Bifurcations on contact manifolds

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Abstract. Consider a 1-parameter compactly supported family of Legendrian submanifolds of the 1-jet bundle of a compact manifold with its natural contact structure and a path of intersection points of the Legendrian family with the 1-jet of a constant function. Since the contact distribution is a symplectic vector bundle, it is possible to assign a Maslov type index to the intersection path. We show that the non-vanishing of the Maslov intersection index implies that there exists at least one point of bifurcation from the given path of intersection points. This result can be viewed as a kind of analogue in bifurcation theory of the Arnold-Sandon conjecture on intersections of Legendrian submanifols. The proof is based on the technique of generating functions, that relates the properties of Hamiltonian diffeomorphisms to the Morse theory of the associated functions.

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1. Introduction

In a joint work with J. Pejsachowicz [9] we showed that intersections of a 1-parameter family of Lagrangian submanifolds of the cotangent bundle of a compact manifold with a given Lagrangian submanifold have stronger bifurcation properties than the intersections of general submanifolds of right codimension. As a corollary we obtained a bifurcation theorem for fixed points of symplectomorphisms (also proved in [8]) that implies that a 1-parameter family of symplectomorphisms bifurcates more often than prescribed by standard bifurcation theory. These theorems can be viewed as a kind of (weak) analogue in bifurcation theory of the classical Arnold conjectures.

Arnold conjectured that a Hamiltonian symplectomorphism φ of a compact symplectic manifold (W, ω) should have at least as many fixed points as the minimal number of critical points of a smooth function $f\colon W \to \mathbb{R}$.

Moreover, Arnold conjectured that if all fixed points are non-degenerate then their number is bounded below by the sum of the Betti numbers of W, while the Lefschetz-Hopf fixed point theorem predicts that φ must have at least as many fixed points as the Euler-Poincaré characteristic.
If φ is a Hamiltonian symplectomorphism of (W, ω) then its graph is a Lagrangian submanifold of the product $(W \times W, -\omega \oplus \omega)$ and fixed points of φ correspond to intersection points of the graph of φ with the diagonal. Thus the Arnold conjecture has an interpretation in terms of the number of intersection points between a compact Lagrangian manifold and its image under a Hamiltonian symplectomorphism.
Contact analogues of Arnold's conjectures were formulated by Sandon in [18] and in [19]. Fixed points of symplectomorphisms had to be replaced by translated points of contactomorphisms. A point q in a contact manifold is called a *translated point for a contactomorphism* ϕ with respect to some fixed contact form if $\phi(q)$ and q belong to the same Reeb orbit and the contact form is preserved at q. A *discriminant point* of ϕ is a translated point which is also a fixed point. Sandon conjectured that every contactomorphism ϕ on a compact contact manifold (M, ξ) that is contact isotopic to the identity map must have at least as many translated points as a function on M must have critical points. The analogue to intersections between Lagrangian submanifolds in symplectic geometry is given in contact geometry by Reeb chords connecting Legendrian submanifolds.

Here we investigate the bifurcation phenomena in the contact situation. The results we obtain can be viewed as (weak) analogues in bifurcation theory of Sandon's conjectures. We consider a 1-parameter compactly supported family $\{L_t\}$ of Legendrian submanifolds of the 1-jet bundle of a compact manifold with its standard contact structure and a path γ of intersection points of $\{L_t\}$ with the 1-jet j^1u_0 of a given constant function $u_0 : B \to \mathbb{R}$. We will refer to j^1u_0 as the u_0-section and we will denote it by $B_{u_0} := j^1u_0 = 0_B \times \{u_0\}$, where 0_B is the 0-section of the cotangent bundle.
Since the contact hyperplane field is a symplectic vector bundle it is possible to assign a Maslov type index $\mu(L_t, B_{u_0}; \gamma)$ to the intersection path (Section 3.1). Our main result is the following

Theorem 1.1. *Let $\{L_t\}_{t\in[0,1]}$ be a compactly supported family of Legendrian submanifolds of the 1-jet bundle J^1B of a compact manifold B, and assume that L_0 has a generating function quadratic at infinity. Let $\gamma\colon [0,1] \to J^1B$ be a path such that $\gamma(t) \in L_t \cap B_{u_0}$ for all t, and suppose that L_t is transverse to the 0-wall $0_B \times \mathbb{R}$ at $\gamma(t)$ for $t = 0, 1$. If the Maslov intersection index $\mu(L_t, B_{u_0}; \gamma) \neq 0$ then there is at least one bifurcation point from the given path γ of intersection points of $\{L_t\}$ with B_{u_0}.*

A *bifurcation point* from a path γ of intersection points of a 1-parameter Legendrian family $\{L_t\}$ with the u_0-section B_{u_0} is a point on the path such that every neighborhood of it contains at least one point of intersection of $\{L_t\}$ with the 0-wall $0_B \times \mathbb{R}$ which does not belong to γ (Definition 3.1).

The proof of Theorem 1.1 follows the same lines as in the symplectic case in [9]: given a family of generating functions quadratic at infinity $S_t\colon E \to \mathbb{R}$ for the Legendrian isotopy $\{L_t\}$, using the invariance of the Maslov index under symplectic reduction it is proved (Proposition 3.5) that $\mu(L_t, B_{u_0};\gamma) = \mu(j^1S_t, E_{u_0};\tau)$ where $\tau\colon [0,1] \to J^1E$ is the path of intersection points of the Legendrian family $\{j^1S_t\}$ with the u_0-section E_{u_0} of J^1E defined by $\tau(t) = j^1S_t(c(t))$, and $c\colon [0,1] \to E$ is the path of critical points for the family of functions $\{S_t\}$ that corresponds to the given path $\gamma\colon [0,1] \to J^1B$; but (Proposition 3.6), $\mu(j^1S_t, E_{u_0};\tau)$ is equal to the difference of the Morse indices of S_0 and S_1 at their respective critical points $c(0)$ and $c(1)$, and (Proposition 3.7) a non-zero difference of Morse indices forces a bifurcation from the path c of critical points, hence a bifurcation from the given intersection path γ of the Legendrian isotopy $\{L_t\}$ with the u_0-section B_{u_0}.
A first approach to this question was discussed in [7] where I gave a different definition of point of bifurcation from intersection points. In that framework one does not always find bifurcations. The idea to relax the definition was suggested to me by J. Pejsachowicz. The right context to formulate our problem is provided by Sandon's work [18] and [19].

Similarly to the symplectic case, we deduce from Theorem 1.1 the following bifurcation result for discriminant points of contactomorphims. Let (M,ξ) be a compact contact manifold and let $Cont_0(M)$ be the connected component of the identity map in the group of contactomorphisms of (M,ξ). We consider a contact isotopy $\{\phi_t\}$ in $Cont_0(M)$ and a smooth path β of discriminant points of $\{\phi_t\}$. Assuming that the end-points of β are non-degenerate, we define the relative Conley-Zehnder index $\mathcal{CZ}(\phi_t,\beta)$ of $\{\phi_t\}$ along β in terms of the relative Maslov intersection index along a corresponding path of intersection points between Legendrian isotopies.

Corollary 1.2. *Let $\{\phi_t\}_{t\in[0,1]}$ be a path of contactomorphisms isotopic to the identity for a compact contact manifold (M,ξ), and let $\beta\colon [0,1] \to M$ be a path such that $\beta(t)$ is a discriminant point of ϕ_t, for all t. Assume that $\beta(t)$ is non-degenerate at $t = 0,1$. If the relative Conley-Zehnder index $\mathcal{CZ}(\phi_t,\beta) \neq 0$ then there is at least one bifurcation point from the given path β of discriminant points of $\{\phi_t\}$.*

A *bifurcation point* from a path β of discriminant points of a contact isotopy $\{\phi_t\}$ is a point on the path such that every neighborhood of it contains at least one translated point of $\{\phi_t\}$ which does not belong to β (Definition 4.1).
This article is organised as follows: we recall basic definitions and the needed theorems in Section 2. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Corollary 1.2.

