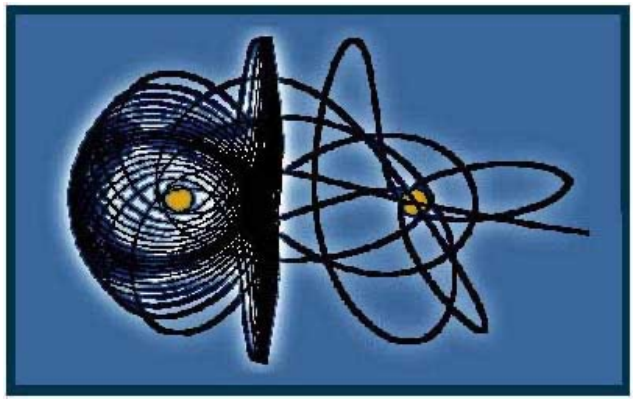
Angles-only relative orbit determination with the Keplerian integrals of motion

G. Romagnoli, G. Baù, G. Bucchioni



CELMEC IX
San Martino al Cimino, Viterbo, Italy

A growing RSO population calls for a complementary observation layer

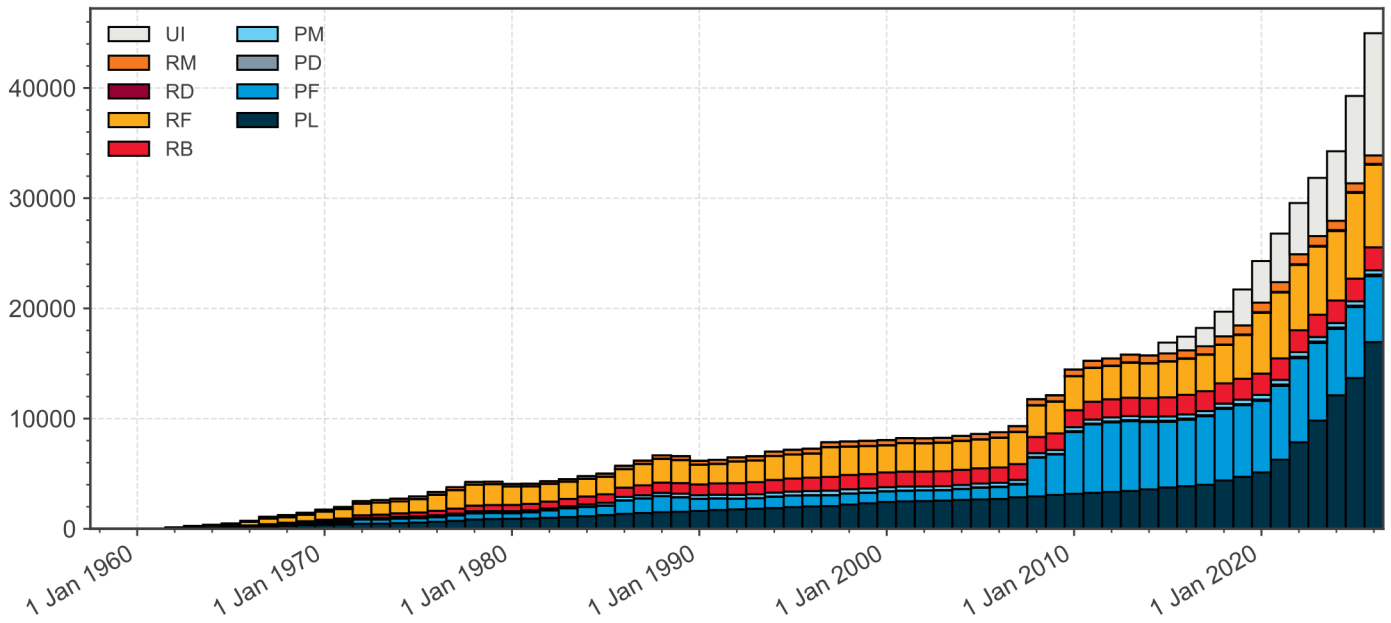
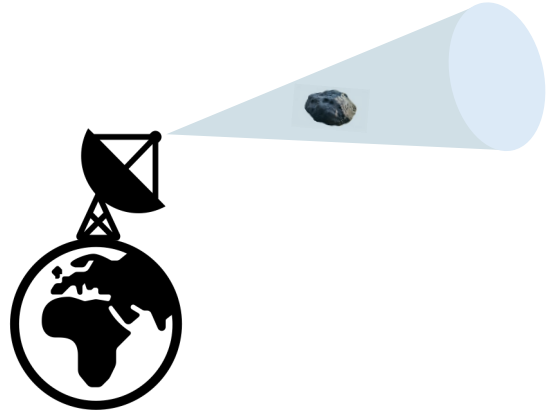


