A numerical exploration of resonances and periodic orbits in the lunar problem

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Lunar months

The lunar months relevant for this talk are:

- the synodic month: the time interval between consecutive new or full moons, whose mean value is $T_S = 29.530\,589$ d;
- the anomalistic month: the time interval between consecutive perigee or apogee passages, whose mean value is $T_A = 27.554\,550$ d;
- the nodical month: the time interval between consecutive passages at the ascending or descending node on the ecliptic, whose mean value is $T_N = 27.212\,221$ d.

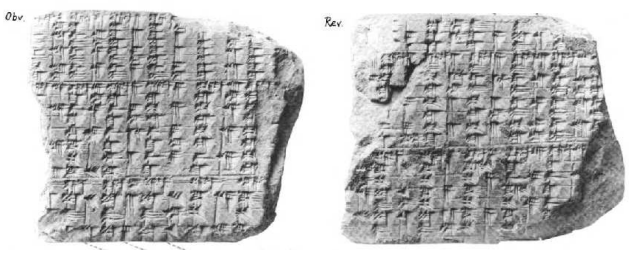
An eclipse cycle involves a near commensurability between the nodical and the synodic month (Steves 1998).

Eclipse cycles and the Saros

Among the many known eclipse cycles the most important is the Saros, known since well over two millennia.

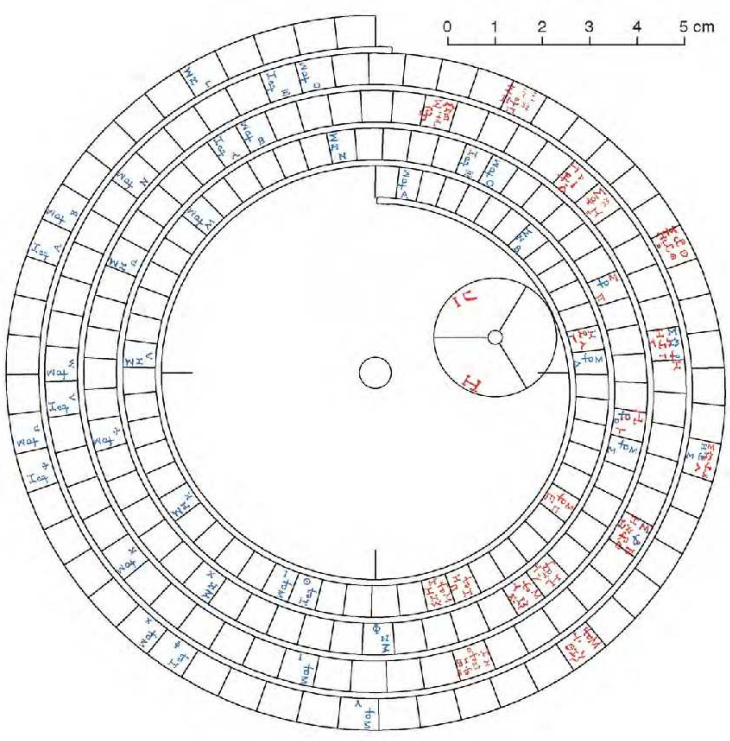
A Saros lasts 223 synodic months, a time interval nearly equal to 242 nodical months, ensuring the eclipse repetitions, and to 239 anomalistic months, ensuring the repetition of the lunar angular diameter.

synodic	$223 \cdot T_S = 6\,585.321\,347$ d
anomalistic	$239 \cdot T_A = 6\,585.537\,450$ d
nodical	$242 \cdot T_N = 6\,585.357\,482$ d



Left: the Saros Canon, a Chaldean tablet showing the eclipse seasons from 400 BC to 271 BC, arranged in 7 columns of 38 lines, with a total of 223 synodic months per column (Aaboe et al. 1991).

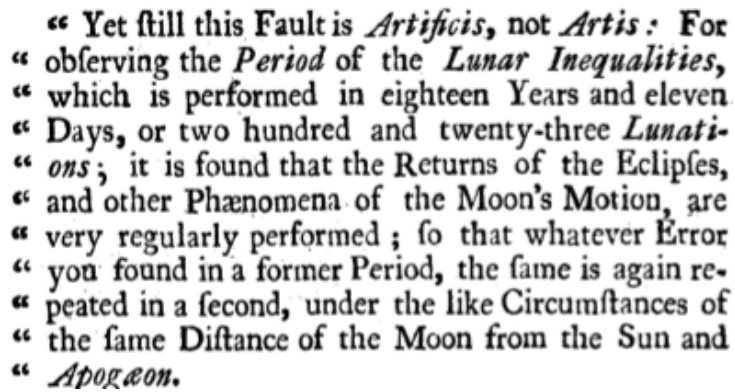
The ancient Greeks knew the Saros





The Saros was well known to the Greeks: one of the dials of the Antikythera mechanism (half of IInd century BC) predicted eclipses using the Saros: in the "user guide" appears the number 223, and one of the gears had 223 teeth (Freeth et al. 2006, 2008).

