Stability of the Laplace-like Resonances

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Motivation and Content

Relative Equilibria of Multi-resonant Planetary Systems

- Equilibrium and stability of the Laplace and Laplace-like resonances
- A simple model and its normal forms

Main results

- de Sitter equilibrium sequences are **linearly stable** for any Planet/Star mass-ratio (P. CMDA 2021)
- Saddle-Node **bifurcation** of 'New' de Sitter sequences beyond a given threshold of the 'resonance proximity parameter' (P. CMDA 2024)
- N-body numerical integrations indicate **instability** of the Laplace resonance above 10^{-4} for the Planet/Star mass-ratio (Lhotka, P. and Dvorak, AA 2026, to appear)

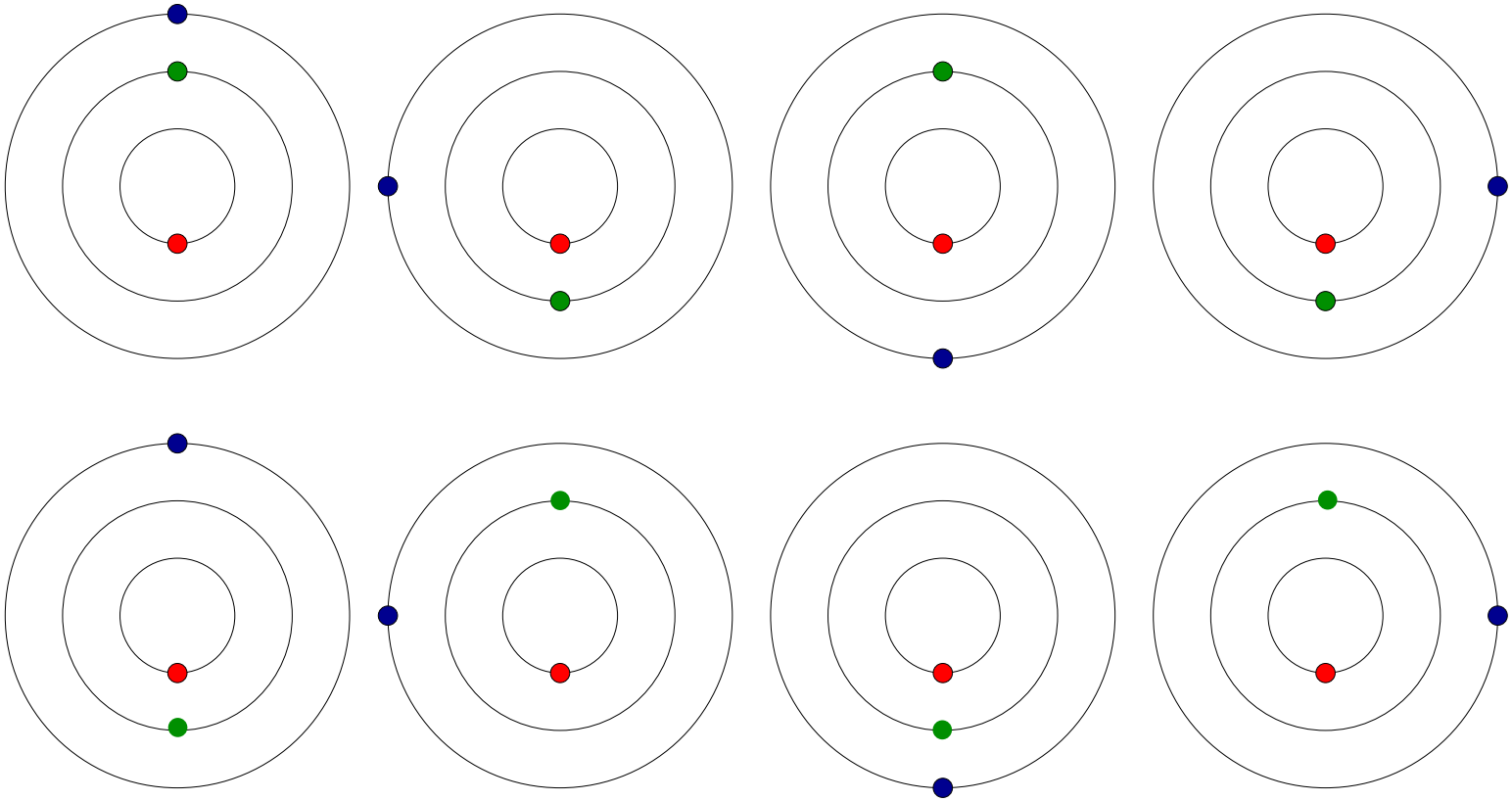


Figure: Galilean system (top panel) compared with GJ-876 (lower panel). Each snapshot correspond to a revolution period of Io or planet-c (red dots); in the upper plots Europa (green dots) and Ganymede (blue dots) can be in conjunction with Io one at a time; in the lower plots, planet-b (green dots) and planet-e (blue dots) can together be in conjunction with planet-c.

