q(\zeta) = \zeta\left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)}\left.\frac{\partial f^i}{\partial\zeta}\right|_\zeta \ . \tag{2.3}$$

The definitions of Θ^f and of $\tilde{f}$ are clearly independent on the choice of ρ (see also[Le4]).

Definition 2.1. A disc $f : \Delta \to D$ is called *critical (or stationary) disc* if

(1) f extends smoothly to $\bar{\Delta}$ so to give a properly embedded disc in D;
(2) there exists an holomorphic embedding $\tilde{f} : \bar{\Delta} \longrightarrow T^{*10}\mathbb{C}^{n+1}$ such that $\tilde{f}|_{\partial\Delta}$ coincides with the embedding (2.2)

If f is a critical disc, we call $\tilde{f}$ the *lift of* f *in* $T^{*10}\bar{D}$ ($\tilde{f}$ is also called *dual map of* f - see [Pa], Def. 2.10). The family $\tilde{\mathcal{D}}$ of complex subspaces at the points of $f(\bar{\Delta}) \subset \bar{D}$ defined by

$$\tilde{\mathcal{D}}_{f(\zeta)} = \{\ V \in T_{f(\zeta)}\mathbb{C}^{n+1}\ : V^{10} \in \operatorname{Ker} \tilde{f}_{f(\zeta)} \cap T^{10}_{f(\zeta)}\mathbb{C}^{n+1}\ \} \subset T_{f(\zeta)}\mathbb{C}^{n+1}$$

is called *inner extension along* f *of* $\mathcal{D}$. By construction, $\tilde{\mathcal{D}}|_{f(\partial\Delta)} = \mathcal{D}|_{f(\partial\Delta)}$, where we denoted by $\mathcal{D}|_{f(\partial\Delta)}$ the restriction of $\mathcal{D}$ to the circle $f(\partial\Delta) \subset S = \partial D$.

In his fundamental paper [Le], L. Lempert showed how critical discs are crucially related with the Kobayashi distance of D. In the following theorem, we recall just a few of their properties, which we will be constantly used in the sequel.

Theorem 2.2. *(*[Le], [CHL]*) Let D be a smoothly bounded, strictly convex domain in $\mathbb{C}^{n+1}$, endowed with the Kobayashi metric κ. Then:*

(1) *a disc $f : \Delta \to D$ is critical if and only if it is a holomorphic, totally geodesic, isometric embedding of the unit disc Δ;*
(2) *for every $x \in D$ the exponential maps givexs a diffeomorphism $exp_x : T_x(D) \to D$*
(3) *for any $(x, y) \in D \times \bar{D}$ with $x \neq y$, there exists a unique critical disc $f_{x,y} : \Delta \to D$ such that*

$$x = f_{x,y}(0) \ , \qquad y = f_{x,y}(t) \qquad \textit{for some } t \in (0, 1] \ ; \tag{2.4}$$

(4) *for any $x \in D$ and any $0 \neq v \in T_xD$ there exists a unique critical disc $f_{x,v} : \Delta \to M$ such that*

$$x = f(0) \ , \qquad v^{10} \in \mathbb{R}f'(0) \ ; \tag{2.5}$$

(5) *for a fixed x, the map $v \mapsto f_{x,v}$ is $\mathcal{C}^\infty$.*
(6) *given a sequence f_{x_n,v_n} of critical discs, then at least one of the following will occur:*

a) there exists a subsequence $\{f_{n_k}\}$ and a critical disc f, so that f_{n_k} converges to f uniformely on $\bar{\Delta}$ together with all its derivatives. b) there exists a subsequence $\{f_{n_k}\}$ which converges uniformily on compacta of Δ to a constant map $f(\zeta) \equiv y \in \partial D$.

Proof. The first five points can be found in [Le], [CHL]. As for the last point, since D is bounded there exists a subsequence f_{n_k} convergng uniformely on the compact subset of Δ to a holomorphic map f_0. If $f_0(\Delta) \cap D = \emptyset$, since D is stricly pseudoconvex f_0 is a constant map with value in ∂D. Otherwise there exists $\zeta_0 \in \Delta$ such that $f_0(\zeta_0) \in D$. Let σ be an authomorphism of the disc such that $\sigma(0) = \zeta_0$. By [CH] we know that $f_0 \circ \sigma_0$ is critical and so is f_0. Since $f_{n_k}(0)$ converges to $f_0(0)$, and $f'_{n_k}(0)$ converges to ${f_0}'(0)$, it follows that f_{n_k} converges to f_0 uniformely on $\bar{\Delta}$ together with all its derivatives. □

In the following, for any pair $(x, y) \in D \times \bar{D}$ with $x \neq y$, and for any $0 \neq v \in T_xM$, we will denote by $f_{x,y} : \Delta \to D$ and $f_{x,v} : \Delta \to D$ the critical discs which verify (2.4) and (2.5), respectively.

2.2 Levi forms, Holomorphic Symmetric forms, and normalized defining functions.

We need now to recall some concepts introduced by M.-Y. Pang in [Pa], which are crucial tools for studying Jacobi vector fields in D.

Definition 2.3. Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc and $\tilde{f}$ the corresponding lift in $T^{*10}D$. Assume also that ρ is a (local) defining function for ∂D on a neighborhood $\mathcal{U}$ of $f(\partial\Delta)$ and that $\xi = (z^0, \ldots, z^n) : \mathcal{U} \to \mathbb{C}^{n+1}$ is a system of complex coordinates on $\mathcal{U}$.

