Torus actions on weakly pseudoconvex spaces

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Abstract

We show that the univalent local actions of the complexification of a compact connected Lie group K on a weakly pseudoconvex space have a universal orbit convex weakly pseudoconvex complexification. We also show that if K is a torus, then every holomorphic action of K on a weakly pseudoconvex space extends to a univalent local action of $K^{\mathbf{C}}$.

1 Introduction

The geometric invariant theory of Mumford and its interpretation in terms of moment maps of hamiltonian actions, is dealing essentialy with the holomorphic (algebraic) action of a reductive complex Lie group on a complex space see [16]. A reductive complex group is the complexification of a compact real Lie group see [12]. Suppose that a complex space X is endowed with an action of a compact Lie group by holomorphic tranformations. In order to study this in the framework of geometric invariant theory it is convenient to embed X as an open K invariant subset of a "good" complex space X^* with an holomorphic action of the reductive complexified group $K^{\mathbf{C}}$. By the results of Palais [17] and Heinzner-Iannuzzi [11], in a broad varaity of situations a canonical space X^* as above can be constructed as a non necessarly Hausdorff complex space. However Heinzner proved in [7] that if X is Stein, then X^* exists is Hausdorff and in fact it is a Stein space. Later Heinzer and Iannuzzi proved that if X is holomorphically convex then X^* exists is Hausdorff and holomorphically convex. Iannuzzi recently dealt in [13] with the case of the action of some non compact group on Taut manifolds. He proved that also in such cases X^* exists and it is Hausdorff. In this paper we consider the case of action of compact groups on comlex spaces having some weak pseudoconvexity. If the action is good enough we again prove that the space X^* exists it is Hausdorff and carries the same weak pseudoconvexity properties of the original space X. We also prove that every action of a torus is as we need. In case X is equivariantly immersed in a $K^{\mathbf{C}}$ Kaeler manifold this weak pseudoconvexity notions are sufficient to prove vanishing theorems for cohomology of holomorphic vector bundles over X^*. See [4]. Moreover the results of [11] allow, in the hamiltonian situation, to construct suitable quotients. Note that our pseudoconvexity condition seems to be close to optimal, at least in the case of S^1 actions, to make sure that the Hausdorff property holds for every complexification X^*. See remark 3.2 We would like to thank professor Jean Pierre Demailly for discussions about weak pseudoconvexity.

2 Preliminaries

2.1 Local actions

In this subsection we give a quick exposition of the Palais theory in [17] on local actions of Lie groups and on its adaptation to the complex analytic setting given in [11]. For more details we refer to the original papers.

Let X be a reduced complex space, let G be a connected Lie group,

A local action of G on X is given by a real analytic map $\Phi : \Omega \to X$ where Ω is an open neighborhood of the neutral section $e \times X$ in $G \times X$ with the following properties:

i) For all $x \in X$ the open subset $\Omega_x = \{g \in G : (g, x) \in \Omega\}$ is connected
ii) For fixed $g \in G$ the map $x \to \Psi(g, x) = gx$ is holomophic when defined
iii)For every $g, h \in G$ and $x \in X$ we have $ex = x$
and if
$(gh, x) \in \Omega,\ (h, x) \in \Omega$, and
$(g, hx) \in \Omega$, then
$(g, hx) = (gh, x)$.

For a complex Lie group the local action is said to be holomorphic if Ψ is holomorphic.

To every local action of G we can associate a linear map ξ from the Lie algebra g of G to the algebra of holomorphic vector fields on X given by $v \to \frac{\partial exp(tv)x)}{\partial t}_{|t=0}$. This map is an homomorphism of Lie algebras. Conversely every such homomorphism induces a local action. If G is a complex Lie group, the local holomorphic G action corresponds to $\mathbf{C}$ linear homomorphisms. A product $N \times U$ where N is an open connected neighborhood of e in G and U is an open set of X is called an elementary slice pair if $N = N^{-1}$, $N^2 \times U \subseteq \Omega$ and $N^2 \times N^2U \subseteq \Omega$. Given an elementary slice pair $N \times U$ and $h \in G$ we define as a slice chart the map $\Phi : Nh \times U \to G \times X$ defined as $\Phi(g, x) = (g, gh^{-1}x)$. The map Φ is then an homoeomorphism of $Nh \times U$ onto an open set of $G \times X$.

