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Perturbative Methods in Studying the Secular Evolution of a Planetary System of Many Bodies with Variable Masses

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Problem of many bodies with constant masses

Consider a system of point bodies P_0, P_1, ..., P_n of constant masses m_0, m_1, ..., m_n, respectively, attracting each other according to Newton's law of universal gravitation. Applying Newton's laws, one can write the equations of motion of the system as

$$m_j \frac{d^2 \vec{R}_j}{dt^2} = \operatorname{grad}_{\vec{R}_j} U, \quad j = 0, 1, \ldots, n,$$

where the force function U is given by

$$U = G \sum_{k=0\ (k\neq j)}^{n} \frac{m_j m_k}{R_{jk}}.$$

Here G is the gravitational constant, $\vec{R}_j$ is the radius-vector of the body P_j in an inertial reference frame, and $R_{jk} = \left| \vec{R}_k - \vec{R}_j \right|$.
Using a relative coordinate system with the origin in the center of P_0, we can write the differential equations of motion of the bodies P_1, ..., P_n in the form

$$\frac{d^2 \vec{r}_j}{dt^2} + G (m_0 + m_j) \frac{\vec{r}_j}{r_j^3} = \vec{F}_j, \quad j = 1, 2, \ldots, n,$$

where the vector $\vec{r}_j = \vec{R}_j - \vec{R}_0$ determines position of the body P_j relative to P_0, and

$$\vec{F}_j = G \sum_{k=1\ (k\neq j)}^{n} m_k \left(\frac{\vec{r}_k - \vec{r}_j}{r_{jk}^3} - \frac{\vec{r}_k}{r_k^3} \right).$$

Planetary system

In the case of $m_0 >> m_j$ (j = 1, 2, ..., n) the mutual gravitational attraction of the bodies P_1, ..., P_n described by the force $\vec{F}_j$ is at least an order of magnitude smaller than that of the body P_0 and may be neglected, in the first approximation. Putting $F_j = 0$, we obtain n independent differential equations

$$\frac{d^2 \vec{r}_j}{dt^2} + G(m_0 + m_j)\frac{\vec{r}_j}{r_j^3} = 0, \quad j = 1, 2, \ldots, n,$$

describing the corresponding problems of two bodies P_0 and P_j, a complete and general solution to which is known and is given by

$$x_j(t) = \rho_j(\cos(f_j + \omega_j)\cos\Omega_j - \sin(f_j + \omega_j)\sin\Omega_j \cos i_j),$$

$$y_j(t) = \rho_j(\cos(f_j + \omega_j)\sin\Omega_j + \sin(f_j + \omega_j)\cos\Omega_j \cos i_j),$$

$$z_j(t) = \rho_j \sin(f_j + \omega_j)\sin i_j, \quad \rho_j = \frac{a_j\left(1 - e_j^2\right)}{1 + e_j \cos f_j},$$

where f_j is the true anomaly determined by the equations

$$\tan\frac{f_j}{2} = \sqrt{\frac{1 + e_j}{1 - e_j}}\tan\frac{E_j}{2}, \quad E_j - e_j \sin E_j = M_j = \frac{\sqrt{G(m_0 + m_j)}}{a_j^{3/2}}(t - \tau_j).$$

Here E_j is the eccentric anomaly, M_j is the mean anomaly, and ***six constants*** $a_j, e_j, i_j, \Omega_j, \omega_j, \tau_j$ that are uniquely found from the initial conditions are known as the ***orbital elements***.

Taking into account the mutual gravitational attraction of the bodies P_1, ..., P_n leads to the dependence of the orbital elements on time which can be investigated with the aid of the perturbation methods which have been developed quite well.

Two-body problem with variable masses

Assume that each body of the two-body system ejects the mass isotropically in all directions and the velocity of ejection is high enough for the particles to escape from the system. Then their interaction with the bodies may be ignored and the only effect of the mass loss is the decrease in the attraction between the two bodies. Then the relative motion of the bodies P_j and P_0 may be described by the Gylden equation

$$\frac{d^2\vec{r}_j}{dt^2} + G\,(m_0(t) + m_j(t))\,\frac{\vec{r}_j}{r_j^3} = 0, \quad j = 1, \ldots, n.$$

This differential equation is not integrable, in general, because of the dependence of masses on time. As its general solution cannot be written in closed form we cannot generalize directly the perturbation methods developed in the case of constant masses to the non-stationary system.