Figure: ©ESA Space Environment Report 2026

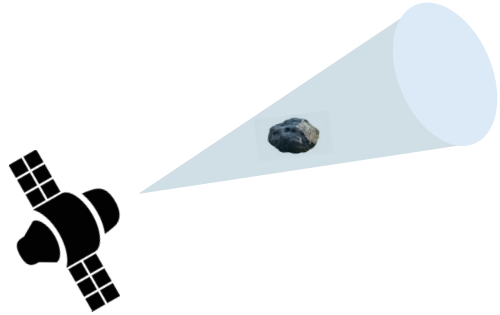
Increasing monitoring demand



Ground-based SSA

+

Space-based SSA



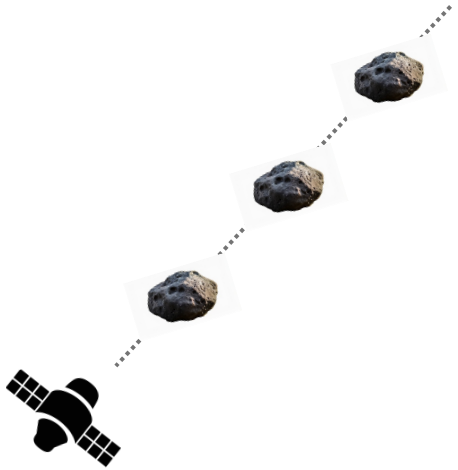
Passive optical sensing is attractive, but range must be recovered indirectly

Passive Optical Sensing

- **Low size, weight and power**
- **Precise** and cost-effective
- **Passive**
- Potentially **already on-board**
- **Angle-Only** measurements

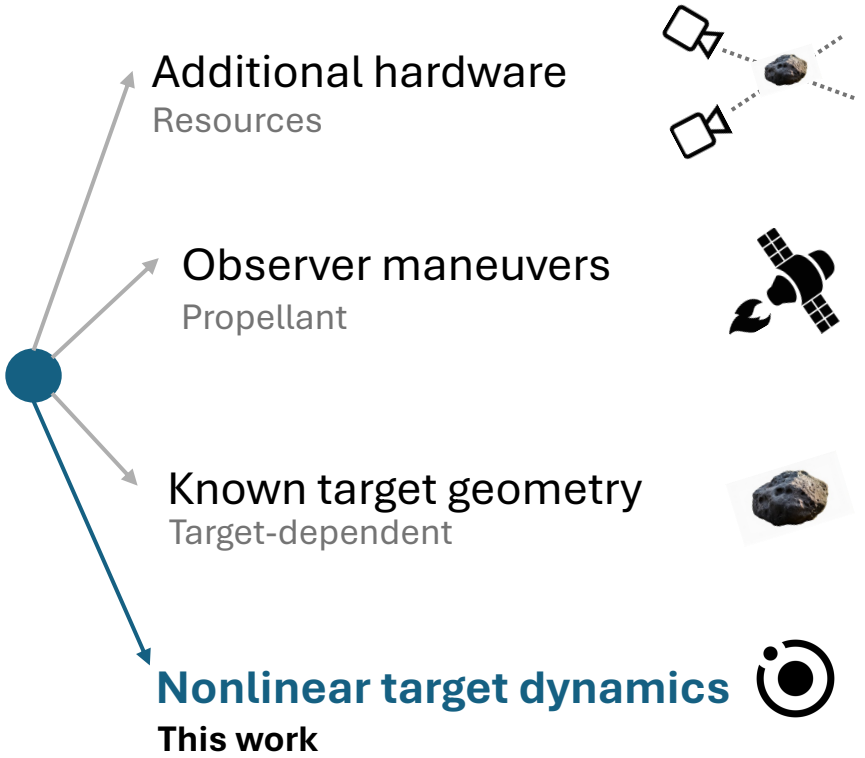


Angle-only limitation



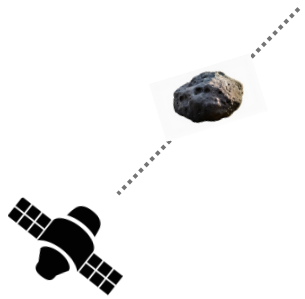
Linearized relative dynamics preserve range ambiguity

How can range information be recovered



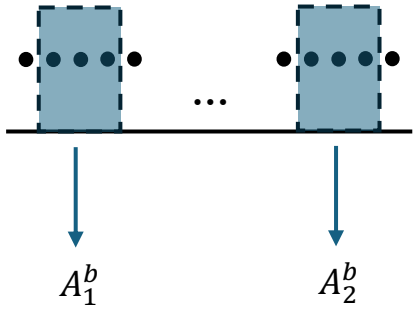
Proposed angle-only relative orbit determination pipeline