(1) The *Levi form of f* is the 1-parameter family of Levi forms $\mathcal{L}^f_\zeta$, parameterized by the points $\zeta \in \partial\Delta$, defined by

$$\mathcal{L}^f_\zeta \overset{\text{def}}{=} \mathcal{L}^{\theta_f}|_{f(\zeta)} \ , \qquad \theta_f \overset{\text{def}}{=} \operatorname{Re}(\Theta^f\big|_\zeta) = \operatorname{Re}(-i\bar{\zeta}\tilde{f}|_\zeta) \ . \tag{2.6}$$

(2) The *Holomorphic Symmetric form of f in the system of coordinates ξ* is the 1-parameter family of symmetric forms $\mathcal{B}^{f,\xi}_\zeta$, parameterized by the points $\zeta \in \partial\Delta$, defined by

$$B^{f,\xi}_\zeta(X,Y) \overset{\text{def}}{=} \frac{1}{4q(\zeta)}\,(\hat{X} - iJ\hat{X})\left((\hat{Y} - iJ\hat{Y})\rho\right)\Big|_\zeta \tag{2.7}$$

where q is the function defined in (2.6) and where the symbols $\hat{X}$ and $\hat{Y}$ denote any two vector fields with values in the real holomorphic distribution $\mathcal{D}$ that are equal to X and Y at the point $f(\zeta)$.

In coordinates, $\mathcal{L}^f_\zeta$ and $\mathcal{B}^{f,\xi}_\zeta$ are given by the expressions

$$\mathcal{L}^f_\zeta = \left[\frac{1}{q(\zeta)}\frac{\partial^2\rho}{\partial z^i\partial\overline{z^j}}(f(\zeta))\right] dz^i\otimes d\overline{z^j}\ ,\ \ \mathcal{B}^{f,\xi}_\zeta = \left[\frac{1}{q(\zeta)}\frac{\partial^2\rho}{\partial z^i\partial z^j}(f(\zeta))\right] dz^i\otimes dz^j\ . \tag{2.8}$$

Since $\{\mathcal{L}^f_\zeta\ ,\ \zeta \in \partial\Delta\}$ is a family of Levi forms, it does not depend on the choice of the system of coordinates nor on the defining function ρ. On the other hand, it can be directly checked that the family $\{\mathcal{B}^{f,\xi}_\zeta\ ,\ \zeta \in \partial\Delta\}$ does not depend on the choice of the defining function ρ but *does depend on the given system of coordinates*. For explicit formulas, which express the dependence on the system of coordinates of $\mathcal{B}^{f,\xi}$, see [Pa], Th.2.17 (3).

We conclude with the following lemma.

Lemma 2.4. *(*[Pa] *Prop. 2.36 and Thm. 2.45) Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc. Then there exists a defining function ρ of D, defined on a neighborhood $\mathcal{U}$ of $f(\bar{\Delta})$, such that for any $\zeta \in \bar{\Delta}$*

$$\tilde{f}(\zeta) = \frac{1}{\bar{\zeta}}\left.\frac{\partial\rho}{\partial z^i}\right|_{f(\zeta)} dz^i \tag{2.9}$$

and for any $\zeta \in \partial\Delta$, the function q defined in (2.3) and $\tilde{f}$ verify

$$q(\zeta) = 1\ ,\quad \tilde{f} = \left.\frac{\partial^2\rho}{\partial z^i\partial\overline{z^j}}\right|_{f(\zeta)} d\overline{z^j}(\overline{f'(\zeta)})dz^i\ ,\quad \frac{\partial\tilde{f}}{\partial\zeta} = \frac{1}{\bar{\zeta}}\left.\frac{\partial^2\rho}{\partial z^i\partial z^j}\right|_{f(\zeta)} dz^j(f'(\zeta))dz^i\ . \tag{2.10}$$

A function ρ which verifies (2.11) *strongly normalized defining function w.r.t. f.*

Lemma 1.6. *(see* [Le], [Le2], [Le3]*) Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc. Then there exists a field of holomorphic frames $(e_1, \ldots, e_n)_\zeta$, $\zeta \in \bar{\Delta}$, for the subspaces $\tilde{\mathcal{D}}^{10}_\zeta \subset T^{10}_{f(\zeta)}\mathbb{C}^{n+1}$ such that*

$$\mathcal{L}^f_{e^{i\phi}}(e_\alpha, e_\beta) = \delta_{\alpha\beta}\ ,$$

for any $e^{i\phi} \in \partial\Delta$.

It follows that, for any $\zeta \in \bar{\Delta}$, the set of holomorphic vectors

$$(f'(\zeta), e_1(\zeta), \ldots, e_n(\zeta))$$

constitutes a holomorphic basis for $T^{10}_{f(\zeta)}\mathbb{C}^{n+1}$. We will call $(f'(\zeta), e_1(\zeta), \dots, e_n(\zeta))$ the *unitary frame field adapted to the disc* f. We will denote by $e_0(\zeta) = f'(\zeta)$ and we will use the symbol $e_i(\zeta)$ to denote any vector of the unitary basis; we will use the symbol $e_\alpha(\zeta)$, with *an index denoted by a greek letter*, to denote the vectors which span the subspace $\tilde{\mathcal{D}}^{10}_\zeta$.

4.2 The Symplectification and the Hyperplane bundle of ∂D.

Consider the real hypersurface $S = \partial D \subset \mathbb{C}^{n+1}$ endowed with its induced CR structure $(\mathcal{D}, J)$. The *symplectification of* $(S, \mathcal{D})$ is the subbundle E_S of $\pi : T^*S \to S$, defined by

$$E_S \overset{\text{def}}{=} \{ \ \theta \in T^*_x S \ , \ x \in S \ : \ \text{Ker}\ \theta = \mathcal{D}_x \ \} \ . \tag{4.3}$$

The bundle $\pi : E_S \to S$ is a principal bundle with structural group $\mathbb{R}^*$. Moreover, since $S = \partial D$ is the boundary of a convex domain, there exists a globally defining function ρ for S, with $\operatorname{grad}\rho$ pointing outwards D at all points. This implies that there exists a (global) associated defining 1-form $\theta^\rho = -\frac{1}{2} d\rho \circ J$ for the distribution $\mathcal{D}$, which represents a global section of $\pi : E_S \to S$. It follows that E_S has exactly two connected components, say E^+_S and E^-_S, each of which with structural group $\mathbb{R}_{>0}$. For simplicity of notation, we will use the symbol E_S to denote just the connected component containing the 1-forms $\theta^\rho|_x$, $x \in S$.