2. Preliminaries

In this section we recall the definitions and basic notions of contact manifolds and Legendrian submanifolds in §2.1. In §2.2 we review the definition of the

Maslov index for paths of Lagrangian subspaces of $\mathbb{R}^{2n}$, which will be used in §3.1 to define an index for a path of intersection points between two families of Legendrian submanifolds, and in Section 4 to define an index for a path of discriminant points of a contact isotopy. In §2.3 we describe the technique of generating functions, which is the main tool for the proof of Theorem 1.1.

2.1. Contact manifolds

A co-oriented contact manifold (M, ξ) is a $(2n+1)$-dimensional manifold M equipped with a smooth maximally non-integrable hyperplane field $\xi \subset TM$, *i.e.* $\xi = \ker \alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. This condition implies that $d\alpha$ restricted to ξ is a non-degenerate 2-form. The hyperplane field ξ is called the contact structure on M and α a contact form for ξ.

Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomophic* if there exists a diffeomorphism $\phi\colon M_1 \to M_2$ with $\phi_*(\xi_1) = \xi_2$, where $\phi_*\colon TM_1 \to TM_2$ is the differential of ϕ. If $\xi_i = \ker \alpha_i$ this condition is equivalent to asking that α_1 and $\phi^*\alpha_2$ determine the same hyperplane field, *i.e.* that there exists a function $g\colon M_1 \to \mathbb{R}$ such that $\phi^*\alpha_2 = e^g \alpha_1$.

On a contact manifold of dimension $2n+1$ the integral manifolds of the contact structure ξ are called *isotropic*. Because of the non-integrability of the hyperplane distribution the maximal dimension of such manifolds equals n. In this case they are called *Legendrian submanifolds*.

An *isotopy* $\{\phi_t\}$ of a manifold M is a smooth family of diffeomorphisms $\phi_t\colon M \to M$, $t \in [0,1]$ with $\phi_0 = id$. It is said to have *compact support* if $\phi_t(x) = x$ outside of a fixed compact subset. A *contact isotopy* is an isotopy of contact diffeomorphisms. A smooth family of embeddings $j_t : L_0 \to (M, \xi)$ is called *isotropic (Legendrian) isotopy* if each $j_t(L_0)$ is an isotropic (Legendrian) submanifold of (M, ξ). It is said to have *compact support* if $j_t \equiv j_0$ outside of a compact subset of L_0.

Chaperon [5] proved that any Legendrian isotopy can be extended to a contact isotopy. More generally, the Isotropic Isotopy Theorem as stated in Geiges [11, Theorem 2.6.2] is the following

Theorem 2.1. *Let $j_t\colon L_0 \to (M, \xi)$, $t \in [0,1]$ be an isotopy of isotropic embeddings of a compact manifold L_0 in a contact manifold (M, ξ). Then there exists a compactly supported contact isotopy $\{\phi_t\}$ of (M, ξ) satisfying $\phi_t \circ j_0 = j_t$.*

2.2. The Maslov-Arnold index in $\Lambda(n)$

We briefly review the construction of the Maslov index in $\mathbb{R}^{2n}$ following Arnold [1] and Duistermaat [10]. The Lagrangian Grassmaniann $\Lambda(n) = \Lambda(\mathbb{R}^{2n})$ consists of all linear Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form. The unitary group $U(n)$ acts transitively on $\Lambda(n)$ with stabiliser group $O(n)$, thus $\Lambda(n)$ is diffeomorphic to the homogeneous space $U(n)/O(n)$. The square of the determinant $\det^2 : U(n) \to S^1$ factors through a smooth map $\det^2 : \Lambda(n) \to S^1$.

Given any loop $l : S^1 \to \Lambda(n)$ the *Maslov index* $\mu(l)$ is the winding number of the closed curve $t \to \det^2\big(l(t)\big)$. The Maslov index induces an isomorphism from the fundamental group $\pi_1(\Lambda(n))$ to the free cyclic group $\mathbb{Z}$, thus from the cohomology group $H^1(\Lambda(n);\mathbb{Z})$ to $\mathbb{Z}$. A generator of $H^1(\Lambda(n);\mathbb{Z})$ is called the *Maslov class.*

To define the Maslov index of a non-closed curve of Lagrangian subspaces relative to a fixed Lagrangian subspace $\ell \in \Lambda(n)$, notice that if ℓ' is any Lagrangian subspace transverse to ℓ then ℓ' can be identified with the graph of a symmetric transformation from ℓ into itself. Thus the set $\Lambda^0(\ell)$ of Lagrangian subspaces ℓ' transverse to ℓ is an affine space diffeomorphic to the space of symmetric bilinear forms on $\mathbb{R}^n$ and hence it is contractible in $\Lambda(n)$. Let $l\colon [0,1] \to \Lambda(n)$ be a smooth path such that its end-points are transverse to ℓ, take any path δ in $\Lambda^0(\ell)$ joining the end-points of the curve l and consider the loop l' obtained by concatenating l with δ. Then the *Maslov index* $\mu(l;\ell)$ *of the path* l *relative to* ℓ is defined to be the Maslov index of the loop l'. The result is independent of the choice of δ, and since $\Lambda^0(\ell)$ is contractible $\mu(l;\ell)$ is invariant under homotopies keeping the end-points in it.

The *train* Λ_ℓ *of vertex* ℓ is the complement of the set $\Lambda^0(\ell)$ in $\Lambda(n)$. Denote by $\Lambda^k(\ell)$ the set of Lagrangian subspaces whose intersection with ℓ has dimension k. Then Λ_ℓ equals $\bigcup_{1\le k\le n}\Lambda^k(\ell)$, thus the train of vertex ℓ is a stratified space of $\Lambda(n)$. We have $\operatorname{codim}\Lambda^1(\ell) = 1$ and the $\operatorname{codim}\Lambda^k(\ell) \ge 3$ for $k > 1$. For $\lambda \in \Lambda_\ell$ define its *multiplicity* as the dimension of $\lambda \cap \ell$.

For $\lambda \in \Lambda(n)$ the tangent space $T_\lambda\Lambda(n)$ can be identified with the space of symmetric bilinear forms on λ in the following way (*cf.* [3]): every curve l in $\Lambda(n)$ such that $l(0) = \lambda$ can be written as $l(t) = A(t)\lambda$ where $A(t)$ is a path of linear symplectic transformations of $\mathbb{R}^{2n}$ with $A(0) = id$. To the vector $\frac{d}{dt}\big|_{t=0} l(t) \in T_\lambda\Lambda(n)$ there corresponds the symmetric bilinear form given by $(v,w) \to \omega_0(v,(\frac{d}{dt}\big|_{t=0}A(t))w)$ for all $v,w \in \lambda$. For $\lambda \in \Lambda^1(\ell)$ identify $T_\lambda\Lambda(n)/T_\lambda\Lambda^1(\ell)$ with the 1-dimensional space of symmetric bilinear forms on $\lambda\cap\ell$. A vector tangent to $T_\lambda\Lambda(n)/T_\lambda\Lambda^1(\ell)$ is said to be ℓ-positive if the corresponding bilinear form is positive definite on $\lambda\cap\ell$. Any such vector is transverse to the train Λ_ℓ. Thus the train Λ_ℓ is transversally oriented by ℓ-positive vectors. The train Λ_ℓ together with the transverse orientation represents a singular cycle which is Poincaré dual to the Maslov class.