1. On-board angular measurement



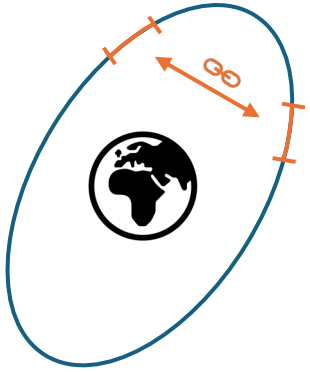
Body-frame measurements

2. Slicing and attributable derivation



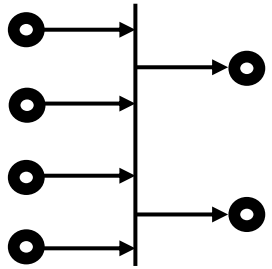
Two body-frame attributables

3. Keplerian linkage



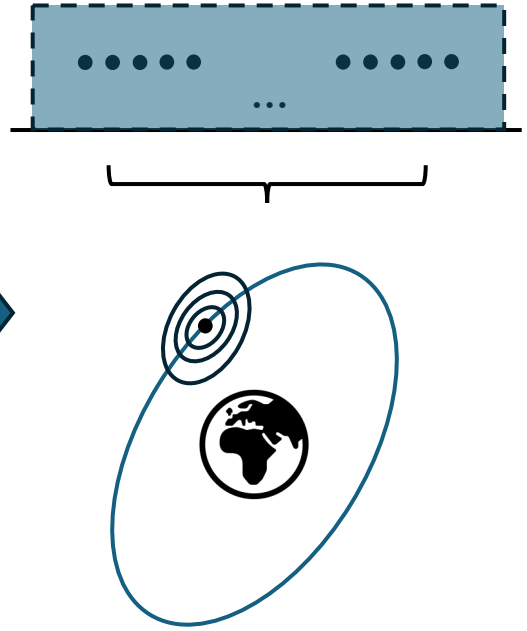
Finite set of candidate solutions

4. Candidate assessment



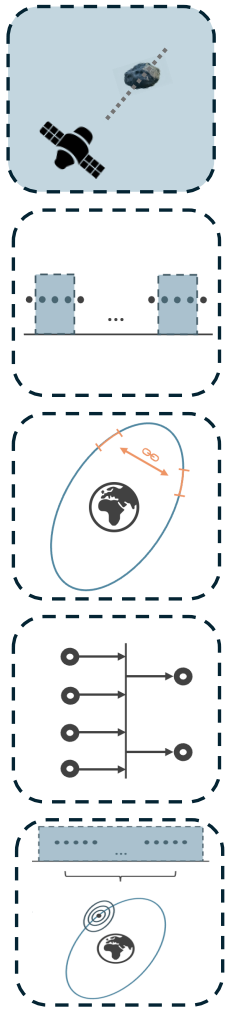
Filtered solutions and associated uncertainty

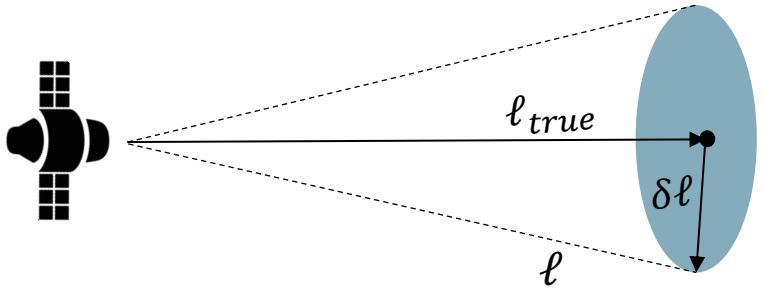
5. Batch refinement



Final orbit and covariance

On-board angular measurements





Angular measurements

$$\alpha_i = atan2\left(\ell_i^y, \ell_i^x\right)$$

$$\delta_i = atan2\left(\ell_i^z, \sqrt{\ell_i^{x\,2} + \ell_i^{y\,2}}\right)$$

Noise generation

Assumptions on the error

- **Purely angular** → $\delta\ell \perp \ell_{true}$
- **Isotropic**
- The magnitude follows a zero-mean **Gaussian distribution**

$$\ell_i = \exp([\theta_i]_\times)\,\ell_{i,true}$$

$\theta_i \sim \mathcal{N}(0, \sigma^2 I_3)$; $\|\theta_i\| \ll 1$

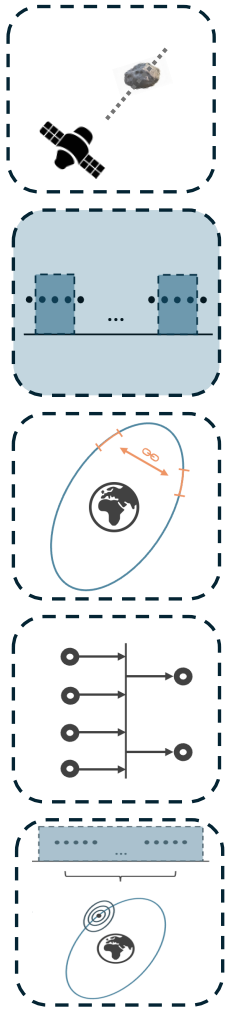
$$\delta\ell_i = \ell_i - \ell_{i,true} \approx [\theta_i]_\times \ell_{i,true}$$

