Quasi-periodic co-orbital motions in the planetary three-body problem

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Collaboration with

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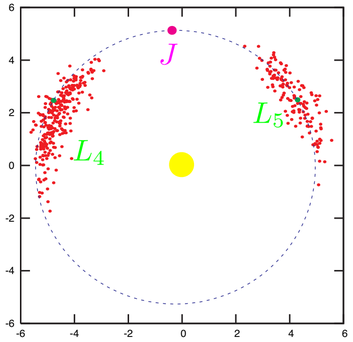
Luigi Chierchia
(Università Roma Tre)

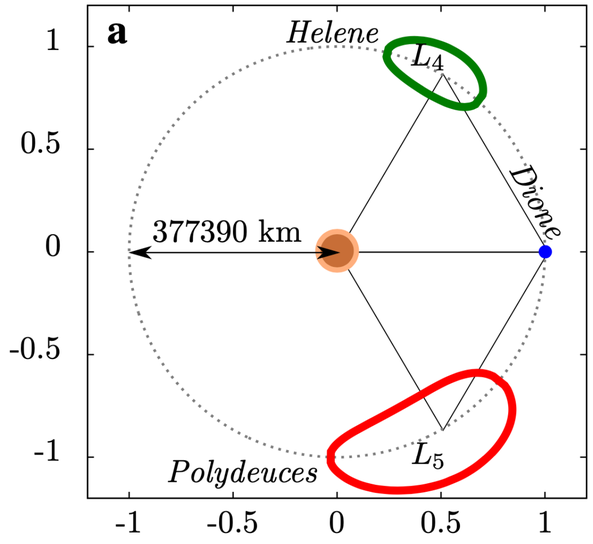
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LMO, Topologie et Dynamique, Université Paris XI)

CO-ORBITAL SYSTEMS

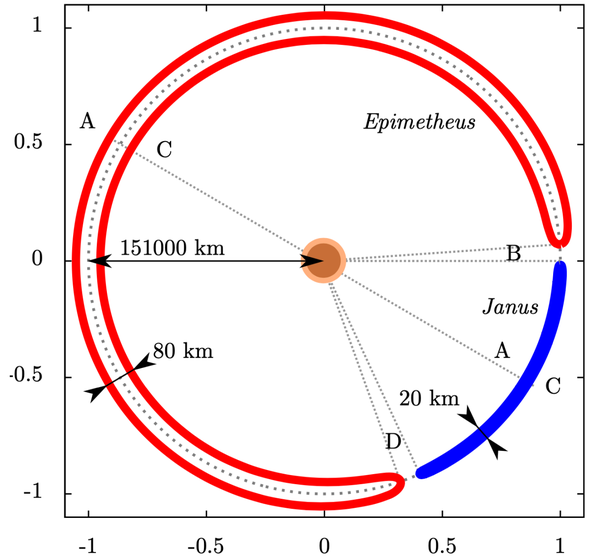
In short: two bodies orbiting a third body (more massive) with the ***same mean orbital period. (1:1 MMR resonance)***

Examples : Jovian Trojans, Saturn's moons Janus and Epimetheus…





Tadpole orbits



Horseshoe orbits

KAM-Stability

Existence of QP trajectories (infinite time stability)

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \kappa H_1(\mathbf{J}, \boldsymbol{\theta}, \kappa) \qquad (\mathbf{J}, \boldsymbol{\theta}) \in \mathbb{R}^n \times \mathbb{T}^n$$

KAM-Stability

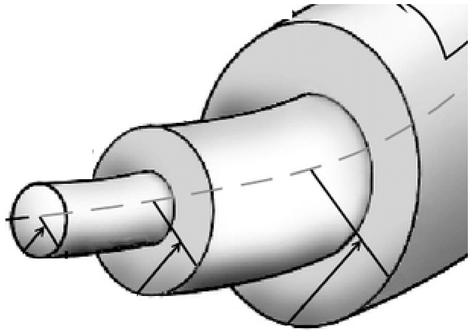
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$(\mathbf{J},\boldsymbol{\theta})$ A-A variables for the unperturbed pb.

Non-degeneracy condition (twist) : $J \longmapsto \omega(J) = \nabla H_0(J)$ Local diffeo.

Kolmogorov-like condition (strong)



KAM-Stability

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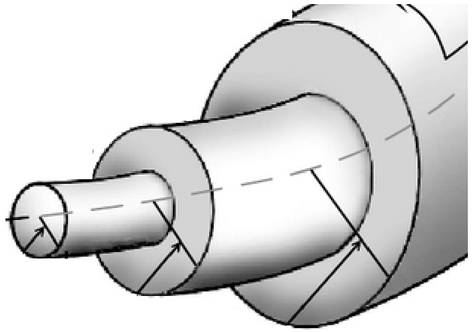
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κ small enough

Kolmogorov-like condition (strong)



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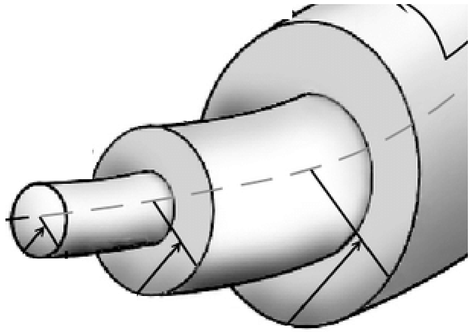
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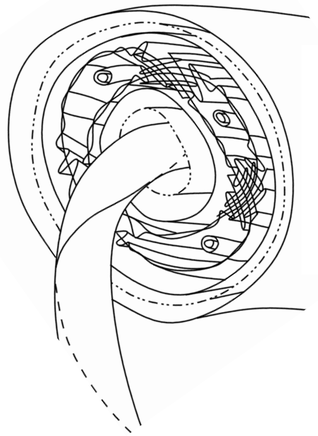
Persistence of a set of Lagrangian tori (max. dim.)
densely covered by QP

Measure of the set >0

Measure of its complementary set tends to zero with κ

$\mathcal{O}(\sqrt{\kappa})$ if only the primary tori are considered

$\mathcal{O}(\kappa|\log\kappa|^{\gamma})$ primary tori + secondary Biasco & Chierchia (2026)

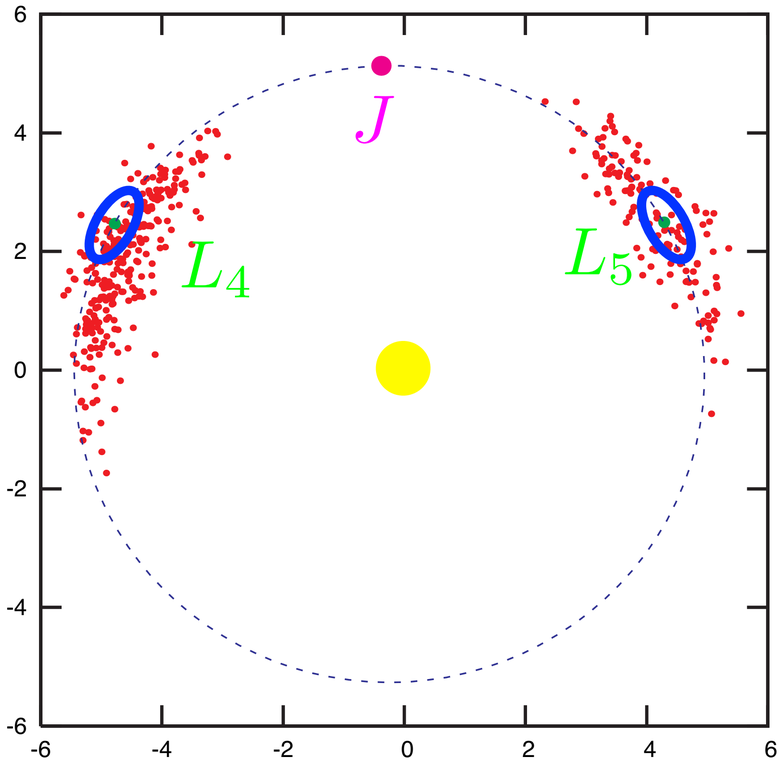


KAM and the co-orbital resonance

Leontovich 1962: quasi-periodic tadpole orbits in the planar restricted TPB in a « small » neighborhood of L4