The *tautological 1-form of* E_S is the 1-form defined at any $\theta \in E_S$ as

$$\omega|_\theta \overset{\text{def}}{=} \pi^*\theta \ . \tag{4.4}$$

Definition 4.3. The *hyperplane bundle of* S is the subbundle

$$H(E_S) = \bigcup_{\theta \in E_S} H(E_S)_\theta \subset TE_S \ , \quad \text{where} \quad H(E_S)_\theta \overset{\text{def}}{=} \{ v \in T_\theta E_S \ : \ \omega_\theta(v) = 1 \} \ . \tag{4.8}$$

If we denote by $\hat{\pi} : H(E_S) \to E_S$ the natural projection of $H(E_S)$ onto E_S, one can realize that the projection $\pi \circ \hat{\pi} : H(E_S) \to S$ makes $H(E_S)$ a fiber bundle over S, with standard fiber diffeomorphic to $\mathbb{R}^{2(n+1)}$.

I.2 Jacobi fields of bounded convex domains.

Let $D \subset \mathbb{C}^{n+1}$ be a strongly convex bounded domain.

Let $f : \Delta \to D$ be a critical disc. We want to determine for any value $V, W \in T_{f(0)}D$ the unique holomorphic Jacobi field $\mathbb{J}$ such that

$$\text{Re}(\mathbb{J}(0)) = V \ , \qquad \text{Re}(\mathbb{J}'(0)) = W \ .$$

First of all, observe that for any 1-parameter family $\mathcal{V}$ of extremal disc centered at f there exists a 1-parameter family $\tilde{V}$ of embedded discs in $T^{*10}\mathbb{C}^{n+1}$ defined by

$$\tilde{V}(s, \zeta) = \tilde{f}^{(s)}(\zeta)$$

where $\tilde{f}^{(s)}(\zeta)$ is the lift in $T^{*10}\mathbb{C}^{n+1}$ of the embedded disc $f^{(s)} = \mathcal{V}(s, *)$.

We call *lift in* $T^{10}(T^{*10}\mathbb{C}^{n+1})$ *of a holomorphic Jacobi field* $\mathbb{J}$ the holomorphic map

$$\tilde{\mathbb{J}} : \Delta \to T^{10}(T^{*10}\mathbb{C}^{n+1})$$

$$\tilde{\mathbb{J}}(\zeta) = \left[\frac{d}{ds} \tilde{V}(s,\zeta)\Big|_{t=0} \right]^{10} .$$

It is clear that if $\pi : T^{*10}\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the standard projection map, then

$$\pi_*(\tilde{\mathbb{J}}(\zeta)) = \mathbb{J}(\zeta) ,$$

for any $\zeta \in \bar{\Delta}$.

Notice also that, since $T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}$ is a vector space, we have the natural identification

$$T^{10}_{\tilde{f}(\zeta)}(T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}) = T^{*10}_{f(\zeta)}\mathbb{C}^{n+1}$$

and hence $\tilde{\mathbb{J}}$ is naturally identified with a map

$$\tilde{\mathbb{J}} : \bar{\Delta} \to T^{*10}\mathbb{C}^{n+1} , \qquad \zeta \mapsto \tilde{\mathbb{J}}(\zeta) \in T^{*10}_{f(\zeta)}\mathbb{C}^{n+1} .$$

Let $f : \bar{\Delta} \to \bar{D}$ be a critical disc and $\mathbb{J}$ a holomorphic Jacobi field. Let also ρ a normalized defining function (see §I.1.2, Lemma 1.7) and $f^{(s)}$ a 1-parameter family of extremal discs associated with $\mathbb{J}$. Then, if we denote by $q_s : \bar{\Delta} \to \mathbb{C}$ the 1-parameter family of functions

$$q_s(\zeta) = \zeta \left.\frac{\partial \rho}{\partial z^i}\right|_{f^{(s)}(\zeta)} \left.\frac{\partial f^{(s)i}}{\partial \zeta}\right|_{\zeta}$$

we have that the values $\tilde{\mathbb{J}}(\zeta)$ for $\zeta = e^{i\phi} \in \partial\Delta$ may be expressed in terms of ρ, q_s and $\mathbb{J}$ as follows:

$$\tilde{\mathbb{J}}(\zeta) =$$

$$= \left[-\frac{\zeta}{q(\zeta)^2} \left.\frac{\partial q_s(\zeta)}{\partial s}\right|_0 \left.\frac{\partial \rho}{\partial z^i}\right|_{f(\zeta)} + \frac{\zeta\overline{\mathbb{J}^j(\zeta)}}{q(\zeta)} \left.\frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j}\right|_{f(\zeta)} + \frac{\zeta \mathbb{J}^j(\zeta)}{q(\zeta)} \left.\frac{\partial^2 \rho}{\partial z^i \partial z^j}\right|_{f(\zeta)} \right] dz^i =$$

$$= \left[- \left.\frac{\partial q_s(\zeta)}{\partial s}\right|_{s=0} \tilde{f}(\zeta) + \zeta\overline{\mathbb{J}^j(\zeta)} \left.\frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j}\right|_{f(\zeta)} + \zeta \mathbb{J}^j(\zeta) \left.\frac{\partial^2 \rho}{\partial z^i \partial z^j}\right|_{f(\zeta)} \right] dz^i$$

Consider now the following decomposition of the vectors $\mathbb{J}(\zeta)$ as

$$\mathbb{J}(\zeta) = \mathbb{J}^{//}(\zeta) + \mathbb{J}^{\perp}(\zeta) , \qquad \mathbb{J}^{//}(\zeta) \in \mathbb{C}f'(\zeta) , \quad \mathbb{J}^{//}(\zeta) \in \mathcal{D}^{10}_{\zeta} .$$

Such unique decomposition can be done by expressing $\mathbb{J}(\zeta)$ in terms of an adapted unitary basis $(e_0 = f', e_1, \ldots e_n)_\zeta$, as given in Lemma 1.6, and by taking as $\mathbb{J}^{\perp}$ the projection into $\operatorname{span}\{e_1, \ldots, e_n\}$.