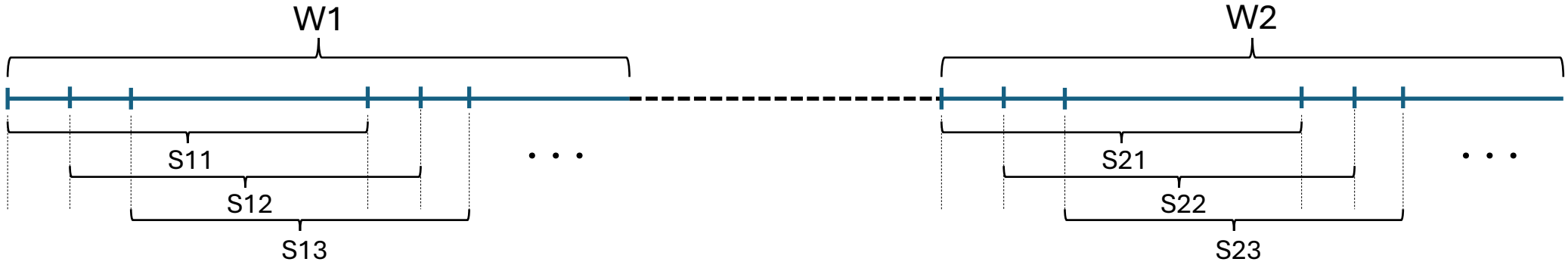
First order approximation

$$\Sigma_{\ell,i} = \mathbb{E}\left[\delta\ell_i \delta\ell_i^T\right] = \sigma_\theta^2 (I_3 - \ell_i \ell_i^T)$$

- **Measurements** are generated when the **relative distance is below the visibility threshold**
  - At each measurement epoch we store: $\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{r_c}, \mathbf{v_c}, \mathbf{q}, \boldsymbol{\omega}$

From observation windows to linkage hypotheses





1. Slice

A short, possibly overlapping, **subset of consecutive measurements** (approximately 10)

Centered at the epoch $t_{c,ij}$

2. Adaptive local fit

LOS and auxiliary time-dependent quantities are locally **fitted with an adaptively selected polynomial order** and evaluated at $t_{c,ij}$

LOS – observer state – attitude – angular velocity

3. Body-frame attributable

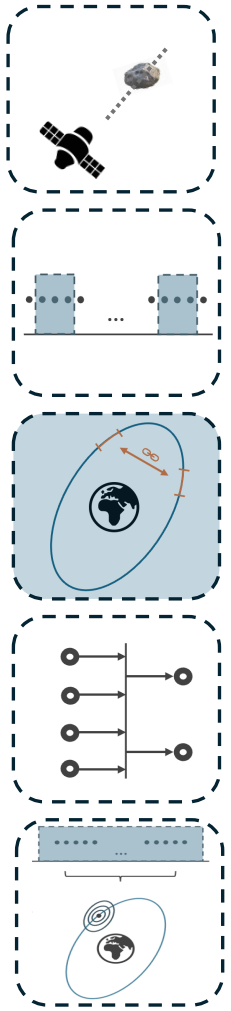
The fitted LOS and its derivative are **converted into body-frame angular position and rates**, defining one attributable per slice.

$$A_i^b = \left[\alpha_i^b \quad \delta_i^b \quad \dot{\alpha}_i^b \quad \dot{\delta}_i^b\right]^{\mathrm{T}}$$

$$\Gamma_i^b = Cov(A_i^b)$$

- **One attributable from W1 + one from W2 = 1 linkage hypothesis**
- **Up to N1 x N2 possible combinations**

Keplerian linkage



Gronchi, G. F., Dimare, L., & Milani, A. (2010). Orbit determination with the two-body integrals. *Celestial Mechanics and Dynamical Astronomy*, *107*(3), 299-318.

Gronchi, G. F., Baù, G., & Maro, S. (2015). Orbit determination with the two-body integrals: III. *Celestial Mechanics and Dynamical Astronomy*, *123*(2), 105-122.

1. State parametrization

Body-frame attributables are mapped into **inertial states**

$$e_{\rho_i} = R_b^i(q_i)\, e_{\rho_i}^b$$

$$\dot{e}_{\rho_i} = R_b^i(q_i)\,(e_{\perp,i}^b + \omega_i^b \times e_{\rho_i}^b)$$

$$r_i = r_{c,i} + \rho_i e_{\rho_i}$$

$$v_i = v_{c,i} + \dot{\rho}_i e_{\rho_i} + \rho_i \dot{e}_{\rho_i}$$

$$R = (\rho_1, \dot{\rho}_1, \rho_2, \dot{\rho}_2)$$

2. Keplerian consistency

The states at t_1 and t_2 are **linked through a combination of first- integral relations**

$$\begin{cases} h_1 = h_2 \\ \Delta K \times \Delta r = 0 \end{cases}$$

$$\Delta K = K_1 - K_2$$

$$\Delta r = r_1 - r_2$$

$$K_i = \mu L_i - \epsilon_i r_i$$

3. Algebraic solution

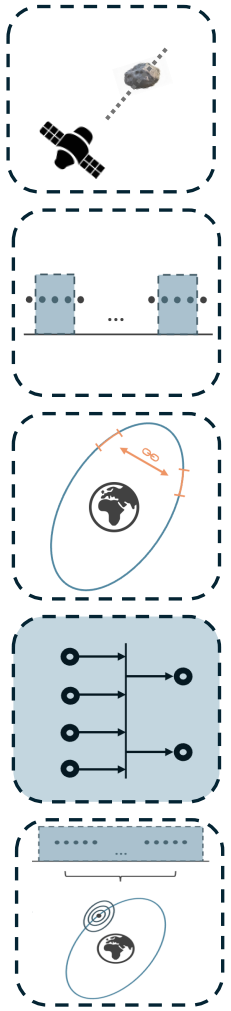
The conservation equations are **reduced** through elimination theory to a **univariate polynomial**