Halley's insight



" Yet ſtill this Fault is *Artificis*, not *Artis*: For " obſerving the *Period* of the *Lunar Inequalities*, " which is performed in eighteen Years and eleven " Days, or two hundred and twenty-three *Lunati-* " *ons*; it is found that the Returns of the Eclipſes, " and other Phænomena of the Moon's Motion, are " very regularly performed; ſo that whatever Error " you found in a former Period, the ſame is again re- " peated in a ſecond, under the like Circumſtances of " the ſame Diſtance of the Moon from the Sun and " *Apogæon*.

Halley wanted to solve the problem of the determination of the longitude at sea using lunar appulses of known bright stars.

In a 1731 paper he wrote: *...observing the Period of the Lunar Inequalities, which is performed in eighteen Years and eleven Days, or two hundred and twenty-three Lunations; it is found that the Return of the Eclipses and other Phaenomena of the Moon's Motion, are very regularly performed; so that whatever Error you found in the former Period, the same is again repeated in a second, under the like Circumstances of the same Distance of the Moon and the Sun and Apogaeon*.

Halley is saying that after a Saros the angular distances of Moon and Sun from the lunar apse repeat.

Roy's insight

Semidiameter of Sun and Moon during eclipse

Year	1898	1916	1934	1952	1970
Date	Jan. 7	Jan. 19	Jan. 30	Feb. 10–11	Feb. 21
Moon	14′52″.00	14′49″.8	14′48″.5	14′47″.3	14′46″.8
Sun	16′15″.87	16′15″.3	16′14″.1	16′12″.4	16′10″.3
Date	Jan. 21	Feb. 3	Feb. 13–14	Feb. 25	Mar. 7
Moon	16′24″.30	16′25″.4	16′27″.3	16′29″.2	16′31″.6
Sun	16′14″.83	16′13″.5	16′11″.6	16′09″.4	16′06″.8
Date	Jul. 3	Jul. 14	Jul. 26	Aug. 5	Aug. 17
Moon	16′43″.32	16′42″.9	16′43″.1	16′43″.2	16′43″.9
Sun	15′43″.86	15′44″.1	15′44″.9	15′46″.2	15′47″.9
Date	Jul. 18	Jul. 29	Aug. 10	Aug. 20	Aug. 31–Sep. 1
Moon	14′45″.87	14′44″.0	14′43″.2	14′42″.5	14′42″.6
Sun	15′44″.36	15′45″.3	15′46″.8	15′48″.6	15′50″.8

Roy (1973) made a modern tabulation showing the repetition of the characteristics of eclipses one Saros apart.

Most importantly, he noted that eclipses occurring with the Moon at perigee or apogee are very nearly Mirror Configurations.

Mirror Configurations

A system of n point-masses acted upon only by their mutual gravitational forces is said to be in a Mirror Configuration (MC) at a certain epoch if each radius vector from the (assumed stationary) centre of mass of the system is perpendicular to every velocity vector (Roy & Ovenden 1955).

Two kinds of MCs are possible in three-dimensional space:

- collinear MCs, in which the point-masses are on a straight line, with all their velocity vectors normal to that line;
- coplanar MCs, in which the point-masses are on a plane, with all their velocity vectors normal to that plane.

Mirror Configurations

In the circular restricted Earth-Moon-Sun problem, MCs occur at syzygies ($\lambda - \lambda' = 0, \pi$), with the Moon at perigee or apogee ($f = 0, \pi$); a collinear MC implies an eclipse ($\omega = 0, \pi$), while a coplanar MC has the Moon at maximum distance from the ecliptic ($\omega = \pi/2, 3\pi/2$).

In total, there are 16 possible MCs in this problem, 8 of which are collinear (i.e., eclipses) and 8 coplanar.

Before Roy and Ovenden, in a paper on the three-body problem Poincaré (1884) had called these configurations *conjonctions symétriques*; in fact, he defined what we call coplanar MCs in three-dimensional space, but considered collinear MCs only in the plane.

The Mirror Theorem

MCs are important for the Mirror Theorem: if n point-masses are acted upon by their mutual gravitational forces only, and at a certain epoch each radius vector from the (assumed stationary) centre of mass of the system is perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch (Roy & Ovenden 1955).