Geometrically the Maslov index $\mu(l;\ell)$ for oriented curves on $\Lambda(n)$ counts with multiplicities the number of intersections of the path l with the train of vertex ℓ, with positive sign if the curve cuts the train in the direction of its co-orientation and with negative sign otherwise. Moreover, if $l : S^1 \to \Lambda(n)$ is an ℓ-positive smooth closed curve then l intersects the train Λ_ℓ in a finite number of points and $\mu(l) = \sum_{t:l(t)\in\Lambda_\ell}\dim(l(t)\cap\ell)$.

The Maslov index is natural, in the sense that if $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectic isomorphism then for any path l transverse to ℓ at the end-points we have $\mu(\phi(l);\phi(\ell)) = \mu(l;\ell)$. This follows from the homotopy invariance and the fact that the symplectic group $Sp(2n,\mathbb{R})$ is connected. This property allows to extend the definition of the Maslov index to any finite dimensional symplectic space and to any symplectic vector bundle of finite rank.

2.3. Generating Functions

The 1-jet bundle J^1B of a smooth manifold B is $T^*B \times \mathbb{R}$ and its standard contact structure is the kernel of the contact form $\alpha_B = du - \lambda_B$, where λ_B is the canonical Liouville 1-form on the cotangent bundle T^*B of B and u is the $\mathbb{R}$-coordinate. The 0-*wall* $0_B \times \mathbb{R}$ of J^1B is defined to be the product of the 0-section 0_B of T^*B with the $\mathbb{R}$-coordinate.

Given a smooth function $f\colon B \to \mathbb{R}$, the graph L_f of its differential is an exact Lagrangian submanifold of the cotangent bundle T^*B and the 1-jet $j^1f = \{(q, df(q), f(q)) \,|q \in B\}$ of the function f is a Legendrian submanifold of the 1-jet bundle J^1B. Note that (non-degenerate) critical points of f correspond to (transverse) intersection points of L_f with the 0-section 0_B of T^*B, and correspond to (transverse) intersections of j^1f with the 0-wall $0_B \times \mathbb{R}$. The submanifolds L_f and j^1f are said to be generated by the function f.

Generating functions, introduced by Hörmander [13], generalise this construction. They are functions associated to Lagrangian submanifolds of the cotangent bundle and to Legendrian submanifolds of the 1-jet bundle that are not necessary sections. We recall the definition of generating function for Lagrangian submanifolds of T^*B and for Legendrian submanifolds of J^1B. We refer to the articles of Sandon [17], [18].
Let E be the total space of a fiber bundle $p\colon E \to B$, and let $S\colon E \to \mathbb{R}$ be a function such that $dS\colon E \to T^*E$ is transverse to the fiber normal bundle

$$N_E := \{(e, \nu) \in T^*E \mid \nu = 0 \text{ on} \ker dp(e)\}.$$

The set $\Sigma_S = (dS)^{-1}(N_E)$ of fiber critical points of S is a submanifold of E of the same dimension of B. To any $e \in \Sigma_S$ one associates an element $v^*(e)$ of $T^*_{p(e)}B$ defined by $v^*(e)(X) := dS(\hat{X})$ for $X \in T_{p(e)}B$, where $\hat{X}$ is any vector in T_eE with $p_*(\hat{X}) = X$. Then $i_S\colon \Sigma_S \to T^*B,\ e \mapsto (p(e), v^*(e))$ and $j_S\colon \Sigma_S \to J^1B,\ e \mapsto (p(e), v^*(e), S(e))$ are respectively a Lagrangian and a Legendrian immersion. The function $S\colon E \to \mathbb{R}$ is called a generating function for the Lagrangian and Legendrian immersions i_S and j_S (or for the Lagrangian and Legendrian submanifolds $i_S(\Sigma_S)$ and $j_S(\Sigma_S)$ when i_S and j_S are embeddings). Furthermore, given a Legendrian immersion $j\colon L_0 \to J^1B$, the function $S\colon E \to \mathbb{R}$ is said to be a generating function for j if there exists a diffeomorphism $h\colon L_0 \to \Sigma_S$ such that $j = j_S \circ h$. If j is an embedding then so is j_S.
The significant feature of this construction is that critical points of S correspond to intersection points of $i_S(\Sigma_S)$ with the 0-section 0_B of T^*B and to intersection points of $j_S(\Sigma_S)$ with the 0-wall of J^1B. Non-degenerate critical points correspond to transverse intersections. Moreover critical points with zero as a critical value correspond to intersection points of the generated Legendrian submanifold with the 0-section B_0 of J^1B.

A generating function $S\colon E \to \mathbb{R}$ is said to be *quadratic at infinity* (*g.f.q.i.*) if $p\colon E \to B$ is a vector bundle of finite rank and if there exists a non-degenerate quadratic form $F\colon E \to \mathbb{R}$ such that the vertical derivative $\partial_v(S - F) : E \to E^*$ is bounded.

Existence results for *g.f.q.i.* on the cotangent bundle were obtained by Sikorav ([20], [21]) using ideas of Chaperon [4] and of Laudenbach-Sikorav [14]. These results have been generalised to the contact case by Chaperon [5] and Chekanov [6]. The existence Theorem of *g.f.q.i.* for Legendrian submanifolds of J^1B states the following

Theorem 2.2. *If B is compact, then any Legendrian submanifold of J^1B contact isotopic to the 0-section has a g.f.q.i.. More generally, if $L_0 \subset J^1B$ has a g.f.q.i. and $\{\phi_t\}$ is a contact isotopy of J^1B, then there exists a continuous family of g.f.q.i. $S_t\colon E \to \mathbb{R}$ such that S_t generates the corresponding $\phi_t(L_0)$.*

3. Bifurcation from a path of intersection points

Let J^1B be the 1-jet bundle of a compact manifold B endowed with the standard contact structure. The 1-jet j^1u_0 of a constant function $u_0 : B \to \mathbb{R}$ will be referred to as the u_0-section B_{u_0} of J^1B ($B_{u_0} := j^1u_0 \equiv 0_B \times \{u_0\}$). We will consider bifurcations from intersections of a given u_0-section B_{u_0} with a 1-parameter family of compactly supported Legendrian submanifolds $\{L_t\}$ of J^1B. More precisely, we consider a family $\{L_t = j_t(L_0)\}$ where $j_t\colon L_0 \to J^1B$, $t \in [0, 1]$ is a Legendrian isotopy with compact support.

Definition 3.1. Let $\gamma\colon [0, 1] \to J^1B$ be a smooth path such that $\gamma(t) \in L_t \cap B_{u_0}$ for all t. A point $\gamma(t_*) \in L_{t_*} \cap B_{u_0}$ is said to be a *bifurcation point* from the path γ of intersection points of $\{L_t\}$ with the u_0-section B_{u_0} if any neighborhood of $(t_*, \gamma(t_*))$ in $[0, 1] \times J^1B$ contains points (t, q) such that q belongs to the intersection of L_t with the 0-wall $0_B \times \mathbb{R}$ and such that $q \neq \gamma(t)$.

It follows from the implicit function theorem that a necessary condition for $\gamma(t_*)$ to be a bifurcation point is that the manifold L_{t_*} fails to be transverse to the 0-wall $0_B \times \mathbb{R}$ at $\gamma(t_*)$. This means that for the point $p_* = \gamma(t_*)$ one has that $T_{p_*}L_{t_*} + T_{p_*}(0_B \times \mathbb{R})$ is a proper subset of $T_{p_*}J^1B$. Since $\dim T_{p_*}L_{t_*} = n$ and $\dim T_{p_*}(0_B \times \mathbb{R}) = n + 1$ this is equivalent to $T_{p_*}L_{t_*} \cap T_{p_*}(0_B \times \mathbb{R}) \neq \{0\}$.