$$P(\rho_2) = 0 \qquad \deg(P) = 9$$

Up to 9 **candidate linkage solutions** for each pair of attributables/slices

Only **real and positive ranges** are retained

Linkage candidate screening



Angular consistency check

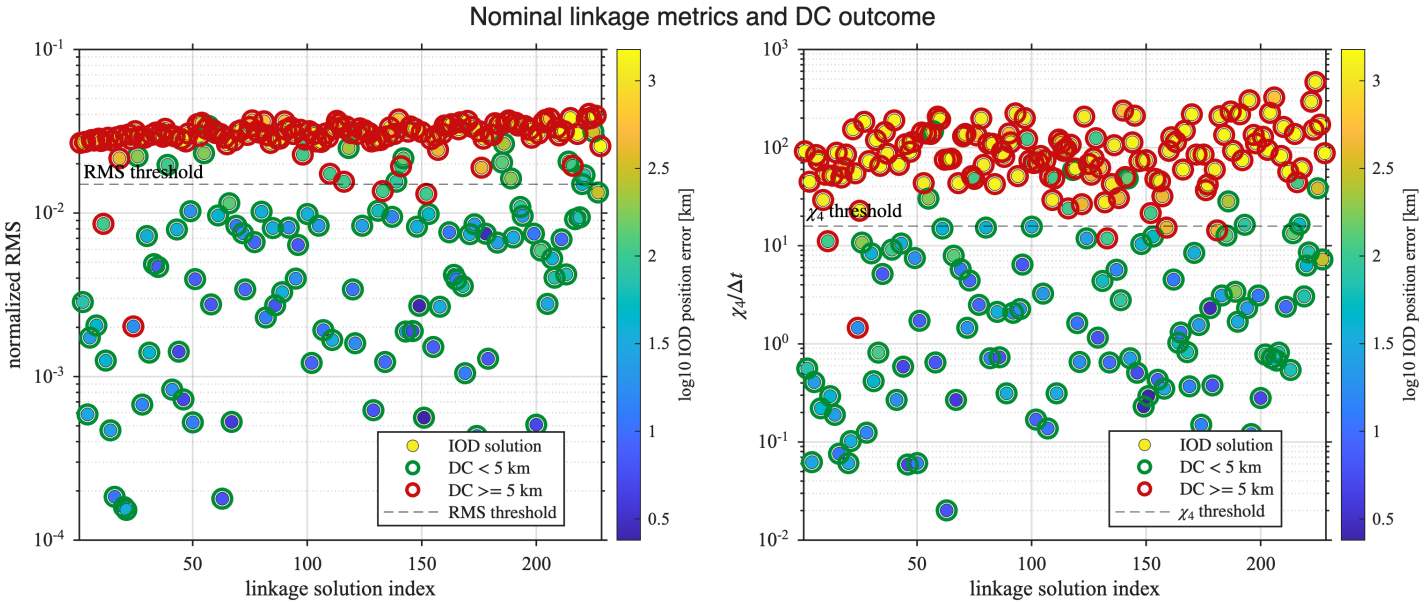
For each candidate solution:

1. **Propagate the reconstructed** state over the observation windows
2. Reconstruct the **predicted angular measurements**
3. Compute the **angular residuals**
4. Evaluate the **normalized RMS**

$$RMS_n = \frac{1}{\Delta t^s}\sqrt{\frac{1}{N}\sum_{k=1}^{N} e_k^2}$$

$$e_k^2 = (\Delta\alpha_k \cos\delta_k)^2 + (\Delta\delta_k)^2$$

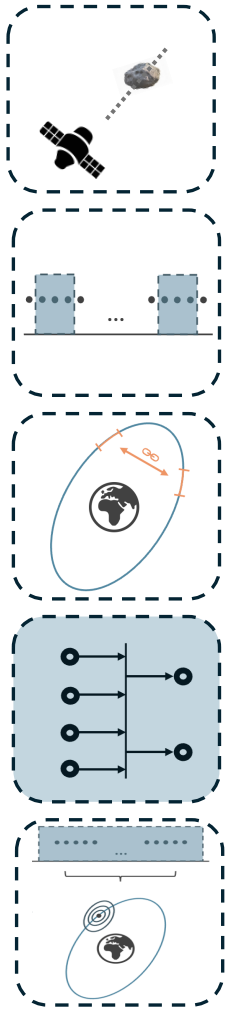
$$RMS_n < RMS_{n,th}$$



Most retained candidates lead to successful refinement

The normalized angular RMS effectively **rejects inconsistent algebraic roots** while **preserving useful initial guesses**

Covariance propagation of the preliminary orbit



Assumptions

- Only **LOS measurement noise** is propagated
- **Observer state, attitude, angular velocity** are treated as exactly **known**
- **Dynamical model errors are not considered**
- **All mappings follow a first-order propagation**

