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PERIODIC SOLUTIONS OF SYMMETRIC BI-KEPLER PERTURBATIONS AND APPLICATIONS

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Organization of the Talk

1. Preliminaries
2. Study of the first problem
3. Application to the planar three-body problem

Section 1 | Preliminaries

Statement of the first problem

The first case is associated to the Hamiltonian function of the form

$$H(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,\epsilon) = H_{01}(\mathbf{q}_1,\mathbf{p}_1)+H_{02}(\mathbf{q}_2,\mathbf{p}_2)+\epsilon^k H_1(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)+ H_R(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,\epsilon), \quad k\in\mathbb{N}, \tag{1}$$

where $\mathbf{q}_1 = (q_{11}, q_{12})$, $\mathbf{q}_2 = (q_{21}, q_{22})$, $\mathbf{p}_1 = (p_{11}, p_{12})$, $\mathbf{p}_2 = (p_{21}, p_{22})$ are points of $\mathbb{R}^2$,

$$H_{01}(\mathbf{q}_1,\mathbf{p}_1) = \frac{\|\mathbf{p}_1\|^2}{2} - \frac{1}{\|\mathbf{q}_1\|}, \quad H_{02}(\mathbf{q}_2,\mathbf{p}_2) = \frac{\|\mathbf{p}_2\|^2}{2} - \frac{1}{\|\mathbf{q}_2\|},$$

correspond to two planar decoupled Kepler problems, the perturbed differentiable function denoted by $H_1(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)$ is the first perturbation part which can or not have coupled terms and the function $H_R(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,\epsilon)$ is differentiable with order $\mathcal{O}(\epsilon^{k+1})$.

Statement of the second problem

The second case is associated to the Hamiltonian function of the form

$$K(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,\epsilon) = K_{01}(\mathbf{q}_1,\mathbf{p}_1)+K_{02}(\mathbf{q}_2,\mathbf{p}_2)+\epsilon^k K_1(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)+ K_R(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,\epsilon), \quad k\in\mathbb{N}, \tag{2}$$

where $\mathbf{q}_1=(q_{11},q_{12})$, $\mathbf{q}_2=(q_{21},q_{22})$, $\mathbf{p}_1=(p_{11},p_{12})$, $\mathbf{p}_2=(p_{21},p_{22})$ are points of $\mathbb{R}^2$,

$$K_{01}(\mathbf{q}_1,\mathbf{p}_1)=\frac{\|\mathbf{p}_1\|^2}{2}+\omega\mathbf{p}_1 J\mathbf{q}_1^T-\frac{1}{\|\mathbf{q}_1\|}, \quad K_{02}(\mathbf{q}_2,\mathbf{p}_2)=\frac{\|\mathbf{p}_2\|^2}{2}+\omega\mathbf{p}_2 J\mathbf{q}_2^T-\frac{1}{\|\mathbf{q}_2\|},$$

correspond to two planar decoupled rotating Kepler problems, the perturbed differentiable function $K_1(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)$ is the first perturbation part which can or not have coupled terms and $K_R(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}$ is differentiable and of order $\mathcal{O}(\epsilon^{k+1})$.

The existence of symmetries

It is also assumed that H in (1) and K in (2) are invariant under some of the following anti-symplectic reflections

$$\begin{aligned}
S_1 &: (q_{11}, q_{12}, q_{21}, q_{22}, p_{11}, p_{12}, p_{21}, p_{22}) \to (q_{11}, -q_{12}, q_{21}, -q_{22}, -p_{11}, p_{12}, -p_{21}, p_{22}),\\
S_2 &: (q_{11}, q_{12}, q_{21}, q_{22}, p_{11}, p_{12}, p_{21}, p_{22}) \to (-q_{11}, q_{12}, -q_{21}, q_{22}, p_{11}, -p_{12}, p_{21}, -p_{22}),\\
S_3 &: (q_{11}, q_{12}, q_{21}, q_{22}, p_{11}, p_{12}, p_{21}, p_{22}) \to (q_{11}, -q_{12}, -q_{21}, q_{22}, -p_{11}, p_{12}, p_{21}, -p_{22}),\\
S_4 &: (q_{11}, q_{12}, q_{21}, q_{22}, p_{11}, p_{12}, p_{21}, p_{22}) \to (-q_{11}, q_{12}, q_{21}, -q_{22}, p_{11}, -p_{12}, -p_{21}, p_{22}).
\end{aligned} \tag{3}$$

The respective fixed sets of the symmetries $S_1, \ldots, S_4$ are the four-dimensional Lagrangian subplanes defined by

$$L_1 = \{(q_{11}, 0, q_{21}, 0, 0, p_{12}, 0, p_{22}) : q_{11}, q_{21}, p_{12}, p_{22} \in \mathbb{R}\},$$

$$L_2 = \{(0, q_{12}, 0, q_{22}, p_{11}, 0, p_{21}, 0) : q_{12}, q_{22}, p_{11}, p_{21} \in \mathbb{R}\},$$

$$L_3 = \{(q_{11}, 0, 0, q_{22}, 0, p_{12}, p_{21}, 0) : q_{11}, q_{22}, p_{12}, p_{21} \in \mathbb{R}\},$$

$$L_4 = \{(0, q_{12}, q_{21}, 0, p_{11}, 0, 0, p_{22}) : q_{12}, q_{21}, p_{11}, p_{22} \in \mathbb{R}\}.$$

Symmetry conditions (Half-time symmetry and Time reversal symmetry

If a solution $\varphi(t)$ starts in the Lagrangian sub-plane L_1 at time $t = 0$. Using the time reversible symmetry S_1 ($S_1^2 = I$) we have that the two solutions $\varphi(t)$ and $\psi(t) = \varphi \circ S_1(t)$ coincide at $t = 0$; then by the theorem of uniqueness of solutions of an ordinary differential equation, they must be the same.

If there is another time $T/2$ such that the solution $\varphi(t)$ crosses the L_1 sub-plane orthogonally, then by the symmetry S_1 the orbit of $\varphi(t)$ must be closed, and $\varphi(t)$ is called an S_1-*symmetric periodic solution* with period T.

Notations and objectives

In order to enunciate our main results, under our approach, we call *circular orbit* of the bi-circular Kepler problem, the solution of the unperturbed problem associated with the Hamiltonian function (1) (resp. (2)), such that the projection of this solution onto $\mathbb{R}^4 = \{(\mathbf{q_1}, \mathbf{p_1})\}$ and $\mathbb{R}^4 = \{(\mathbf{q_2}, \mathbf{p_2})\}$ are circular solutions of the Kepler problem $H_{01}(\mathbf{q_1}, \mathbf{p_1})$ and $H_{02}(\mathbf{q_2}, \mathbf{p_2})$, respectively (resp. $K_{01}(\mathbf{q_1}, \mathbf{p_1})$ or $K_{02}(\mathbf{q_2}, \mathbf{p_2})$).

Poincaré-Delaunay variables

Furthermore, we consider conveniently modified Poincaré-Delaunay variables.

The main reasons to consider these types of variables can be summarized as follows:

Firstly, in these coordinates the characterization of "reflection" symmetries is simpler,

Secondly, because the periodicity equation (equation that characterizes the initial conditions of symmetric periodic solutions) can be reduced to a minimum number of equations,

In third place, the elimination of the degeneracy due to periodicity (maximal rank).