Model for a planar 1+3 body multi-resonant system

Hamiltonian

$$H = H_{Kep} + H_{Pert}$$

where

- Keplerian interaction of the masses m_1, m_2, m_3 with a 'large' central spherical $m_0 \gg m_k$.
- Disturbing function: Coupling terms of the minor bodies ($m_k,\ k = 1, 2, 3$) truncated to e.g. 2nd order in eccentricity expressed in terms of modified Delaunay variables

$$H_{Pert} = -\mathcal{R}(L_k, P_k, \lambda_k, p_k).$$

$$L_k \simeq m_k\sqrt{m_0 a_k}, \quad P_k \simeq \frac{1}{2} L_k e_k^2.$$

$$\lambda_k = \ell_k + \varpi_k, \quad p_k = -\varpi_k.$$

Laplace-like resonance: Keplerian part

$$H_{Kep} \simeq \sum_{k=1}^{3} \left[\overline{n}_k (L_k - \overline{L}_k) - \frac{3}{2} \eta_k (L_k - \overline{L}_k)^2 \right]$$

$\overline{L}_k$ are three 'nominal' values of the first actions and

$$\overline{n}_k = \sqrt{\frac{m_0}{\overline{a}_k{}^3}} = \frac{m_0^2 m_k^3}{\overline{L}_k^3}, \qquad \eta_k \doteq \frac{\overline{n}_k}{\overline{L}_k}$$

are evaluated at nominal values, namely, for first-order **Laplace-like resonances**:

$$k_1 \overline{n}_2 = (k_1 - 1)\overline{n}_1, \quad k_2 \overline{n}_3 = (k_2 - 1)\overline{n}_2,$$

with $k_1, k_2 \in \mathbb{N}$ and $|k_1 - k_2| \leq 1$ ($k_1 = k_2 = 2$ for the standard Laplace resonance).

First Normalisation: averaging with respect to 'fast' variables

$$\begin{aligned}\mathcal{R}_{avg} \quad = \quad & \frac{m_1 m_2}{a_2} \times \{ f_{12,1} e_1 \cos(k_1\lambda_2 - (k_1-1)\lambda_1 - \varpi_1) \\ & \quad + f_{12,2} e_2 \cos(k_1\lambda_2 - (k_1-1)\lambda_1 - \varpi_2)\} + \\ & \frac{m_2 m_3}{a_3} \times \{ f_{23,1} e_2 \cos(k_2\lambda_3 - (k_2-1)\lambda_2 - \varpi_2) \\ & \quad + f_{23,2} e_3 \cos(k_2\lambda_3 - (k_2-1)\lambda_2 - \varpi_3)\} + \mathrm{O}(e_k^2) \ ,\end{aligned}$$

where

$$a_k = \frac{L_k^2}{m_0 m_k^2}, \quad e_k = \sqrt{\frac{2P_k}{L_k}}, \quad \varpi_k = -p_k,$$

and the $f_{ij,\alpha}$ $(i,j = 1,2,3,\ \alpha \in \mathbb{N})$, are the 'Laplace coefficients'.

Variables adapted to the resonance

We perform the canonical transformation
$(L_j, P_j, \lambda_j, p_j) \longrightarrow (Q_a, q_a), \quad a = 1, \ldots, 6$:

$$
\begin{aligned}
q_j &= k_1\lambda_2 - (k_1 - 1)\lambda_1 + p_j, \\
Q_j &= P_j,\ j = 1, 2, 3, \\
q_4 &= (1 - k_1)\lambda_1 + (k_1 + k_2 - 1)\lambda_2 - k_2\lambda_3\ , \\
Q_4 &= \tfrac{k_1}{3(k_1-1)} L_1 + \tfrac{1}{3} L_2 + \tfrac{k_2-4}{3k_2} L_3\ , \\
Q_5 &= \tfrac{k_1+k_2-1}{3(k_1-1)} L_1 + \tfrac{1}{3} L_2 + \tfrac{k_2-1}{3}\Gamma\ , \\
Q_6 &= L_1 + L_2 + L_3 - \Gamma
\end{aligned}
$$

$\Gamma = \sum_k P_k = \sum_k Q_k$ is the *angular momentum deficit*.
Q_6 is the total angular momentum.
$Q_L = 3Q_5 + (k_2 - 1)Q_6 = \frac{k_1 k_2}{k_1 - 1} L_1 + k_2 L_2 + (k_2 - 1)L_3$ is a scaling invariant.

Reduction to 4 degrees of freedom

Keplerian part (the expansion parameters $C > B > A > 0$ are linear functions of the η_j):

$$H_{Kep} = \kappa_1 \Gamma + \kappa_4 Q_4 - \frac{3}{2} A \Gamma^2 - 3 B \Gamma Q_4 - \frac{3}{2} C Q_4^2 .$$

Resonant perturbing part:

$$\begin{aligned} H_{Pert} \quad = \quad & -\alpha \sqrt{2Q_1} \cos q_1 - \beta_1 \sqrt{2Q_2} \cos q_2 \\ & -\beta_2 \sqrt{2Q_2} \cos(q_2 - q_4) - \gamma \sqrt{2Q_3} \cos q_3 + \mathrm{O}(Q_k). \end{aligned}$$

Q_5, Q_6 are **integrals of motion** and provide the frequencies

$$\begin{aligned} \kappa_1 \quad &= \quad \kappa_{11}(\eta_j) Q_5 + \kappa_{12}(\eta_j) Q_6 \ , \\ \kappa_4 \quad &= \quad \kappa_{41}(\eta_j) Q_5 + \kappa_{42}(\eta_j) Q_6 . \end{aligned}$$

The coupling constants $\alpha, \beta_2 < 0, \beta_1, \gamma > 0$ depend on the mass ratios m_k/m_0.