A subset V is a neighborhood of (h_0, y_0) for the leaf topology in $G \times X$ if and only if there exists a slice chart Φ such that $(h_0, y_0) \in \Phi(Nh \times x_0) \subseteq V$ for some $x_0 \in U$. A connected component of $G \times X$ in the leaf topology is called a leaf of the local action. If X is smooth then the leaves are the maximal integral submanifolds of the distribution $\xi(g)$ coming from the diagonal local action of G on $G \times X$ obtained from the left action of G on itself. Let us consider the space

$$X^* = G \times X \ / \ \text{ modulo leaves}$$

with the quotient topology. We call X^* the Palais globalization of the local action. Note that the leaves are invariant with respect to the diagonal local action. Moreover the right action of G on the first factor of $G \times X$ induces a global action of G on X^*. Let $\Pi : G \times X \to X^*$ the natural projection and let $j : X \to X^*$ given by $j(x) = \Pi(e, x)$. Then both Π and j are continuous open mappings, see [11]. Moreover if G is complex each leaf is a complex locally closed submanifold of $G \times X$ and the restriction to the leaf of the projection on the first factor is a locally biholomorphic map. The local action is said to be univalent if for each leaf the above local biholomorphic map is injective. In other words (since the right action of G on the first factor of $G \times X$ tranforms leaves into leaves) the action is univalent if and only if j is injective. For a univalent local action we can use the slice charts to define an holomorphic atlas on X^* which makes X^* a possibly non Hausodorff complex space. For notational convenience a non necessarly Hausdorff complex space will be called in this paper a quasi complex space. The map j becomes then an holomorphic open locally G equivariant embedding. Note that if Y is a quasi complex space with a global G action the map $(g, y) \rightarrow g^{-1}y$ induces a G equivariant biholomorphism of Y^* onto Y. We also have the following

Lemma 2.1 *The space X^* has the following universal property. If Y is a quasi complex space with an holomorphic G action and $f : X \rightarrow Y$ a locally G equivariant holomorphic map, then there exists a unique holomorphic G equivariant map $f^* : X^* \rightarrow Y$ such that $f^* \circ j = f$. Moreover $Gj(X) = X^*$.*

Proof.
see Theorem 2 of section 3 in [11]. ■

Any quasi complex space with a G action satisfing the condition of the Lemma is called a universal globalization of X.

A local G action is univalent if and only if it admits some globalization, in such case it admits a universl globalization, i.e. X^*, in the category of quasi complex spaces. There are however example of spaces with a local G action which admit a universal globalization in the category of Hausdorff complex spaces which is different from X^*.

We now consider the case where there exists a global G action on the complex space X by holomorphic tranformation. Assume moreover that the Lie algebra of $G^{\mathbf{C}}$ is the complexification of the Lie algebra of G. This is the case for example if G is compact of simply connected. The global action of G induces a local holomorphc action of the complexified group $G^{\mathbf{C}}$ see [14].

Lemma 2.2 *If there exists a global action on X of the group $(\mathbf{R}, +)$, then the local $(\mathbf{C}, +)$ action is univalent. Similarly if there exists a global action of S^1 then the local action of $\mathbf{C}^*$ is univalent.*

Proof. In the case of $(\mathbf{R}, +)$ let F be an $(\mathbf{R}, +)$ equivariant connected Riemann domain over $\mathbf{C}$, then the action of $(\mathbf{R}, +)$ on F is free and proper. The quotient $F mod$ $\mathbf{R}$ is then a connected non compact one dimensional real manifold and the induced map from $F mod$ $\mathbf{R}$ into $\mathbf{R}$ is a local diffeomorphism, hence it is injective. Therefore by $(\mathbf{R}, +)$ equivariance and freeness the original map is also injective. A similar proof works for the case of S^1. ■

Definition 2.3 *Let G be a connected Lie group with Lie algebra g. The group G is said to by polar if the map from $G \times g$ into $G^{\mathbf{C}}$ defined as $(g, v) \rightarrow gexp(iv)$ is a diffeomorphism.*

For example $(\mathbf{R}^n, +)$ is polar and every compact connected Lie group is polar.