Corollary: if n point-masses are moving under their mutual gravitational forces only, their orbits are periodic if at two separate epochs two MCs occur.

Near Mirror Configurations in the Saros

As a consequence of the near commensurabilities involved in the Saros, if at a certain time the Moon is either new or full, and is very close to both the line of apsides and the line of nodes of its geocentric orbit (a collinear near MC, corresponding to an eclipse), it will be in the same situation a full Saros cycle later (Roy 1973; Perozzi et al. 1991).

For the real Moon, as we have seen, 223 synodic months last 238.992 anomalistic months and 241.999 nodical months.

If the commensurability relations involved in the Saros were exactly fulfilled, i.e. the above numbers were 223, 239 and 242, and the geocentric orbit of the Sun were circular, then we could have periodic orbits.

Saros-related periodic orbits

In fact, Valsecchi et al. (1993a) found periodic orbits in the circular restricted three-dimensional three-body problem in which the exact Saros commensurabilities are present.

There are 8 such orbits, differing from each other for the starting phases; each of these orbits involves a pair of the 16 possible MCs.

Of these orbits, 4 can be obtained from the other 4 by exploiting the symmetry of the problem with respect to the ecliptic plane.

In each of these periodic orbits the Moon makes 223 anticlockwise revolutions in the synodic reference frame; in that frame the lunar perigee makes 16 clockwise revolutions and the lunar node 19.

For the real Moon the same numbers, in a Saros, are 223, 15.992 and 18.999.

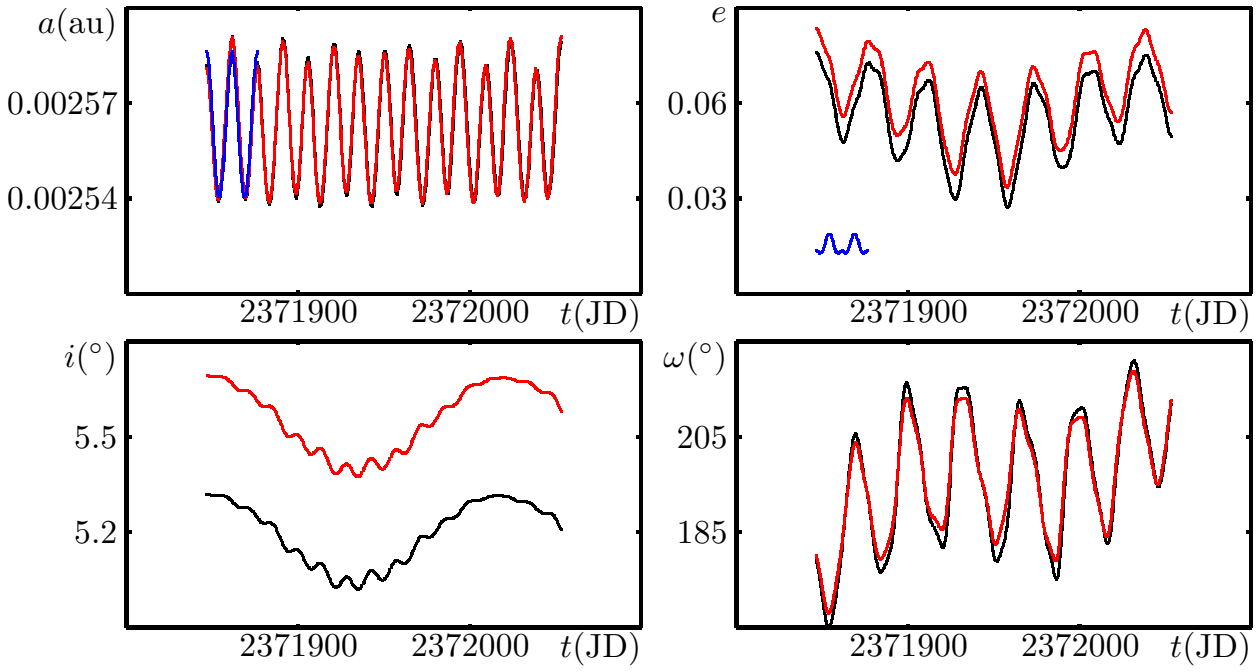
Saros-related periodic orbits

The total solar eclipse of 17.372 October 1781 was a near MC; another near MC took place half a Saros later, on 23.033 October 1790, with a lunar eclipse; the Saros ended with the near MC corresponding to the total solar eclipse of 28.693 October 1799.