Poincaré variables x_j, y_j $(j = 1, 2, 3)$:

$$x_j = \sqrt{2Q_j}\cos q_j, \quad y_j = \sqrt{2Q_j}\sin q_j \quad \Gamma(x,y) = \frac{1}{2}\sum_{j=1}^{3}\left(x_j^2 + y_j^2\right).$$

'Book-keeping' order of the Hamiltonian

$$H(Q_4, q_4, x_j, y_j) = H_0 + \epsilon H_1 + \epsilon^2 H_2 :$$

$$\begin{aligned}
H_0 &= \kappa_4 Q_4 - \frac{3}{2} C Q_4^2, \\
H_1 &= \kappa_1 \Gamma(x,y) - 3B\Gamma(x,y)Q_4 - \alpha x_1 - \beta_1 x_2 \\
&\quad -\beta_2(x_2\cos q_4 + y_2 \sin q_4) - \gamma(x_3\cos q_4 + y_3 \sin q_4), \\
H_2 &= -\frac{3}{2} A \Gamma(x,y)^2.
\end{aligned}$$

Assuming Q_4 oscillating with small amplitude around an equilibrium value $Q_4^\star$ given by

$$\dot{q}_4 = 0 \rightarrow \frac{\partial H_0}{\partial Q_4} = 0 \longrightarrow Q_4^\star = \frac{\kappa_4}{3C}$$

and introducing

$$\Lambda = Q_4 - Q_4^\star, \quad \lambda = q_4,$$

we get

$$H(\mathbf{x}, \Lambda, \mathbf{y}, \lambda) = \omega\Gamma - \frac{3}{2}A\Gamma^2 - 3B\Lambda\Gamma - \frac{3}{2}C\Lambda^2 + H_{Res}(\mathbf{x}, \mathbf{y}, \lambda)$$

where (the unit of time is the period of the first body)

$$\omega(Q_5, Q_6) = \frac{1}{C}\left[3(AC - B^2)\Gamma - (C - B)\Delta_{12} - B\Delta_{23}\right]$$

is the **proximity parameter to the resonance** with

$$\Delta_{12} = (k_1 - 1)n_1 - k_1 n_2, \quad \Delta_{23} = (k_2 - 1)n_2 - k_2 n_3.$$

EQUILIBRIA: $\nabla H(x_E, \Lambda_E, y_E, \lambda_E) = 0$

Considering only the unperturbed part linear in the momenta we get the so called 'de Sitter-Sinclair' equilibria:

$$H(\mathbf{x}, \Lambda, \mathbf{y}, \lambda) = \omega\Gamma + H_{Res} + \epsilon H_{pert}.$$

$$\begin{aligned}
\Lambda_E &= 0, & \lambda_E &= 0, \pi, \\
x_{1E} &= \frac{\alpha}{\omega}, & & \\
x_{2E} &= \frac{\beta_1 \pm \beta_2}{\omega}, & y_{jE} &= 0,\ j = 1, 2, 3, \\
x_{3E} &= \pm\frac{\gamma}{\omega}. & &
\end{aligned}$$

Equilibria including nonlinear terms in the unperturbed part

$$H(\mathbf{x},\Lambda,\mathbf{y},\lambda)=\omega\Gamma-\tfrac{3}{2}(A\Gamma^2+2B\Lambda\Gamma+C\Lambda^2)+H_{Res}(\mathbf{x},\mathbf{y},\lambda)+\epsilon H_{pert}.$$

$\Lambda_E=-\frac{B}{C}\Gamma_E, \lambda_E=0,\pi$ and the solutions of the cubic system:

$$\begin{aligned}
x_{1E} =& \sqrt[3]{\tfrac{\alpha}{2\hat{K}}\left(-1+\sqrt{1-(\omega/\omega_b)^3}\right)}+\sqrt[3]{\tfrac{\alpha}{2\hat{K}}\left(-1-\sqrt{1-(\omega/\omega_b)^3}\right)},\\
x_{2E} =& \frac{\beta_1\mp\beta_2}{\alpha}x_{1E},\\
x_{3E} =& \pm\frac{\gamma}{\alpha}x_{1E}, \quad y_{jE}=0,\ j=1,2,3,
\end{aligned}$$

where ω_b is the *bifurcation threshold*

$$\omega_b = 3\left(\frac{1}{4}\alpha^2\hat{K}\right)^{1/3}=\frac{3}{2}\left(\frac{3(AC-B^2)}{C}(\alpha^2+(\beta_1\mp\beta_2)^2+\gamma^2)\right)^{1/3}.$$