Let us express $\mathbb{J}^{//}$ as

$$\mathbb{J}^{//}(\zeta) = \lambda^J(\zeta) f'(\zeta) . \tag{I.2.1}$$

Now, observe that by Lemma 1.7,

$$\overline{\mathbb{J}^{//j}(\zeta)}\ \frac{\partial^2\rho}{\partial z^i\partial\bar z^j}\bigg|_{f(\zeta)} dz^i = \overline{\lambda^j(\zeta)}\tilde f(\zeta)\ , \qquad \mathbb{J}^{//j}(\zeta)\ \frac{\partial^2\rho}{\partial z^i\partial z^j}\bigg|_{f(\zeta)} dz^i = \bar\zeta\lambda^J(\zeta)\ \frac{\partial\tilde f}{\partial\zeta}\bigg|_\zeta$$

and hence, recalling the definition of $\mathcal{L}^f$ and $\mathcal{B}^{f,\xi}$, we may write $\tilde{\mathbb{J}}(\zeta)$, for any $\zeta\in\partial\Delta$ as

$$\tilde{\mathbb{J}}(\zeta) =$$

$$= \left[-\ \frac{\partial q_s(\zeta)}{\partial s}\bigg|_{s=0}\ \tilde f(\zeta) + \zeta\overline{\mathbb{J}^{//\mathbb{J}}(\zeta)}\ \frac{\partial^2\rho}{\partial z^i\partial\bar z^j}\bigg|_{f(\zeta)} + \frac{\zeta\mathbb{J}^{//j}(\zeta)}{q(\zeta)}\ \frac{\partial^2\rho}{\partial z^i\partial z^j}\bigg|_{f(\zeta)}\right] dz^i +$$

$$+\zeta\mathcal{L}^f_\zeta(*,\mathbb{J}^\perp) + \zeta\mathcal{B}^{f,\xi}_\zeta(*,\mathbb{J}^\perp) =$$

$$= \left[-\ \frac{\partial q_s(\zeta)}{\partial s}\bigg|_{s=0}\ \tilde f(\zeta) + \zeta\overline{\lambda^J(\zeta)}\right]\tilde f(\zeta) + \lambda^J(\zeta)\ \frac{\partial\tilde f}{\partial\zeta}\bigg|_\zeta + \zeta\mathcal{L}^f_\zeta(*,\mathbb{J}^\perp) + \zeta\mathcal{B}^{f,\xi}_\zeta(*,\mathbb{J}^\perp) \tag{I.2.2}$$

Since $\tilde{\mathbb{J}}:\bar\Delta\to T^{*10}\mathbb{C}^{n+1}$ is holomorphic, for any holomorphic section of $\tilde{\mathcal{D}}^{10}$ which vanishes at the origin

$$\tau_o = \zeta\cdot\tau : \bar\Delta\to\tilde{\mathcal{D}}^{10}\subset T^{10}\mathbb{C}^{n+1}\ ,$$

we have that

$$0 = \tilde{\mathbb{J}}_0[\tau_o] = \frac{1}{2\pi}\int_0^{2\pi}\tilde{\mathbb{J}}_{e^{i\phi}}[\tau_o]d\phi = \frac{1}{2\pi}\int_0^{2\pi}\tilde{\mathbb{J}}_{e^{i\phi}}[\tau]e^{i\phi}d\phi\ . \tag{I.2.3}$$

Notice that, since the function

$$\lambda^J(\zeta)\ \frac{\partial\tilde f}{\partial\zeta}\bigg|_\zeta\ (\tau)$$

is holomorphic for any holomorphic function λ^J and any holomorphic section $\tau:\bar\Delta\to\tilde{\mathcal{D}}^{10}$ and since $\tilde f(\tau)\equiv 0$ for any such section τ, we have that (I.2.2) and (I.2.3) give to the following condition on the component $\mathbb{J}^\perp$ of the holomorphic Jacobi field $\mathbb{J}$:

$$0 = \frac{1}{2\pi}\int_0^{2\pi} e^{i2\phi}\left\{\mathcal{L}^f_{e^{i\phi}}(\tau,\mathbb{J}^\perp) + \mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau,\mathbb{J}^\perp)\right\}d\phi \tag{I.2.4}$$

for any holomorphic section $\tau:\bar\Delta\to\tilde{\mathcal{D}}^{10}$.

We may now prove the following.

Theorem 2.1. *Let $D \subset \mathbb{C}^{n+1}$ be a smoothly bounded, strongly convex domain and let $f : \bar{\Delta} \to \bar{D}$ be an extremal disc. Let $\mathbb{J} = \lambda^{\mathbb{J}} f' + \mathbb{J}^{\perp} : \bar{\Delta} \to T^{10}\mathbb{C}$ be a holomorphic Jacobi field, of which $\mathbb{J}^{\perp} : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ is the corresponding holomorphic section $\tilde{\mathcal{D}}^{10}$. Let also $(e_i)_\zeta$ a unitary frame field adapted to f (see Lemma 1.6) and, for any holomorphic map $H : \bar{\Delta} \to \tilde{D}^{10}$ of the form $H = H_\alpha e^\alpha$, with H_α holomorphic, let us denote by $\nabla H = H'_\alpha e^\alpha$.*

Then:

a) *for any pair of vector $v = v^\alpha e_\alpha(0)$ and $w = w^\alpha e_\alpha(0) \in \mathbb{C}^n$, let us denote by $j_{v,w} : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ the section of the form*

$$j_{v,w}(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta) \; ; \tag{I.2.5}$$

then there exists a unique holomorphic section $j^{(2)}_{v,w} : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\mathcal{L}^f_{e^{i\phi}}(\tau, j^{(2)}_{v,w}) + e^{i4\phi}\mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, j^{(2)}_{v,w})\right\} d\phi =$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{e^{i2\phi}\mathcal{B}^{f,\xi}_{e^{i\phi}}(\tau, j_{v,w})\right\} d\phi \tag{I.2.6}$$

for any holomorphic section $\tau : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$;

b) *if $\mathbb{J}^\perp(0) = v = v^\alpha e_\alpha(0)$ and $\nabla \mathbb{J}^\perp\big|_0 = w = w^\alpha e_\alpha(0)$, then $\mathbb{J}^\perp(\zeta)$ is equal to*

$$\mathbb{J}^\perp(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta) + \zeta^2 j^{(2)}_{v,w}(\zeta) \; , \tag{I.2.7}$$

where $j^{(2)}_{v,w}(\zeta) : \bar{\Delta} \to \tilde{\mathcal{D}}^{10}$ is the unique section which verifies (I.2.6) with $j_{v,w}(\zeta) = (v^\alpha + \zeta w^\alpha) e_\alpha(\zeta)$.