for almost all mass ratios satisfying the Gascheau condition

Condition improved by Deprit & Deprit-Bartholomé (1966)



Via Birkhoff Normal Form around L4/L5
he proved Kolmogorov nondegeneracy

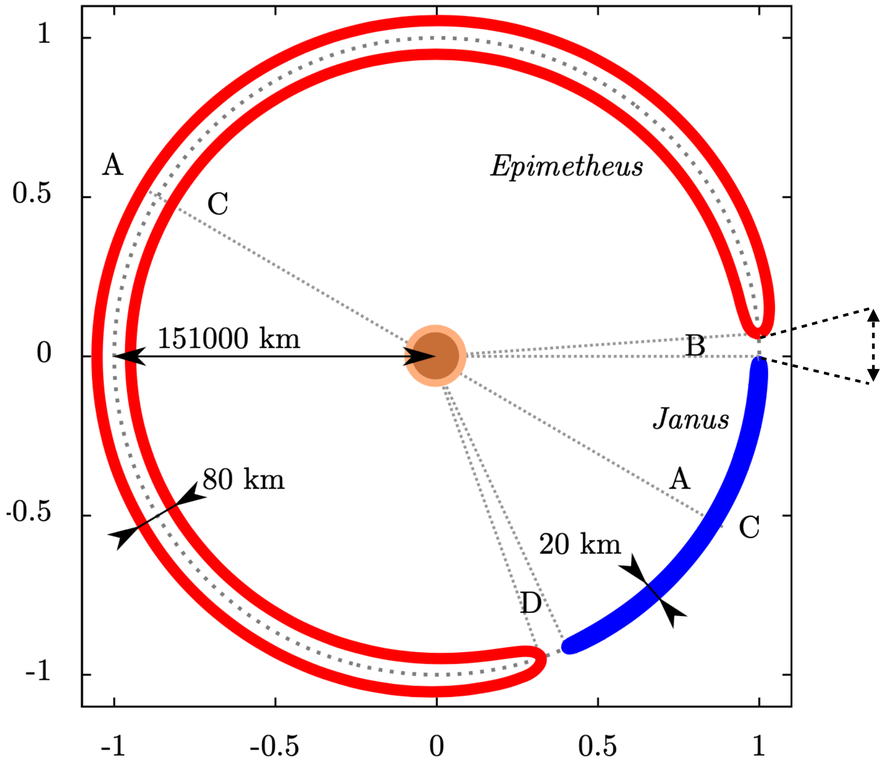
None of the observed Trojan is included this small domain

Improvements of the domain size

Giorgilli et al. (1989), Simo (1989), Celletti & Giorgilli (1990), Giorgilli & Skokos (1997), ….

KAM and the co-orbital resonance

Cors, Palacian, Yanguas (2019) give a sketch of a proof for the existence of Lagrangian tori associated with co-orbital orbits where there are close approaches of the 2 small bodies



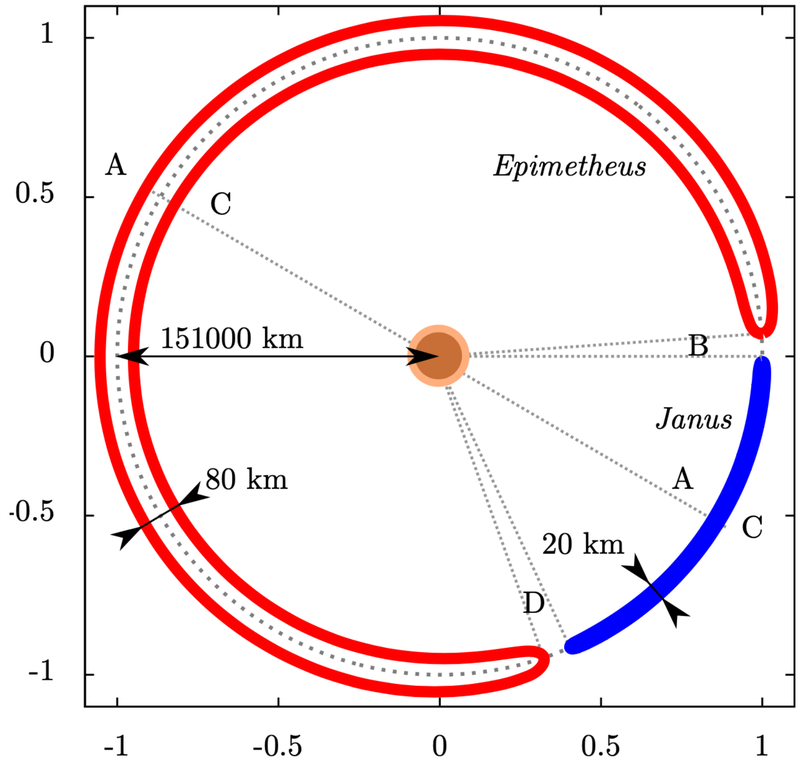
Small parameter:

ε : mass of the co-orbital bodies

$\varepsilon^c (c > 0)$: min distance between the co-orbitals *(Close to the Hill sphere)*

Hill-like approximation

KAM and the co-orbital resonance



Niederman, Pousse & Robutel (2020) rigorously proved, for the TBP, the existence of quasi-periodic horseshoe-type orbits densely filling 2-dimensional elliptic invariant tori