Poincaré-Delaunay variables

$$\begin{aligned} Q_{11} &= l_1 + g_1, & P_{11} &= L_1, \\ Q_{12} &= -\sqrt{2(L_1 - G_1)}\sin(g_1), & P_{12} &= \sqrt{2(L_1 - G_1)}\cos(g_1), \end{aligned} \quad \text{(PD-1-1)} \tag{4}$$

$$\begin{aligned} Q_{21} &= l_2 + g_2, & P_{21} &= L_2, \\ Q_{22} &= -\sqrt{2(L_2 - G_2)}\sin(g_2), & P_{22} &= \sqrt{2(L_2 - G_2)}\cos(g_2), \end{aligned} \quad \text{(PD-1-2)}, \tag{5}$$

where (l_i, g_i, L_i, G_i) are the classical Delaunay variables, associated with the Kepler problem $\mathcal{H}_{0i}$, and given by

$$\begin{aligned} l_i &= E_i - e_i \sin E_i & L_i &= \sqrt{\mathbf{a}_i}, \\ g_i &= \theta_i - f_i & G_i &= \sqrt{\mathbf{a}_i(1 - e_i^2)}, \end{aligned} \tag{6}$$

where $(\mathbf{a}_i, e_i, g_i)$ are the classical Keplerian elements of an elliptical orbit of the Kepler problem $\mathcal{H}_{0i}$. Moreover, the angle θ_i is the latitude argument, f_i is the true anomaly and E_i is the eccentric anomaly of the corresponding orbit, see more details in K. Meyer

Section 2 | Study of the first problem

The first problem in Poincaré-Delaunay variables

It is verified that in the (PD-1-1)-(PD-1-2) variables (that we call of the Poincaré-Delaunay variables or PD variables) the Hamiltonian (1) takes the form

$$\mathcal{H}(\mathbf{Q},\mathbf{P},\epsilon)=\mathcal{H}_{01}(\mathbf{Q}_1,\mathbf{P}_1)+\mathcal{H}_{02}(\mathbf{Q}_2,\mathbf{P}_2)+\epsilon^k\mathcal{H}_1(\mathbf{Q},\mathbf{P})+\mathcal{H}_R(\mathbf{Q},\mathbf{P},\epsilon), \tag{7}$$

where $\mathbf{Q}=(\mathbf{Q}_1,\mathbf{Q}_2)$, $\mathbf{P}=(\mathbf{P}_1,\mathbf{P}_2)$, $\mathbf{Q}_1=(Q_{11},Q_{12})$, $\mathbf{Q}_2=(Q_{21},Q_{22})$, $\mathbf{P}_1=(P_{11},P_{12})$, $\mathbf{P}_2=(P_{21},P_{22})$, $\mathcal{H}_R(\mathbf{Q},\mathbf{P},\epsilon)=\mathcal{O}(\epsilon^{k+1})$, $k\in\mathbb{N}$, and the Hamiltonian of the bi-Kepler problem assumes the form

$$\mathcal{H}_0(\mathbf{Q},\mathbf{P})=\mathcal{H}_{01}(\mathbf{Q}_1,\mathbf{P}_1)+\mathcal{H}_{02}(\mathbf{Q}_2,\mathbf{P}_2), \tag{8}$$

$$\mathcal{H}_{01}(\mathbf{Q}_1,\mathbf{P}_1)=-\frac{1}{2P_{11}^2},\quad \mathcal{H}_{02}(\mathbf{Q}_2,\mathbf{P}_2)=-\frac{1}{2P_{21}^2}. \tag{9}$$

The Hamiltonian system of the first problem in Poincaré-Delaunay variables

The Hamiltonian system associated to the Hamiltonian (7) is written as

$$\dot{\mathbf{Q}} = \mathcal{H}_{\mathbf{P}}, \quad \dot{\mathbf{P}} = -\mathcal{H}_{\mathbf{Q}}. \tag{10}$$

whose equations of motion are defined as

$$\begin{aligned} \dot{Q}_{j1} &= \frac{1}{P_{j1}^3} + \epsilon^k \frac{\partial \mathcal{H}_1}{\partial P_{j1}} + \mathcal{O}(\epsilon^{k+1}), & \dot{P}_{j1} &= -\epsilon^k \frac{\partial \mathcal{H}_1}{\partial Q_{j1}} + \mathcal{O}(\epsilon^{k+1}), \\ \dot{Q}_{j2} &= \epsilon^k \frac{\partial \mathcal{H}_1}{\partial P_{j2}} + \mathcal{O}(\epsilon^{k+1}), & \dot{P}_{j2} &= -\epsilon^k \frac{\partial \mathcal{H}_1}{\partial Q_{j2}} + \mathcal{O}(\epsilon^{k+1}), \end{aligned} \tag{11}$$

for $j = 1, 2$ and $k \in \mathbb{N}$.

Periodicity conditions of the first problem

Let us now assume that the Hamiltonian in (1) is invariant under the symmetry S_1. Additionally, note that in the set of PD variables an orbit is S_1-symmetric and T-periodic, if and only if, it is characterized by the following relations

Periodicity conditions

$$\begin{aligned} Q_{11}\left(\frac{T}{2}\right) = 0\,(mod\,\pi),\; Q_{12}\left(\frac{T}{2}\right) = 0, \\ Q_{21}\left(\frac{T}{2}\right) = 0\,(mod\,\pi),\; Q_{22}\left(\frac{T}{2}\right) = 0. \end{aligned} \tag{12}$$

Suitable initial conditions

In this section, we will denote by $\varphi_K(t, \mathbf{X})$ the solution (in PD-variables) of the bi-Kepler problem (8) with initial condition $\mathbf{X}$. Next, we consider a suitable initial condition $\mathbf{X}_0^{(1)}$ for the S_1-symmetric bi-circular solution of the system defined in (11) with $\epsilon = 0$,

$$\mathbf{X}_0^{(1)} = \left(n_1\pi, 0, n_2\pi, 0, \alpha_1^{-1/3}, 0, \alpha_2^{-1/3}, 0\right),\ n_1, n_2 \in \mathbb{Z},\ \alpha_1, \alpha_2 \in \mathbb{R}^+, \tag{13}$$

and we perturbed the previous initial condition in the following convenient form

$$\mathbf{X}^{(1)} = \mathbf{X}_0^{(1)} + (0, 0, 0, 0, \delta P_{11}, \delta P_{12}, \delta P_{21}, \delta P_{22}), \tag{14}$$

where $\delta Q_{j2}, \delta P_{j1}, \delta P_{j2}$ are sufficiently small (i.e., $|\delta P_{j2}| < \alpha_j^{-1/3} + |\delta P_{j1}|, j = 1, 2$ and will be considered as independent variables. Note that $\alpha_1 = \alpha_2$ corresponds to a bicircular solution with equal radii for the initial condition (13), while $\alpha_1 \neq \alpha_2$ corresponds to a bicircular solution with different radii.

Suitable initial conditions

Observe that n_1 and n_2 represent the number of orbital revolutions in each of the decoupled Kepler problems. Hence, without loss of generality, we shall assume from this point onward that $n_1, n_2 \in \{0, 1\}$.