Equilibria of the Galilean system

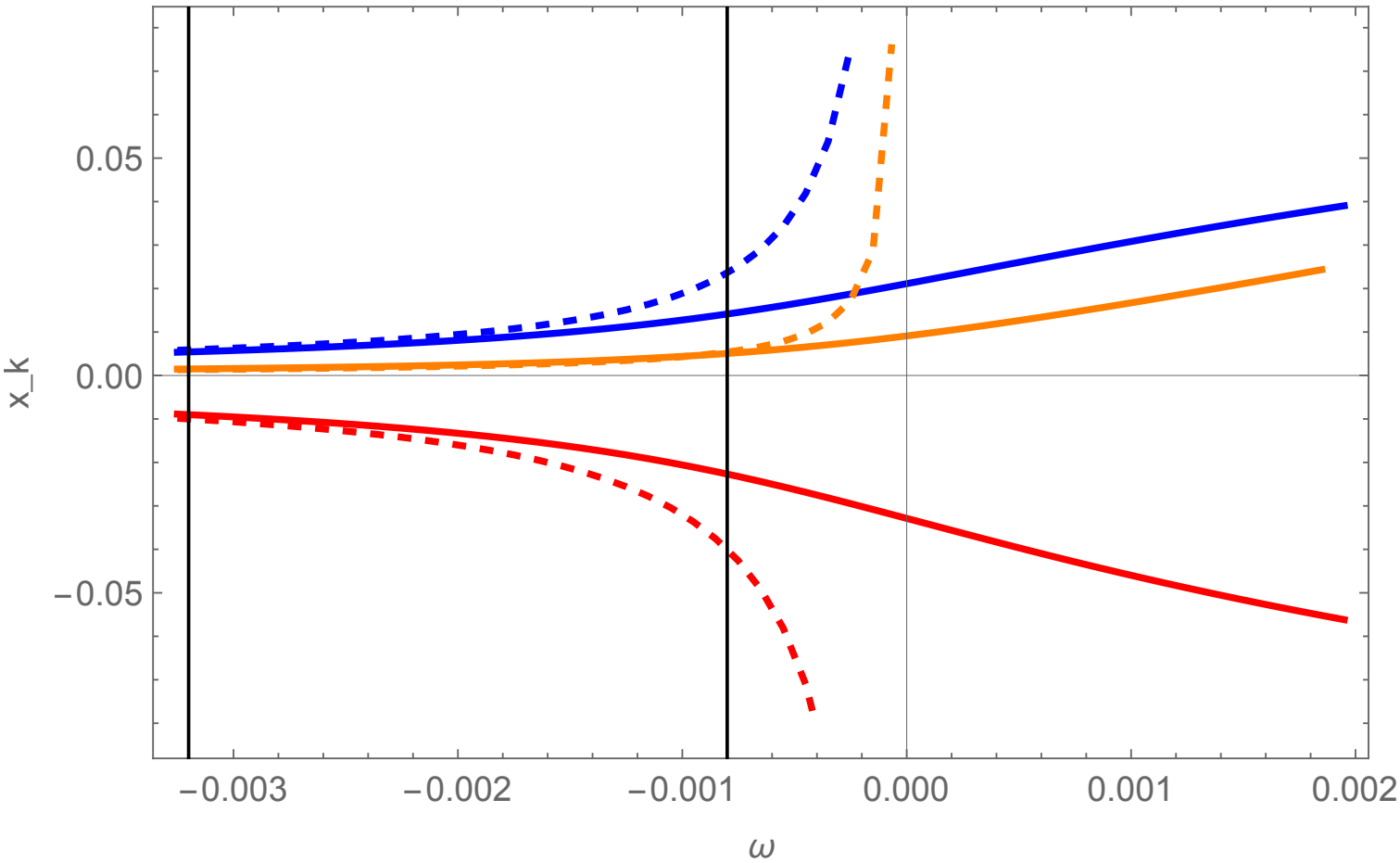


Figure: de Sitter-Sinclair equilibria of the Galilean system at first order in the resonant terms: cubic solution (continuous lines); linear solution (dashed lines). Io is depicted in Blue; Europa in Red and Ganymede in Orange.