Definition 2.4 *Let G be a real connected polar Lie group with Lie algebra g. Let $G^{\mathbf{C}}$ be the complexification of G. Let Y be a quasi complex space with a $G^{\mathbf{C}}$ action such that X is a G invariant open subset of Y and $G^{\mathbf{C}}X = Y$. We say that X is orbit connected in Y if for every $x \in X$ the set*

$$\Omega_x = \{g \in G^{\mathbf{C}} : gx \in X\}$$

is connected. We say that X is orbit convex in Y if for every $x \in X$ and $v \in g$ such that $exp(iv)x \in X$ one has $exp(itv)x \in X$ for every t in the interval $[0, 1]$.

Lemma 2.5 *Let G be a real connected polar Lie group with Lie algebra g. Let $G^{\mathbf{C}}$ be the complexification of G. Let Y be a quasi complex space with a $G^{\mathbf{C}}$ action such that X is a G invariant open subset of Y and $G^{\mathbf{C}}X = Y$. Then Y is $G^{\mathbf{C}}$ equivariantly biholomorphic to the Palais complexification X^* of X if and only if X is orbit connected in Y.*

Proof. See [1]. ■

Lemma 2.6 *Let K be a compact connected Lie group, let F be a connected complex manifold with a K action. Let $\pi : F \to K^{\mathbf{C}}$ be a K equivariant Riemann domain projection, (where K acts on $K^{\mathbf{C}}$ by left multiplication). Then π is injective if and only if F is holomorphically separeted.*

Proof.

See the proof of Proposition 3 of [11]. ■

Corollary 2.7 *Let X be a holomorphically separeted complex space, let K be a compact Lie group acting on X by biholomorphisms, then the local action of $K^{\mathbf{C}}$ is univalent.*

2.2 P-pseudoconvex spaces

Definition 2.8 *As in [3] and [15] we say that a complex space X is P-pseudoconvex If it is convex with respect to the family of plurisubharmonic functions, i.e. if for every compact set H the hull $\tilde{H}$ of H with respect to the family of plurisubharmonic functions is relatively compact. The complex space X is said to be weakly pseudoconvex if it carries a plurisubharmonic exaustion function, and for $0 \leq k \leq \infty$ it is C^k weakly pseudoconvex if it carries a C^k plurisubharmonic exaustion function.*

Note that X is C^0 weakly pseudoconvex if and only if it is convex with respect to the family of continuous plurisubharmonic functions, see [18].

Examples 2.9 *Every Stein space is C^∞ weakly pseudoconvex, more generally by using the Remmert reduction one can see that every holomorphically convex space is C^∞ weakly pseudoconvex.*

Example 2.10 *Let D be a relativly compact domain in a complex manifold X and assume that for a given $k \geq 2$ the domain D has a C^k defining function on X which is plurisubharmonic in a neighborhood of ∂D, then D is C^0 weakly pseudoconvex. In [6] Grauert constructs an example af a D as above which is not holomorphically convex.*

Example 2.11 *Let X be a compact complex manifold and L an holomorphic line bundle on X. Let $k \geq 2$ and h be a C^k hermitian metric on L with non negtive curvature. Let D^* be the set of vectors in the total space of L^* having norm smaller then 1 with respect to the dual metric h^*. Then D^* is C^0 weakly pseudoconvex. Note that, as observed by Grauert, given an holomorphic function f on D^*, the degree h homogeneus component of the Taylor expansion of f along the fibers gives an holomorphic section of L^h.*