Auxiliary relation

Next, we consider the following relation

$$\Xi_1 = \int_0^{T/2} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{12} \partial P_{12}} (\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt \int_0^{T/2} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{22}} (\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt -$$
$$\int_0^{T/2} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{12}} (\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt \int_0^{T/2} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{12} \partial P_{22}} (\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt. \quad (15)$$

The first main result: Existence of S_1-symmetric periodic solution

Theorem

[Existence of S_1-symmetric periodic solution] Let $\alpha_j \in \mathbb{R}^+$ arbitrary, $n_1, n_2 \in \{0, 1\}$ and $a_j \in \mathbb{Z}$, $j = 1, 2$ with $a_1 a_2 > 0$. Suppose that the Hamiltonian function H in (1) *is S_1-symmetric and let $\varphi_K(t, \mathbf{X}^{(1)})$ be a solution of the bi-Kepler problem with initial condition $\mathbf{X}^{(1)}$, where $\mathbf{X}^{(1)}$ is given by* (14) *which is a perturbation of $\mathbf{X}_0^{(1)}$ as in* (13)*. Moreover, assume the following conditions*

$$A) \int_0^{T/2} \frac{\partial \mathcal{H}_1}{\partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(i)}=\mathbf{X}_0^{(1)}} dt = 0,$$

$$B) \int_0^{T/2} \frac{\partial \mathcal{H}_1}{\partial P_{22}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(i)}=\mathbf{X}_0^{(1)}} dt = 0,$$

$$C) \ \Xi_1 \neq 0.$$

Continuation of the theorem

Theorem

(a) *If* $\alpha_1 \neq \alpha_2$ *such that* $\frac{\alpha_1}{\alpha_2} = \frac{a_1}{a_2} \in \mathbb{Q}$, *then for* ϵ *sufficiently small there exists a family of initial conditions parametrized by* ϵ, $\mathbf{X}_\epsilon^{(1)} = \mathbf{X}_0^{(1)} + \mathbf{X}^{(1)}(\epsilon)$ *with*

$$\mathbf{X}^{(1)}(\epsilon) = (0, 0, 0, 0, \delta P_{11}(\epsilon), \delta P_{12}(\epsilon), \delta P_{21}(\epsilon), \delta P_{22}(\epsilon)),$$

such that $\varphi(t, \mathbf{X}_\epsilon^{(1)}; \epsilon) = \varphi_K(t, \mathbf{X}_0^{(1)}) + \mathcal{O}(\epsilon)$ *is a* S_1*-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian* (1) *with fixed period* $T = \frac{2a_1\pi}{\alpha_1} = \frac{2a_2\pi}{\alpha_2} = \frac{2\pi(a_2-a_1)}{\alpha_2-\alpha_1}$.

(b) *If* $\alpha_1 = \alpha_2 = \alpha$, *then for* ϵ *sufficiently small there exists a family of initial conditions parametrized by* ϵ, $\mathbf{X}_\epsilon^{(1)} = \mathbf{X}_0^{(1)} + \mathbf{X}^{(1)}(\epsilon)$ *with*

$$\mathbf{X}^{(1)}(\epsilon) = (0, 0, 0, 0, \delta P_{11}(\epsilon), \delta P_{12}(\epsilon), \delta P_{21}(\epsilon), \delta P_{22}(\epsilon)),$$

such that $\varphi(t, \mathbf{X}_\epsilon^{(1)}; \epsilon) = \varphi_K(t, \mathbf{X}_0^{(1)}) + \mathcal{O}(\epsilon)$ *is a* S_1*-symmetric periodic solution of the Hamiltonian system associated to the Hamiltonian* (1) *with fixed period* $T = \frac{2a\pi}{\alpha}$ *where* $a = a_1 = a_2$.

Main ideas of the proof

To prove part (a), let us note that in the PD variables, the flow of the unperturbed system is given by

$$\begin{aligned} Q_{j1}^{(0)}(t) &= \frac{t}{P_{j1}^3(0)} + Q_{j1}(0), \quad & Q_{j2}^{(0)}(t) &= Q_{j2}(0), \\ P_{j1}^{(0)}(t) &= P_{j1}(0), & P_{j2}^{(0)}(t) &= P_{j2}(0). \end{aligned} \tag{16}$$

It should be noted that this solution includes the circular solutions that occur when $Q_{j2} = P_{j2} = 0$, $j = 1, 2$, that is, the circular solutions of the bi-circular Kepler problem with initial condition $\mathbf{X}_0^{(i)}$ as in (13), are given by

$$\varphi_K(t, \mathbf{X}_0^{(i)}) = \left(\frac{t}{P_{11}^3(0)} + \frac{(i-1)\pi}{2} + n_1\pi, 0, \frac{t}{P_{21}^3(0)} + \frac{(i-1)\pi}{2} + n_2\pi, 0, P_{11}(0), 0, P_{21}(0), 0 \right), \tag{17}$$

for $n_1, n_2 \in \{0, 1\}$.

Main ideas of the proof

Furthermore, by defining $P^3_{j1}(0) = \alpha_j^{-1}$, $j = 1, 2$, it is verified that this circular Keplerian orbit is of course S_1-symmetric, has period $T = \frac{2\pi a_1}{\alpha_1} = \frac{2\pi a_2}{\alpha_2}$, if and only if, $\frac{\alpha_1}{\alpha_2} = \frac{a_1}{a_2} \in \mathbb{Q}$. Moreover, we have

$$T = \frac{2\pi a_1}{\alpha_1} = \frac{2\pi a_2}{\alpha_2} = \frac{2\pi(a_2 - a_1)}{\alpha_2 - \alpha_1}.$$

Now, considering the initial condition $\mathbf{X}^{(1)}$ given in (14) and denoting by $\varphi(t, \mathbf{X}^{(1)}; \epsilon)$ the flow of the unperturbed system (11).

Main ideas of the proof

We verify that the coordinate components of the solution $\varphi(t, \mathbf{X}^{(1)}; \epsilon)$ can be approximated by

$$
\begin{aligned}
Q_{11}(t, \mathbf{X}^{(1)}; \epsilon) &= \frac{t}{(\alpha_1^{-1/3} + \delta P_{11})^3} + \mathcal{O}(\epsilon^{k+1}),\\
Q_{12}(t, \mathbf{X}^{(i)}; \epsilon) &= \epsilon^k Q_{12}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
Q_{21}(t, \mathbf{X}^{(i)}; \epsilon) &= \frac{t}{(\alpha_2^{-1/3} + \delta P_{21})^3} + n_2\pi + \epsilon^k Q_{21}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
Q_{22}(t, \mathbf{X}^{(i)}; \epsilon) &= \epsilon^k Q_{22}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
P_{11}(t, \mathbf{X}^{(i)}; \epsilon) &= \alpha_1^{-1/3} + \delta P_{11} + \epsilon^k P_{11}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
P_{12}(t, \mathbf{X}^{(i)}; \epsilon) &= \delta P_{12} + \epsilon^k P_{12}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
P_{21}(t, \mathbf{X}^{(i)}; \epsilon) &= \alpha_2^{-1/3} + \delta P_{21} + \epsilon^k P_{21}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),\\
P_{22}(t, \mathbf{X}^{(i)}; \epsilon) &= \delta P_{22} + \epsilon^k P_{22}^{(1)}(t) + \mathcal{O}(\epsilon^{k+1}),
\end{aligned}
\tag{18}
$$

where $k \in \mathbb{N}$, $n_1, n_2 \in \{0, 1\}$ and

Main ideas of the proof

where

$$Q_{jl}^{(1)}(t) = \int_0^t \frac{\partial \mathcal{H}_1}{\partial P_{jl}}(\varphi_K(s, \mathbf{X}^{(i)}))ds,$$
$$P_{jl}^{(1)}(t) = -\int_0^t \frac{\partial \mathcal{H}_1}{\partial Q_{jl}}(\varphi_K(s, \mathbf{X}^{(i)}))ds,$$

for $j, l = 1, 2$.