Note. In the definition of bifurcation point from a path of intersection points given in [7] the points q were required to belong to the intersection of L_t with the 0-section $B_0 = 0_B \times \{0\}$.

Applying the generating function construction, the problem of finding bifurcation points from a path of intersection points of a Legendrian isotopy with a u_0-section translates into the problem of finding bifurcation points from a path of critical points of a family of generating functions.
Putting together Theorem 2.1 and Theorem 2.2 we obtain the following result.

Proposition 3.2. *For any smooth compactly supported family of Legendrian submanifolds L_t of J^1B satisfying the hypothesis of Theorem* 1.1 *there exists a smooth 1-parameter family of function quadratic at infinity $S_t\colon E = B\times\mathbb{R}^k \to \mathbb{R}$, $t \in [0,1]$ such that S_t generates L_t, i.e. $L_t = j_{S_t}(\Sigma_{S_t})$.*

Consider the 1-jet bundle J^1E with the standard contact structure. Observe that the 1-parameter family of Legendrian submanifolds $\{j^1S_t\}$ of J^1E intersects the u_0-section $E_{u_0} = 0_E \times \{u_0\}$ (0_E denotes the 0-section of the cotangent bundle T^*E).
To see this, recall that $\gamma\colon [0,1] \to J^1B$ is a path such that $\gamma(t) \in L_t \cap B_{u_0}$, for all $t \in [0,1]$. Call $c\colon [0,1] \to E$ the path of critical points of the 1-parameter family of generating functions $\{S_t\}$ that corresponds to the given path γ of intersection points, *i.e.* for each t, $dS_t(c(t)) = 0$ and $j_{S_t}(c(t)) = \gamma(t)$. Recall that $j_{S_t}(c(t)) = (p(c(t)), v^*(c(t)), S_t(c(t))$. Since $c(t)$ is a critical point of S_t we have $v^*(c(t)) = 0$. And since $\gamma(t)$ belongs to the u_0-section B_{u_0} of J^1B, the critical value of S_t at the critical point $c(t)$ is u_0. Hence

$$\gamma(t) = j_{S_t}(c(t)) = (p(c(t)), 0, u_0). \tag{3.1}$$

Define $\tau\colon [0,1] \to J^1E$ to be the path $t \mapsto j^1S_t(c(t))$. Since the 1-jet of S_t at $c(t)$ is $j^1S_t(c(t)) = (c(t), dS_t(c(t)), S_t(c(t)))$ we have that

$$\tau(t) = j^1S_t(c(t)) = (c(t), 0, u_0) = j^1u_0(c(t)), \tag{3.2}$$

where, in the last term, j^1u_0 is the 1-jet of the constant function $u_0 : E \to \mathbb{R}$, *i.e.* $j^1u_0 = E_{u_0}$. Hence $\tau(t) \in j^1S_t \cap E_{u_0}$, for all $t \in [0,1]$.

3.1. The Maslov index for an intersection path

In this section we define a Maslov type index of two families of Legendrian submanifolds $\{L_t\}$ and $\{B_t\}$ of a contact manifold $(M, \xi = \ker\alpha)$ along a path $\gamma\colon [0,1] \to M$ such that $\gamma(t) \in L_t \cap B_t$, for all t and such that the tangent spaces to the Legendrian submanifolds intersect transversally in the contact hyperplane at the end-points of γ.
The contact structure ξ is a $2n$-dimensional subbundle of the tangent bundle TM of M. The restriction $(d\alpha)|_\xi$ of $d\alpha$ to ξ defines on every fiber a symplectic form. Hence $(\xi, (d\alpha)|_\xi) \to M$ is a symplectic vector bundle. Since L_t and B_t are Legendrian submanifolds their respective tangent spaces $T_{\gamma(t)}L_t$ and $T_{\gamma(t)}B_t$ at $\gamma(t)$ are Lagrangian subspaces of $(\xi_{\gamma(t)}, (d\alpha)|_{\xi_{\gamma(t)}})$. Consider the pullback bundle $\gamma^*\xi$ of the symplectic vector bundle $(\xi, (d\alpha)|_\xi)$ along the path γ. Since the interval $[0,1]$ is contractible, $\gamma^*\xi$ is a trivial symplectic bundle whose fiber over t is the symplectic vector space $(\xi_{\gamma(t)}, (d\alpha)|_{\xi_{\gamma(t)}})$. Choose a symplectic trivialisation of $\gamma^*\xi$, *i.e.* a bundle isomorphism $\Phi\colon \gamma^*\xi \to [0,1]\times\mathbb{R}^{2n}$ such that the isomorphism Φ_t between the fibers satisfies $\Phi_t^*\omega_0 = (d\alpha)|_{\xi_{\gamma(t)}}$, where ω_0 denotes the standard symplectic form on $\mathbb{R}^{2n}$. The images of $T_{\gamma(t)}L_t$ and $T_{\gamma(t)}B_t$ under the mappings Φ_t are Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. Call the resulting Lagrangian paths $\{t \mapsto l_t\}$ and $\{t \mapsto b_t\}$. By the trivialisation Φ all the Lagrangian vector spaces b_t can be identified with a fixed Lagrangian subspace $b_0 = \mathbb{R}^n \times 0$. Since by assumption l_0 and l_1 are transverse to b_0 and b_1 respectively, the relative Maslov index $\mu(l; b_0)$

is defined. And from the invariance under the action of the symplectic group it is independent of the choice of the trivialisation. We call this index the *Maslov intersection index of the Legendrian families* $\{L_t\}$ *and* $\{B_t\}$ *along* γ, and we denote it by $\mu(L_t, B_t; \gamma)$.

3.2. Symplectic reduction

We recall the notion of symplectic reduction to give a symplectic description of the generating function construction (see [23]). A submanifold Q of a symplectic manifold (W, ω) is coisotropic if at every point $q \in Q$ the symplectic orthogonal $(T_qQ)^\perp$ of the tangent space to Q at q is contained in T_qQ. The distribution $\ker(\omega|_Q)$ is integrable and gives rise to a foliation of Q by isotropic leaves. If the quotient W' of Q by the isotropic foliation is a manifold then it has a symplectic form ω' induced by ω. The symplectic manifold (W', ω') is called the symplectic reduction of (W, ω) relative to Q. If L is a Lagrangian submanifold of W transverse to Q, the quotient L' of $L \cap Q$ by the isotropic foliation is a Lagrangian submanifold of (W', ω').

In the generating function construction the fiber normal bundle N_E of the vector bundle $p\colon E \to B$ is a coisotropic submanifold of the cotangent bundle T^*E. The isotropic leaves are the fibers of E. Thus the cotangent bundle T^*B is naturally identified to the symplectic reduction of T^*E relative to N_E. The 0-section 0_E of T^*E is a Lagrangian submanifold whose symplectic reduction is the 0-section 0_B of T^*B.

In what follows identify J^1E with $T^*(B \times \mathbb{R}^k) \times \mathbb{R} = T^*B \times \mathbb{R}^k \times (\mathbb{R}^k)^* \times \mathbb{R}$, and include N_E in J^1E as $T^*B \times \mathbb{R}^k \times 0 \times 0$. Then consider the compactly supported Legendrian isotopy $\{L_t\}$ of $(J^1B, \xi = \ker \alpha_B)$ and the path $\gamma\colon [0,1] \to J^1B$ such that $\gamma(t) \in L_t \cap B_{u_0}$, for all t. Let $S_t\colon E \to \mathbb{R}$, $t \in [0,1]$ be the 1-parameter family of functions such that S_t generates L_t as in Proposition 3.2. For each t, $dS_t(E)$ is a Lagrangian submanifold of T^*E transverse to the fiber normal bundle N_E, thus its symplectic reduction is the Lagrangian submanifold $i_{S_t}(\Sigma_{S_t})$ generated by S_t.