c) *Then $\lambda^J(\zeta)$ is of the form*

$$\lambda^J(\zeta) = \alpha + ib\zeta - \bar{\alpha}\zeta^2 \tag{I.2.9}$$

where $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}$.

Proof. [1] a) Observe that the left hand side of (I.2.6) can be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\{\mathcal{L}^f_{e^{i\phi}}(e^{i2\phi}\tau, e^{i2\phi} j^{(2)}_{v,w}) + \mathcal{B}^{f,\xi}_{e^{i\phi}}(e^{i2\phi}\tau, e^{i2\phi} j^{(2)}_{v,w})\right\} d\phi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Hess(\rho)_{f(e^{i\phi})}(\mathrm{Re}(e^{i2\phi}\tau), \mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w})) d\phi \tag{I.2.8}$$

where

$$Hess(\rho)_{f(e^{i\phi})}(\mathrm{Re}(e^{i2\phi}\tau), \mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w}))$$

denotes the real Hessian of the defining function ρ evaluated at the point $f(e^{i\phi})$ along the vectors $\mathrm{Re}(e^{i2\phi}\tau)$, $\mathrm{Re}(e^{i2\phi} j^{(2)}_{v,w}) \in T_{f(e^{i\phi})}\mathbb{C}^{n+1}$. By hypothesis of strict linear convexity, the Hessian $Hess(\rho)$ is positive definite at all tangent spaces

[1] The proof consists in a suitable adaptation to a more general case of a few arguments of the proof of Thm. 3.6 in [Pa1]

$T_{f(e^{i\phi})}\partial D$. From this it follows that (I.2.8) is vanishing for any section τ if and only if the section $e^{i2\phi}j^{(2)}_{v,w}$ is identically equal to zero. Same conclusion holds for the section $j^{(2)}_{v,w}$ and this proves the uniqueness part. For the existence part, see [Pa].

b) It is clear that (I.2.4) holds for any holomorphic section τ if and only if

$$0 = \mathrm{Re}\left\{ \frac{1}{2\pi}\int_0^{2\pi} e^{i2\phi}\left\{ \mathcal{L}^f_{e^{i\phi}}(\tau, \mathbb{J}^\perp) + \mathcal{B}^{f;\xi}_{e^{i\phi}}(\tau, \mathbb{J}^\perp)\right\} d\phi \right\} \qquad \text{(I.2.8')}$$

for any holomorphic section τ. From the hypothesis, it follows that $\mathbb{J}^\perp(\zeta)$ is of the form (I.2.7) for some holomorphic function $j^{(2)}_{v,w}$. Substituting (I.2.7) into (I.2.8) it follows that $j^{(2)}_{v,w}$ has to verify (I.2.6) for any τ and this concludes the proof. □c) Let ρ be a normalized defining form for ∂D (see Lemma 1.7) and let $f^{(s)}$ a 1-parameter family of critical discs, centered at f, associated with the Jacobi field $\mathbb{J}$.

This implies that for any s we have that

$$\rho(f^{(s)}(\zeta))\Big|_{\partial\Delta} = 0 \quad \Rightarrow \qquad d\rho(\mathrm{Re}(\mathbb{J}))|_{\partial\Delta} = \mathrm{Re}\left[d^{10}\rho(\mathbb{J})\big|_{\partial\Delta}\right] \equiv 0 \ .$$

This can be rewritten as

$$\mathrm{Re}\left[\bar\zeta\tilde f(\mathbb{J})\Big|_{\partial\Delta}\right] = \mathrm{Re}\left[\bar\zeta\lambda^J(\zeta)\big|_{\partial\Delta}\right] \equiv 0 \qquad \text{(I.2.10)}$$

By developing $\lambda^J(\zeta)$ into Fourier series, we obtain from (I.2.10) that $\lambda^J(\zeta)$ is exactly of the form (I.2.9). □

I.4 Busemann functions and horospheres of strongly convex domains.

Let $D \subset \mathbb{C}^n$ be a smoothly bounded, strongly convex domain in $\mathbb{C}^n$. By Lempert's results (see Theorem 1.9 (2)), we know that the Kobayashi metric k_D makes D a complex Finsler manifold, as defined in §I.3. Let us denote by $S(D) = \{v \in T(D) : k_D(v) = 1\}$ and by $S_x(D)$ the fiber of x for the natural projection $proj : S(D) \to D$. Let us also denote by $\mathfrak{D}$ the (finite) Kobayashi distance of D, and let $d_x : D \to \mathbb{R}$ be the function $y \to \mathfrak{D}(x, y)$.