Main ideas of the proof

Now we impose that the solution $\varphi(t, \mathbf{X}^{(1)}; \epsilon)$ is S_1-symmetric. For this, we impose that the solution (starting in L_1) at the instant $t = T/2$ intercepts orthogonally the subspaces L_1, that is, the following four equations in PD variables must be satisfied

$$\begin{aligned} Q_{11}\left(\tfrac{T}{2}, \mathbf{X}^{(1)}; \epsilon\right) &= 0\,(mod\,\pi), \\ Q_{21}\left(\tfrac{T}{2}, \mathbf{X}^{(1)}; \epsilon\right) &= 0\,(mod\,\pi), \\ Q_{12}\left(\tfrac{T}{2}, \mathbf{X}^{(1)}; \epsilon\right) &= 0,, \\ Q_{22}\left(\tfrac{T}{2}, \mathbf{X}^{(1)}; \epsilon\right) &= 0. \end{aligned} \tag{19}$$

Main ideas of the proof

In this last equation, considering

$$0, (mod\ \pi) = m_j\pi, m_j \in \mathbb{Z}, m_j \neq n_j,$$

then from the equations $Q_{jl}(t)$, $P_{j2}(t)$ for $j, l = 1, 2$, it follows that the periodicity equations which we must solve for $\epsilon \neq 0$ as function of $\delta Q_{j2}, \delta P_{j1}, \delta P_{j2}$ are $f_{j1} = 0$ and $f_{j2} = 0$, where

$$\begin{aligned} f_{11}(x^{(1)};\epsilon) = & \frac{T/2}{(\alpha_1^{-1/3}+\delta P_{11})^3} - a_1\pi + \mathcal{O}(\epsilon^k), \\ f_{21}(x^{(1)};\epsilon) = & \frac{T/2}{(\alpha_2^{-1/3}+\delta P_{21})^3} - a_2\pi + \mathcal{O}(\epsilon^k), \\ f_{12}(x^{(1)};\epsilon) = & Q_{12}^{(1)}(T/2) + \mathcal{O}(\epsilon), \\ f_{22}(x^{(1)};\epsilon) = & P_{22}^{(1)}(T/2) + \mathcal{O}(\epsilon), \end{aligned} \tag{20}$$

with $x^{(1)} = (\delta P_{11}, \delta P_{12}, \delta P_{21}, \delta P_{22})$ and $a_j = m_j - n_j$, $j = 1, 2$.

Section 3 | Application to the planar three-body problem

The planar three-body problem

In this section, we apply our results to the planar 3-body problem whose equations of motion are

$$\ddot{\mathbf{x}}_l = \sum_{j=0, j\neq l}^{2} \frac{\mu_j(\mathbf{x}_j - \mathbf{x}_l)}{||\mathbf{x}_j - \mathbf{x}_l||^3}, \quad l = 0, 1, 2, \tag{21}$$

where $\mathbf{x}_j \in \mathbb{R}^2$ describes the position on the particle with mass μ_j. Our first step is to eliminate the center of mass using Heliocentric variables, see S. Ferraz, 1992 and J. Moser, 1970. In contrast to the restricted 3-body problem, we do not assume that one of the particles has zero mass but we assume instead that the masses of the two bodies μ_1, μ_2 are small compared with μ_0. We follow the ideas of J. Moser and E. Zehnder in Notes on dynamical systems. *American Mathematical Soc.*, vol. 12, 2005.

The planar three-body problem

Next, we shall normalize

$$\mu_0 = 1 \quad \text{with} \quad M = \sum_{l=0}^{2} \mu_l \quad \text{and} \quad \sum_{l=0}^{2} \mu_l \dot{\mathbf{x}}_l = 0.$$

The center of mass is

$$\mathbf{q}_0 = \frac{1}{M} \sum_{l=0}^{2} \mu_l \mathbf{x}_l,$$

and we introduce the relative coordinates of the first and second particle with respect to the center of mass setting

$$\mathbf{q}_1 = \mathbf{x}_1 - \mathbf{q}_0, \; \mathbf{q}_2 = \mathbf{x}_2 - \mathbf{q}_0,$$

so that with $\mu_0 = 1$,

$$\mathbf{x}_0 = M\mathbf{q}_0 - \sum_{l=1}^{2} \mu_l \mathbf{q}_l, \quad \mathbf{x}_1 = \mathbf{q}_0 + \mathbf{q}_1, \mathbf{x}_2 = \mathbf{q}_0 + \mathbf{q}_2,$$

Equations of motion of the planar three-body problem using Heliocentric variables

and we obtain the equations

$$\ddot{\mathbf{q}}_0 = 0,$$

$$\ddot{\mathbf{q}}_1 = \ddot{\mathbf{x}}_1 = \frac{\mathbf{x}_0 - \mathbf{x}_1}{||\mathbf{x}_0 - \mathbf{x}_1||^3} + \mu_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{||\mathbf{x}_2 - \mathbf{x}_1||^3},$$

$$\ddot{\mathbf{q}}_2 = \ddot{\mathbf{x}}_2 = \frac{\mathbf{x}_0 - \mathbf{x}_2}{||\mathbf{x}_0 - \mathbf{x}_2||^3} + \mu_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{||\mathbf{x}_1 - \mathbf{x}_2||^3}.$$

Equations of motion of the planar three-body problem using Heliocentric variables

The first equation expresses that the center of mass $\mathbf{q}_0$ moves on a straight line and can be ignored. The other two equations with

$$\begin{aligned}\mathbf{x}_1 - \mathbf{x}_2 &= \mathbf{q}_1 - \mathbf{q}_2,\\ \mathbf{x}_1 - \mathbf{x}_0 &= (1+\mu_1)\mathbf{q}_1 + \mu_2\mathbf{q}_2,\\ \mathbf{x}_2 - \mathbf{x}_0 &= (1+\mu_2)\mathbf{q}_2 + \mu_1\mathbf{q}_1,\end{aligned}$$

can be written of the form

$$\ddot{\mathbf{q}}_j = -\frac{(1+\mu_j)\mathbf{q}_j + \mu_l\mathbf{q}_l}{||(1+\mu_j)\mathbf{q}_j + \mu_l\mathbf{q}_l||^3} + \mu_l\frac{\mathbf{q}_l - \mathbf{q}_j}{||\mathbf{q}_l - \mathbf{q}_j||^3};\ j \neq l;\ j,l = 1,2.$$

Equations of motion of the planar three-body problem using Heliocentric variables

Defining $\mathbf{p}_j = \dot{\mathbf{q}}_j$, we obtain that the corresponding Hamiltonian function is given by

$$H_j = \frac{1}{2}||\mathbf{p}_j||^2 - \frac{1}{(1+\mu_j)||(1+\mu_j)\mathbf{q}_j + \mu_l\mathbf{q}_l||} - \frac{\mu_l}{||\mathbf{q}_l - \mathbf{q}_j||}; \; j \neq l; \; j, l = 1 \tag{22}$$

Therefore, the Hamiltonian associated with the planar 3-body problem in Heliocentric variables is invariant through the symmetries S_l, $l = 1, 2$ and it is of the form

$$H = H_1(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) + H_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2), \tag{23}$$

where $H_j(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2)$ is defined as in (22) for $j = 1, 2$.