Let $\tau\colon [0,1] \to (J^1E, \eta = \ker \alpha_E)$ be the path of intersection points of the Legendrian isotopy $\{J^1S_t\}$ with the u_0-section E_{u_0} of J^1E that corresponds to the given intersection path γ. Let $\tau^*\eta$ be the pullback bundle of the symplectic vector bundle $(\eta, (d\alpha_E)|_\eta)$ along the path τ. And let $\gamma^*\xi$ be the pullback bundle of the symplectic vector bundle $(\xi, (d\alpha_B)|_\xi)$ along the path γ.

Since the interval $[0,1]$ is contractible the pullback bundles $\tau^*\eta$ and $\gamma^*\xi$ are trivial vector bundles over $[0,1]$. The respective fibers over t are the symplectic vector spaces $(\eta_{\tau(t)}, (d\alpha_E)|_{\eta_{\tau(t)}})$ and $(\xi_{\gamma(t)}, (d\alpha_B)|_{\xi_{\gamma(t)}})$. Call

$$\Psi\colon \tau^*\eta \to [0,1] \times \mathbb{R}^{2n+2k}$$

the symplectic trivialisation of $\tau^*\eta$ and

$$\Phi\colon \gamma^*\xi \to [0,1] \times \mathbb{R}^{2n}$$

the symplectic trivialisation of $\gamma^*\xi$. We choose the trivialisations Ψ and Φ to be adapted to the surjective submersion $\pi : J^1E \to J^1B$.

Recall now the notion of symplectic reduction in the category of vector bundles [23]. If K is a coisotropic subbundle of a symplectic vector bundle (W,ω) then the reduced bundle $K/K^{\perp}$ exists and it is naturally a symplectic vector bundle. If L is a Lagrangian subbundle of W such that $L\cap K$ is a subbundle then $L\cap K^{\perp}$ is also a subbundle and the reduced bundle exists and is a Lagrangian subbundle of $K/K^{\perp}$. We claim that

Lemma 3.3. *The symplectic reduction of the symplectic vector bundle $\tau^*\eta$ relative to the coisotropic subbundle $\tau^*(TN_E)$ can be identified with the symplectic vector bundle $\gamma^*\xi$.*

Proof. Choose charts of E and B adapted to the submersion $p\colon E\to B$. Denote the corresponding local coordinate systems by $q^1,\dots,q^n$, $x^1,\dots,x^k$ and by $q^1,\dots,q^n$. Then p is expressed as $p\colon (q^1,\dots,q^n,x^1,\dots,x^k)\mapsto(q^1,\dots,q^n)$. Let $q^1,\dots,q^n,x^1,\dots,x^k,p_1,\dots,p_n,y_1,\dots y_k,u$ be local coordinates of J^1E and let $q^1,\dots,q^n,p_1,\dots,p_n,u$ be local coordinates of J^1B associated to the charts of E and B chosen above. In these coordinates the submanifold N_E is given by the equations $\{y_1=\dots=y_k=u=0\}$; the map $\pi : J^1E\to J^1B$ sends $(q^1,\dots,q^n,\ x^1,\dots,x^k,\ p_1,\dots,p_n,\ y_1,\dots y_k,u)$ to $(q^1,\dots,q^n,\ p_1,\dots,p_n,u)$; and the Legendrian embedding $j_{S_t}\colon\Sigma_{S_t}\to J^1B$ sends $(q^1,\dots,q^n,x^1,\dots,x^k)$ to $(q^1,\dots,q^n,\frac{\partial S_t}{\partial q^1},\ \dots,\ \frac{\partial S_t}{\partial q^n},S_t(q,x))$.

The contact forms of J^1E and J^1B are $\alpha_E=du-\sum_{i=1}^n p_idq^i-\sum_{i=1}^k y_idx^i$ and $\alpha_B=du-\sum_{i=1}^n p_idq^i$ respectively.

At a point z whose coordinates $p_1=\dots=p_n=y_1=\dots=y_k=0$ the contact hyperplane η_z is horizontal ($\eta_z=\ker du$). Hence $T_zN_E\cap\eta_z=T_zN_E$. Furthermore since $d\alpha_E=d\lambda_E$, for such a point z, T_zN_E contains its symplectic orthogonal complement, *i.e.* $(T_zN_E)^{\perp}\subset T_zN_E\subset\eta_z$.

Recall that the path τ is contained in the u_0-section E_{u_0} of J^1E (see (3.2)) and that by hypothesis the path γ is contained in the u_0-section B_{u_0} of J^1B (so $\xi_{\gamma(t)}=\ker du$). Therefore along the paths τ and γ the contact hyperplanes do not rotate. It follows that the pullback $\tau^*(TN_E)$ of the tangent vector bundle TN_E to the submanifold N_E along the path τ is a coisotropic subbundle of the symplectic vector bundle $\tau^*\eta$.

Since the space $(T_{\tau(t)}N_E)^{\perp}$ is generated by $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^k}$, one can see that at $\tau(t)$ the quotient of $T_{\tau(t)}N_E$ by $(T_{\tau(t)}N_E)^{\perp}$ is the fiber $\xi_{\gamma(t)}$ of $\gamma^*\xi$ at t. Thus $\gamma^*\xi$ is the reduced bundle of $\tau^*\eta$. $\square$

Furthermore,

Lemma 3.4. *The Lagrangian subbundles $\gamma^*(TB_{u_0})$ and $\gamma^*(TL_t)$ of $\gamma^*\xi$ are the reduced bundles of the Lagrangian subbundles $\tau^*(TE_{u_0})$ and $\tau^*(Tj^1S_t)$ of $\tau^*\eta$ respectively.*

Proof. The Lagrangian subbundle $\tau^*(Tj^1S_t)$ is transverse to the coisotropic subbundle $\tau^*(TN_E)$ in $\tau^*\eta$. Therefore the intersection $\tau^*(Tj^1S_t)\cap\tau^*(TN_E)$ is a subbundle of $\tau^*\eta$, thus the reduced bundle of $\tau^*(Tj^1S_t)$ relative to $\tau^*(TN_E)$ exists. Since $T_{\tau(t)}J^1S_t$ intersects $(T_{\tau(t)}N_E)^{\perp}$ in $\{0\}$ its projection into $\xi_{\gamma(t)}$ is $T_{\gamma(t)}L_t$.

By inspection in the coordinates constructed above one can see that also $T_{\tau(t)}E_{u_0}$ intersects $(T_{\tau(t)}N_E)^{\perp}$ in $\{0\}$ and that its projections into $\xi_{\gamma(t)}$ is $T_{\gamma(t)}B_{u_0}$. □

3.3. Invariance of the Maslov index under symplectic reduction

A fundamental property that we need is Viterbo's result on the invariance of the Maslov class under symplectic reduction [22]. Let (W,ω) be a symplectic vector space, let K be a non-Lagrangian coisotropic subspace and let $K/K^{\perp}$ be its symplectic reduction. Denote by $\Lambda_K(W)$ the set of Lagrangian subspaces of W transverse to K and consider the inclusion $\iota\colon \Lambda_K(W)\to\Lambda(W)$ and the projection $\rho\colon \Lambda_K(W)\to\Lambda(K/K^{\perp})$ that to $T\in\Lambda(W)$ transverse to K associates its projection $\rho(T)=T\cap K/T\cap K^{\perp}$. Note that ρ is smooth on the set $\{T\in\Lambda(W)$ such that $K^{\perp}\not\subset T\}$ [12, Proposition 3.2].