JPL DE440

Date	a(au)	e	$i(^\circ)$	$\omega(^\circ)$	$f(^\circ)$	$\Delta\lambda(^\circ)$
1781	0.002 581 9	0.075 80	5.32	179.0	0.0	0.0
1790	0.002 590 7	0.048 42	5.28	0.0	178.6	180.1
1799	0.002 582 0	0.075 96	5.31	180.4	357.9	359.6
Saros-related (223-239-242) Periodic Orbit						
Rev.	a(au)	e	$i(^\circ)$	$\omega(^\circ)$	$f(^\circ)$	$\Delta\lambda(^\circ)$
0.0	0.002 581 0	0.083 36	5.69	180.0	0.0	0.0
111.5	0.002 590 6	0.056 49	5.69	0.0	180.0	180.0
223.0	0.002 581 0	0.083 36	5.69	180.0	0.0	0.0

Saros-related periodic orbits



Black: osculating a, e, i and ω of the real lunar orbit (from JPL DE440) for 7 months starting from 17.372 October 1781.

Red: the same elements for the closest Saros-related periodic orbit.

Blue: osculating a and e of Hill's variational orbit.

Saros-like periodic orbits

In the Saros-related periodic orbits the following relationships hold (Valsecchi et al. 1993a):

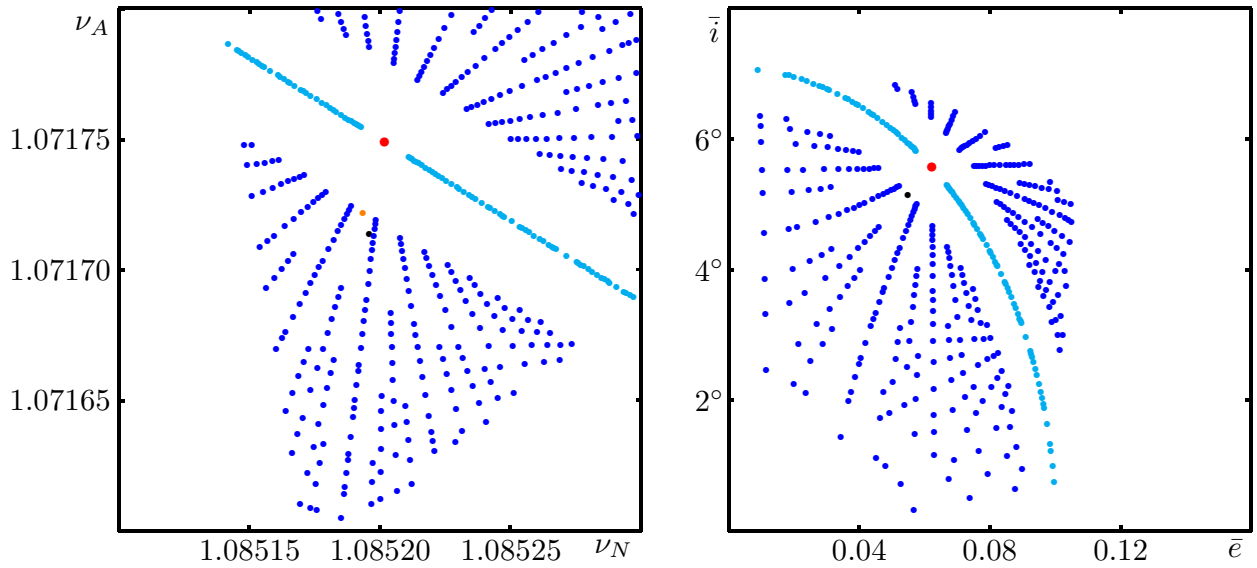
$$\begin{aligned}(242-223)\dot{\lambda}-242\dot{\lambda}'+223\dot{\Omega} &= 0\\ (239-223)\dot{\lambda}-239\dot{\lambda}'+223\dot{\varpi} &= 0.\end{aligned}$$

These angular relationships can be generalized; given three integers N_S, N_A, N_N, we can write:

$$\begin{aligned}(N_N-N_S)\dot{\lambda}-N_N\dot{\lambda}'+N_S\dot{\Omega} &= 0\\ (N_A-N_S)\dot{\lambda}-N_A\dot{\lambda}'+N_S\dot{\varpi} &= 0.\end{aligned}$$

Using the expressions for $\dot{\varpi}$ and $\dot{\Omega}$ given by Delaunay (1872), Valsecchi et al. (1993b) computed how such periodic orbits would be arranged in the space of lunar mean eccentricity and inclination, for a fixed duration of the synodic month.