1. LOS measurements

$$\left\{\ell_j^b, \Sigma_{\ell,j}^b\right\}_{j\in S_i}$$

Angular sensor uncertainty

Polynomial fitting

2. Body-frame attributable

$$\Gamma_{A_i}^b$$

Angular position and angular rate uncertainty

Body to inertial mapping

3. Inertial attributables

$$\Gamma_{A_i} = J_{b\to i}\Gamma_{A_i}^b J_{b\to i}^T$$

$$\Gamma_A = \begin{bmatrix}\Gamma_{A1} & 0\\ 0 & \Gamma_{A2}\end{bmatrix}$$

Joint attributable covariance

Implicit linkage map

4. Linkage solution

$$\Phi(R, A) = 0$$

$$J_{RA} = -\Phi_R^{-1}\Phi_A$$

Sensitivity of ranges and range rates

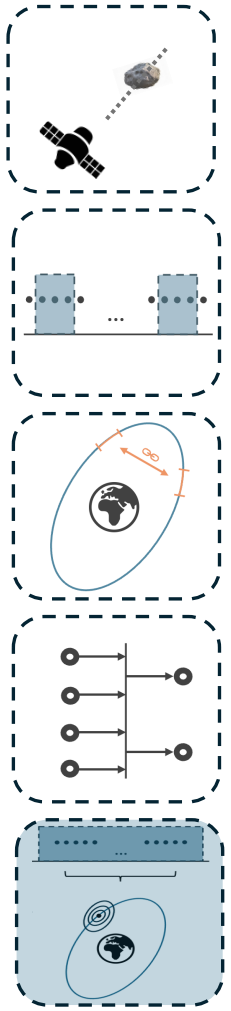
Cartesian reconstruction

5. Covariance of the preliminary orbit

$$J_{xA} = J_{xA}^{dir} + J_{xR}J_{RA}$$

$$\boxed{P_0 = J_{xA}\Gamma_A J_{xA}^T}$$

Batch refinement



- **Each** RMS-selected **linkage solution** initializes an **independent batch refinement.**
- **All available LOS measurements are used**, not only those employed to construct the linked attributables.

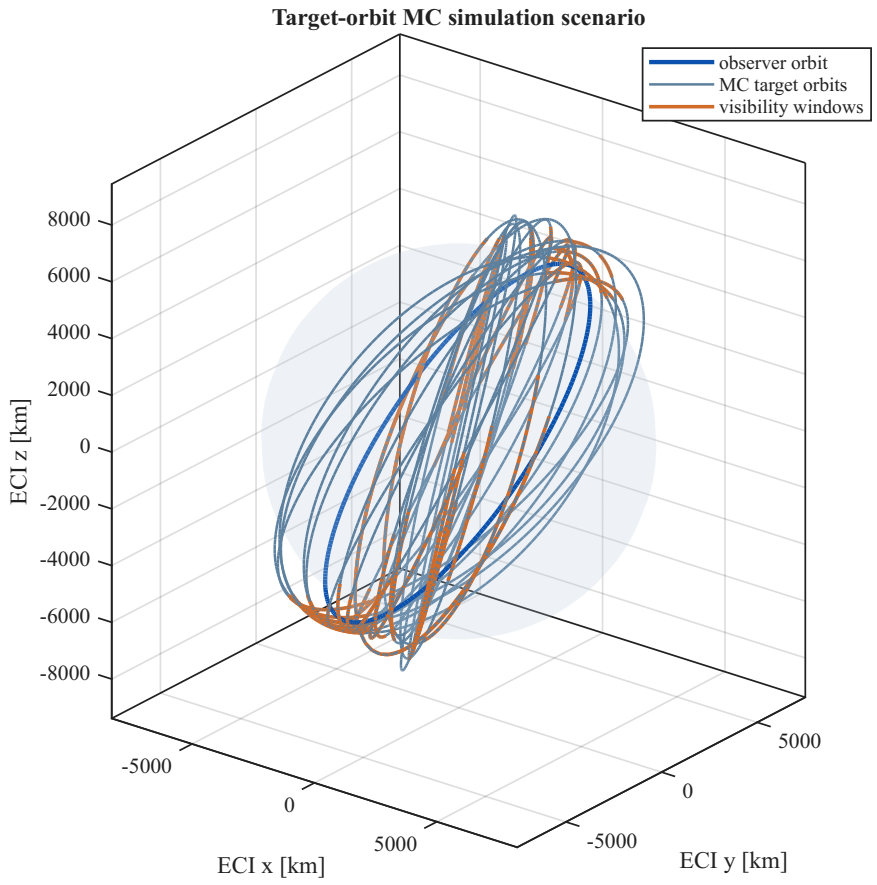
$$\hat{x}_0 = \underset{x_0}{argmin}\, J(x_0) = \underbrace{\frac{1}{2}(x_0 - \bar{x}_0)^T \frac{1}{s} P_0^{-1}(x_0 - \bar{x}_0)}_{\text{Prior residual}} + \underbrace{\frac{1}{2}\sum_{j=1}^{N}\left(y_j - h_j(x_0)\right)^T R_j^{-1}\left(y_j - h_j(x_0)\right)}_{\text{LOS measurement residual}}$$

$$(\hat{x}_{0,DC}, P_{0,DC})$$

- **Levenberg–Marquardt** optimization
- State propagation: **Keplerian dynamics + J_2**