Viterbo proved that if $\bar\mu\in H^1(\Lambda(W);\mathbb{Z})$ and $\bar\mu'\in H^1(\Lambda(K/K^{\perp});\mathbb{Z})$ denote the respective Maslov classes then

$$\iota^*\bar\mu=\rho^*\bar\mu'.$$

To do this he uses the geometric interpretation of the Maslov index. For that purpose he chooses a Lagrangian subspace T_0 such that $K^{\perp}\subset T_0\subset K$ and considers the train Λ_{T_0} of vertex T_0, *i.e.* the Maslov singular cycle that is Poincaré dual of the Maslov class $\bar\mu$. Recall that the Maslov class is the element of $H^1(\Lambda(n);\mathbb{Z})$ that corresponds to the element $\zeta\mapsto\zeta\cdot\Lambda_{T_0}$ of $\mathrm{Hom}(H_1(\Lambda(n),\mathbb{Z});\mathbb{Z})$ that associates to ζ its number of algebraic intersections with Λ_{T_0}. Then using that the maps ι and ρ are submersions Viterbo shows that $\iota^{-1}(\Lambda_{T_0})=\rho^{-1}(\Lambda_{\rho(T_0)})$. From this equality he obtains that

$$\iota(\zeta)\cdot\Lambda_{T_0}=\zeta\cdot\iota^{-1}(\Lambda_{T_0})=\zeta\cdot\rho^{-1}(\Lambda_{\rho(T_0)})=\rho(\zeta)\cdot\Lambda_{\rho(T_0)}. \tag{3.3}$$

We shall apply Viterbo's calculations to the situation of Lemma 3.3 and Lemma 3.4.

Now we define the relative Maslov intersection indices $\mu(L_t,B_{u_0};\gamma)$ and $\mu(j^1S_t,E_{u_0};\tau)$ of the pair of Legendrian families $\{L_t\}$ and $\{B_t\equiv B_{u_0}\}$ of J^1B along the path γ and of the pair of Legendrian families $\{j^1S_t\}$ and $\{E_{u_0}\}$ of J^1E along the path τ as in Section 3.1.

By the hypothesis of Theorem 1.1 L_0 and L_1 are transverse to the 0-wall $0_B\times\mathbb{R}$. This implies that their respective tangent spaces are transverse in the contact hyperplane $\xi_{\gamma(t)}$ to the tangent space to the u_0-section B_{u_0}, for $t=0,1$. Thus their images $l_t=\Phi_t(T_{\gamma(t)}L_t)$ and $b_t=\Phi_t(T_{\gamma(t)}B_{u_0})$ under the trivialisation mappings Φ_t, for $t=0,1$ are transverse. Hence the Maslov intersection index $\mu(L_t,B_{u_0};\gamma)$ is defined.

Similarly, since j^1S_t and E_{u_0} are Legendrian submanifolds of J^1E their respective tangent spaces $T_{\tau(t)}j^1S_t$ and $T_{\tau(t)}E_{u_0}$ at $\tau(t)$ are Lagrangian subspaces of the symplectic vector space $(\eta_{\tau(t)},(d\alpha_E)|_\eta)$. Hence their images under the trivialisation mappings Ψ_t, call them λ_t and ε_t, are Lagrangian subspaces of $(\mathbb{R}^{2n+2k},\Omega_0)$, where Ω_0 denotes the standard structure in $\mathbb{R}^{2n+2k}$. By the trivialisation Ψ, along the path τ identify all the Lagrangian vector spaces ε_t with the fixed Lagrangian space $\varepsilon_0=\mathbb{R}^{n+k}\times 0$. Since l_t and b_t

intersect transversally for $t = 0, 1$, so do λ_t and ε_t. Hence $\mu(\lambda_t, \varepsilon_0)$ is defined. We call this index the Maslov intersection index of the Legendrian families $\{j^1 S_t\}$ and $\{E_{u_0}\}$ along the path τ and we denote it by $\mu(j^1 S_t, E_{u_0}; \tau)$.

Let $\Lambda(\xi)$ be the bundle of Lagrangian vector subspaces of the fibers of the symplectic vector bundle $(\xi, (d\alpha_B)|_\xi)$ and let $\gamma^*\Lambda(\xi) \to [0,1]$ be its pullback bundle along γ. Let $\Lambda(\eta)$ be the bundle of Lagrangian vector subspaces of the fibers of the symplectic vector bundle $(\eta, (d\alpha_E)|_\eta)$ and let $\tau^*\Lambda(\eta) \to [0,1]$ be its pullback bundle along τ.

Denote by $\Lambda_{\tau^* N_E}(\tau^*\eta)$ the bundle of Lagrangian vector subspaces of the fibers of the symplectic bundle $\tau^*\eta$ transverse to the fibers of the coisotropic subbundle $\tau^* N_E$ of $\tau^*\eta$. By the trivialisation Ψ, along the path τ all the tangent spaces $T_{\tau(t)} N_E$ to the submanifold N_E can be identified with a fixed subspace $\kappa \simeq \mathbb{R}^{2n+k} \times 0$.

The inclusion

$$\iota \colon \Lambda_{\tau^* N_E}(\tau^*\eta) \longrightarrow \tau^*\Lambda(\eta)$$

and the projection

$$\rho \colon \Lambda_{\tau^* N_E}(\tau^*\eta) \longrightarrow \gamma^*\Lambda(\xi)$$

induce corresponding maps on each fiber, which by abuse of notation we still call ι and ρ.

Since the paths $t \mapsto T_{\tau(t)} E_{u_0}$ and $t \mapsto T_{\tau(t)}(j^1 S_t)$ belong to $\Lambda_{\tau^* N_E}(\tau^*\eta)$, their respective projections $t \mapsto T_{\gamma(t)} B_{u_0}$ and $t \mapsto T_{\gamma(t)} L_t$ belong to $\gamma^*\Lambda(\xi)$ (see Lemma 3.4); and since the trivialisations have been chosen to be adapted to the map $\pi \colon J^1 E \to J^1 B$, one has that $\rho(\varepsilon_t) = b_t$ and that $\rho(\lambda_t) = l_t$ for all t. Clearly $\iota(\lambda_t) = \lambda_t$. Therefore by (3.3) and by the continuity of all the maps involved one obtains that

$$\lambda_t \cdot \Lambda_{\varepsilon_0} = \iota(\lambda_t) \cdot \Lambda_{\varepsilon_0} = \rho(\lambda_t) \cdot \Lambda_{\rho(\varepsilon_0)} = l_t \cdot \Lambda_{b_0}.$$

From the geometric interpretation of the relative Maslov intersection index of a path as the number of algebraic intersections with the Maslov singular cycle one has that $\mu(l_t, b_0) = l_t \cdot \Lambda_{b_0}$ and that $\mu(\lambda_t, \varepsilon_0) = \lambda_t \cdot \Lambda_{\varepsilon_0}$. Then it follows that

Proposition 3.5. $\mu(j^1 S_t, E_{u_0}; \tau) = \mu(L_t, B_{u_0}; \gamma)$.

3.4. Non-vanishing of the Maslov intersection index

To conclude the proof of Theorem 1.1 we need two more propositions. We first show that generating functions allow to relate the Maslov and the Morse indices. The second result deals with bifurcations from a path of critical points of a family of smooth functions.

Recall that the Hessian $H(S, c)$ of a function $S \colon E \to \mathbb{R}$ at a critical point c is a bilinear form defined on the tangent space $T_c E$ of E at c. A critical point c is said to be non-degenerate if $H(S, c)$ is non-singular. The *Morse index* $m(S, c)$ of a function S at a critical point c is the dimension of the negative eigenspace of $H(S, c)$ (*cf.* [15]).