Equilibria of the Galilean system

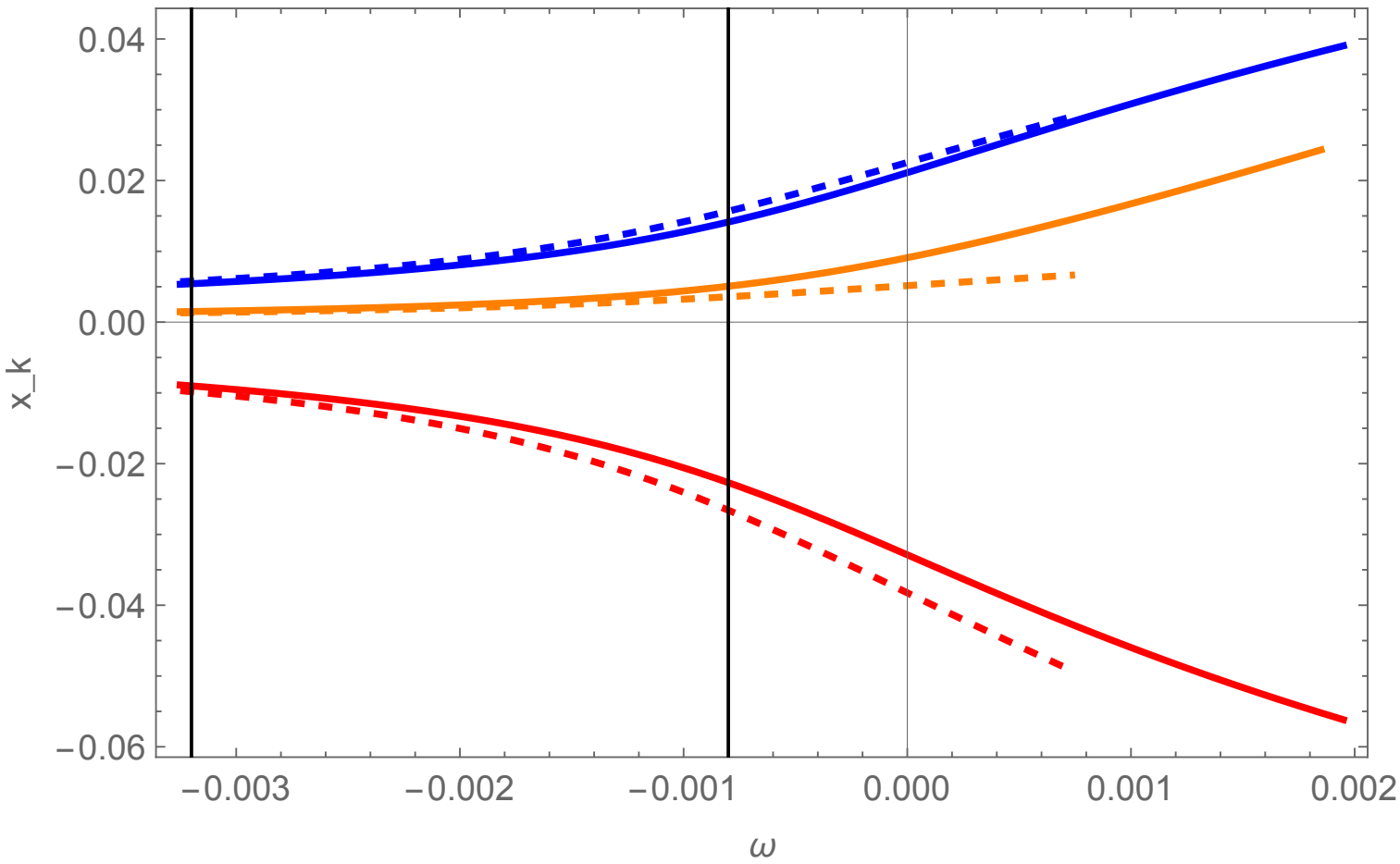


Figure: Comparison of the de Sitter-Sinclair equilibria of the Galilean system: 2nd-order in the resonant terms (continuous lines); first-order in the resonant terms (dashed lines).

Bifurcation of the new-de Sitter equilibria:

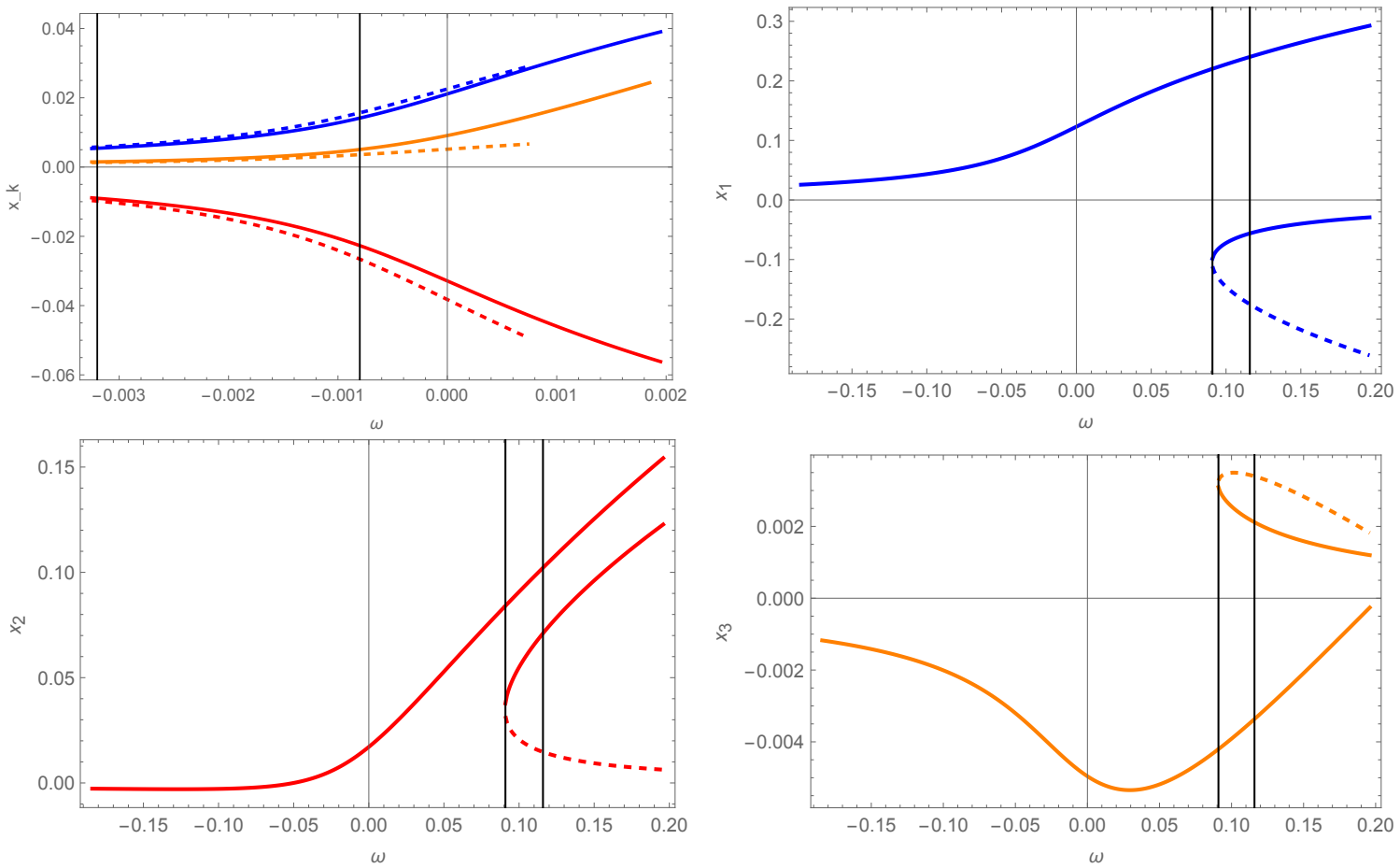


Figure: GJ-876 system: Anti de Sitter ($\lambda_E = 0$) equilibria, bifurcation value of the proximity parameter, $\omega_b = 0.00158$.

de Sitter-Sinclair (dS) and anti de Sitter (AdS) equilibria

	e_1	e_2	e_3	q_1	q_2	q_3	q_4
observed	0.0042	0.0094	0.0015	0	π	rotating	π
dS1	0.0058	0.0118	0.0008	0	π	π	π
dS2	0.0042	0.0095	0.0011	0	π	π	π
AdS1	0.0055	0.0113	0.0008	0	0	π	0

Table: Galilean system ($\omega = -0.00352$).

	e_1	e_2	e_3	q_1	q_2	q_3	q_4
observed	0.253	0.037	0.031	0	0	rotating?	0
dS1	0.048	0.005	0.006	π	0	0	π
AdS1	0.254	0.053	0.038	0	0	π	0

Table: GJ-876 system ($\omega = 0.1159$).