Main question: ***which solution may be considered as the unperturbed one***?

1. Periodic motion on conic section

Let us rewrite the equation above in the form

$$\frac{d^2\vec{r}_j}{dt^2} + G\,(m_{00} + m_{j0})\,\frac{\vec{r}_j}{r_j^3} = \vec{Q}_j,$$

where $m_{00} = m_0(t_0)$, $m_{j0} = m_j(t_0)$ are the initial values of the masses, and

$$\vec{Q}_j = G\,(m_{00} + m_{j0})\left(1 - \frac{m_0(t) + m_j(t)}{m_{00} + m_{j0}}\right)\frac{\vec{r}_j}{r_j^3} = G\,(m_{00} + m_{j0})\left(1 - \frac{1}{\gamma_j(t)}\right)\frac{\vec{r}_j}{r_j^3},\ \gamma_j(t) = \frac{m_{00} + m_{j0}}{m_0(t) + m_j(t)}.$$

In the case of $\vec{Q}_j = 0$ we obtain the classical two-body problem with constant masses a general solution to which is known. However, this solution is perturbed by the force $\vec{Q}_j$ even in the framework of the two-body problem and this perturbation increases with time.

Two-body problem with variable masses

2. Aperiodic motion on conic section

The equation of motion may be also written in the form

$$\frac{d^2\vec{r}_j}{dt^2} + \frac{G\,(m_{00}+m_{j0})}{\gamma_j\,(t)}\,\frac{\vec{r}_j}{r_j^3} + \frac{1}{2\,\gamma_j\,(t)}\,\frac{d\gamma_j}{dt}\,\frac{d\,\vec{r}_j}{dt} = \vec{Q}_j,$$

where the perturbing force is

$$\vec{Q}_j = \frac{1}{2\,\gamma_j\,(t)}\,\frac{d\gamma_j}{dt}\,\frac{d\,\vec{r}_j}{dt}.$$

In the case of $\vec{Q}_j = 0$ we obtain integrable problem a general solution to which looks similar to the case of constant masses

$$x_j\,(t) = \rho_j\,(\cos\,(f_j+\omega_j)\,\cos\Omega_j - \sin\,(f_j+\omega_j)\,\sin\Omega_j\,\cos i_j),$$

$$y_j\,(t) = \rho_j\,(\cos\,(f_j+\omega_j)\,\sin\Omega_j + \sin\,(f_j+\omega_j)\,\cos\Omega_j\,\cos i_j),$$

$$z_j\,(t) = \rho_j\,\sin\,(f_j+\omega_j)\,\sin i_j,\quad \rho_j = \frac{a_j\,\left(1-e_j^2\right)}{1+e_j\,\cos f_j},$$

where the eccentric anomaly E_j is determined by the equation

$$E_j - e_j\,\sin E_j = M_j = \frac{\sqrt{G\,(m_{00}+m_{j0})}}{a_j^{3/2}}\,(\Psi_j\,(t) - \Psi_j\,(\tau)),\ \Psi_j\,(t) = \int_0^t \frac{dt}{\sqrt{\gamma_j\,(t)}}.$$

2.1. Osculating elements of aperiodic motion on conic section

The equation of motion may be also written in the form

$$\frac{da_j}{dt} = \frac{a_j\,(1 + e_j \cos E_j)}{1 - e_j \cos E_j}\,\frac{1}{\gamma_j\,(t)}\,\frac{d\gamma_j}{dt},\quad \frac{de_j}{dt} = \frac{\left(1 - e_j^2\right)\cos E_j}{1 - e_j \cos E_j}\,\frac{1}{\gamma_j\,(t)}\,\frac{d\gamma_j}{dt},$$

$$\frac{di_j}{dt} = 0,\quad \frac{d\Omega_j}{dt} = 0,\quad \frac{d\omega_j}{dt} = \frac{\sqrt{1 - e_j^2}\,\sin E_j}{e_j\,(1 - e_j \cos E_j)}\,\frac{1}{\gamma_j\,(t)}\,\frac{d\gamma_j}{dt},$$

$$\frac{dE_j}{dt} = \frac{\sqrt{G\,(m_{00} + m_{j0})}}{a_j^{3/2}\,(1 - e_j \cos E_j)}\,\frac{1}{\sqrt{\gamma_j\,(t)}} - \frac{\sin E_j}{e_j\,(1 - e_j \cos E_j)}\,\frac{1}{\gamma_j\,(t)}\,\frac{d\gamma_j}{dt}.$$

There exists an integral of motion: $a_j\left(1 - e_j^2\right)\big/\gamma_j(t) = \mathbf{const}$.