Proposition 3.6. *Let $S_t\colon E \to \mathbb{R}$, $t \in [0,1]$ be a 1-parameter family of functions and let $c\colon [0,1] \to E$ be a path such that for each t, $c(t)$ is a critical points of S_t. Assume that the critical points $c(t)$ of S_t, for $t = 0, 1$ are non-degenerate. Consider the family of Legendrian submanifols $\{j^1 S_t\}_{t\in[0,1]}$ of J^1E, and let $\tau\colon [0,1] \to J^1E$ be the path $\tau(t) = j^1S_t(c(t))$. Then*

$$\mu(j^1S_t, E_{u_0}; \tau) = m(S_0, c(0)) - m(S_1, c(1)).$$

Proof. Since the interval $[0,1]$ is contractible, a tubular neighbourhood of the embedded path $t \mapsto (t, c(t))$ in $[0,1] \times E$ is diffeomorphic to $\mathbb{R}^{d+1}$ ($d = n+k$, where $d = \dim E$ and $n = \dim B$), and the general case can be reduced to the case $S\colon [0,1] \times \mathbb{R}^d \to \mathbb{R}$, $c(t) \equiv 0$ with 0 being non-degenerate as critical points of S_0 and S_1. Since j^1S_t is transverse to the 0-wall $0_E \times \mathbb{R}$ at $t = 0, 1$, the tangent space $T_{\tau(t)}(j^1S_t)$ at $\tau(t)$ to j^1S_t is a Lagrangian subspace of $\mathbb{R}^{2d}$ transverse to $\mathbb{R}^d \times 0$ at $t = 0, 1$.
Associated to the Hessian $H(S_t, c(t))$ of the function S_t at the critical point $c(t)$ there is a symmetric operator $A_t\colon T_{c(t)}E \to T^*_{c(t)}E$ defined by $A_t(v) = H\big(S_t, c(t)\big)(v, \cdot)$. Thus the tangent space to $dS_t(E)$ at $\tau(t)$ can be identified with the graph of the symmetric operator A_t. On the other hand the Jacobian of the Lagrangian projection $\Pi^E\colon J^1E \to T^*E$ is non-degenerate when restricted to the contact hyperplanes in TJ^1E and so it defines an isomorphism between the contact hyperplanes and the tangent space of T^*E. Through this isomorphism the tangent space $T_{\tau(t)}(j^1S_t)$ at $\tau(t)$ to the Legendrian submanifold j^1S_t can be identified to the tangent space $T_{\Pi^E(\tau(t))}dS_t(E)$ at $\Pi^E(\tau(t))$ of the Lagrangian submanifold $dS_t(E)$ in T^*E. Hence $T_{\tau(t)}(j^1S_t)$ can be thought of as the graph of the symmetric operator A_t. In this case by the localisation properties the Maslov index [16, Theorem 2.3] can be expressed as the difference of the signatures of A at the endpoints $\mu(GrA) = \frac{1}{2}(\mathrm{sign}A(1) - \mathrm{sign}A(0))$, where the signature is the number of positive eigenvalues minus the number of negative eigenvalues. The number n_1^- of negative eigenvalues of $A(1)$ and the number n_0^- of negative eigenvalues of $A(0)$ are $m(S_1, c(1))$ and $m(S_0, c(0))$ respectively. Since the critical points $c(0)$ and $c(1)$ are assumed to be non-degenerate the Hessians at those points are non-singular. Thus the nullity of $A(1)$ and of $A(0)$ is 0 and the number of positive eigenvalues are $d - n_1^-$ and $d - n_0^-$ respectively. Therefore $\frac{1}{2}(\mathrm{sign}A(1) - \mathrm{sign}A(0)) = \frac{1}{2}(((d - n_1^-) - n_1^-) - ((d - n_0^-) - n_0^-))) = (n_0^- - n_1^-)$. ☐

Using an argument due to Berger [2] we showed in [9] the following

Proposition 3.7. *Let $S_t\colon E \to \mathbb{R}$, $t \in [0,1]$ and $c\colon [0,1] \to E$ be as in the above Proposition. If $m(S_1, c(1)) \neq m(S_0, c(0))$ then there exists $t_* \in (0,1)$ and a convergent sequence $(t_i, q_i) \in [0,1] \times E$ such that q_i is a critical point of S_{t_i} different from $c(t_i)$ and $t_i \to t_*$, $q_i \to c(t_*)$.*

Berger's argument says that in absence of bifurcation points the relative homology groups of the critical sets are independent of t, contradicting the inequality between the Morse indices.

We can now prove that the non-vanishing of the Maslov intersection index forces a bifurcation from the path of intersection points of the given Legendrian families.

Proof of Theorem 1.1. By Proposition 3.6, Proposition 3.5 and the hypothesis of Theorem 1.1 one has that

$$m(S_0, c(0)) - m(S_1, c(1)) = \mu(j^1 S_t, E; \tau) = \mu(L_t, B; \gamma) \neq 0.$$

Thus Proposition 3.7 implies that there exists $t_* \in (0,1)$ and a convergent sequence $(t_i, q_i) \in (0,1) \times E$ such that q_i is a critical point of S_{t_i} different from $c(t_i)$ and $t_i \to t_*$, $q_i \to c(t_*)$.
By hypothesis each L_t is an embedded submanifold, thus each j_{S_t} is an embedding. Therefore there is a bijection between critical points of S_t and intersection points of L_t with the 0-wall $0_B \times \mathbb{R}$. Hence $\gamma(t_*) = j_{S_{t_*}}(c(t_*))$ is a point of bifurcation from the given path γ of intersection points of the Legendrian isotopy $\{L_t\}$ with the u_0-section B_{u_0} of J^1B. $\square$

To explicit the relation of Theorem 1.1 with Sandon's conjectures recall that a contact form α determines an associated Reeb vector field R by the conditions $\imath_R \alpha = 1$ and $\imath_R d\alpha = 0$ and that the *Reeb chords* of a Legendrian submanifold L are defined to be the Reeb flow segments which begin and end on L. Then, observe that the Reeb vector field of the standard contact form $\alpha_B = du - \lambda_B$ of $(J^1B, \xi = \ker \alpha_B)$ is $R = \partial_u$, so the Reeb flow is given by translations in the $\mathbb{R}$-direction. Thus critical points of a function S_t generating the embedded Legendrian submanifold L_t correspond to Reeb chords connecting L_t to the u_0-section B_{u_0} of J^1B.

4. Bifurcation from a path of discriminant points

In contrast to the case of a Hamiltonian symplectomorphism, contactomorphisms contact isotopic to the identity do not need to have fixed points. For example, since the Reeb vector field never vanishes, for small t the time-t map of the Reeb flow does not have any fixed point. Sandon [18] discovered that in contact geometry the role of fixed points of Hamiltonian diffeomorphisms is played by translated points. Let ϕ be a contactomorphism of a contact manifold $(M, \xi = \ker \alpha)$ and let $g \colon M \to \mathbb{R}$ be the function defined by $\phi^* \alpha = e^g \alpha$. A point q of M is called a *translated point* of ϕ (with respect to the contact form α) if q and its image $\phi(q)$ belong to the same Reeb orbit, and if moreover $g(q) = 0$ (*i.e.* the contact form is preserved at q). A point q is called a *discriminant point* of ϕ if it is a translated point which is also a fixed point, *i.e.* if $\phi(q) = q$ and $g(q) = 0$. A discriminant point q is said to be *non-degenerate* if there are no tangent vectors X of M at q satisfying simultaneously $\phi_* X = X$ and $X(g) = 0$.