Example 2.12 *Let X and L as above, assume that $c_1(L)$ belongs to the closure of the cone of (Chern classes of) effective line bundles in $H^2(X, \mathbf{R})$. Then L has a semipostive singular metric, see [4]. Then the domain D^* defined as above is weakly pseudoconvex. Suppose moreover that the domain D^* admits some non constant holomorphic function, hence for some k the bundle L^k admits some non trivial holomorphic section s. We can consider on L the singular metric $h = (\frac{1}{\sum_j |\theta(s_j)|^2})^{1/k}$, where θ is a local trivialization, and $s_1, \dots s_N$ is a base of $H^0(X, L^k)$. So D^* is C^0 weakly pseudoconvex.*

Lemma 2.13 *Every weakly pseudoconvex Riemann domain X over a Stein space Y is Stein. Moreover if Y is smooth and X is only P-pseudoconex then it is Stein.*

Proof.

Let Y be a Stein space with a Riemann domain projection $\pi : X \to Y$. Let ϕ a plurisubharmonic exaustion of X, and let α be a smooth stictly plurisubharmonic function on Y. The function $\phi + exp(\alpha \circ \pi)$ is a stricly plurisubharmonic exaustion of X. By [2] the space X is holomorphically convex. Since however $\alpha \circ \pi$ is a smooth strictly plurisubharmonic function of X, the space X can not contain positive dimensional compact analytic subsets. Hence X is Stein. Assume now that Y is smooth and X is P-pseudoconvex. If we consider Y as a closed analytic subvariety of some $\mathbf{C}^N$ we can find a domain of holomorphy D in $\mathbf{C}^N$ containing Y and an holomorphic retraction $\rho : D \to Y$. That is ρ is holomorphic and $\rho_{|Y} = Id_{|Y}$. Let Z be the fiber product of (X, π) and (D, ρ). Then Z is a closed submanifold of $X \times D$, hence it is a P-pseudoconvex and the projection on the second factor makes it a Riemann domain over D. Moreoer X is biholomorphic to the closed analytic subvariety $Z \cap (X \times Y)$ of Z. It follows easily from the maximum principle for plurisubharmonic functions, that the manifold Z satisfies the Oka "Kontinuitatssatz" condition, hence Z is Stein and so is X. See [19]. ■

Lemma 2.14 *If X is a complex space with an holomorphic action of a compact Lie group K and H is a compact K invariant subset of X, then the hull of H with respect to the set of plurisubharmonic functions on X coincides with the hull with respect to the set of K invriant once.*

Proof. The hull with respect to the plurisubharmonic functions is clearly contained in the hull with respect to the K invariant once. Now to every plurisubharmonic function ϕ on X we associate the plurisubharmonic K invariant function $\psi(x) = max_K \phi(kx)$. Then $\phi \leq \psi$ and if H is a K invariant compact subset of X then $max_H \phi = max_H \psi$. Therefore if x belongs to the hull of H with respect to the K invariant plurisubharmonic function, then $\phi(x) \leq \psi(x) \leq max_H \psi = max_H \phi$. ■

Lemma 2.15 *Let Y be a Stein space with an holomorphic K action, and let ϕ be a plurisubharmonic K invariant function of X. Then for every $x \in X$ and every vector ξ in the Lie algebra of K, the function $\Phi(t) = \phi(exp(it\xi x)$ is convex in t.*

Proof.

The result is known in the case where ϕ is smooth, see [7]. In the general case, since Y is Stein we know from [5] that there is a sequence ϕ_n of smooth strictly plurisubharmonic functions on Y pointwise decreasing to ϕ. Integration on the K orbits and using Dini theorem, we find a sequence of smooth stricly plurisubharmonic K invariant functions pointwise decreasing to ϕ. The result follows. ■

3 Main Theorem

Theorem 3.1 *Let X be a P-pseudoconvex space, let K be a compact connected Lie group acting on X by biholomorphisms. Assume that the local action of $K^{\mathbf{C}}$ is univalent. Than the Palais globalization X^* of X is a Hausdorff complex space. Moreover X is orbit convex in X^*, and the holomorphic functions on X can be approximated by holomorphic functions on X^* in the compact-open topology. If X is weakly pseudoconvex or C^0 weakly pseudoconvex then so is X^*. Moreover if X is smooth and C^k weakly pseudoconvex with $k \geq 2$, then so is X^*.*

Proof.