Let $\gamma :]0, \infty[$ a unit speed geodesics which will be callled from now on an oriented half line. By triangular inequality the functions $\mathcal{D}(\gamma(t), x) - t$ is decresing in t and uniformely bunded by $\mathcal{D}(\gamma(0), x)$, thereofre it has a limit as t goes to $= \infty$ such a limit is called "Bussmann function of γ and it is denoted by B_γ. We clearly have

$$B_\gamma(\gamma(s)) = -s \ . \qquad (4.1.1)$$

and

$$|B_\gamma(x) - B_\gamma(y)| \le \mathcal{D}(x, y). \ . \qquad (4.1.2)$$

The level sets $B_\gamma = c$ are called horospheres or limit spheres, and the sublever sets $B_\gamma < c$ are called horoballs or limit balls. Let us denote by $\Omega_x(r)$ the ball of radius

r centered at x. The terminology is justifieb by the following equality which is valid for every real number c [Bu]

$$\cup_{t\in\mathbb{R},t>-c}\,\Omega_{\gamma(t)}(t+c)=\{x\in D : B_\gamma(x)<c\}.$$

Let $C(D)$ be the space of continuous real functions on D with the topology of uniform convergence on compact sets, let us take the quotient space $C_*(D)$ of $C(D)$ modulo the constant functions with the quotient topology. There is a continuous embeding i of D into $C_*(D)$ given by $x \to [\mathcal{D}_x]$ here ([ ] denotes the equivalence class). According to Gromov, see, [BGS] we compacitify by embedding it into $\overline{i(D)}$ and define the boundary of D by $B(D) = \overline{i(D)} \setminus i(D)$. Note that the equivalence class of a Busemann function is in $B(D)$, in fact $B_\gamma(x) = \lim_{s\to\infty} \mathcal{D}(\gamma(s), x) - s$. On the other hand, as one can see from formula 4.1.2, the function B_γ does not have a minimum, hence it does not belong to $i(D)$.

When this construction is applied to a simply connected complete Riemanian manifold with non positive curvature, it can be showen that the Busmann functions are $\mathcal{C}^2$ and that the boundary $B(D)$ coincides with the set of Busemann functions. Moreoves it also coincides with the usual definition of boundaary as the set of equivalence classes of gedoesics, where two geodesics are conisdered to be equivalent if their distance is bounded. [BGS].

We do not know weather the Kobayashi metric on a stricly convex bounded domain D enjoyes some reasonable notion of non positive curvature, however it is proved in [A] (Notes page 280) that, in this case given a fixed $x_0 \in D$ the map from ∂D to $B(D)$ given by $y \to B_\gamma$, where γ is the half-line connecting x_0 and y, is bijective. We will give an explicit expression of the Busemann functions in terms of the dual maps of critical discs. this will show that the Busemann functions are $\mathcal{C}^\infty$,

Proposition 4.1. *For any $x \in D$, let us denote by $\mathfrak{D}_x : D \to \mathbb{R}$ the distance function $\mathfrak{D}_x(y) \overset{\text{def}}{=} \mathfrak{D}(x, y)$. Then, for any $y \in D$, with $y \neq x$,*

$$d\mathfrak{D}_x|_y = -Re\left[\tilde{f}_{y,x}|_0\right] \tag{I.4.1}$$

Proof. Consider a vector $v \in T_yD$ and let $\sigma : [-\varepsilon, \varepsilon] \to D$ be a smooth curve with $\sigma_0 = x$ and $\sigma_0' = v$ and consider the 1-parameter family of critical discs

$$f_t : \Delta \to D\ , \qquad f_t \overset{\text{def}}{=} f_{\sigma_t,x}\ .$$

From the definition of the critical discs f_t and Corollary 1.10, it follows that $f_t\left(\tanh\left(\mathfrak{D}_x(\sigma_t)\right)\right) = x$, for any t, and hence that

$$0 = \mathbb{J}|_{(\tanh(\mathfrak{D}_x(y))} + f_0'|_{\tanh(\mathfrak{D}_x(y))} \cdot \frac{1}{\cosh^2\left(\mathfrak{D}_x(y)\right)} \cdot \left.\frac{d\mathfrak{D}_x(\sigma_u)}{du}\right|_{u=0} \tag{I.4.2}$$

where we denoted by $\mathbb{J}$ the holomorphic Jacobi vector field for f_0 defined by

$$\mathbb{J} = \left.\frac{\partial f_t}{\partial t}\right|_{t=0}\ . \tag{I.4.3}$$

Consider now the family of lifts $\tilde{f}_t : \Delta \to T^{*10}D$ and recall that by definition, $\tilde{f}_t(f'_t) \equiv 1$ for any t. Moreover $\mathbb{J} = \lambda f' + \mathbb{J}^\perp$, where $\mathbb{J}^\perp$ is a holomorphic Jacobi vector field such that $\tilde{f}_0(\mathbb{J}^\perp) \equiv 0$ and λ is a function of the form

$$\lambda : \Delta \to D \ , \qquad \lambda(\zeta) = \alpha + ib\zeta - \bar{\alpha}\zeta^2$$

where $\alpha \in \mathbb{C}$ and $b \in \mathbb{R}$. From (1.42) we find $\mathbb{J}^\perp(tanh(d_x(y)) = 0$ and

$$\lambda(tanh(d_x(y)) = \frac{-1}{\cosh^2(\mathfrak{D}_x(y)} \frac{d\mathfrak{D}_x(\sigma_u)}{du}|_{u=0}$$

which is a real number. This implies that

$$Re(\alpha)(1 - tanh^2(d_x(y)) = \frac{-1}{cosh^2(d_x(y)}(\frac{d\mathfrak{D}_x(\sigma_u)}{du})|_{u=0}),$$

which is $-Re(\alpha) = (\frac{d\mathfrak{D}_x(\sigma_u)}{du}|_{u=0})$.

Notice now that, since $f_t(0) = \sigma_t$ for any t, so

$$v^{10} = \mathbb{J}(0) \in T^{10}_x D \ . \tag{I.4.4}$$

Then

$$\tilde{f}_{y,x}(v) = \tilde{f}_0(v^{10}) = \tilde{f}_0(\mathbb{J}(0)) = \alpha.$$

□

Note that if $\gamma : \mathbb{R} \to D$ is an oriented straight line and f_{γ_0,γ'_0} is the critical disc so that (I.1.17) holds, then

$$\lim_{s\to\infty} \gamma_s = f_{\gamma_0,\gamma'_0}(1) \ . \tag{I.4.8}$$

For this reason, for any oriented straight line γ and for the corresponding ray $\gamma|_{[0,+\infty)}$, we call the point (I.4.7) *the point at infinity of* γ and we denote it by γ_∞.