Saros-like possible periodic orbits

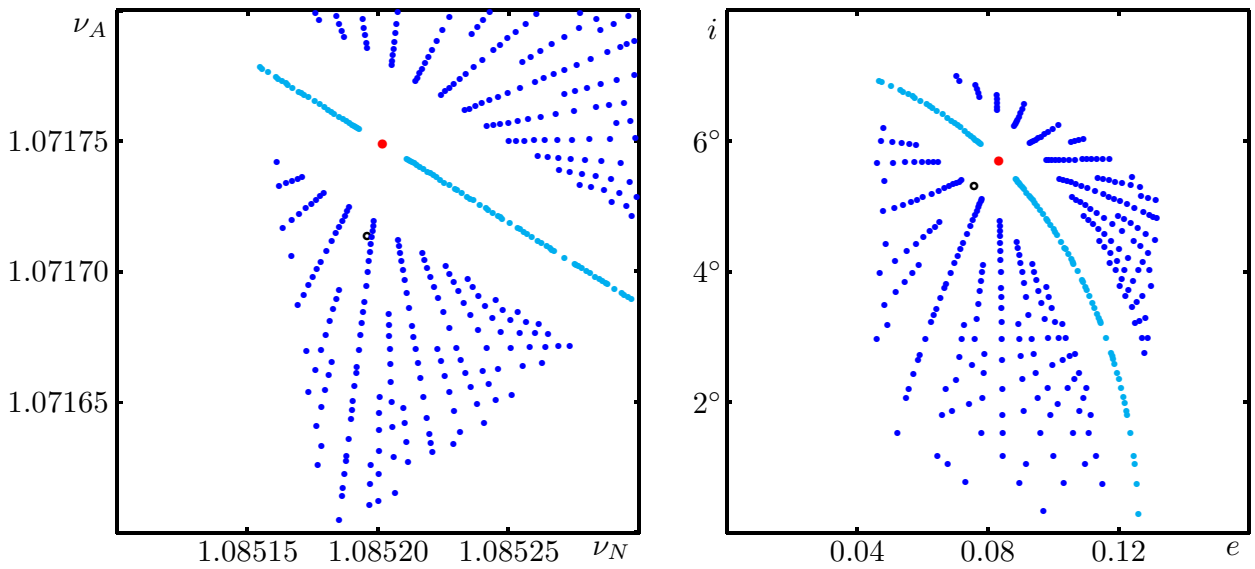


Arrangement in frequency space (left, $\nu_N = T_S/T_N$, $\nu_A = T_S/T_A$) and in $\bar{e}$-$\bar{i}$ space (right) of possible Saros-like periodic orbits.

Small black dots: the Moon; big red dots: the Saros periodic orbit; blue and cyan dots: other possible periodic orbits lasting up to 4 000 T_S.

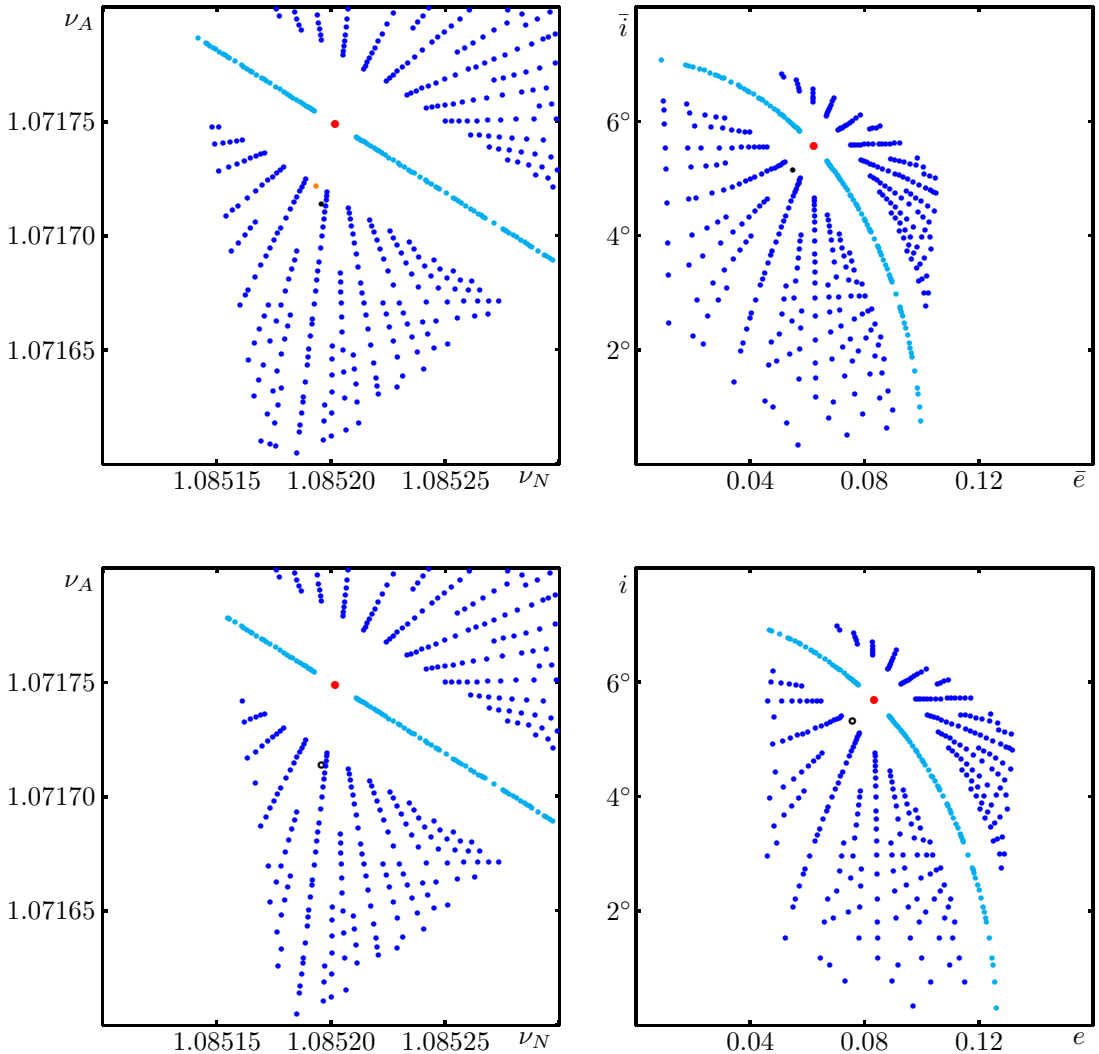
The straight lines on the left are resonances; in cyan the resonance $\dot{\lambda} - 14\dot{\lambda}' + 8\dot{\varpi} + 5\dot{\Omega} = 0$; the orange dot shows Delaunay's frequency ratio for $e' = 0$.