New de Sitter equilibria: a saddle-node bifurcation

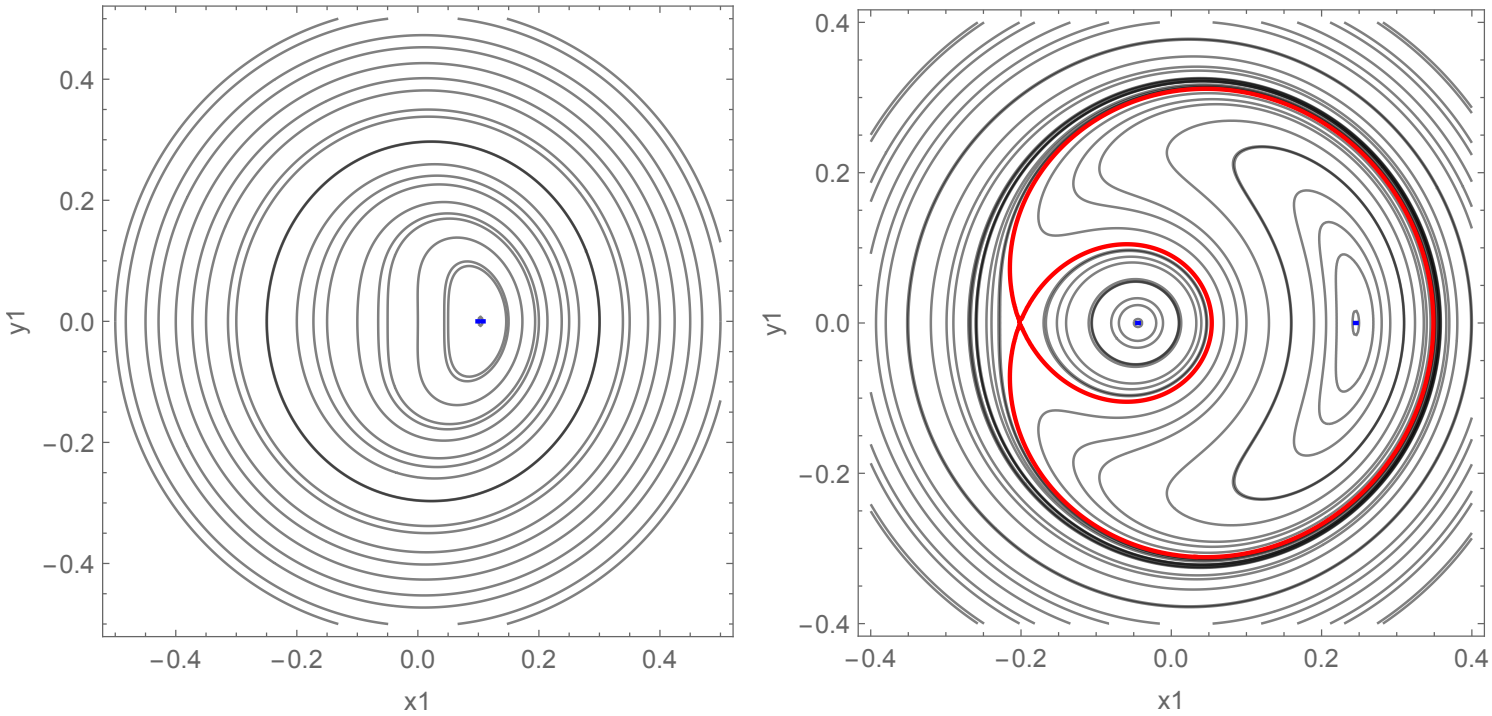


Figure: Left: de Sitter-Sinclair equilibrium. Right: New de Sitter equilibria.

Stability of the equilibria ($z = (Q_4, q_4, x_j, y_j)$):

$$\dot{\delta z} = \mathbf{J} H_{zz}\big|_E \delta z$$

All eigenvalues $\pm\sigma_k (k = 1, ..., 4)$ pure imaginary for de Sitter-Sinclair equilibria.

Second normalisation: Normal form for the Laplace libration (linear integrable part)

Assuming ω 'fast' with respect to the libration of the Laplace argument $\lambda = q_4 - q_{4E}$, we can normalise

$$H = \omega\Gamma + \epsilon\left(-3B\Lambda\Gamma - \frac{3}{2}C\Lambda^2 - \frac{3}{2}A\Gamma^2 + H_{res}(\mathbf{Q},\mathbf{q},\lambda) + ...\right),$$

with respect to the 'isotropic oscillator' $H_I = \omega(Q_1 + Q_2 + Q_3)$ obtaining

$$K = 3B\Gamma\Lambda + \frac{3}{2}C\Lambda^2 \pm \frac{\beta_1\beta_2}{\omega}\cos\lambda + K_{n\geq 2} + H_{remainder}$$

where $K_{n\geq 2} = \sum_{n\geq 2} c_n \cos n\lambda$ plus resonant terms of H_I and

$$H_{rem} = \Sigma_{k,k+1,\ell m} c_{k,k+1,\ell m} (Q_i Q_j)^{\ell/2} \cos(\ell(q_k - q_{k+1}) + m\lambda).$$