Hamiltonian of the planar three-body problem using Heliocentric variables and putting $\mu_1 \to \epsilon$, $\mu_2 \to \epsilon$

Now, considering

$\mathbf{q}_1 = (q_{11}, q_{12})$, $\mathbf{p}_1 = (p_{11}, p_{12})$, $\mathbf{q}_2 = (q_{21}, q_{22})$, $\mathbf{p}_2 = (p_{21}, p_{22})$ and by putting $\mu_1 \to \epsilon$, $\mu_2 \to \epsilon$,

after expanding in series, the Hamiltonian (23) associated with the planar 3-body problem is given by

$$H = H_{01}(\mathbf{q}_1, \mathbf{p}_1) + H_{02}(\mathbf{q}_2, \mathbf{p}_2) + \epsilon(F_{01} + F_{02})(\mathbf{q}_1, \mathbf{q}_2) + \mathcal{O}(\epsilon^2), \tag{24}$$

Hamiltonian of the planar three-body problem using Heliocentric variables

where

$$H_{0j}(\mathbf{q}_j, \mathbf{p}_j) = \frac{1}{2}(p_{j1}^2 + p_{j2}^2) - \frac{1}{\sqrt{q_{j1}^2 + q_{j2}^2}}, \; j = 1, 2,$$

and

$$F_{01}(\mathbf{q}_1, \mathbf{q}_2) = \frac{2q_{11}^2 + 2q_{12}^2 + q_{11}q_{21} + q_{12}q_{22}}{(q_{11}^2 + q_{12}^2)^{3/2}} - \frac{1}{\sqrt{(q_{21} - q_{11})^2 + (q_{22} - q_{12})^2}},$$

$$F_{02}(\mathbf{q}_1, \mathbf{q}_2) = \frac{2q_{21}^2 + 2q_{22}^2 + q_{11}q_{21} + q_{12}q_{22}}{(q_{21}^2 + q_{22}^2)^{3/2}} - \frac{1}{\sqrt{(q_{11} - q_{21})^2 + (q_{12} - q_{22})^2}}.$$

Hamiltonian of the planar three-body problem in polar-nodal variables

Thus, we observe that the perturbed Hamiltonian is a perturbation of the planar bi-Kepler problem. Then, we express the Hamiltonian function in polar-nodal variables given by

$$q_{j1} = r_j \cos(\theta_j), \quad p_{j1} = R_j \cos(\theta_j) - \frac{\Theta_j \sin(\theta_j)}{r_j}, j = 1, 2,$$

$$q_{j2} = r_j \sin(\theta_j), \quad p_{j2} = R_j \sin(\theta_j) + \frac{\Theta_j \cos(\theta_j)}{r_j}, j = 1, 2.$$

In these coordinates, the Hamiltonian (24) admits the form

$$\widetilde{H} = \widetilde{H}_{01}(r_1, \theta_1, R_1, \Theta_1) + \widetilde{H}_{02}(r_2, \theta_2, R_2, \Theta_2) + \epsilon(\widetilde{F}_{01} + \widetilde{F}_{02})(r_1, r_2, \theta_1, \theta_2) + \mathcal{O}(\epsilon^2), \tag{25}$$

where

$$\widetilde{H}_{0j}(r_j, \theta_j, R_j, \Theta_j) = \frac{R_j^2}{2} - \frac{1}{r_j} + \frac{\Theta_j^2}{2r_j^2}, \; j = 1, 2.$$

$$\widetilde{F}_{01}(r_1, r_2, \theta_1, \theta_2) = \frac{2}{r_1} + \frac{r_2 \cos(\theta_1 - \theta_2)}{r_1^2} - \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}},$$

Hamiltonian of the planar three-body problem using Delaunay variables

Now, we need to consider the Delaunay elements (g_j, l_j, G_j, L_j), $j = 1, 2$ given by the relations

$$\theta_j = f_j + g_j, \quad \Theta_j = G_j, \quad r_j = r(l_j, L_j, G_j), \quad \frac{R_j^2}{2} - \frac{1}{r_j} + \frac{\Theta_j^2}{2r_j^2} = -\frac{1}{2L_j}.$$

In these coordinates, the Hamiltonian (25) takes the form

$$\overline{H} = -\frac{1}{2L_1} - \frac{1}{2L_2} + \epsilon(\overline{F}_{01} + \overline{F}_{02})(\mathbf{z}) + \mathcal{O}(\epsilon^2), \tag{26}$$

where $\mathbf{z} = (r(l_1, L_1, G_1), f_1, g_1, r(l_2, L_2, G_2), f_2, g_2)$ and

Hamiltonian of the planar three-body problem using Delaunay variables

$$\widetilde{F}_{01}(\mathbf{z}) = \frac{2}{r_1} + \frac{r_2 \cos(f_1 + g_1 - f_2 - g_2)}{r_1^2} - \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(f_1 + g_1 - f_2 - g_2)}},$$

$$\widetilde{F}_{02}(\mathbf{z}) = \frac{2}{r_2} + \frac{r_1 \cos(f_1 + g_1 - f_2 - g_2)}{r_2^2} - \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(f_1 + g_1 - f_2 - g_2)}}.$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

Our next step is to introduce (symplectic) convenient Poincaré-Delaunay variables (PD-1-1) and (PD-1-2) defined in (4) and (5), respectively. Nevertheless, we do not have an explicit expression for the variables f_j and r_j in the coordinates (PD-1-1) and (PD-1-2). To achieve this, we use the Kepler equation

$$l_j = E_j - e_j \sin(E_j), \tag{27}$$

where l_j is the mean anomaly, E_j is the eccentric anomaly, and e_j is the eccentricity of the orbit. Furthermore, if f_j is the true anomaly, then the following equation holds

$$\tan \frac{E_j}{2} = \sqrt{\frac{1-e_j}{4}} \tan \frac{f_j}{2}, \; j = 1, 2. \tag{28}$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

Now, considering the eccentricity as a small parameter and truncating at order two in e_j, from the equation (27), we have that

$$E_j = l_j + e_j \sin(l_j) + e_j^2 \sin(l_j)\cos(l_j) + \mathcal{O}(e_j^3),\ j = 1, 2. \quad (29)$$

Then, using (29) we develop $r_j = L_j^2(1 - e_j \cos(E_j))$ up to order two in function of the mean anomaly, that is,

$$r_j = L_j^2 \left(1 - e_j \cos(l_j) + e_j^2 \sin^2(l_j)\right) + \mathcal{O}(e_j^3),\ j = 1, 2. \quad (30)$$

Furthermore, using (28) and (29) it is verified that the expressions for $\sin(f_j)$ and $\cos(f_j)$ up to order two as a function of the mean anomaly are given by the relations

$$\begin{aligned}
\sin(f_j) &= \sin(l_j) + 2e_j \cos(l_j)\sin(l_j) + \frac{1}{2}e_j^2 \sin(l_j)(5 - 9\sin^2(l_j)) + \mathcal{O}(e_j^3),\ j = 1, 2,\\
\cos(f_j) &= \cos(l_j) - 2e_j \sin^2(l_j) - \frac{9}{2}e_j^2 \sin^2(l_j)\cos(l_j) + \mathcal{O}(e_j^3),\ j = 1, 2.
\end{aligned} \quad (31)$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