Saros-like periodic orbits found numerically



Same as before for the corresponding periodic orbits found numerically.

Comparison



Top: Possible periodic orbits computed with Delaunay's secular theory.

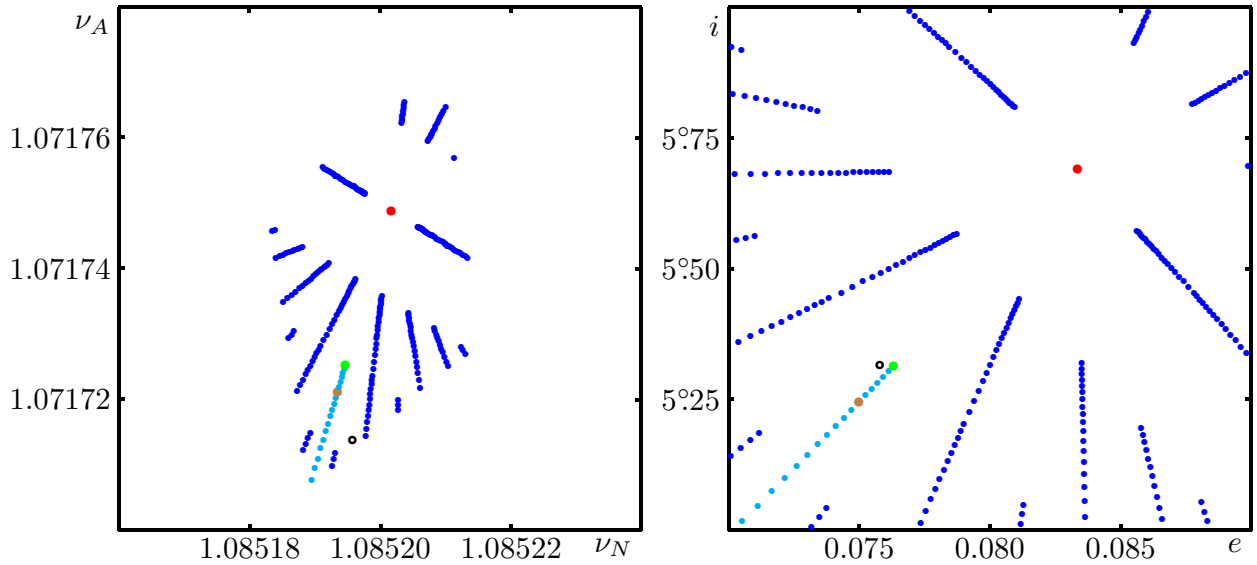
Bottom: initial conditions of periodic orbits found numerically.