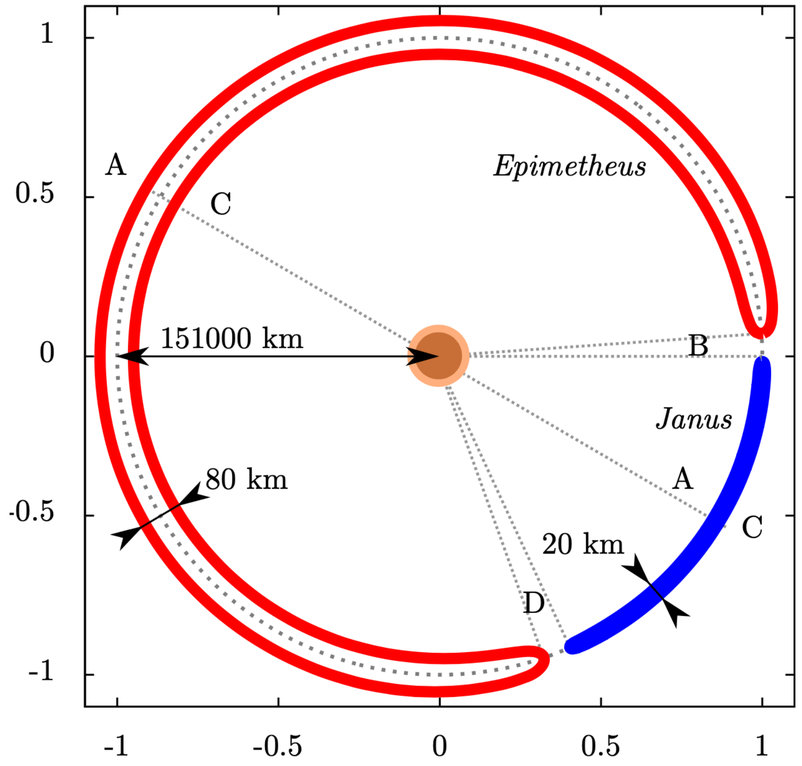
Since these tori are two-dimensional in a system with four degrees of freedom, their union has zero measure

Small parameter:

ε : mass of the small bodies

$\varepsilon^d \, (d > 0)$: distance to the separatrix of L_3

KAM and the co-orbital resonance



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It is possible to construct a set **Lagrangian tori in a larger domain** (HS + Tadpole) with positive measure

Planetary 3-body problem in Poincaré variables (planar case)

2 "planets" : $\varepsilon m_1, \varepsilon m_2 \ll m_0$ primary ε small parameter Gascheau criterium is satisfy

Poincaré variables

$$(\boldsymbol{\Lambda}, \boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\xi}) = (\Lambda_j, \lambda_j, \xi_j, \eta_j)_{j\in\{1,2\}}$$

$$\lambda = nt \qquad \Lambda = \beta\sqrt{\mu a}$$

$$\xi = -\sqrt{2\Lambda}\sqrt{1-\sqrt{1-e^2}}\sin\varpi = -\sqrt{2\Lambda}e\sin\varpi + \mathcal{O}(e^3)$$

$$\eta = \sqrt{2\Lambda}\sqrt{1-\sqrt{1-e^2}}\cos\varpi = \sqrt{2\Lambda}e\cos\varpi + \mathcal{O}(e^3)$$

$$H = H_K + \varepsilon H_p = -\frac{\beta_1^3\mu_1^2}{2\Lambda_1^2} - \frac{\beta_2^3\mu_2^2}{2\Lambda_2^2} + \varepsilon H_p(\boldsymbol{\Lambda}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

$$\beta_j = \frac{m_0 m_j}{m_0 + \varepsilon m_j} \qquad \mu_j = \mathcal{G}(m_0 + \varepsilon m_j)$$

The co-orbitale resonance

Resonant variables: 3 time scales

$\zeta_2 = \lambda_2$	$\Lambda_1 + \Lambda_2$	$\dot{\zeta}_2 = \mathcal{O}(1)$	Fast
$\zeta_1 = \lambda_1 - \lambda_2$	Λ_1	$\dot{\zeta}_1 = \mathcal{O}(\sqrt{\varepsilon})$	Semi-fast
	η_j, ξ_j	$\dot{\omega}_j = \mathcal{O}(\varepsilon)$	Slow (secular)

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Resonant domain around the 1:1 resonance

Same Keplerian mean-motions $\dfrac{\beta_1^3 \mu_1^2}{\Lambda_{\star,1}^3} = \dfrac{\beta_2^3 \mu_2^2}{\Lambda_{\star,2}^3}$

$$Z_2 = \Lambda_1 + \Lambda_2 - \Lambda_{\star,1} - \Lambda_{\star,2} = \mathcal{O}(\sqrt{\varepsilon})$$

$$Z_1 = \Lambda_1 - \Lambda_{\star,1} = \mathcal{O}(\sqrt{\varepsilon})$$

(close to 1:1 MMR)

$$-\pi + \delta < \zeta_1 < \pi - \delta \qquad \delta > 0 \quad \text{independent of } \varepsilon$$

(Far from the collision)

Averaging over the fast angle ζ_2

$$\overline{H}(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{\xi},\boldsymbol{\eta}) = H_K(\boldsymbol{Z}) + \varepsilon\overline{H}_p(\boldsymbol{Z},\zeta_1,\boldsymbol{\xi},\boldsymbol{\eta}) + \mathcal{O}(\varepsilon e^{-c/\sqrt{\varepsilon}})$$

$$\overline{H}_p(\boldsymbol{Z},\zeta_1,\boldsymbol{\xi},\boldsymbol{\eta}) = \langle H_p\rangle_{\zeta_2}(\boldsymbol{Z},\zeta_1,\boldsymbol{\xi},\boldsymbol{\eta}) + \mathcal{O}(\sqrt{\varepsilon}) \qquad \langle H_p\rangle_{\zeta_2} = \frac{1}{2\pi}\int_0^{2\pi} H_p(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{\xi},\boldsymbol{\eta})d\zeta_2$$

Conservation of the angular momentum implies (d'Alembert rule)

$$\overline{H}_p(\boldsymbol{Z},\zeta_1,\boldsymbol{\xi},\boldsymbol{\eta}) \text{ is even in } (\boldsymbol{\eta},\boldsymbol{\xi})$$

$$\overline{H}(\boldsymbol{Z},\boldsymbol{\zeta},\boldsymbol{\xi},\boldsymbol{\eta}) = H_K(\boldsymbol{Z}) + \varepsilon\overline{H}_p(\boldsymbol{Z},\zeta_1,\mathbf{0},\mathbf{0}) + \varepsilon\overline{H}_{\geq 2}(\boldsymbol{Z},\zeta_1,\boldsymbol{\xi},\boldsymbol{\eta}) + \mathcal{O}(\varepsilon e^{-c/\sqrt{\varepsilon}})$$

1 d. f. : Integrable

↓

AA variables

Even and at least quadratic

Standard form

$$H_K(Z_1, Z_2) + \varepsilon \overline{H}_p(Z_1, Z_2, \zeta_1, \mathbf{0}, \mathbf{0})$$

Can be simplistically put in **Standard Form** (Biasco, Chierchia 2023)

$$\big(1 + \nu(Z_1, Z_2, \zeta_1)\big) Z_1^2 + G(Z_2, \zeta_1)$$
$$|\nu| < \sqrt{\varepsilon}$$

The standard form allows to get uniforms estimates with respect to ε in the vicinity of the separatrix of L3 where we are going to reduce the Hamiltonian

Close to a **mechanical system**

$$Z_1^2 + \varepsilon V(\zeta_1) \qquad\qquad V(\zeta_1) = \cos\zeta_1 - \frac{1}{\sqrt{2 - 2\cos\zeta_1}}$$

The mechanical system is convenient for the construction of action-angle variables and averaging

Anything that can be done with the mechanical system can be done with the standard form (but more tricky!)