Since the local action of $K^{\mathbf{C}}$ is assumed to be univalent, we know that the space X^* exists as a quasi complex space. Moreover the leaves of the Palais foliation in $K^{\mathbf{C}} \times X$ are closed subspaces. On the other hand $K^{\mathbf{C}}$ is a Stein manifold, see [8], therefore $K^{\mathbf{C}} \times X$ is P-pseudoconex and so is every leaf of the foliation. By Lemma 2.13 every leaf is therefore Stein. By univalence the image of the leaf by the projection on the first factor is also Stein. Hence by [7] for every $x \in X$ the set $\Omega_x = \{g \in K^{\mathbf{C}} : gx \in X\}$ is orbit convex in $K^{\mathbf{C}}$, this is to say that X is orbit convex in X^*. Let η be a vector in the Lie algebra of K. Consider the action of $(\mathbf{R}, +)$ on X given by $t * x = exp(t\eta)x$. By Lemma 2.2 the local action of $\mathbf{C}$ is univalent. Let X_η the Palais globalization of X with respect to this action. Since X is orbit convex in X^* it follows from Lemma 2.5 that each X_η can be identified with an open set of X^*. Moreover, since K is polar, given two poins p, q in X^*, there exists an element $g \in K^{\mathbf{C}}$ such that gp belongs to X and gq belong to some X_η. Therefore to prove that X^* is Hausdorff is equivalent to proving that each X_η is. Fix η in the Lie algebra of K and consider the associated action of $(\mathbf{R}, +)$ as above. Assme by contraddiction that X_η is not Hausdorff. Let x_1, and x_2 be two distinct points in X_η which have no disjoint neighborhoods. Let $\Delta_{x_k} = \{t \in \mathbf{R} : it * x_k \in X\}$, with $k = 1, 2$. If there existed $t_0 \in \Delta_{x_1} \bigcap \Delta_{x_2}$ then we would have $it_0 * x_1 \in X$ and $it_0 * x_2 \in X$. Since X is Hausodrff and open in X_η this is not possible. So $\Delta_{x_1} \bigcap \Delta_{x_2} = \emptyset$. Since the Palais complexification is allways orbit connected, we have $\Delta_{x_1} =]\alpha, \beta[$ and $\Delta_{x_2} =]\gamma, \delta[$. We can assume $\alpha < \beta \leq \gamma < \delta$. Let $t_1 \in]\alpha, \beta[$ and $t_2 \in]\gamma, \delta[$. Since x_1 and x_2 have no disjioint neighborhhods, there exists a sequence p_n in X_η such that p_n converges to x_1 and p_n converges to x_2. So that for $n \geq n_0$ we have that $it_1 * p_n$ converges to $it_1 * x_1$ and $it_2 * p_n$ converges to $it_2 * x_2$ in X. Since X is orbit connected in X_η we know that $i\beta * p_n$ is in X. Now let $H \subseteq X$ defined by

$$H = K\left(\bigcup_n (it_1 * p_n)\ \bigcup (it_1 * x_1)\ \bigcup_n (it_2 * p_n) \bigcup (it_2 * x_2)\right).$$

Then H is a compact K invariant subset of X. Let ϕ be a plurisubharmonic K invariant function on X, and let M be the maximum of ϕ in H. Let us fix $n \geq n_0$ and restrict ϕ to the Stein local $\mathbf{C}$ orbit of x_n. It follows that the function $\Psi(t) = \phi(exp(it\eta)p_n) = \phi(it * p_n)$ is convex. Since $\Psi(t_1) \leq M$ and $\Psi(t_2) \leq M$ we have $\Psi(\beta) = \phi(\imath\beta * p_n) \leq M$. In other words by Lemma 2.2 the point $i\beta * p_n$ lies in $\tilde{H}$. Since $\tilde{H}$ is relativly compact the sequence $i\beta p_n$ must have a subsequence which converges in X. So we can assume, by renaming the sequence, that $i\beta * p_n$ converges to $\tilde{x}$ in X. For small positive ε we then have $i(\beta - \varepsilon) * p_n$ converges to $-i\varepsilon * \tilde{x}$ in X. However $i(\beta - \varepsilon) * p_n$ converges to $i(\beta - \varepsilon) * x_1$ in X. Therefore $i\beta * x_1 = \tilde{x} \in X$, since X is open in X^* for small positive ε $i(\beta + \varepsilon) * x_1 \in X$, This is in contrast with the definition of Δ_{x_1}. So we found an absurd. Therefore X^* is an actual Hausdorff complex space.