On the other hand, from equations (4) and (5), we obtain the following relations

$$l_j = Q_{j2} + \arccos\left[\frac{P_{j2}}{\sqrt{Q_{j2}^2+P_{j2}^2}}\right], g_j = -\arccos\left[\frac{P_{j2}}{\sqrt{Q_{j2}^2+P_{j2}^2}}\right], L_j = P_{j1}, G_j = P_{j1} - \frac{P_{j2}^2}{2} - \frac{Q_{j2}^2}{2}; j =$$

leading to equations

$$\cos(l_j) = \frac{P_{j2}\cos(Q_{j1}) + Q_{j2}\sin(Q_{j1})}{\sqrt{Q_{j2}^2+P_{j2}^2}}, \quad \cos(g_j) = \frac{P_{j2}}{\sqrt{Q_{j2}^2+P_{j2}^2}}; j = 1, 2,$$

$$\sin(l_j) = \frac{P_{j2}\sin(Q_{j1}) - Q_{j2}\cos(Q_{j1})}{\sqrt{Q_{j2}^2+P_{j2}^2}}, \quad \sin(g_j) = \frac{Q_{j2}}{\sqrt{Q_{j2}^2+P_{j2}^2}}; j = 1, 2.$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

Moreover, we seek to express the perturbation in PD variables as a series expansion in the vicinity of a bi-circular solution. Specifically, we must determine all quadratic terms in Q_{j2} and P_{j2}, as we will be working with second-order partial derivatives. To achieve this, it is essential to expand the eccentricity in a series up to second order. Since the eccentricities are given by

$$e_j = \frac{\sqrt{Q_{j2}^2 + P_{j2}^2}\sqrt{4P_{j1} - P_{j2}^2 - Q_{j2}^2}}{2P_{j1}}, \, j = 1, 2,$$

it follows that the expressions for r_j, $\sin(f_j)$ and $\cos(f_j)$ in PD variables up to second order correspond to

$$r_j = P_{j1}^2 - \frac{1}{2}P_{j1}Q_{j2}\sin(Q_{j1})\sqrt{4P_{j1} - P_{j2}^2 - Q_{j2}^2} - \frac{1}{2}P_{j1}P_{j2}\cos(Q_{j1})\sqrt{4P_{j1} - P_{j2}^2 - Q_{j2}^2} +$$
$$P_{j1}\left[Q_{j2}\cos(Q_{j1}) - P_{j2}\sin(Q_{j1})\right]^2 - \frac{1}{4}\left(P_{j2}^2 + Q_{j2}^2\right)\left[Q_{j2}\cos(Q_{j1}) - P_{j2}\sin(Q_{j1})\right]^2,$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

$$\sin(f_j) = \frac{(Q_{j2}\cos(Q_{j1}) - P_{j2}\sin(Q_{j1}))}{16P_{j1}^2\sqrt{P_{j2}^2+Q_{j2}^2}}\Big[(P_{j2}^2+Q_{j2}^2)^2 - 16P_{j1}^2 - 16P_{j1}P_{j2}\cos(Q_{j1})\sqrt{4P_{j1}-P_{j2}^2-Q$$

$$2Q_{j2}\left(9P_{j2}\sin(2Q_{j1})(4P_{j1}-P_{j2}^2-Q_{j2}^2) - 8P_{j1}\sin(Q_{j1})\sqrt{4P_{j1}-P_{j2}^2-Q_{j2}^2}\right) +$$

$$9(Q_{j2}^2-P_{j2}^2)\cos(2Q_{j1})\left(4P_{j1}-P_{j2}^2-Q_{j2}^2\right) - 4P_{j1}\left(P_{j2}^2+Q_{j2}^2\right)\Big],$$

$$\cos(f_j) = \frac{1}{32P_{j1}^2\sqrt{P_{j2}^2+Q_{j2}^2}}\Big[Q_{j2}\sin(Q_{j1})\left(8P_{j1}-3(P_{j2}^2+Q_{j2}^2)\right)\left(4P_{j1}-3(P_{j2}^2+Q_{j2}^2)\right) +$$

$$32P_{j1}P_{j2}Q_{j2}\sin(2Q_{j1})\sqrt{4P_{j1}-P_{j2}^2-Q_{j2}^2} - 16P_{j1}(Q_{j2}^2-P_{j2}^2)\cos(2Q_{j1})\sqrt{4P_{j1}-P_{j2}^2-Q_{j2}^2}+$$

$$P_{j2}\cos(Q_{j1})\left(8P_{j1}-3(P_{j2}^2+Q_{j2}^2)\right)\left(4P_{j1}-3(P_{j2}^2+Q_{j2}^2)\right) - 16P_{j1}(P_{j2}^2+Q_{j2}^2)\sqrt{4P_{j1}-P_{j2}^2-$$

$$9\left(4P_{j1}-P_{j2}^2-Q_{j2}^2\right)\left(Q_{j2}(Q_{j2}^2-3P_{j2}^2)\sin(3Q_{j1}) - P_{j2}(P_{j2}^2-3Q_{j2}^2)\cos(3Q_{j1})\right)\Big].$$

Hamiltonian of the planar three-body problem using Delaunay-Poincaré variables

Therefore, the Hamiltonian $\overline{H}$ given by (26) in coordinates (PD-1-1) and (PD-1-2) assumes the form of the Hamiltonian (7) with $k = 1$, that is,

$$\mathcal{H}(\mathbf{Q},\mathbf{P},\epsilon) = -\frac{1}{2P_{11}^2} - \frac{1}{2P_{21}^2} + \epsilon\mathcal{H}_1(\mathbf{Q},\mathbf{P}) + \cdots \qquad (32)$$

Remark

It is important to note that the expression for $\mathcal{H}_1(\mathbf{Q},\mathbf{P})$ can be determined explicitly using MATHEMATICA, however, since this expression is sufficiently extensive, it has not been included in this work since for our analysis only the expressions of certain partial derivatives evaluated in the solutions of the bi-Kepler problem are needed.

Main result applied to the planar three-body problem: Existence of S_1-symmetric periodic solution

Theorem

[Existence of S_1-symmetric periodic solution] We consider the initial condition $\mathbf{X}_0^{(1)}$ in (13) *with $n_1 = 1$, $n_2 = 0$, and $\alpha_1 = \alpha$, $\alpha_2 = \beta = 3\alpha$ with $\alpha > 0$ arbitrary. Then for the Hamiltonian system associated to the Hamiltonian as* (23) *with $\mu_l = \epsilon$, $l = 1, 2$, there exists a family of initial conditions parametrized by ϵ, such that each of them gives us a S_1-symmetric periodic solution with fixed period $T = 2\pi/\alpha$ close to a bi-circular Keplerian solution of period T with radius $\alpha^{-2/3}$ and $\beta^{-2/3}$.*

Main ideas of the proof

Step 1: The proof of this result is obtained by directly applying Theorem 2.2(a) with $i = 1$ by considering $\alpha_1 = \alpha$, $\beta = 3\alpha$ and assuming that $n_1 = 1$, $n_2 = 0$ in (13).
Step 2: That is, considering the initial condition $\mathbf{X}^{(1)}$ of each S_1-circular solution of each Kepler problem, by

$$\mathbf{X}_0^{(1)} = (\pi, 0, 0, 0, \alpha^{-1/3}, 0, \beta^{-1/3}, 0),\ \alpha \in \mathbb{R}^+, \tag{33}$$