Implicit averaging

Mechanical System $E(Z_1, \zeta_1) = Z_1^2 + \varepsilon V(\zeta_1)$

A-A transformation $\psi : (Z_1, \zeta_1) \longmapsto (J_1, \theta_1)$

$$J_1(E(Z_1, \zeta_1)) \propto \int_{\zeta_{1,min}}^{\zeta_{1,max}} \sqrt{E(Z_1, \zeta_1) - \varepsilon V(\zeta_1)} d\zeta_1 \qquad \theta_1(Z_1, \zeta_1)) \propto \int_{\zeta_{1,min}}^{\zeta_1} \frac{d\zeta_1}{\sqrt{E(Z_1, \zeta_1) - \varepsilon V(\zeta_1)}}$$

Averaging

$$\langle g \rangle_{\theta_1} = \frac{1}{2\pi} \int_0^{2\pi} g(Z_1, \theta_1) d\theta_1 \propto \int_{\zeta_{1,min}}^{\zeta_{1,max}} \frac{g \circ \psi(Z_1, \zeta_1) d\zeta_1}{\sqrt{E(Z_1, \zeta_1) - \varepsilon V(\zeta_1)}}$$

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Standard form $\big(1 + \nu(Z_1, Z_2, \zeta_1)\big) Z_1^2 + G(Z_2, \zeta_1)$

Averaging

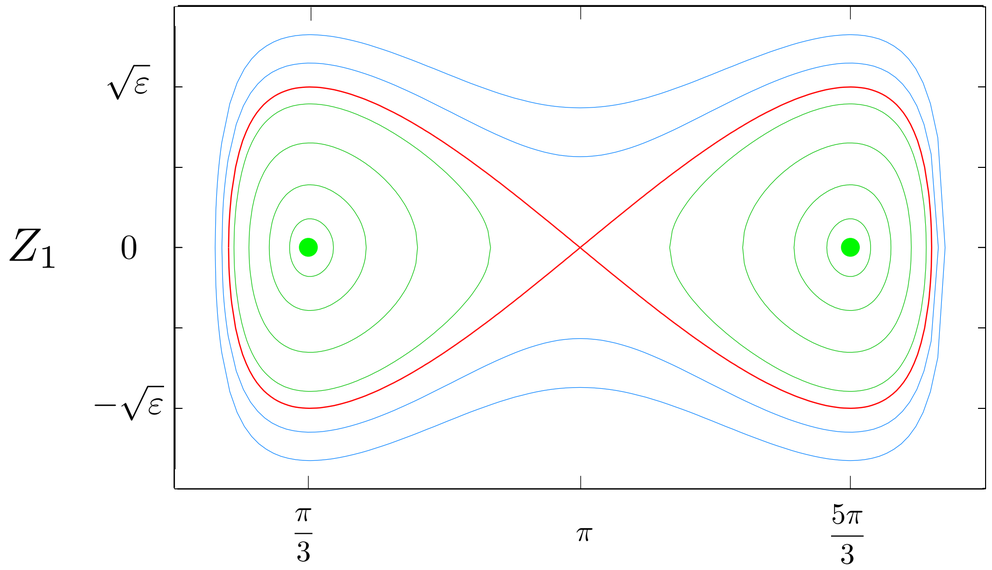
$$\langle g(\theta_1) \rangle_{\theta_1} \propto \int_{\zeta_{1,min}}^{\zeta_{1,max}} (1 + u(Z_1,\zeta_1)) \frac{g \circ \psi(Z_1,\zeta_1) d\zeta_1}{\sqrt{E(Z_1,\zeta_1) - G(\zeta_1)}}$$

u: small $\qquad$ G: close to εV

Approximation by a mechanical system

$$Z_1^2 + \varepsilon \left(\cos \zeta_1 - \frac{1}{\sqrt{2 - 2\cos \zeta_1}} \right)$$