The Fourier series expansion together with the universal property of X^*, proves that the holomorphic functions on X^* are dense in the holomorphic functions on X in the compact open topology, See [7]. We are left to show that if X is weakly pseudoconvex then so is X^*. Let ϕ be a positive plurisubharmonic exaustion of X. By avareging we can assume that ϕ is K invariant. Since K is compact if the original ϕ is an exaustion the avareged one is also an exaustion. The group $K^{\mathbf{C}}$ is Stein manifold, let ρ be a positive smooph K invariant stricly plurisubharmonic exaustion in $K^{\mathbf{C}}$. The function $\tau(g,x) = \rho(g) + \phi(x)$ is a plurisubarmonic exaustion function on $K^{\mathbf{C}} \times X$. Let $\tau^* : X^* \to \mathbf{R} \cup \{-\infty\}$ be defined by

$$\tau^*(p) = inf_{(g,x)\in\Pi^{-1}(p)}\ \tau(g,x).$$

Here $\Pi : K^{\mathbf{C}} \times X \to X^*$ is the natural projection. Since Π is an open mapping, τ^* is easily seen to be an upper semicontinuous exaustion function. we need to show that it is plurisubharmonic. Consider the map $F : K^{\mathbf{C}} \times X \to K^{\mathbf{C}} \times X^*$ defined by $F(g,x) = (g, \Pi(g,x))$, then F is holomorphic and injective. Moreover the set $\Omega = F(K^{\mathbf{C}} \times X)$ is given by $\Omega = \{(g,p) : gp \in X\}$. So Ω is open and $F^{-1} : \Omega \to K^{\mathbf{C}} \times X$ is the map $F^{-1}(g,p) = (g, gp)$. Therefore F is a holomorphic embedding and the diagonal action of K on $K^{\mathbf{C}} \times X$ trasformes via F in the left multiplication on the first factor in $K^{\mathbf{C}} \times X^*$. Moreover the leaves of the Palais foliation transform via F in the fibers of the restriction to Ω of the projection P_2 on the second factor. From now on we identify $K^{\mathbf{C}} \times X$ with Ω.

Let U be a Stein neighborhood of p in X^*. Then $\Omega \cap (K^{\mathbf{C}} \times U)$ is a weakly pseudoconvex open subset of a Stein space, hence it is Stein. By [5], there exists a decresing sequence τ_n of smooth stricly plurisubharmonic functions which pointwise converge to τ on $\Omega \cap (K^{\mathbf{C}} \times U)$. By avareging we may assume that each τ_n is K invariant. (Since τ is constant on the K orbits, by Dini theorem this avareged τ_n is still pointwise converging to τ). Note that for each fixed n the map τ_n is an exaustion on Ω. Therefore the map $\tau_n \times P_2 : \Omega \cap (K^{\mathbf{C}} \times U) \to \mathbf{R} \times U$ is proper. From [10] we know that the functions

$$\tau^*{}_n(p) = inf_{(g,p)\in\Omega}\ \tau_n(g,p)$$

are stricly plurisubharmonic in U. On the other hand ${\tau^*}_n$ is poinwise decresing to τ_*, hence τ_* is plurisubharmonic.