Step 3: we perturbed the previous initial condition in the following convenient form

$$\mathbf{X}^{(1)} = (\pi, 0, 0, 0, \alpha^{-1/3} + \delta P_{11}, \delta P_{12}, \beta^{-1/3} + \delta P_{21}, \delta P_{22}). \tag{34}$$

Step 4: Therefore, the Keplerian S_1-symmetric solution with initial condition $\mathbf{X}^{(1)}$ is given by

$$\varphi_K(t, \mathbf{X}^{(1)}) = \left(\frac{t}{(\alpha^{-1/3} + \delta P_{11})^3} + \pi, 0, \frac{t}{(\beta^{-1/3} + \delta P_{21})^3}, 0, \alpha^{-1/3} + \delta P_{11}, \delta P_{12}, \beta^{-1/3} + \delta P_{21}, \delta P_{22} \right. \tag{35}$$

Main ideas of the proof

Step 5: Furthermore, considering $m_1 = 2$ and $m_2 = 3$ in the periodicity equation, we obtain that $a_1 = 1$ and $a_2 = 3$. Therefore, it is verified that $T = \frac{2a_1\pi}{\alpha_1} = \frac{2a_2\pi}{\alpha_2} = \frac{2\pi}{\alpha}$.

Step 6: Now, clearly the Hamiltonian (23) is S_1-invariant, so we will focus on verifying hypotheses $A)$, $B)$, and $C)$ in Theorem 2.2(a) with $i = 1$. After differentiating $\mathcal{H}_1$ with respect to P_{j2} and evaluating at the Keplerian solution defined in (35), we obtain that

$$\frac{\partial \mathcal{H}_1}{\partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = -\frac{2}{3}\alpha^{5/6}\cos(\alpha t)\left[3 - 19\sqrt[3]{3} + 27\sqrt[3]{3}\cos(2\alpha t)\right] + \frac{18\alpha^{5/6}\cos(\alpha t)\left[3 - 2\sqrt[3]{3} + 3\sqrt[3]{3}\cos(2\alpha t)\right]}{\left[9 + 3^{2/3} + 6\sqrt[3]{3}\cos(2\alpha t)\right]^{3/2}},$$

$$\frac{\partial \mathcal{H}_1}{\partial P_{22}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = \frac{\alpha^{5/6}\left[3\cos(\alpha t) + 12\sqrt[3]{3}\cos(3\alpha t) - 37\cos(5\alpha t)\right]}{2\sqrt{3}} - \frac{6\sqrt{3}\alpha^{5/6}\cos(\alpha t)\left[3 - \sqrt[3]{3} + \left(3 + 2\sqrt[3]{3}\right)\cos(2\alpha t) - 3\cos(4\alpha t)\right]}{\left[9 + 3^{2/3} + 6\sqrt[3]{3}\cos(2\alpha t)\right]^{3/2}}.$$

Main ideas of the proof

Thus, we verify that

$$\int_0^{\pi/\alpha} \frac{\partial \mathcal{H}_1}{\partial P_{j2}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt = 0,\ j = 1, 2.$$

Therefore, hypotheses $A)$ and $B)$ have been verified. Next, we proceed to demonstrate hypothesis $C)$. To do this, we define the following parameters

$$a = 9 + \sqrt[3]{9},\ b = 6\sqrt[3]{3},\ \kappa_{11}^{(1)} = 54\left(1 - 9\sqrt[3]{3} - 2\sqrt[3]{9}\right)\alpha,\ \kappa_{11}^{(2)} = -54\left(19 + 3\sqrt[3]{3}\right)\alpha,$$

$$\kappa_{11}^{(3)} = -27\left(1 - 57\sqrt[3]{3} + 5\sqrt[3]{9}\right)\alpha,\ \kappa_{11}^{(4)} = 675\sqrt[3]{9}\alpha,$$

$$\kappa_{22}^{(1)} = 54\sqrt[3]{9}\left(3 - 2\sqrt[3]{3} - \sqrt[3]{9}\right)\alpha,\ \kappa_{22}^{(2)} = -27\sqrt[3]{9}\left(39 + 5\sqrt[3]{3} + 3\sqrt[3]{9}\right)\alpha,\ \kappa_{22}^{(3)} = 36\sqrt[3]{9}\left(27 + 7\sqrt[3]{9}\right)\alpha,$$

$$\kappa_{22}^{(4)} = -18\sqrt[3]{9}\left(11 + \sqrt[3]{9}\right)\alpha,\ \kappa_{22}^{(5)} = -9\left(279 - 16\sqrt[3]{9}\right)\alpha,\ \kappa_{22}^{(6)} = 972\alpha,$$

$$\kappa_{12}^{(1)} = -18\sqrt{3}\left(9 - 18\sqrt[3]{3} + \sqrt[3]{9}\right)\alpha,\ \kappa_{12}^{(2)} = 9\sqrt{3}\left(36 - 63\sqrt[3]{3} + 4\sqrt[3]{9}\right)\alpha,\ \kappa_{12}^{(3)} = -162 \times 3^{5/6}\alpha,$$

$$\kappa_{12}^{(4)} = 54\sqrt{3}\left(9 - 2\sqrt[3]{3} + \sqrt[3]{9}\right)\alpha,\ \kappa_{12}^{(5)} = -27 \times 3^{5/6}\left(1 + 8\sqrt[3]{3}\right)\alpha.$$

Main ideas of the proof

On the other hand, let us define the following integrals

$$I_{11}^{(1)} = \int_0^{\pi/\alpha} \frac{dt}{(a+b\cos(2\alpha t))^{5/2}}$$
$$= \frac{\sqrt{a-b}\,[4aE(\xi_2)-(a+b)K(\xi_2)]+\sqrt{a+b}\,[(b-a)K(\xi_1)+4aE(\xi_1)]}{3\alpha(a-b)^2(a+b)^2},$$

$$I_{11}^{(2)} = \int_0^{\pi/\alpha} \frac{\cos(2\alpha t)dt}{(a+b\cos(2\alpha t))^{5/2}}$$
$$= \frac{\left(a^2+3b^2\right)\left[(b-a)\sqrt{a+b}E(\xi_2)-\sqrt{a-b}(a+b)E(\xi_1)\right]+a(a^2-b^2)\left[\sqrt{a-b}K(\xi_1)+\sqrt{a+b}K\right.}{3\alpha b\left(a^2-b^2\right)^{5/2}}$$

$$I_{11}^{(3)} = \int_0^{\pi/\alpha} \frac{\sin^2(2\alpha t)dt}{(a+b\cos(2\alpha t))^{5/2}}$$
$$= -\frac{2\left[\left[(a^2-b^2)\left(\sqrt{a-b}K(\xi_1)+\sqrt{a+b}K(\xi_2)\right)\right]-a(a+b)\sqrt{a-b}E(\xi_1)-a(a-b)\sqrt{a+b}E(\xi_2)\right.}{3\alpha b^2\left(a^2-b^2\right)^{3/2}}$$

$$I_{11}^{(4)} = \int_0^{\pi/\alpha} \frac{\cos(2\alpha t)\sin^2(2\alpha t)dt}{(a+b\cos(2\alpha t))^{5/2}}$$
$$= \frac{2(4a^2-3b^2)\left[(b-a)\sqrt{a+b}E(\xi_2)-\sqrt{a-b}(a+b)E(\xi_1)\right]+8a(a^2-b^2)\left[\sqrt{a-b}K(\xi_1)+\sqrt{a+b}\right.}{3\alpha b^3((a-b)(a+b))^{3/2}}$$

$$I_{22}^{(1)} = \int_0^{\pi/\alpha} \frac{\sin^4(2\alpha t)dt}{(a+b\cos(2\alpha t))^{5/2}}$$
$$= -\frac{4\left[4a\left[(a+b)\sqrt{a-b}E(\xi_1)+(a-b)\sqrt{a+b}E(\xi_2)\right]-\left(4a^2-b^2\right)\left[\sqrt{a-b}K(\xi_1)+\sqrt{a+b}K(\xi_2)\right.\right.}{3\alpha b^4\left(a^2-b^2\right)^{3/2}}$$