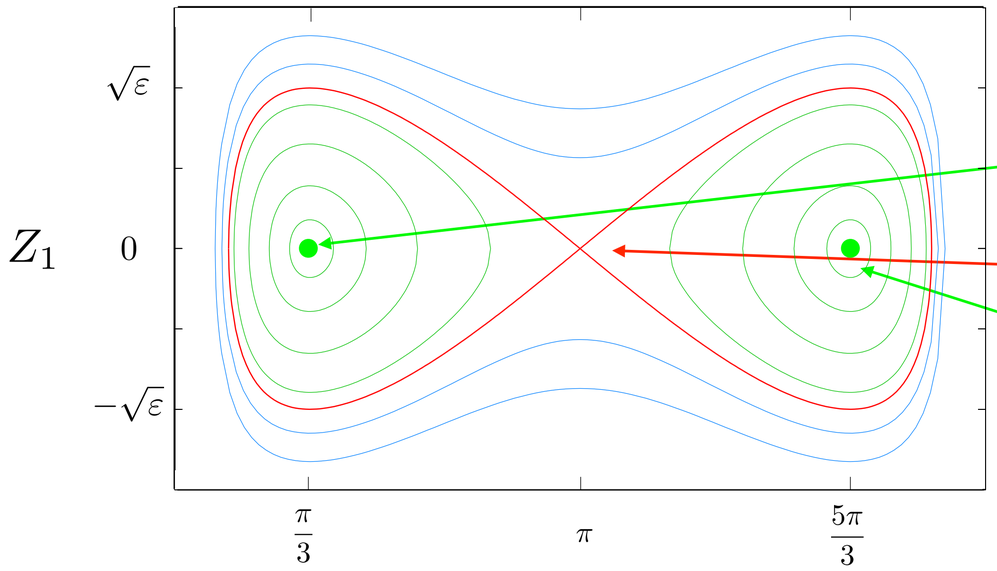
Phase portrait



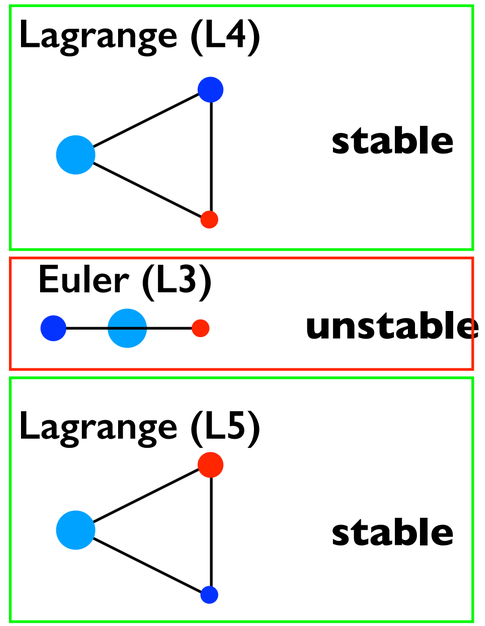
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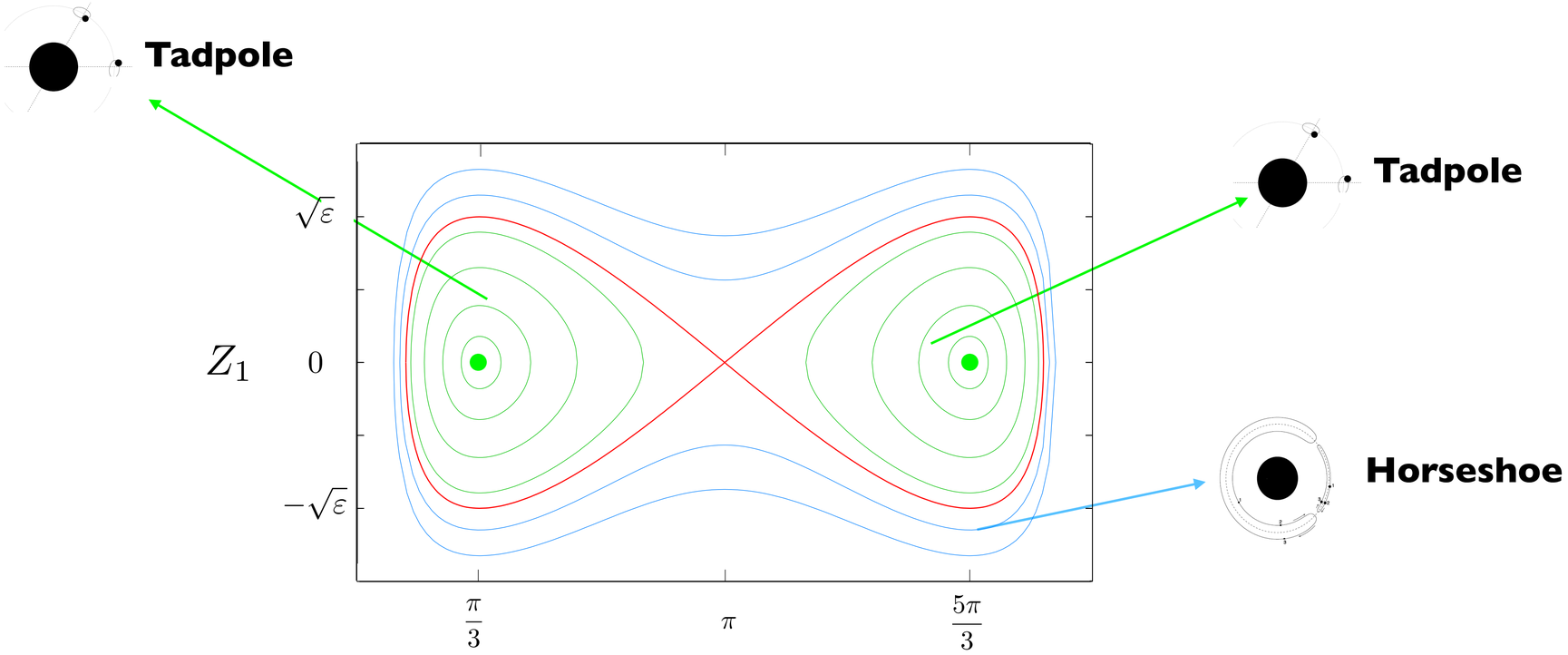


3 fixed points



Approximation by a mechanical system

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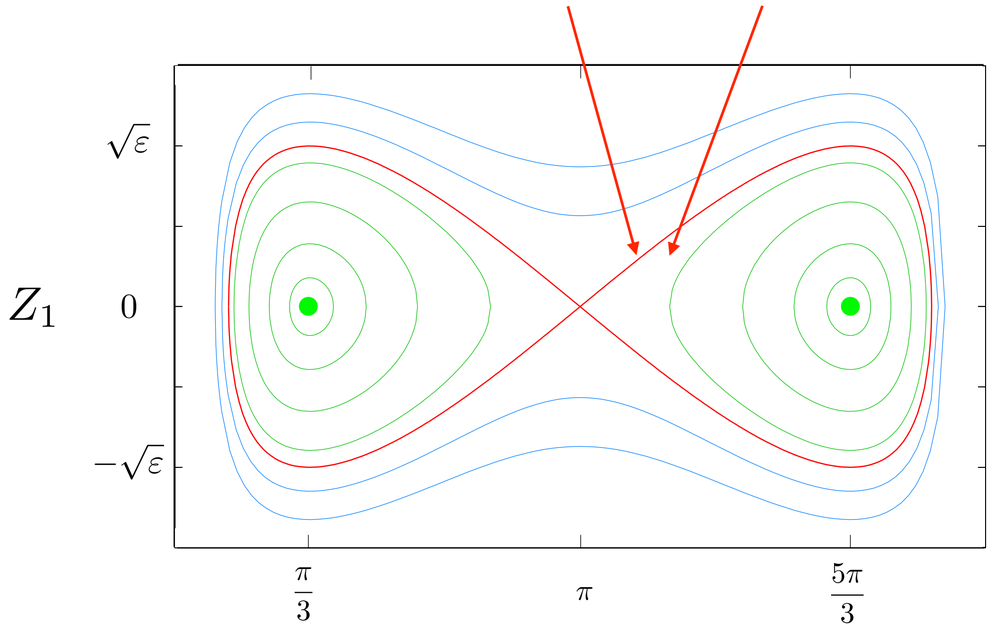


First step: close to the separatrix

As L3 is hyperbolic, a trajectory close to its separatrix spends most of its time in the vicinity of L3

The dominant part of the quantities calculated along such a trajectory will be evaluated in the vicinity of L3: where we are able to carry out the calculations.

First step: prove the twist condition in a neighborhood of the separatrix (both sides)



$$\langle g \rangle_{\theta_1} = g|_{L_3} + \mathcal{O}\left(\frac{1}{|\log d|}\right)$$

d measures the distance to the separatrix

Averaging over the semi-fast angle θ_1

(J_1, θ_1) Action-Angle of the standard form

$$F_K(J_2) + F_0(J_1, J_2) + \varepsilon f(J_1, J_2, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}) + \mathcal{O}(\varepsilon e^{-c/\sqrt{\varepsilon}})$$

Fast | Semi-fast | Semi-fast + secular | Fast but small

$\mathcal{O}_2(\boldsymbol{\xi}, \boldsymbol{\eta})$

Averaging over the semi-fast angle θ_1

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Fast | Semi-fast | Semi-fast + secular $\mathcal{O}_2(\boldsymbol{\xi}, \boldsymbol{\eta})$ | Fast but small

Averaging over the semi-fast angle up to un arbitrary degree L

$$F_K(J_2) + F_0(J_1, J_2) + \varepsilon G(J_1, J_2, \boldsymbol{\xi}, \boldsymbol{\eta}) + \mathcal{O}(\varepsilon^L) + \mathcal{O}(\varepsilon e^{-c/\sqrt{\varepsilon}})$$

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Need to know the exact expression of the terms of de 2 and 4 in $(\boldsymbol{\xi}, \boldsymbol{\eta})$ only

4 is the minimum degree to be able to satisfy the twist condition for the Birkhoff normal form

Twist condition

… After numerous additional transformations with accurate control of the remainders …

$$F_K(J_2) \;+\; F_0(J_1, J_2) \;+\; \varepsilon G(J_1, J_2, \boldsymbol{\xi}, \boldsymbol{\eta}) \;+\; \mathcal{O}(\varepsilon^L) + \mathcal{O}(\varepsilon e^{-c/\sqrt{\varepsilon}})$$

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$$(\mathbf{J}, \boldsymbol{\theta}) \; \in \mathbb{R}^4 \times \mathbb{T}^4 \qquad\qquad G(\mathbf{J}) \; + \; R(\mathbf{J}, \boldsymbol{\theta})$$

$$G(\mathbf{J}) = G_K(J_2) \; + \; G_0(J_1, J_2) \; + G_2(J_1, J_2, J_3, J_4) + \cdots$$