Assume now that ϕ is continuous, hence τ is also continuous. Since τ is an exaustion the infimium over the leaves is a minimum, so that τ^* will never take the value $-\infty$. Moreover τ^* is upper semicontinuous, i.e. for every $p \in X^*$, we have $\tau^*(p) \geq \limsup_{x\to p} \tau^*(x)$. Let now $x_n \in X^*$ converging to p with $\tau^*(x_n)$ converging to $l = \liminf_{x\to p} \tau^*(x)$. Since τ is a continuous exaustion, there exists $q_n \in \Omega$ with $\tau(q_n) = \tau^*(x_n)$. Since for large n we have $\tau(q_n) < l+1$, up to subsequence (which we can rename as above) we may assume that q_n converges to $q_\infty \in \Omega$ with $P_2(q_\infty) = p$. By continuity of τ we have $l = \tau(q_\infty) \geq \tau^*(p)$. Hence τ^* is continuous.

If X is smooth and ϕ is C^k with $k \geq 2$ then so is τ. To show that τ^* is C^k we proceed as in [9]. Note that τ is strictly plurisubharmonic along the fibers of P_2 in particular in every fiber of P_2 over U the function τ has a unique K orbit of critical points and they are absolute minima. On the other hand given a smooth strictly plurisubharmonic function Ψ on U we find that the function $\Theta = \tau + \Psi \circ P_2$ is C^k and strictly plurisubharmonic on $Y = (K^{\mathbf{C}} \times U) \cap \Omega$. To the two forms $\omega_\tau = 2i\partial\overline{\partial}\tau$ and $\omega_\Theta = 2i\partial\overline{\partial}\Theta$ we associate the correponding moment maps $\mu_\tau(x,\xi) = d\tau_x(J\xi_Y)$ and $\mu_\Theta(x,\xi) = d\Theta_x(J\xi_X)$. Here J is the complex structure, ξ is a vector in the Lie algebra of K, and ξ_Y is the vector field associated to ξ via the action of K on $K^{\mathbf{C}} \times U$. See for example [9]. However $\mu_\tau = \mu_\Theta$ and ω_Θ is a Kaeler form. Since X is smooth and the K action is free, it follows that the set $R = {\mu_\tau}^{-1}(0) = {\mu_\Theta}^{-1}(0)$. is C^{k-1} submanifold and coincides with the set of minima of τ and Θ along the fibers of P_2. We can choose local coordinates (z,w) around a point (g,p) in $K^{\mathbf{C}} \times U$ so that $z = x+iy$ and the vectors $\frac{\partial}{\partial x_\alpha}$ are tangent to the K orbits. We think of τ and Θ as functions of y and w only. There is a C^{k-1} function $y(w)$ defined in a neighborhhod of p such that, for w in such neighborhod, $(iy(w),w) \in R/K$. This means that $\tau^*(w) = \tau(iy(w),w)$ is C^{k-1}. To show that τ^* is in fact C^k we note that R/K is the zero set of $\frac{\partial\tau}{\partial y}$, it follows that $\frac{\partial\tau^*}{\partial w} = \frac{\partial\tau}{\partial w}(iy(w),w)$ is C^{k-1}, hence τ^* is C^k. ∎

Remark 3.2 *With a proof as above one can show the following: Let K be a compact connected Lie group so that $K^{\mathbf{C}}$ is acting localy and in a univalent way on the complex space X. Assume that there exists on $K^{\mathbf{C}} \times X$ an exaustion function which is plurisubharmonic when restricted to each leaf of the Palais foliation, then the Palais complexfication is Hausdorff. Moreover in the case of the group S^1 acting on smooth manifold, we can reverse this statement. More precisely if X is smooth, $K = S^1$ and X^* is Hausdorff then there exists a continuous exaustion on $\mathbf{C}^* \times X$ which is subharmonic when restricted to each leaf of the Palais foliation.*

4 Univalence Criteria

Beside the general criterion for univalence in Corollary 2.7, we have in the case of P-pseudoconvex spaces the following criteria:

Proposition 4.1 *Let K be a compact connected Lie group. Let $K^{\mathbf{C}}$ be the complexification of K. Let Y be a quasi complex space with a $K^{\mathbf{C}}$ action. Let X be a*

P-pseudoconvex space with an action of K, assume that there exists a locally injective holomorphic K equivariant map $f : X \to Y$. Then the local $K^{\mathbf{C}}$ action on X is univalent.