Simulation setup

LEO orbital scenario



Simulation parameters:

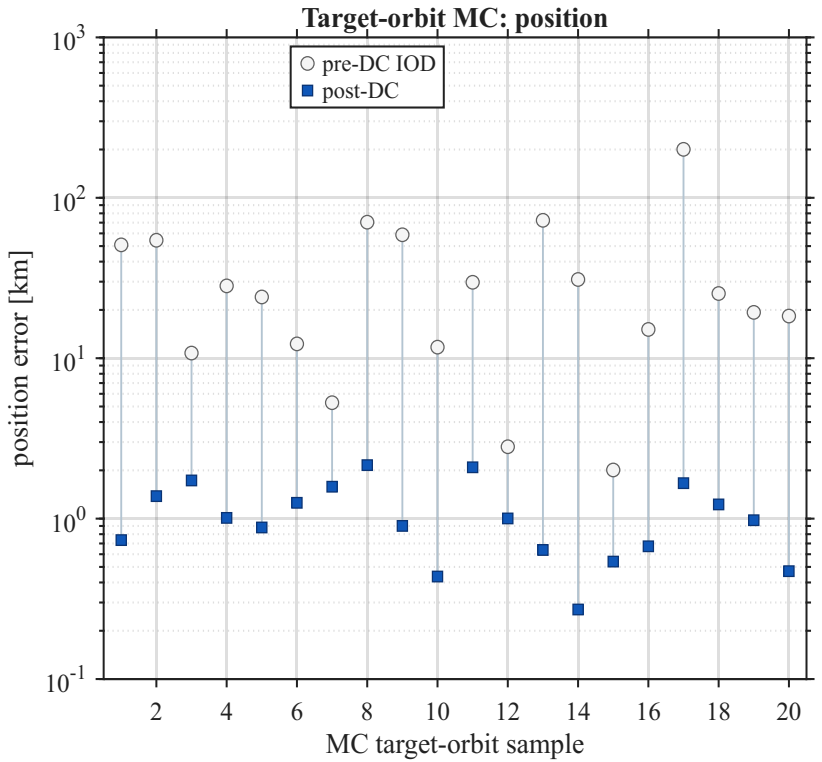
- $\sigma_\theta = 10^{-3}$ rad
- $\Delta t_{meas} = 30$ s
- Detection range = 2000 km
- Tested targets = 20

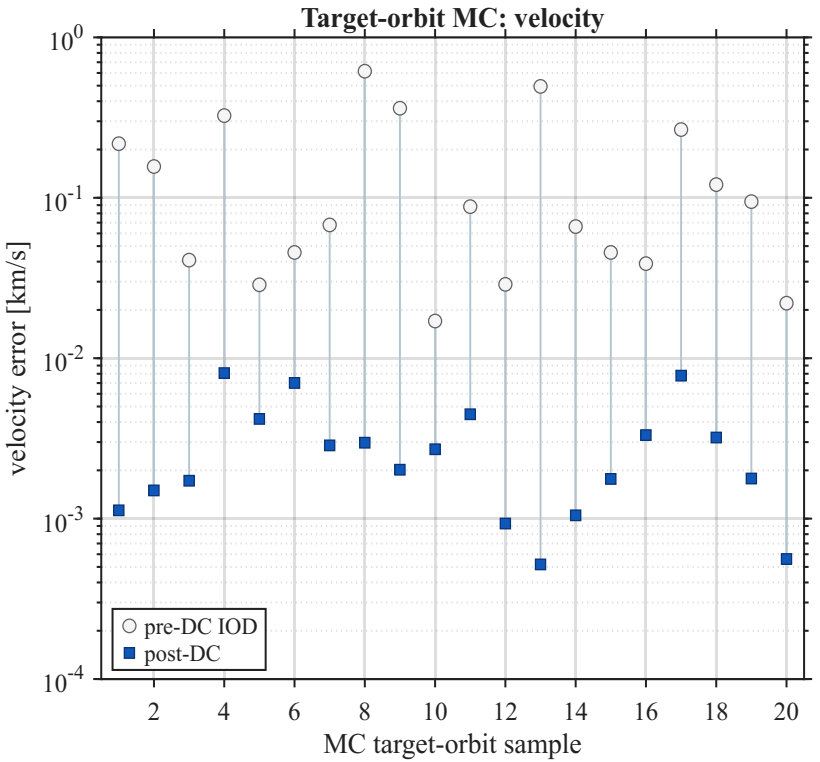
Dynamic model: SGP4

- Earth oblateness up to J_4
- Secular and periodic perturbations
- Simplified atmospheric drag

Main Results

All surviving candidates are refined, and the **converged solution with the lowest cost** is selected as the final orbit.