Main ideas of the proof

Thus, After some routine but lengthy calculations, we verify that

$$\frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{12} \partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = \frac{\alpha}{6}\left[4\left(6+5\sqrt[3]{3}\right)\cos(2\alpha t) - \sqrt[3]{3}(5\cos(4\alpha t)+27)\right] + \frac{\kappa_{11}^{(1)} + \kappa_{11}^{(2)}\cos(2\alpha t) + \kappa_{11}^{(3)}\sin^2(2\alpha t) + \kappa_{11}^{(4)}\sin^2(2\alpha t)\cos(2\alpha t)}{[a+b\cos(2\alpha t)]^{5/2}},$$

$$\frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{22}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = \frac{\alpha\left[20\cos(2\alpha t) - 5\cos(4\alpha t) + 24\sqrt[3]{3}\cos(6\alpha t) - 123\cos(8\alpha t)\right]}{2\sqrt[3]{3}} + \frac{\kappa_{22}^{(1)} + \kappa_{22}^{(2)}\sin^2(2\alpha t) + \kappa_{22}^{(3)}\sin^4(2\alpha t) + \kappa_{22}^{(4)}\cos(2\alpha t)}{[a+b\cos(2\alpha t)]^{5/2}} + \frac{\kappa_{22}^{(5)}\cos(2\alpha t)\sin^2(2\alpha t) + \kappa_{22}^{(6)}\sin^4(2\alpha t)\cos(2\alpha t)}{[a+b\cos(2\alpha t)]^{5/2}},$$

Main ideas of the proof

$$\frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = -\frac{\alpha[3\cos(2\alpha t) - 10\cos(4\alpha t) + 27\cos(6\alpha t)]}{\sqrt{3}} + \frac{\kappa_{12}^{(1)} + \kappa_{12}^{(2)}\sin^2(2\alpha t) + \kappa_{12}^{(3)}\sin^4(2\alpha t) + \kappa_{12}^{(4)}\cos(2\alpha t)}{[a + b\cos(2\alpha t)]^{5/2}} + \frac{\kappa_{12}^{(5)}\cos(2\alpha t)\sin^2(2\alpha t)}{[a + b\cos(2\alpha t)]^{5/2}},$$

$$\frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} = \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{12} \partial P_{22}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}}.$$

Main ideas of the proof

From what, we get that

$$\int_0^{\pi/\alpha} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{12} \partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt = -\frac{9\sqrt[3]{3}\pi}{2} + \kappa_{11}^{(1)} I_{11}^{(1)} + \kappa_{11}^{(2)} I_{11}^{(2)} + \kappa_{11}^{(3)} I_{11}^{(3)} + \kappa_{11}^{(4)} I_{11}^{(4)},$$

$$\int_0^{\pi/\alpha} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{22}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt = \kappa_{22}^{(1)} I_{11}^{(1)} + \kappa_{22}^{(4)} I_{11}^{(2)} + \kappa_{22}^{(2)} I_{11}^{(3)} + \kappa_{22}^{(5)} I_{11}^{(4)} + \kappa_{22}^{(3)} I_{22}^{(1)} +$$

$$\int_0^{\pi/\alpha} \frac{\partial^2 \mathcal{H}_1}{\partial \delta P_{22} \partial P_{12}}(\varphi_K(t, \mathbf{X}^{(1)}))\Big|_{\mathbf{X}^{(1)}=\mathbf{X}_0^{(1)}} dt = \kappa_{12}^{(1)} I_{11}^{(1)} + \kappa_{12}^{(4)} I_{11}^{(2)} + \kappa_{12}^{(2)} I_{11}^{(3)} + \kappa_{12}^{(5)} I_{11}^{(4)} + \kappa_{12}^{(3)} I_{22}^{(1)}.$$

Main ideas of the proof

Therefore, the computation carried out in *Mathematica* yields $\Xi_1 = -215.67229755178545` \neq 0$, where Ξ_1 is defined in (15). Hence, all the hypotheses of Theorem 2.2(a) with $i = 1$ are verified, which completes the proof.

An illustration of the S_1-symmetric periodic solution established in Theorem 3.1 is presented in Figure 1

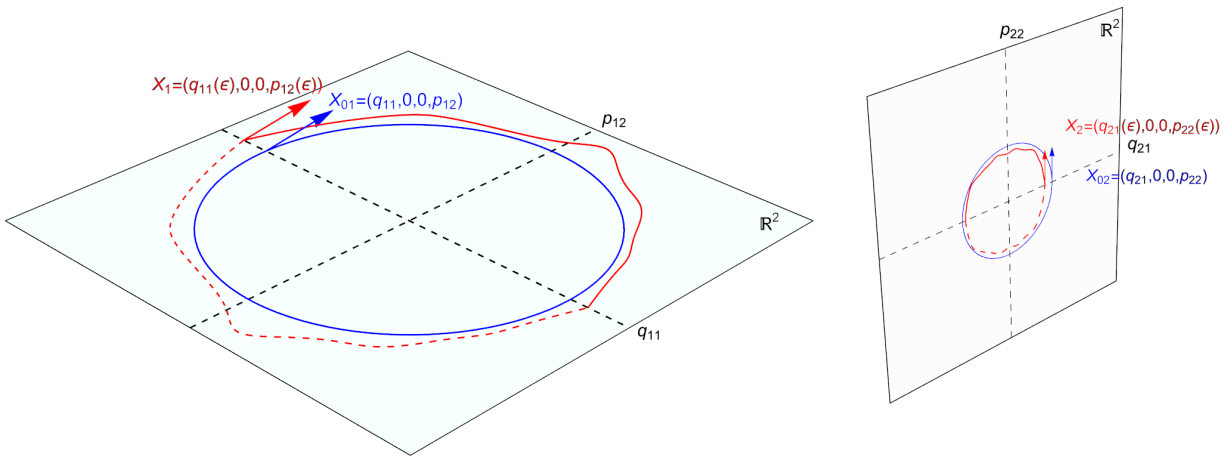


Figure: Illustration of the bi-circular Keplerian solution and the near bi-circular S_1-symmetric periodic solution (in blue color). The initial condition (in Cartesian coordinates) of each circular Keplerian orbit are $X_{01} = (q_{11}, 0, 0, p_{12})$, $X_{02} = (q_{21}, 0, 0, p_{22})$, where $q_{11} = \alpha^{-2/3}$ and $q_{21} = (3\alpha)^{-2/3}$. We denote by $X_1 = X_1(\epsilon)$ and $X_2 = X_2(\epsilon)$ the projection of the initial condition of the S_1-symmetric periodic solution with initial condition $X = (q_{11}(\epsilon), 0, q_{21}(\epsilon), 0, p_{12}(\epsilon), 0, p_{22}(\epsilon))$.

THANK YOU VERY MUCH