Proof.
Let $f^* : X^* \to Y$ the map induced by f as in Lemma 2.1. Since $f = f^* \circ j$ is locally injective, so is j. Since j is an open mapping and $X^* = K^{\mathbf{C}}X$, then the space X^* becames locally euclidean space. In particular X^* is locally Hausdorff, hence each point is closed, i.e. each leaf in $K^{\mathbf{C}} \times X$ is closed. Therefore each leaf is P-pseudoconvex. By Lemma 2.13 each leaf is Stein, and by Lemma 2.6 the local action of $K^{\mathbf{C}}$ is univalent. ■

Definition 4.2 *Let E be an holomorphic vector bundle over a complex n dimensional manifold Y, and let s be an integr with $1 \leq s \leq n$. The vector bundle E is said to be s positive if it has a smooth hermitian metric whose curvature is positive when evaluated on non zero tensors of rank not greater then s. (The 1 postivity is the Griffith positivity, whereas the n positivity is the Nakano positivity).*

Corollary 4.3 *Let K be a compact connected Lie group. Let $K^{\mathbf{C}}$ be the complexification of K. Let Y be an n dimensional complex Kaeler manifold with a $K^{\mathbf{C}}$ action. Let X be a weakly pseudoconvex manifold. Assume that K acts on X. Assume moreover that there exists a K equivariant Riemann domain projection $\pi : X \to Y$. Then X^* is a Kealer manifold, moreover both X and X^* admit complete Kaeler metrics. Let E be an s positive vector bundle over X^*, then $H^{n,q}(X^*, E) = 0$ for $q \geq n - s + 1$.*

Proof.
By Proposition 4.1 the local $K^{\mathbf{C}}$ action on X is univalent. Therefore by the main theorem X^* exists and it is weakly pseudoconvex. Moreover the map π extends to a Riemann domian projection $\pi^* : X^* \to Y$ and we can pull back the Kaeler structure on Y to obtain a Kaeler structure on X^*. The proof that X^* has a complete Kaeler matric and the proof of the vanishing theorem for cohomology of vector bundles is in [4]. ■

Proposition 4.4 *Let Y be a P-pseudoconvex space, let K be a compact connected Lie group so that $K^{\mathbf{C}}$ is acting localy in a univalent way on Y. Let X be a complex space with an action of K. Let $\phi : X \to Y$ be a proper holomorphic K equivariant map. Then X is P-pseudoconvex, and the local action of $K^{\mathbf{C}}$ on X is univalent.*

Proof. Clearly X is P-pseudoconvex, moreover because of the above Theorem the Palais globalization Y^* of Y with respect to the local action of $K^{\mathbf{C}}$ exists and Y is orbit convex in Y^*. The result then follows from the theorem in section 6 of [11]. ■

Proposition 4.5 *Let X be a P-pseudoconvex space, let T be a torus acting on X by biholomorphisms. Then the local action of $T^{\mathbf{C}}$ is univalent.*

Proof.
We prove the claim by induction on the dimension n of the torus. If $n = 1$ the result follows from lemma 2.2. If $n > 1$ we write $T = T' \times S^1$ where T' is

an $n-1$ dimensional torus. By inductive hypotesis we can construct the Palais complexification X' of X with respect to the action of T'. Since T is commutative, the action of S^1 on X is T' equivariant. By universality this action extends to an action of S^1 on X'. Since X is P-pseudoconvex then X' is an actual complex space, then by Lemma 2.2 the Palais globalization X'' of X' with respect to the S^1 action exists. It is easy to see that X'' is a complexification of X with respect to the action of T. Therefore by Lemma 4.1 the local action of $T^{\mathbf{C}}$ is univalent. ■

Because of the results of [11] we have existence of Hausdorff complexification of principal bundles over a P-pseudoconvex space X with an equivariant K action. Moreover again because of [11] we have existence of categorical quotients of hamiltonian K actions on a P-pseudoconvex space X.

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