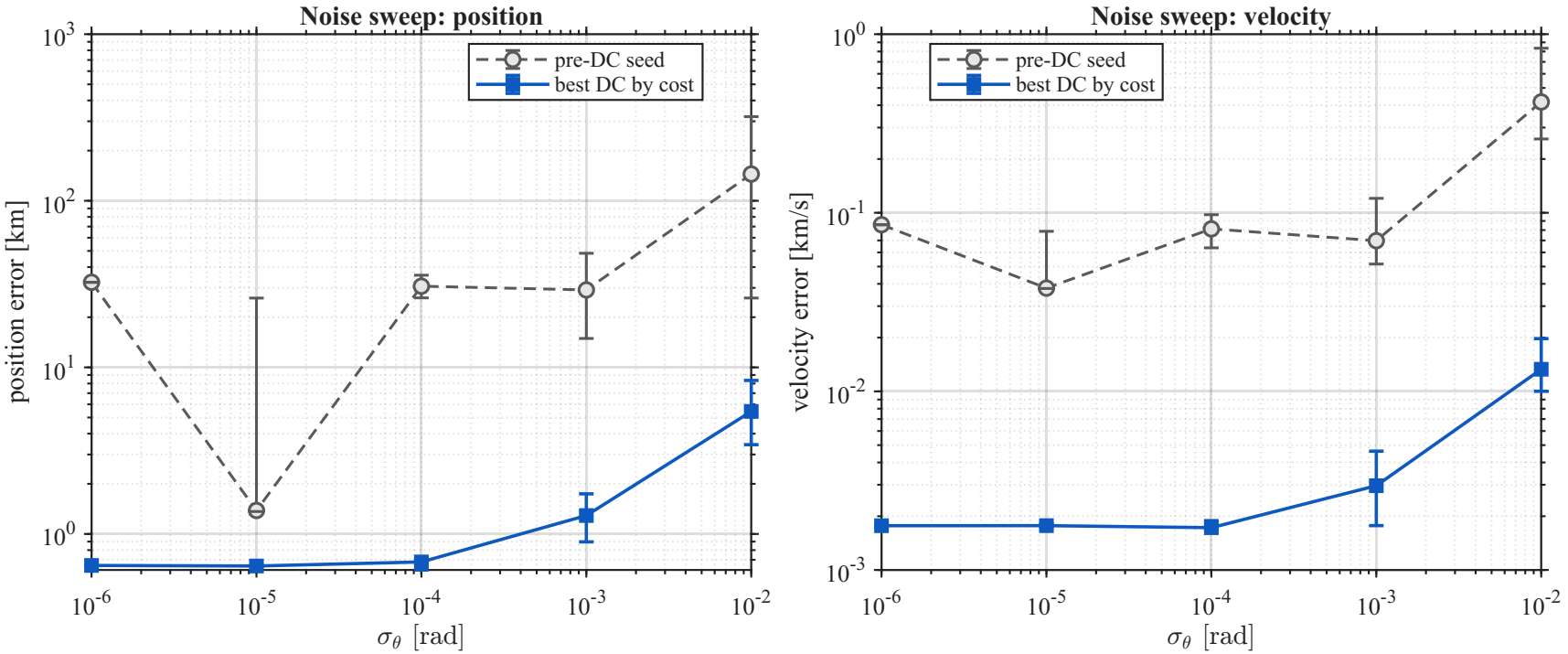




Position error
24.7 km → 0.99 km
Median improvement: ×27

Velocity error
77.9 m/s → 2.36 m/s
Median improvement: ×36

Noise sensitivity



Mean post-refinement errors over 10 Monte Carlo runs per noise level

$$\sigma_\theta = 10^{-4}\ rad : 0.65\ \text{km}, 1.79\ \text{m/s}$$

$$\sigma_\theta = 10^{-3}\ rad : 1.37\ \text{km}, 3.29\ \text{m/s}$$

$$\sigma_\theta = 10^{-2}\ rad : 6.49\ \text{km}, 15.5\ \text{m/s}$$

Accuracy degrades progressively with angular noise; below $\sigma_\theta = 10^{-4}\ rad$ model mismatch and unmodelled effects become dominant

Conclusions and ongoing work

Contribution

Keplerian linkage adapted to onboard body-frame angular observations

•Passive LOS measurements generate a finite set of **dynamically consistent preliminary orbit hypotheses**
•Range ambiguity addressed without **active ranging or target cooperation**
•Candidate screening followed by batch refinement using **all available observations**

Tested performance

20/20 usable target-orbit cases

Median position error: $24.7\ \mathrm{km} \rightarrow \mathbf{0.99\ km}$

Median velocity error: $77.9\ \mathrm{m/s} \rightarrow \mathbf{2.36\ m/s}$

Ongoing work: onboard feasibility

The current implementation evaluates multiple combinations of attributable slices and refines all surviving linkage hypotheses.

•Select **informative and sufficiently separated slices**, accounting for attributable quality and observation geometry.
•Avoid processing every possible slice pair and reduce the number of batch initializations.

Objective: preserve estimation accuracy while reducing the onboard computational workload.

Thank you for your attention

Q&A

Giovanni Romagnoli
giovanni.romagnoli@phd.unipi.it

Giulio Baù
giulio.bau@unipi.it

Giordana Bucchioni
giordana.bucchioni@unipi.it

IN SUPREMÆ DIGNITATIS
1343

AGENZIA SPAZIALE ITALIANA