Normal form for the Laplace libration

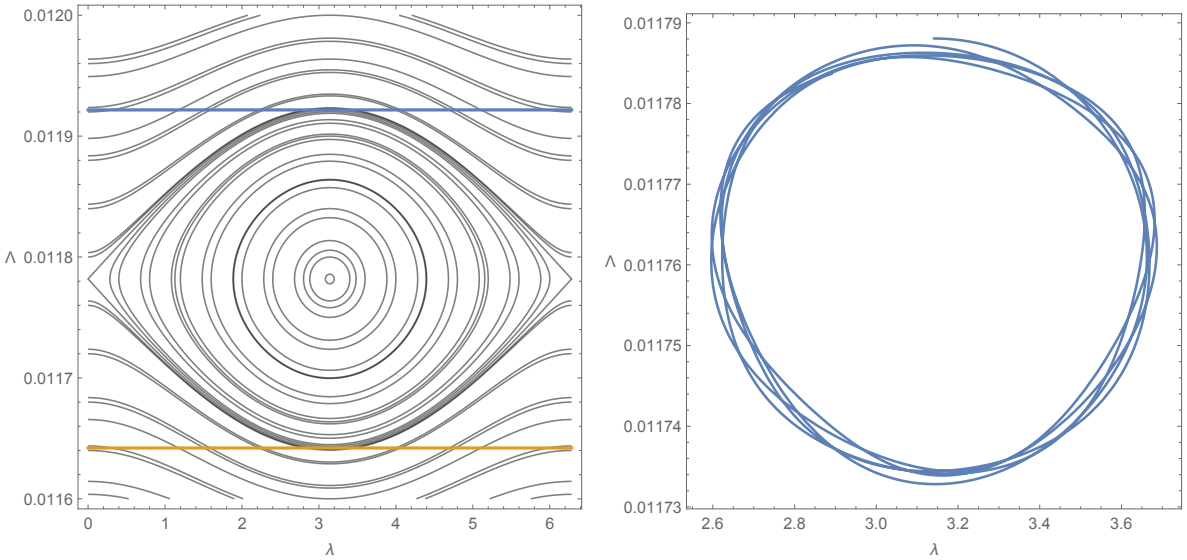


Figure: Left: Normal form around the de Sitter equilibrium for the Galilean toy model ($|\omega| \sim 0.004, \Omega_L \sim 0.001$). Right: projection of numerical phase-curve.

The modified pendulum gives the libration frequency and the resonance width:

$$\Omega_L = \sqrt{\frac{3C\beta_1\beta_2}{\omega}}, \quad \Delta\Lambda = 4\sqrt{\frac{\beta_1\beta_2}{3C\omega}}.$$

Second normalisation with respect to the nonlinear integrable part

In $H = H_0 + \epsilon H_1$ we now assume that the whole integrable part (including quadratic terms in the actions) is used as unperturbed Hamiltonian with respect to which to normalise

$$H_0 = \omega\Gamma - \tfrac{3}{2}A\Gamma^2 - 3B\Lambda\Gamma - \tfrac{3}{2}C\Lambda^2, \quad \Gamma = Q_1 + Q_2 + Q_3.$$

The normal form now contains many more additional harmonics of the angle combinations

$$\sum \ell_k q_k \pm m\lambda, \quad k = 1, 2, 3, \quad |\ell_k|, m \geq 1.$$

The new prediction of the libration frequency is

$$\Omega_L = \sqrt{\frac{3C\beta_1\beta_2}{\omega - 3A\Gamma_E - 3B\Lambda_E}}.$$