The main difference is a shift towards larger values of e.

Poincaré's Conjecture (1892)

Il y a même plus: voici un fait que je n'ai pu démontrer rigoureusement, mais qui me paraît très vraisemblable. Étant données des équations de la forme définie dans le n° 13 et une solution particulière quelconque de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut. D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.

Longer periodic orbits



A zoom in the vicinity of the Moon: the plots contain numerically found periodic orbits with duration up to 9 000 T_S.

In cyan the resonance $3\dot{\lambda} - 36\dot{\lambda}' - 14\dot{\varpi} + 47\dot{\Omega} = 0$, that does not pass very close to the Moon in ν_N-ν_A, but passes very close in e-i.

In brown the orbit 7 571-8 114-8 216 from Valsecchi et al. (1995), in green the orbit 8 909-9 548-9 668.

Longer periodic orbits

JPL DE440

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Saros-related (223-239-242) Periodic Orbit

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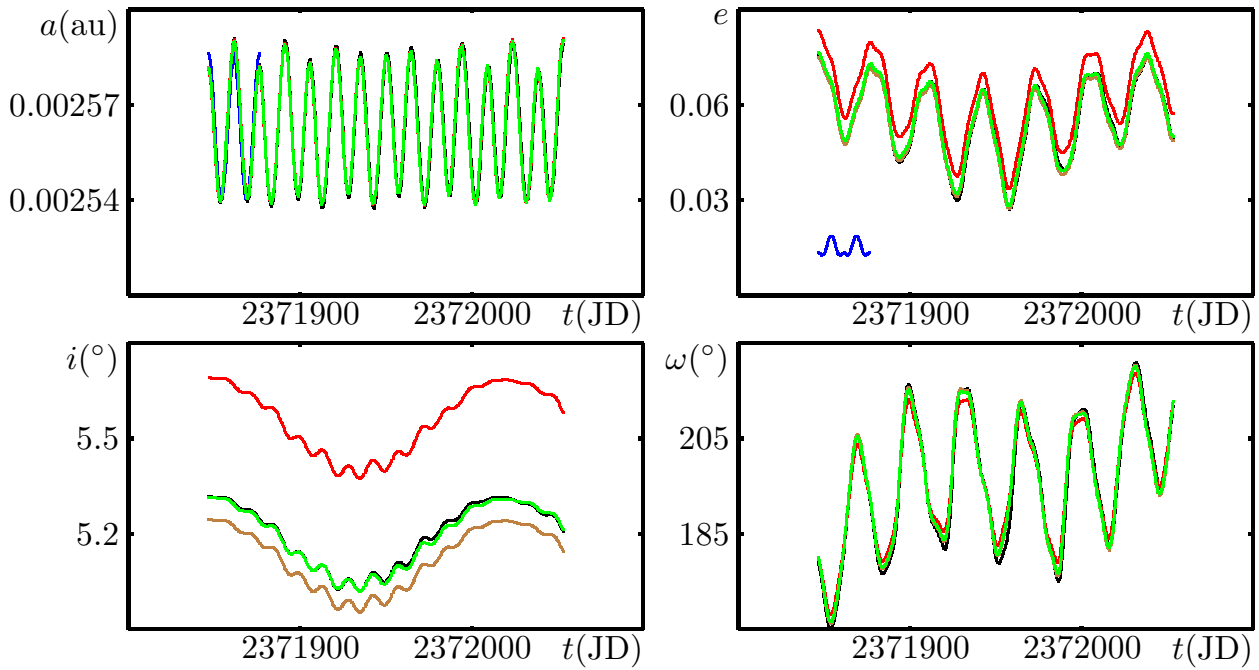
Periodic Orbit 7 571-8 114-8 216

Rev.	a(au)	e	$i(^\circ)$	$\omega(^\circ)$	$f(^\circ)$	$\Delta\lambda(^\circ)$
0.0	0.002 581 6	0.074 99	5.24	180.0	0.0	0.0
111.5	0.002 590 1	0.048 08	5.24	0.7	179.0	180.1
223.0	0.002 581 6	0.074 96	5.24	180.7	358.4	359.8

Periodic Orbit 8 909-9 548-9 668

0.0	0.002 581 5	0.076 32	5.31	180.0	0.0	0.0
111.5	0.002 590 2	0.049 42	5.31	0.6	179.2	180.1
223.0	0.002 581 5	0.076 30	5.31	180.6	358.7	359.8

Longer periodic orbits



Black: osculating a, e, i and ω of the Moon for 7 months starting from 17.372 October 1781.

Blue: osculating a and e of Hill's variational orbit.