We consider an isotopy $\phi \colon [0,1] \to Cont_0(M)$ and a smooth path $\beta \colon [0,1] \to M$ such that $\beta(t)$ is a discriminant point of ϕ_t, for all t.

Definition 4.1. A discriminant point $\beta(t_*)$ of $\phi(t_*)$ is said to be a *bifurcation point* from a path of discriminant points β of a contact isotopy $\{\phi_t\}$ if any neighbourhood of $(t_*, \beta(t_*))$ in $[0,1] \times M$ contains points (t, q) where q is a translated point of ϕ_t such that $q \neq \beta(t)$.

We shall define the Conley-Zehnder index $\mathcal{CZ}(\phi_t, \beta)$ for the contact isotopy $\{\phi_t\}$ along the path β of discriminant points. Our aim is to show that the non-vanishing of the Conley-Zehnder index forces a bifurcation from the given path of discriminant points (Corollary 1.2). In order to apply Theorem 1.1 we need to associate to a contactomorphism of M a Legendrian submanifold in a 1-jet bundle.

The contact product of M with itself is defined to be the manifold $M \times M \times \mathbb{R}$ endowed with the contact structure given by the kernel of the product contact form $A = e^\theta \alpha_1 - \alpha_2$, where α_1 and α_2 are the pullback of α with respect to the projections of $M \times M \times \mathbb{R}$ into the first and second factors respectively and θ is the $\mathbb{R}$-coordinate. The contact diagonal $\Delta = \{(q, q, 0) \mid q \in M\}$ and the contact graph of ϕ, $\mathrm{gr}(\phi) = \{(q, \phi(q), g(q)) \mid q \in M\}$ are Legendrian submanifolds of $M \times M \times \mathbb{R}$.

By Weinstein's neighbourhood theorem for Legendrian submanifolds, a neighbourhood of Δ in the product contact manifold $M \times M \times \mathbb{R}$ is contactomorphic, by a contactomorphism Θ that preserves also the contact form, to a neighbourhood of the 0-section in the 1-jet bundle $J^1\Delta$ of Δ. (See Theorem 2.5.8 and Example 2.5.11 in [11]).

Assuming that the discriminant points $\beta(t)$ of $\{\phi_t\}$, for $t = 0, 1$ are non-degenerate, we define the relative Conley-Zehnder index of $\{\phi_t\}$ along the path of discriminant points β to be the relative Maslov intersection index of the 1-parameter family of Legendrian submanifolds $\{\Theta(\mathrm{gr}(\phi_t))\}$ with Δ along the path $t \to \Theta(\beta(t), \beta(t), 0)$, *i.e.*

$$\mathcal{CZ}(\phi_t; \beta) = \mu(\Theta(\mathrm{gr}(\phi_t)), \Delta; \Theta(\beta, \beta, 0)).$$

Recall that in the symplectic case fixed points of Hamiltonian symplectomorphisms correspond to intersection points of their graphs with the diagonal; instead in the contact case translated points of a contactomorphism ϕ are in one to one correspondence with Reeb chords between the contact graph of ϕ and the contact diagonal in $M \times M \times \mathbb{R}$. Indeed, if q is a translated point of ϕ then the corresponding point in $\mathrm{gr}(\phi)$ is of the form $(q, \varphi^R_{t_0}(q), 0)$ for some t_0, where φ^R_t is the Reeb flow generated by the Reeb vector field R defined by the contact form α. The Reeb flow of the contact form A is generated by the vector field $(0, -R, 0)$. Thus $(q, \varphi^R_{t_0}(q), 0)$ and $(q, q, 0)$ belong to the same Reeb orbit.

A time-dependent Hamiltonian function H_t uniquely defines a (contact) vector field X_t by $\imath_{X_t} d\alpha = dH_t(X_t) - dH_t$ and $\alpha(X_t) = H_t$. Observe that, unlike in the symplectic case, all contact isotopies can be written as the flow induced by a time-dependent function on M. Moreover the contact Hamiltonian function of a contact isotopy is uniquely defined, since by adding a constant to a function the generated isotopy changes (because $\alpha(X_t) = H_t$). Hence there is a 1-1 correspondence between contact isotopies and Hamiltonian functions (which however depend on the choice of a contact form α for ξ).

Proof of Corollary 1.2. We follow the idea of the proof given in [9] for the symplectic case. Let $\{\tilde{H}_t\}$ be the family of time-dependent Hamiltonian functions associated to the given contact isotopy $\{\phi_t\}$. Let U be a Weinstein neighbourhood of the diagonal Δ in $M \times M \times \mathbb{R}$, V the corresponding neighbourhood of the 0-section of $J^1\Delta$ and $\Theta\colon U \to V$ a contactomorphism preserving the contact form. In order to be in the hypothesis of Theorem 1.1 we modify the Hamiltonian $\tilde{H}$. To simplify, for the contractibility of $[0,1]$, we can assume the path $\beta(t)$ to be constant, *i.e.* $\beta(t) \equiv p$. Then take any metric on M and consider a ball $B(p,2r)$ and a positive ϵ such that the closure of $B(p,2r) \times B(p,2r) \times (-\epsilon,\epsilon)$ is contained in U. Let $f\colon M \to \mathbb{R}$ be such that $0 \leq f \leq 1, f \equiv 1$ on $B(p,r)$ and $f \equiv 0$ outside $B(p,2r)$. Define $H : [0,1] \times M \to \mathbb{R}$ by $H(t,v) = f(v)\tilde{H}(t,v)$. Let $X_t = X(H_t)$ and let ψ_t be the corresponding flow. By definition of H, $\psi_t(v) = v$ outside of $B(p,2r)$ and therefore the manifold $\mathrm{gr}(\psi_t) \cap U$ coincide with the contact diagonal Δ outside the closure of $B(p,2r) \times B(p,2r) \times (-\epsilon,\epsilon)$. Thus $L_t = \Theta(\mathrm{gr}(\psi_t) \cap U)$ is a Legendrian submanifold of $J^1\Delta$ with compact support. Since L_0 has a generating function, being ψ_0 contact isotopic to the identity map of M, if $\mathcal{CZ}(\phi,\beta) \neq 0$ we can apply Theorem 1.1 to $\{L_t\}$ and to the 0-section Δ of $J^1\Delta$. Thus we obtain at least one point of bifurcation for intersection points of $\{L_t\}$ with Δ from the intersection path $t \mapsto \gamma = \Theta(\beta(t),\beta(t),0)$.
Intersection points of L_t with the 0-wall of $J^1\Delta$ correspond to Reeb chords between L_t and the 0-section of $J^1\Delta$, which correspond to Reeb chords between the $\mathrm{gr}(\psi_t) \cap U$ and the diagonal in $M \times M \times \mathbb{R}$, hence to translated points of ψ_t.
Therefore the point of bifurcation from the path γ of intersection points of the 1-parameter Legendrian family $\{L_t\}$ with the 0-section Δ corresponds to a point of bifurcation from the given path β of discriminant points of the contact isotopy $\{\psi_t\}$. Namely, we find a convergent sequence $(t_i,q_i) \in (0,1) \times M$ with $t_i \to t_*$, $q_i \to p_*$, such that q_i is a translated point of ψ_{t_i} and $q_i \neq \beta(t_i)$ for all i. By continuous dependence on initial conditions and parameters for solutions of differential equations and compactness of $[0,1]$ it is possible to find a ball $B(p,\delta)$ such that for all $v \in B(p,\delta)$ and all $t \in [0,1]$ $\psi_t(v) \in B(p,r)$. Since on $B(p,\delta)$ we have that $\psi_t(v) = \phi_t(v)$ for all t then for i large we have that q_i is a translated point of ϕ_i. $\square$

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