Numerical integrations of the N-body code

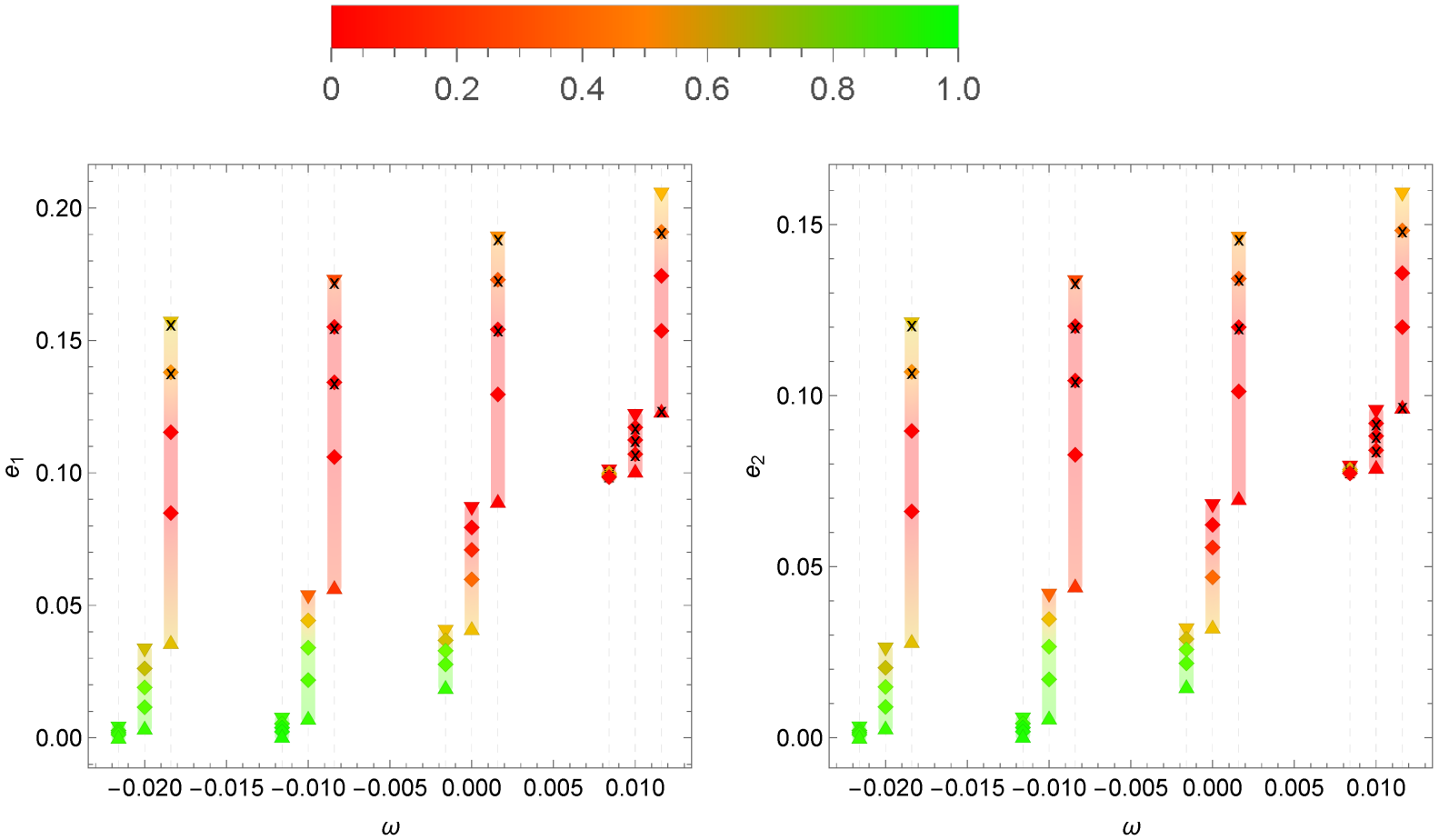


Figure: Stability of the 2:1-2:1 Laplace resonance (green→libration of $\lambda - \pi$; red→rotation): each group of three coloured columns is centred on a specific value of ω, for masses 10^{-5} (left), 10^{-4} (middle), and 10^{-3} (right).

Numerical integrations of the N-body code

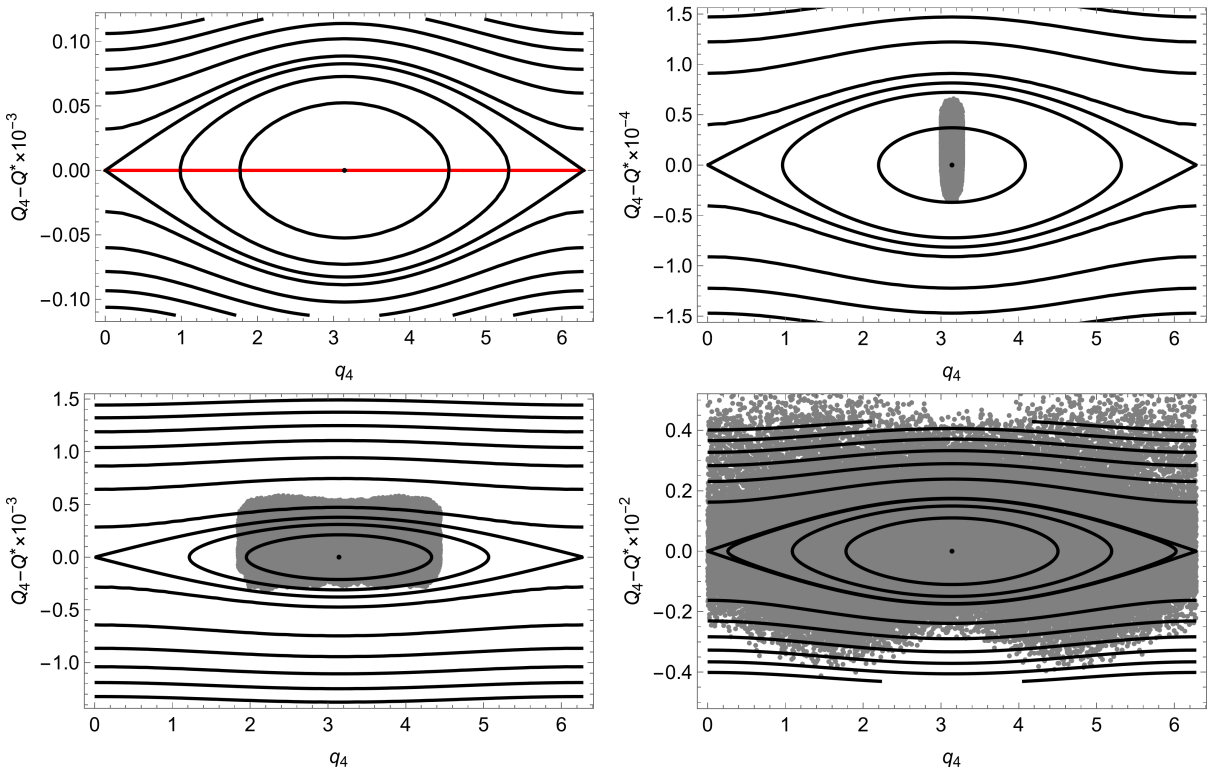


Figure: Projection of numerical trajectories over the level curves of the integrable normal form at $\omega = 0$ for different mass values: 10^{-5} (upper right); 10^{-4} (lower left); 10^{-3} (lower right).

THANK YOU FOR THE ATTENTION

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Normal forms for the Laplace resonance (P. CMDA 2021, 133, 11 & 2024, 136, 51)