The 7 months evolution for the periodic orbits 223-239-242 (red), 7 571-8 114-8 216 (brown), 8 909-9 548-9 668 (green).

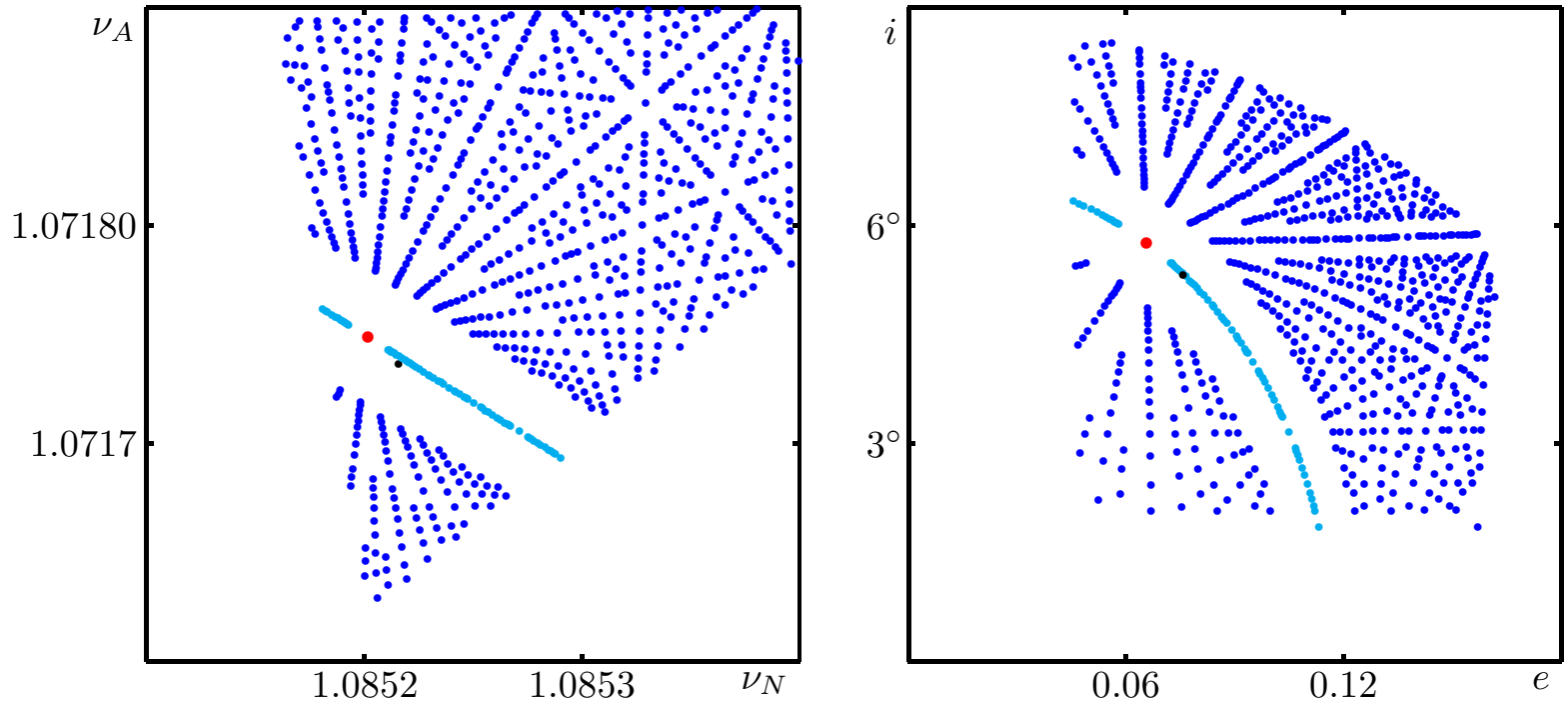
The expanding lunar orbit

Due to the tides raised by the Moon on the Earth, the lunar orbit is currently expanding; this implies an increase of the duration of the synodic month, and resonances may be crossed in the process (Kovalevsky 1983).

The current rate of expansion is of about 4 $\mathrm{cm/y}$; in a few million years, the duration of the synodic month, currently of 29.530 589 d, will grow to, say, 29.538 d.

We can look for the periodic orbits that will be in the vicinity of the lunar orbit for that duration of the synodic month.

The vicinity of the future lunar orbit



Periodic orbits computed for a duration of the synodic month of 29.538 d.

When the synodic month will have become $T_S = 29.538$ d the Moon will be close to the $\dot{\lambda} - 14\dot{\lambda}' + 8\dot{\varpi} + 5\dot{\Omega} = 0$ resonance.

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