Lemma 4.7. *For any oriented straight line γ, the associated Busemann function B_γ is $\mathcal{C}^1$ and, for any $x \in D$*

$$dB_\gamma|_x = -Re(\tilde{f}_{x,\gamma_\infty}|_0) \ , \tag{I.4.9}$$

Proof. Consider a point $x \in D$, a vector $v \in T_xD$ and a regular curve $\sigma : [-\varepsilon, +\varepsilon] \to D$ such that $\sigma_0 = x$ and $v = \dot{\sigma}_0$. We want to prove that B_γ is $\mathcal{C}^1$ and

$$v(B_\gamma) = \left.\frac{dB_\gamma(\sigma_t)}{dt}\right|_{t=0} = -\mathrm{Re}((\tilde{f}_{x,\gamma_\infty}|_0)(v)) \ . \tag{I.4.10}$$

By (I.4.1'), we know that for any $s \in \mathbb{R}$

$$\left.\frac{d\mathfrak{D}_{\gamma_s}(\sigma_t)}{dt}\right|_{t=0} = d\mathfrak{D}_{\gamma_s}|_x(v) = -\mathrm{Re}(\left[\tilde{f}_{x,\gamma_s}|_0)(v)\right] \ ,$$

and, since $f_{x,\gamma_s}(0) = x$ for every $s \in \mathbb{R}$, from Theorem 2.2 and well know facts on exchanging of limits under uniform convergence, we may infer that

$$\lim_{s\to+\infty} \left.\frac{d\mathfrak{D}_{\gamma_s}(\sigma_t)}{dt}\right|_{t=0} = \lim_{s'\to 1} \left.\frac{d\mathfrak{D}_{\gamma_{\tanh(s')}}(\sigma_t)}{dt}\right|_{t=0} = -\mathrm{Re}\left[(\tilde{f}_{x,\gamma_\infty}|_0)(v)\right] \ . \tag{I.4.11}$$

On the other hand, by definition,

$$B_\gamma(\sigma_t) = \lim_{s\to\infty}(\mathfrak{D}_{\gamma_s}(\sigma_t) - s) \ .$$

□

Lemma 4.8. *Let γ be an oriented half line of D with $\gamma(0) = x_0$ and $\gamma_\infty = y$. then the function*

$$x \to \tilde{f}_{x_0,y}(f'_{x,y}|_1$$

takes real positive values.

Proof.

Recall that $\frac{\partial f_{x_0,y}(e^{i\phi})}{\partial\phi}_{|\phi=0} = 2Re(if'_{x_0,y}|1) \in T_y(\partial D)$. Moreover

$$(-i\tilde{f}_{x_0,y}|1)((\tilde{\mathcal{D}})_y) = \{0\}$$

and $\tilde{f}_{x_0,y}(f'_{x_0,y}|_1 = 1$. However $T_y(\partial D) = \frac{\partial f_{x_0,y}(e^{i\phi})}{\partial\phi}_{|\phi=0} \oplus \tilde{D}_y(\partial D)$. It follows that for every $v \in T_y(\partial D)$ the number $(-i\tilde{f}_{x_0,y}|1)(v)$ is real. For every $x \in D$ the vector $if'_{x,y}|_1$ never belongs to the holomorphic distribution, so that $x \to \tilde{f}_{x_0,y}(f'_{x,y}|_1$ is real never vanishing. Since $\tilde{f}_{x_0,y}(f'_{x_0,y}|_1 = 1$ the statement follows by continuity. □

Teorem 4.9. *Let γ be an oriented half line of D with $\gamma(0) = x_0$ and $\gamma_\infty = y$. then*

$$B_\gamma(x) = 1/2\log((\tilde{f}_{x_0,y}(f'_{x,y})|_1).$$

Proof.

We have $1/2\log((\tilde{f}_{x_0,y}(f'_{x_0,y})|_1) = 1/2\log(1) = 0 = B_\gamma(x_0)$.

Therefore, by Lemma 4.7, it is sufficient to show that for every $x \in D$

$$d(1/2\log(\tilde{f}_{x_0,y}(f'_{x,y}|_1))_x = dB_\gamma|_x = -Re(\tilde{f}_{x,y}|_0).$$

Consider a vector $v \in T_xD$ and let $\sigma : [-\varepsilon, \varepsilon] \to D$ be a smooth curve with $\sigma_0 = x$ and $\sigma'_0 = v$ and consider the 1-parameter family of critical discs

$$f_t : \Delta \to D\ , \qquad f_t \stackrel{\text{def}}{=} f_{\sigma_t,y}\ .$$

Let $\mathbb{J}$ be the holomorphic Jacobi vector field for f_0 defined by

$$\mathbb{J} = \left.\frac{\partial f_t}{\partial t}\right|_{t=0}.$$

Let us write

$$\mathbb{J} = (\alpha + ib\zeta - \bar{\alpha}\zeta^2)f'_{x,y} + \sum_{k=1}^{n}\phi_k(\zeta)e_k(\zeta)$$

where $\phi_1, \dots \phi_n$ are holomorphic on Δ and smooth on $\bar{\Delta}$ and the $e_1, \dots e_n$ are given in Lemma 1.6. Since $f_t(1) = y$ for every $t \in [-\varepsilon, \varepsilon]$ we find $\mathbb{J}(1) = 0$ i.e. $\alpha + ib - \bar{\alpha} = 0$ and $\phi_j(1) = 0$ for $1 \le j \le n$. Now

$$d(1/2\log(\tilde{f}_{x_0,y}(f'_{x_0,y})|1)_x(v) =$$

$$1/2\frac{\tilde{f}_{x_0,y}(\mathbb{J}'(1))}{\tilde{f}_{x_0,y}(f'_{x_0,y})(1)}.$$