In the case of small eccentricity $e_j << 1$ the evolutionary equations take form

$$\frac{da_j^{(sec)}}{dt} = a_j^{(sec)}\,\frac{1}{\gamma_j\,(t)}\,\frac{d\gamma_j}{dt},\quad \frac{de_j^{(sec)}}{dt} = 0,\quad \frac{d\omega_j^{(sec)}}{dt} = 0,\quad \frac{dE_j^{(sec)}}{dt} = \frac{dM_j^{(sec)}}{dt} = \frac{\sqrt{G\,(m_{00} + m_{j0})}}{a_j^{3/2}\,\sqrt{\gamma_j\,(t)}}$$

Solving these equations, we obtain

$$a_j^{(sec)} = a_{j0}\,\gamma_j\,(t),\quad E_j^{(sec)} = M_j^{(sec)} = \frac{\sqrt{G\,(m_{00} + m_{j0})}}{a_{j0}^{3/2}}\,(\Phi_j\,(t) - \Phi_j\,(t_0)),\quad \Phi_j\,(t) = \int_0^t \frac{dt}{\gamma_j^2\,(t)},$$

where a_{j0} is an initial value of the semi-major axis a_j at the time $t = t_0$.

Two-body problem with variable masses

3. Aperiodic motion on quasi-conic section

The equation of motion may be also written in the form

$$\frac{d^2\vec{r}_j}{dt^2} + \frac{G\,(m_{00}+m_{j0})}{\gamma_j\,(t)}\,\frac{\vec{r}_j}{r_j^3} - \frac{1}{\gamma_j\,(t)}\,\frac{d^2\gamma_j}{dt^2}\,\vec{r}_j = \vec{Q}_j,$$

where the perturbing force is

$$\vec{Q}_j = -\frac{1}{\gamma_j\,(t)}\,\frac{d^2\gamma_j}{dt^2}\,\vec{r}_j.$$

In the case of $\vec{Q}_j = 0$ we obtain integrable problem a general solution to which is

$$x_j\,(t) = \gamma_j\,(t)\,\rho_j\,(\cos\,(f_j+\omega_j)\,\cos\Omega_j - \sin\,(f_j+\omega_j)\,\sin\Omega_j\,\cos i_j),$$

$$y_j\,(t) = \gamma_j\,(t)\,\rho_j\,(\cos\,(f_j+\omega_j)\,\sin\Omega_j + \sin\,(f_j+\omega_j)\,\cos\Omega_j\,\cos i_j),$$

$$z_j\,(t) = \gamma_j\,(t)\,\rho_j\,\sin\,(f_j+\omega_j)\,\sin i_j,\quad \rho_j = \frac{a_j\,\left(1-e_j^2\right)}{1+e_j\cos f_j},$$

where the eccentric anomaly E_j is determined by the equation

$$E_j - e_j\sin E_j = M_j = \frac{\sqrt{G\,(m_{00}+m_{j0})}}{a_j^{3/2}}\,(\Phi_j\,(t)-\Phi_j\,(\tau)),\quad \Phi_j\,(t) = \int_0^t \frac{dt}{\gamma_j^2\,(t)}.$$