The frequency map is a local diffeo. (Twist map)

$$G(\mathbf{J}) \longmapsto Hess(G(\mathbf{J})) = \frac{\partial^2 G(\mathbf{J})}{\partial \mathbf{J}^2}$$

$$|Det(Hess(G))| = \tau > 0 \qquad \text{Close to the separatrix}$$

Twist condition

… After numerous additional transformations with accurate control of the remainders …

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∞ when $d \longrightarrow 0$

The frequency map is a local diffeo. (Twist map)

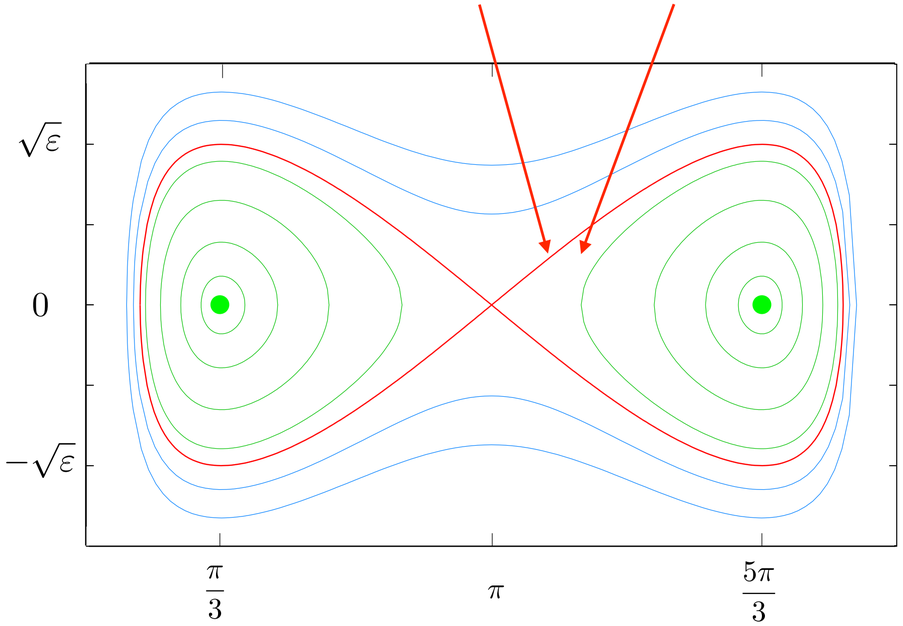
$$G(\mathbf{J}) \longmapsto Hess(G(\mathbf{J})) = \frac{\partial^2 G(\mathbf{J})}{\partial \mathbf{J}^2}$$

$$|Det(Hess(G))| = \tau > 0$$

Close to the separatrix (but not too close …)

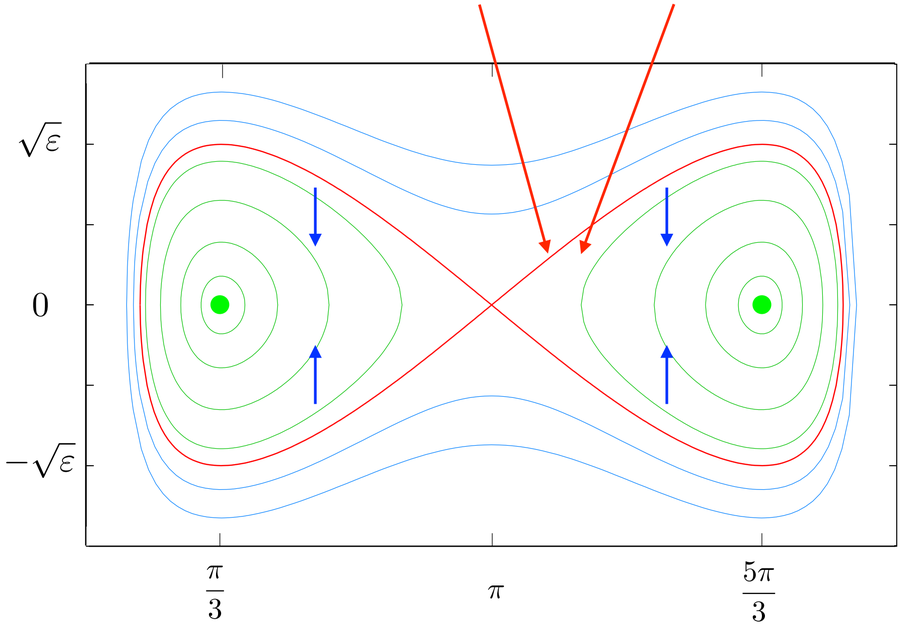
Extension to the entier tadpole region

There exist Lagrangian tori a neighborhood of the separatrix (both sides)



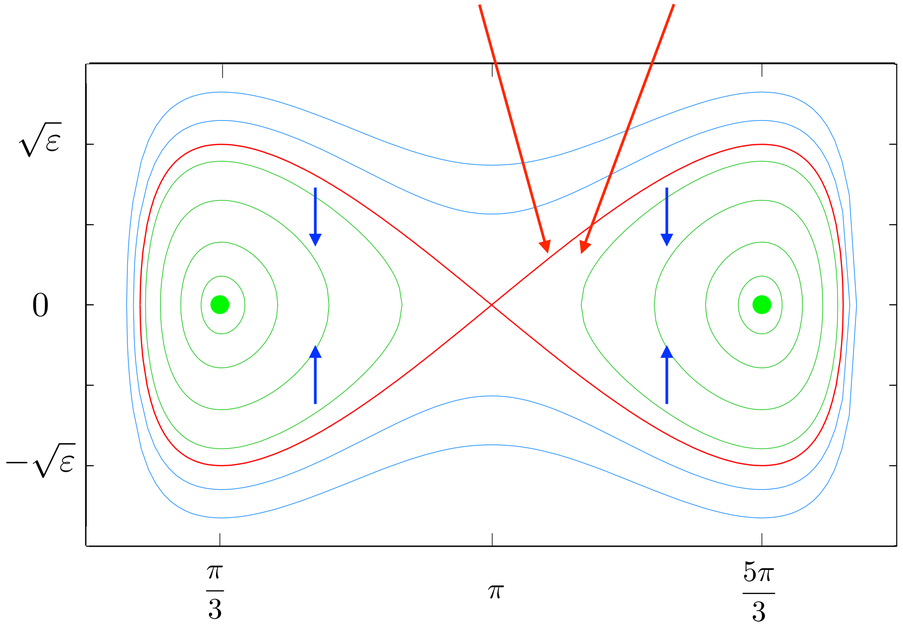
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Extension to the entier tadpole region

There exist Lagrangian tori a neighborhood of the separatrix (both sides)



After a few adjustments: our construction also applies in the tadpole region (not the estimates)

Twist condition close to the separatix $|Det(Hess(G))| = \tau > 0$

All our expressions are analytic in a complex domain

Using standard results from complex analysis it can be shown that

$$\exists\, r > 0 \;\; s.t. \;\; \forall\, \delta \,\text{the twist is} > \delta^r \,\text{outside a set of measure}\, \delta$$

find a balance between the smallness of the excluded zone and the one of the twist