Since $\tilde{f}_{x_0,y}|(e_j) \equiv 0$ for every j from 1 to n, we find

$$\tilde{f}_{x_0,y}|_1(\mathbb{J}'(1)) = (ib - 2\bar{\alpha})\tilde{f}_{x_0,y}(f'{}_{x,y})(1)$$

$$= (\bar{\alpha} - \alpha - 2\bar{\alpha})\tilde{f}_{x_0,y}f'{}_{x,y})(1)$$

$$= -2Re(\alpha)\tilde{f}_{x_0,y}(f'{}_{x,y})(1)$$

In summary

$$1/2\frac{\tilde{f}_{x_0,y}(\mathbb{J}'(1))}{\tilde{f}_{x_0,y}(f'_{x_0,y})(1)} = -Re(\alpha).$$

Now $f_t(0) = \sigma_t$ for any t, so

$$v^{10} = \mathbb{J}(0) \in T^{10}_x D\ .$$

Then

$$\tilde{f}_{y,x}(v) = \tilde{f}_0(v^{10}) = \tilde{f}_0(\mathbb{J}(0)) = \alpha.$$

□

Remark 4.1. Since for every fixed $x \in D$ and $y \in \partial D$ the linear map $-i\tilde{f}_{x,y}$ is real on $T_y(\partial D)$ vanisches on $\mathcal{D}_y$ and it is positive on the tangent vector to critical discs passing through y, we see that by changing x and keeping y fixed the Busemann function of the geodesics half-line connecting x with y changes by a constant.

Remark 4.2. Let $x_0 \in D$ be fixed then for y and z in D we have

$$\mathcal{D}(y,z) = (1/2)\ max_{w\in\partial D}|\log((\tilde{f}_{x_0,w}(f'_{y,w})|_1) - \log((\tilde{f}_{x_0,w}(f'_{z,w})|_1)|$$

$$= (1/2)\ max_{w\in\partial D}|\log((|\frac{\partial\rho}{\partial Z}(w)(f'_{y,w})(1)|) - \log((|\frac{\partial\rho}{\partial Z}(w)(f'_{z,w})(1)|)|.$$

Where ρ is a defining function of ∂D. In fact the inequality

$$\mathcal{D}(y,z) \geq (1/2)\ max_{w\in\partial D}|\log((\tilde{f}_{x_0,w}(f'_{y,w})|_1) - \log((\tilde{f}_{x_0,w}(f'_{z,w})|_1)|$$

comes form the Theorem 4.9 together with formula 4.1.1, To find equality we use formula 4.1.2 applied to the geodesics half line connecting y to z.

5. The eversive map and horospheres.

In [Sp. T.] we prove the following :

Let $E_{\partial D}$ and $H(E_{\partial D})$ be the symplectification and the hyperplane bundle of ∂D, respectively, and let $\alpha_{x,v} : \mathbb{R} \to (T^*\partial D)^{\mathbb{C}}$ be the curve

$$\alpha_{x,v}(t) = \left[\Theta^{f_{x,v}}|_{f_{x,v}(e^{it})}\right]\Big|_{T\partial D} = \left[\Theta^{f_{x,exp(Jt)v}}|_{f_{x,exp(Jt)v}(1)}\right]\Big|_{T\partial D}\ . \tag{5.1}$$

Then the curve $\alpha_{x,v}$ takes values in $E_{\partial D}$ and the tangent vector $\alpha'_{x,v}|_{t=0}$ is in $H(E_{\partial D})$.

If moreover we define the map $\mathcal{E} : S(D) \to H(E_{\partial D})$ by

$$\mathcal{E}(v) = \alpha'_{x,v}|_0 \in H(E_{\partial D})$$

then $\mathcal{E}$ is a biholomorphism equivariant diffeomorphism. The reason for the name eversive map, is that $\mathcal{E}$ maps the fibers of $proj$ onto section of π, and some sections of $proj$ onto the fibers of π.

The foliation of $S(D)$ given by the fibers of $proj$ defines via $\mathcal{E}$ a foliation $\mathcal{F}_1$ of $H(E_{\partial D})$ by spheres. Similarly the foliation of $H(E_{\partial D})$ given by fibers of $\hat{\pi} : H(E_{\partial D}) \to E_S$ defines via $\mathcal{E}$ a foliation $\mathcal{F}_2$ of $S(D)$ by closed submanifolds diffeomorphism to $\mathbb{R}^{2n+1}$. We are going to describe the foliation $\mathcal{F}_2$ in more details. For a fixed $y \in S$ consider the section of $proj$ given by $s_y : D \to S(D)$ given by $s_y(x) = {f_{x,y}}'(0)$.

Then we have the following

Proposition 5.1. *Let $v \in S(D)$, $x_0 = proj(v)$ and $y = \pi \circ \hat{\pi}(\mathcal{E}(v)) = f_{x_0,v}(1) \in \partial D$. Then the leaf of $\mathcal{F}_2$ through v is the image by s_y of the horosphere in D passing through x_0 and with point at infinity y.*

Proof.

Let γ be the geodesics half-line connecting x_0 and y, let F be the leaf of $\mathcal{F}_2$ through the point v and let $\Gamma = \{x : B_\gamma(x) = 0\}$ be the horoshere through x_0 with point at infinity y. Clearly $F = s_y(proj(F))$, so we need to show that $proj(F) = \Gamma$. Let $x \in proj(F)$, then $-i\tilde{f}_{x_0,y}(1) = -i\tilde{f}_{x,y}(1)$. Therefore

$$1 = \tilde{f}_{x,y}({f'}_{x,y})(1) = \tilde{f}_{x_0,y}({f'}_{x,y})(1),$$

so $B_\gamma(x) = 0$ because of Theorem 4.9. Viseversa let $x \in \Gamma$, since $\pi(-i\tilde{f}_{x,y}(1)) = \pi(-i\tilde{f}_{x_0,y})(1)) = y$, there exists a positive real λ such that $-i\tilde{f}_{x,y}(1) = -i\lambda\tilde{f}_{x_0,y}(1)$.

On the other hand, since $x \in \Gamma$, by Theorem 4.9 again we have

$$1 = \tilde{f}_{x_0,y}({f'}_{x,y})(1) = \tilde{f}_{x,y}({f'}_{x,y})(1)$$

so that $-i\tilde{f}_{x,y}(1) = -i\tilde{f}_{x_0,y}(1)$. and $x \in proj(F)$.

□

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