3.1. Osculating elements of aperiodic motion on quasi-conic section

The equation of motion may be also written in the form

$$\frac{da_j}{dt} = -\frac{2\,a_j^{5/2}\,e_j \sin E_j}{\sqrt{G\,(m_{00}+m_{j0})}}\,\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2},\quad \frac{de_j}{dt} = -\frac{a_j^{3/2}\left(1-e_j^2\right)\sin E_j}{\sqrt{G\,(m_{00}+m_{j0})}}\,\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2},$$

$$\frac{di_j}{dt} = 0,\quad \frac{d\Omega_j}{dt} = 0,\quad \frac{d\omega_j}{dt} = \frac{a_j^{3/2}\sqrt{1-e_j^2}\,(\cos E_j - e_j)}{e_j\sqrt{G\,(m_{00}+m_{j0})}}\,\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2},$$

$$\frac{dE_j}{dt} = \frac{\sqrt{G\,(m_{00}+m_{j0})}}{a_j^{3/2}\,(1-e_j\cos E_j)}\,\frac{1}{\gamma_j^2(t)} + \frac{a_j^{3/2}\sin E_j}{e_j\sqrt{G\,(m_{00}+m_{j0})}}\left(2\,e_j - \left(1-e_j^2\right)\cos E_j\right)\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2}.$$

There exists an integral of motion: $a_j\left(1-e_j^2\right) = \mathbf{const}$.

In the case of small eccentricity $e_j << 1$ the evolutionary equations take form

$$\frac{da_j^{(sec)}}{dt} = 0,\quad \frac{de_j^{(sec)}}{dt} = 0,\quad \frac{d\omega_j^{(sec)}}{dt} = -\frac{3\,a_j^{3/2}\left(2-e_j^2\right)}{4\sqrt{G\,(m_{00}+m_{j0})}}\,\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2},$$

$$\frac{dE_j^{(sec)}}{dt} = \frac{\sqrt{G\,(m_{00}+m_{j0})}}{a_j^{3/2}\,\gamma_j^2(t)} + \frac{a_j^{3/2}}{2\sqrt{G\,(m_{00}+m_{j0})}}\left(7+3\,e_j^2\right)\gamma_j(t)\,\frac{d^2\gamma_j}{dt^2}.$$

Proposition: Aperiodic motion along a quasi-conic section gives very good approximation for the solution to the two-body problem with variable masses and may be successfully used as the unperturbed solution in the many-body problem with variable masses.

Many-body problem with variable masses

Consider a system of point bodies $P_0,\ P_1,\ \ldots,\ P_n$ of variable masses $m_0(t),\ m_1(t),\ \ldots,\ m_n(t)$, respectively, attracting each other according to Newton's law of universal gravitation. Using a relative coordinate system with the origin in the center of P_0 and taking into account the reactive forces which appear in the case of anisotropic mass change, one can write the equations of motion of the system as

$$\frac{d^2\vec{r}_j}{dt^2} + G\,(m_0(t) + m_j(t))\,\frac{\vec{r}_j}{r_j^3} = \operatorname{grad}_{\vec{r}_j} U_j + \frac{1}{m_j}\frac{dm_j}{dt}\vec{V}_j - \frac{1}{m_0}\frac{dm_0}{dt}\vec{V}_0, \quad j = 1, 2, \ldots, n,$$

where $\vec{V}_j$ is the relative velocity of the particles leaving (or absorbed by) the body P_j, and

$$U_j = G\sum_{k=1\ (k\neq j)}^{n} m_k\left(\frac{1}{r_{jk}} - \frac{\vec{r}_j\cdot\vec{r}_k}{r_k^3}\right).$$

Neglecting the right-hand side in the equations of motion, we obtain n two-body problems with variable masses which are not integrable. To represent the equations of motion in the form being appropriate for application of the perturbation methods, we rewrite them as

$$\frac{d^2\vec{r}_j}{dt^2} + G\,(m_0(t) + m_j(t))\,\frac{\vec{r}_j}{r_j^3} - \frac{1}{\gamma_j(t)}\frac{d^2\gamma_j}{dt^2}\vec{r}_j = \operatorname{grad}_{\vec{r}_j} W_j, \quad j = 1, 2, \ldots, n,$$

where W_j, the perturbing function, has an expression

$$W_j = G\sum_{k=1\ (k\neq j)}^{n} m_k\left(\frac{1}{r_{jk}} - \frac{\vec{r}_j\cdot\vec{r}_k}{r_k^3}\right) + \frac{1}{m_j}\frac{dm_j}{dt}\vec{V}_j\vec{r}_j - \frac{1}{m_0}\frac{dm_0}{dt}\vec{V}_0\vec{r}_j - \frac{1}{2\gamma_j(t)}\frac{d^2\gamma_j}{dt^2}\vec{r}_j^{\,2}.$$

In the case of $W_j = 0$, we obtain n independent integrable two-body problems.