Conclusion

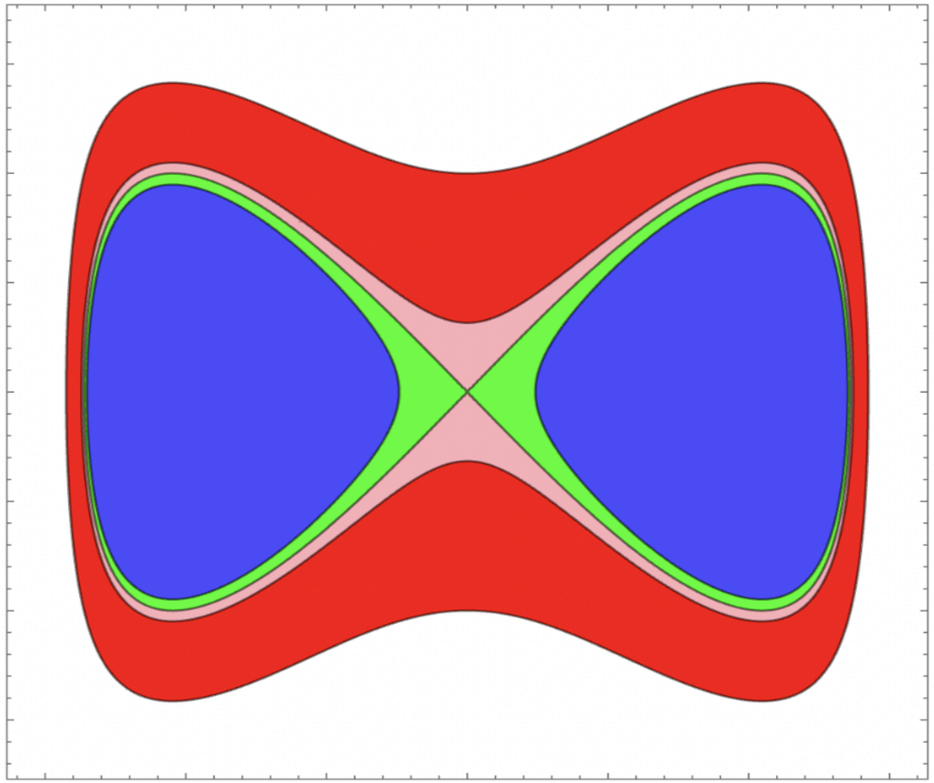
We find quasi-periodic orbits on Lagrangian torus

Of horseshoe type (red)

Of tadpole type (blue)

No tori in the green and pink regions
since they are to close to the separatrix
(splitting phenomena, chaos, …)

The density of the tori is $1 - \mathcal{O}(\varepsilon^c)$
for some $c > 0$



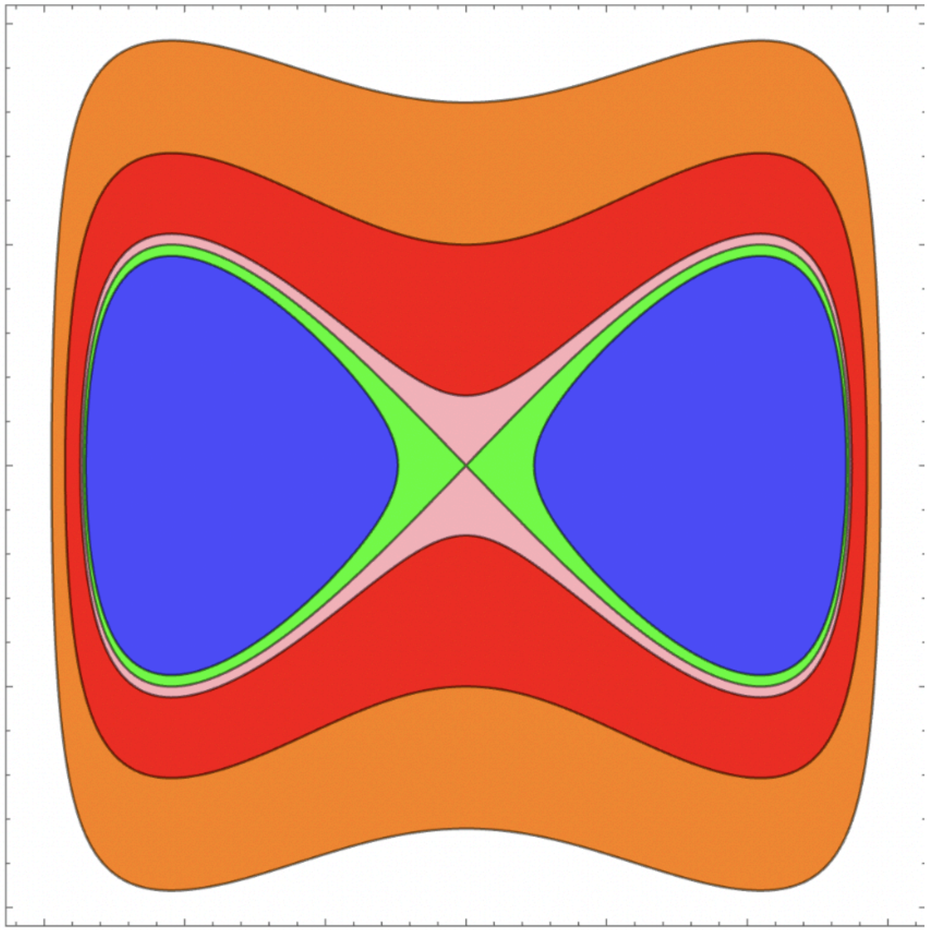
Open question

Is it possible to extend these results to the entire horseshoe region, in particular, close to the Hill sphere?

Difficult due to the singularity of the potential

Distance to the collision $\mathcal{O}(\varepsilon^{\frac{1}{3}})$

Potential $\mathcal{O}(\varepsilon^{-\frac{1}{3}})$



2 main ingredients from Bisaco & Chierchia (2023)

Standard form

$$\big(1+\nu(Z_1,Z_2,\zeta_1)\big)\,Z_1^2+G(Z_2,\zeta_1)$$

Accurate estimates of the analyticity domain for A-A

$$\text{Action}\quad \rho\sim\sqrt{\varepsilon}d|\log d|$$

$$\text{Angle}\quad \sigma\sim\frac{1}{|\log d|}$$

Averaging over the semi-fast angle θ_1

Transformation by the time-1 flow of εS gives

$$F_0 + \varepsilon\left(f + \{F_0, S\}\right) + \varepsilon^2\left(\{f, S\} + \frac{1}{2}\{\{F_0, S\}, S\}\right) + \cdots$$

Averaging over the semi-fast angle θ_1

Transformation by the time-1 flow of εS gives

$$F_0 + \boxed{\varepsilon\,(f + \{F_0, S\})} + \varepsilon^2 \left(\{f, S\} + \frac{1}{2}\{\{F_0, S\}, S\} \right) + \cdots$$



$$\varepsilon\langle f^{(2)}\rangle_{\theta_1} + \varepsilon\langle f^{(4)}\rangle_{\theta_1} + \cdots \qquad \text{degree } 2,\ 4 \ \ldots \text{ in } (\boldsymbol{\xi}, \boldsymbol{\eta})$$

$$S = \frac{2\pi}{\partial_{J_1} F_0} \int_0^1 t\,(f(\mathbf{J}, \theta_1 + 2\pi t, \boldsymbol{\xi}, \boldsymbol{\eta}) - \langle f(\mathbf{J}, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\theta_1})\,dt$$