Perturbed motion in the osculating elements of aperiodic motion on quasi-conic section

In the case of $W_j = 0$, we obtain n independent differential equations

$$\frac{d^2 \vec{r}_j}{dt^2} + \frac{\kappa_j}{\gamma_j(t)} \frac{\vec{r}_j}{r_j^3} - \frac{1}{\gamma_j(t)} \frac{d^2 \gamma_j}{dt^2} \vec{r}_j = 0, \quad j = 1, 2, \ldots, n,$$

where $\kappa_j = G(m_{00} + m_{j0})$. A general solution to this equation describes aperiodic motion on quasi-conic section and gives good approximation for the relative motion of the body P_j around the star P_0. Using this solution and doing quite standard symbolic calculations we obtain the equations of the perturbed motion in the osculating elements of aperiodic motion on quasi-conic section:

$$\frac{da_j}{dt} = \frac{2\sqrt{a_j}}{\sqrt{\kappa_j}} \frac{\partial \tilde{W}_j}{\partial M_j}, \quad \frac{de_j}{dt} = \frac{(1-e_j^2)}{e_j \sqrt{a_j \kappa_j}} \frac{\partial \tilde{W}_j}{\partial M_j} - \frac{\sqrt{1-e_j^2}}{e_j \sqrt{a_j \kappa_j}} \frac{\partial \tilde{W}_j}{\partial \omega_j},$$

$$\frac{di_j}{dt} = \frac{\cot i_j}{\sqrt{a_j \kappa_j}\sqrt{1-e_j^2}} \frac{\partial \tilde{W}_j}{\partial \omega_j} - \frac{1}{\sqrt{a_j \kappa_j}\sqrt{1-e_j^2}\sin i_j} \frac{\partial \tilde{W}_j}{\partial \Omega_j},$$

$$\frac{d\Omega_j}{dt} = \frac{1}{\sqrt{a_j \kappa_j}\sqrt{1-e_j^2}\sin i_j} \frac{\partial \tilde{W}_j}{\partial i_j}, \quad \frac{d\omega_j}{dt} = \frac{\sqrt{1-e_j^2}}{e_j \sqrt{a_j \kappa_j}} \frac{\partial \tilde{W}_j}{\partial e_j} - \frac{\mathrm{Cot}[i]}{\sqrt{a}\sqrt{1-e_j^2}\sqrt{\kappa}} \frac{\partial \tilde{W}_j}{\partial i_j},$$

$$\frac{dM_j}{dt} = \frac{\sqrt{\kappa_j}}{a_j^{3/2} \gamma_j^2(t)} - \frac{2\sqrt{a_j}}{\sqrt{\kappa_j}} \frac{\partial \tilde{W}_j}{\partial a_j} - \frac{(1-e_j^2)}{e_j \sqrt{a_j \kappa_j}} \frac{\partial \tilde{W}_j}{\partial e_j},$$

where $\tilde{W}_j$ is obtained in terms of the orbital elements by substitution of the unperturbed solutions into the expression for the perturbing function W_j. These equations are known in the literature as ***Lagrange's planetary equations***.

Secular evolution of the orbital elements

Assume that the relative velocity $\vec{V}_0$ of the particles ejected by the star P_0 is directed to the most massive planet P_k while the masses of the planets are constant. To simplify the calculations we consider the case of small eccentricities and inclinations and replace the perturbing functions $\tilde{W}_j$ by the corresponding power series in terms of small parameters $e_j << 1$ and $i_j << 1$. Averaging the perturbing functions over the mean longitudes, we obtain

$$\tilde{W}_j^{(sec)} =$$

$$G \sum_{s=1}^{j-1} \frac{m_s}{a_j \gamma_j} \left(\frac{A_0(\alpha_{sj})}{2} + \frac{1}{2} \Pi_{11}(\alpha_{sj}) e_s^2 + \frac{1}{2} \Pi_{22}(\alpha_{sj}) e_j^2 + \Pi_{12}(\alpha_{sj}) e_s e_j \cos(\omega_s - \omega_j + \Omega_s - \Omega_j) - \frac{1}{8} B_1(\alpha_{sj}) \left(i_s^2 + i_j^2 - 2 i_s i_j \cos(\Omega_s - \Omega_j) \right) \right) +$$

$$G \sum_{s=j+1}^{n} \frac{m_s}{a_s \gamma_s} \left(\frac{A_0(\alpha_{js})}{2} + \frac{1}{2} \Pi_{11}(\alpha_{js}) e_j^2 + \frac{1}{2} \Pi_{22}(\alpha_{js}) e_s^2 + \Pi_{12}(\alpha_{js}) e_j e_s \cos(\omega_j - \omega_s + \Omega_j - \Omega_s) - \frac{1}{8} B_1(\alpha_{js}) \left(i_j^2 + i_s^2 - 2 i_j i_s \cos(\Omega_j - \Omega_s) \right) \right) -$$

$$\frac{\ddot{\gamma}_j}{2} a_j^2 \gamma_j \left(1 + \frac{3}{2} e_j^2\right) - \frac{\dot{m}_0}{m_0} V_0 a_j \gamma_j \left(\left(1 - e_k^2\right) \delta_{jk} + \frac{3}{2} e_j e_k \cos(\omega_j - \omega_k + \Omega_j - \Omega_k) \right),$$

$$\Pi_{12} = \frac{9}{8} B_0(\alpha_{sj}) + \frac{1}{8} B_2(\alpha_{sj}) + \frac{21}{16} C_1(\alpha_{sj}) + \frac{3}{16} C_3(\alpha_{sj}) - 9 C_0(\alpha_{sj}) \frac{1 + \alpha_{sj}^2}{8 \alpha_{sj}} + 3 C_2(\alpha_{sj}) \frac{1 + \alpha_{sj}^2}{8 \alpha_{sj}};$$

$$\Pi_{11} = \frac{6 \alpha_{sj}^2 + 15}{8} C_0(\alpha_{sj}) - \frac{3}{4} \alpha_{sj} \left(B_0(\alpha_{sj}) + 2 C_1(\alpha_{sj}) \right) - \frac{9}{8} C_2(\alpha_{sj}) - \frac{1}{2} B_1(\alpha_{sj});$$

$$\Pi_{22} = \frac{6 + 15 \alpha_{sj}^2}{8 \alpha_{sj}^2} C_0(\alpha_{sj}) - \frac{3}{4 \alpha_{sj}} \left(B_0(\alpha_{sj}) + 2 C_1(\alpha_{sj}) \right) - \frac{9}{8} C_2(\alpha_{sj}) - \frac{1}{2} B_1(\alpha_{sj});$$

Here $A_i(\alpha_{sj})$, $B_i(\alpha_{sj})$, $C_i(\alpha_{sj})$, $i = 0, 1, \ldots$, are the Laplace coefficients and parameter $\alpha_{sj} = a_s \gamma_s / (a_j \gamma_j) < 1$.

Evolutionary equations

The evolutionary equations determining the long-term behavior of the orbital elements are obtained from the equations of the perturbed motion by substituting the secular part $\tilde{W}_j^{(\mathrm{sec})}$ instead of the perturbing function $\tilde{W}_j$. Since $\tilde{W}_j^{(\mathrm{sec})}$ do not depend on the mean longitudes $\lambda_j = M_j + \Omega_j + \omega_j$, the corresponding derivatives of $\tilde{W}_j^{(\mathrm{sec})}$ with respect to the mean anomaly M_j vanish and, therefore, the secular perturbations of the analogs of the semi-major axes a_j do not change with time. The secular parts of the rest orbital elements are determined by the equations

$$\frac{de_j^{(\mathrm{sec})}}{dt} = -\frac{G}{\sqrt{a_j \kappa_j}} \sum_{s=1}^{j-1} \frac{m_s e_s}{a_j \gamma_j} \Pi_{12}(\alpha_{sj}) \sin(\omega_s - \omega_j + \Omega_s - \Omega_j) + \frac{G}{\sqrt{a_j \kappa_j}}$$