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$$S = \frac{2\pi}{\partial_{J_1} F_0} \int_0^1 t\,(f(\mathbf{J}, \theta_1 + 2\pi t, \boldsymbol{\xi}, \boldsymbol{\eta}) - \langle f(\mathbf{J}, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\theta_1})\,dt$$

$\partial_{J_1} F_0 = \mathcal{O}(\sqrt{\varepsilon})$ Semi-fast frequency

$$S = S^{(2)} + S^{(4)} = \frac{\mathcal{O}_2(\boldsymbol{\xi}, \boldsymbol{\eta})}{\sqrt{\varepsilon}} + \frac{\mathcal{O}_4(\boldsymbol{\xi}, \boldsymbol{\eta})}{\sqrt{\varepsilon}}$$

Averaging over the semi-fast angle θ_1

Transformation by the time-1 flow of εS gives

$$F_0 + \varepsilon\left(f + \{F_0, S\}\right) + \varepsilon^2\left(\{f, S\} + \frac{1}{2}\{\{F_0, S\}, S\}\right) + \cdots$$

Averaging over the semi-fast angle θ_1

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$$F_0 + \boxed{\varepsilon\,(f + \{F_0, S\})} + \varepsilon^2 \left(\{f, S\} + \frac{1}{2}\{\{F_0, S\}, S\} \right) + \cdots$$



$$\varepsilon\langle f^{(2)}\rangle_{\theta_1} + \varepsilon\langle f^{(4)}\rangle_{\theta_1} + \cdots \qquad \text{degree } 2,\ 4 \ \ldots \text{ in } (\boldsymbol{\xi}, \boldsymbol{\eta})$$

$$S = \frac{2\pi}{\partial_{J_1} F_0} \int_0^1 t\,(f(\mathbf{J}, \theta_1 + 2\pi t, \boldsymbol{\xi}, \boldsymbol{\eta}) - \langle f(\mathbf{J}, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\theta_1})\,dt$$

Averaging over the semi-fast angle θ_1

Transformation by the time-1 flow of εS gives

$$F_0 + \boxed{\varepsilon\,(f + \{F_0, S\})} + \varepsilon^2 \left(\{f, S\} + \frac{1}{2}\{\{F_0, S\}, S\} \right) + \cdots$$



$$\varepsilon\langle f^{(2)}\rangle_{\theta_1} + \varepsilon\langle f^{(4)}\rangle_{\theta_1} + \cdots \qquad \text{degree } 2,\ 4 \ \ldots \text{ in } (\boldsymbol{\xi}, \boldsymbol{\eta})$$

$$S = \frac{2\pi}{\partial_{J_1} F_0} \int_0^1 t\,(f(\mathbf{J}, \theta_1 + 2\pi t, \boldsymbol{\xi}, \boldsymbol{\eta}) - \langle f(\mathbf{J}, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\theta_1})\,dt$$

$\partial_{J_1} F_0 = \mathcal{O}(\sqrt{\varepsilon})$ Semi-fast frequency

$$S = S^{(2)} + S^{(4)} + \cdots = \frac{\mathcal{O}_2(\boldsymbol{\xi}, \boldsymbol{\eta})}{\sqrt{\varepsilon}} + \frac{\mathcal{O}_4(\boldsymbol{\xi}, \boldsymbol{\eta})}{\sqrt{\varepsilon}} + \cdots$$

Averaging over the semi-fast angle θ_1

$$F_0 + \varepsilon \left(\langle f^{(2)} \rangle_{\theta_1} + \varepsilon \langle f^{(4)} \rangle_{\theta_1} \right) + \varepsilon^2 \left(\{f, S\} + \frac{1}{2} \{\{F_0, S\}, S\} \right) + \cdots$$

Averaging over the semi-fast angle θ_1

$$F_0 + \varepsilon\left(\langle f^{(2)}\rangle_{\theta_1} + \varepsilon\langle f^{(4)}\rangle_{\theta_1}\right) + \boxed{\varepsilon^2\left(\{f,S\} + \frac{1}{2}\{\{F_0,S\},S\}\right)} + \cdots$$

$$\mathcal{O}\left(\varepsilon\right)$$

$$\{.,.\} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

Averaging over the semi-fast angle θ_1

$$F_0 + \varepsilon\left(\langle f^{(2)}\rangle_{\theta_1} + \varepsilon\langle f^{(4)}\rangle_{\theta_1}\right) + \boxed{\varepsilon^2\left(\{f,S\} + \frac{1}{2}\{\{F_0,S\},S\}\right)} + \cdots$$

$$\mathcal{O}\left(\varepsilon\right)$$

$$\{.,.\} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

Only S_2 is necessary to get the degree 4 in $(\boldsymbol{\xi},\boldsymbol{\eta})$

$$F_0 + \varepsilon\left(\langle f^{(2)}\rangle_{\theta_1} + \langle f^{(4)}\rangle_{\theta_1}\right) + \boxed{\varepsilon^2\langle\{f^{(2)},S^{(2)}\} + \frac{1}{2}\{\{F_0,S^{(2)}\},S^{(2)}\}\rangle_{\theta_1}} + \mathcal{O}(\varepsilon^{3/2}) + \varepsilon\mathcal{O}_6(\boldsymbol{\xi},\boldsymbol{\eta})$$

Averaging over the semi-fast angle θ_1

$$F_0 + \underbrace{\varepsilon\left(\langle f^{(2)}\rangle_{\theta_1} + \langle f^{(4)}\rangle_{\theta_1}\right)}_{\varepsilon\mathcal{O}_2(\boldsymbol{\xi},\boldsymbol{\eta}) + \varepsilon\mathcal{O}_4(\boldsymbol{\xi},\boldsymbol{\eta})} + \underbrace{\varepsilon^2\langle\{f^{(2)}, S^{(2)}\} + \frac{1}{2}\{\{F_0, S^{(2)}\}, S^{(2)}\}\rangle_{\theta_1}}_{\varepsilon\mathcal{O}_4(\boldsymbol{\xi},\boldsymbol{\eta})} + \mathcal{O}(\varepsilon^{3/2}) + \varepsilon\mathcal{O}_6(\boldsymbol{\xi},\boldsymbol{\eta})$$

$$S^{(2)} = \frac{2\pi}{\partial_{J_1} F_0}\int_0^1 t\left(f^{(2)}(\mathbf{J}, \theta_1 + 2\pi t, \boldsymbol{\xi}, \boldsymbol{\eta}) - \langle f^{(2)}(\mathbf{J}, \theta_1, \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\theta_1}\right) dt$$

Close to the separatrix

$$\partial_{J_1} F_0 \longrightarrow \infty \qquad S_2 \longrightarrow \infty \qquad \varepsilon\langle f^{(4)}\rangle_{\theta_1} \ll \varepsilon^2\langle\{f^{(2)}, S^{(2)}\} + \frac{1}{2}\{\{F_0, S^{(2)}\}, S^{(2)}\}\rangle_{\theta_1}$$

Dominant terms close to the separatrix

$$F_0 + \varepsilon\langle f^{(2)}\rangle_{\theta_1} + \varepsilon^2\langle\{f^{(2)}, S^{(2)}\} + \frac{1}{2}\{\{F_0, S^{(2)}\}, S^{(2)}\}\rangle_{\theta_1}$$

Only deg 2 is needed