$$\sum_{s=1}^{j-1} \frac{m_s e_s}{a_s \gamma_s} \Pi_{12}(\alpha_{js}) \sin(\omega_j - \omega_s + \Omega_j - \Omega_s) - \frac{3\dot{m}_0}{2 m_0} \sqrt{\frac{a_j}{\kappa_j}} V_0 \gamma_j e_k \sin(\omega_j - \omega_k + \Omega_j - \Omega_k);$$

$$\frac{di_j^{(\mathrm{sec})}}{dt} =$$

$$-\frac{G}{4\sqrt{a_j \kappa_j}} \sum_{s=1}^{j-1} \frac{m_s i_s}{a_j \gamma_j} B_1(\alpha_{sj}) \sin(\Omega_s - \Omega_j) + \frac{G}{4\sqrt{a_j \kappa_j}} \sum_{s=1}^{j-1} \frac{m_s i_s}{a_s \gamma_s} B_1(\alpha_{js}) \sin(\Omega_j - \Omega_s);$$

$$\frac{d\Omega_j^{(\mathrm{sec})}}{dt} = -\frac{G}{4\sqrt{a_j \kappa_j}} \sum_{s=1}^{j-1} \frac{m_s}{a_j \gamma_j} B_1(\alpha_{sj}) \left(1 - \frac{i_s}{i_j} \cos(\Omega_s - \Omega_j)\right) -$$

$$\frac{G}{4\sqrt{a_j \kappa_j}} \sum_{s=1}^{j-1} \frac{m_s}{a_s \gamma_s} B_1(\alpha_{js}) \left(1 - \frac{i_s}{i_j} \cos(\Omega_j - \Omega_s)\right);$$

$$\frac{d\omega_j^{(sec)}}{dt} =$$

$$\frac{3\,\ddot{\gamma}_j}{2\sqrt{\kappa_j}}\,a_j^{3/2}\,\gamma_j - \frac{\dot{m}_0}{m_0}\sqrt{\frac{a_j}{\kappa_j}}\,V_0\,\gamma_j\left(-\frac{1}{2}\,\delta_{jk} + \frac{3\,e_k}{2\,e_j}\cos\left(\omega_j - \omega_k + \Omega_j - \Omega_k\right)\right) + \frac{G}{\sqrt{a_j\,\kappa_j}}\sum_{s=1}^{j-1}\frac{m_s}{a_j\,\gamma_j}$$

$$\left(\Pi_{22}\left(\alpha_{sj}\right) + \frac{1}{4}\,B_1\left(\alpha_{sj}\right)\left(1 - \frac{i_s}{i_j}\cos\left(\Omega_s - \Omega_j\right)\right) + \frac{e_s}{e_j}\,\Pi_{12}\left(\alpha_{sj}\right)\cos\left(\omega_s - \omega_j + \Omega_s - \Omega_j\right)\right) +$$

$$\frac{G}{\sqrt{a_j\,\kappa_j}}\sum_{s=1}^{j-1}\frac{m_s}{a_s\,\gamma_s}\left(\Pi_{11}\left(\alpha_{js}\right) + \frac{1}{4}\,B_1\left(\alpha_{js}\right)\left(1 - \frac{i_s}{i_j}\cos\left(\Omega_j - \Omega_s\right)\right) +\right.$$

$$\left.\frac{e_s}{e_j}\,\Pi_{12}\left(\alpha_{js}\right)\cos\left(\omega_j - \omega_s + \Omega_j - \Omega_s\right)\right);$$

where $\vec{V}_0$ is the relative velocity of the particles leaving the star P_0.

Simulation

As an example, we consider six planets, namely, the Mercury, Venus, Earth, Mars, Jupiter, and Saturn orbiting the Sun.

We define initial values of the solar system parameters.

Mass of the Sun, Mercury, Venus, Earth, Mars, Jupiter, Saturn:
$m_{00} = M_{\text{Sun}} = 1.98911 \times 10^{30}$ kg, $m_1 = 0.3302 \times 10^{24}$ kg, $m_2 = 4.8685 \times 10^{24}$ kg; $m_3 = 5.9736 \times 10^{24}$ kg; $m_4 = 0.64185 \times 10^{24}$ kg;
$m_5 = 1.8986 \times 10^{27}$ kg; $m_6 = 5.6846 \times 10^{26}$ kg;

Orbital parameters:

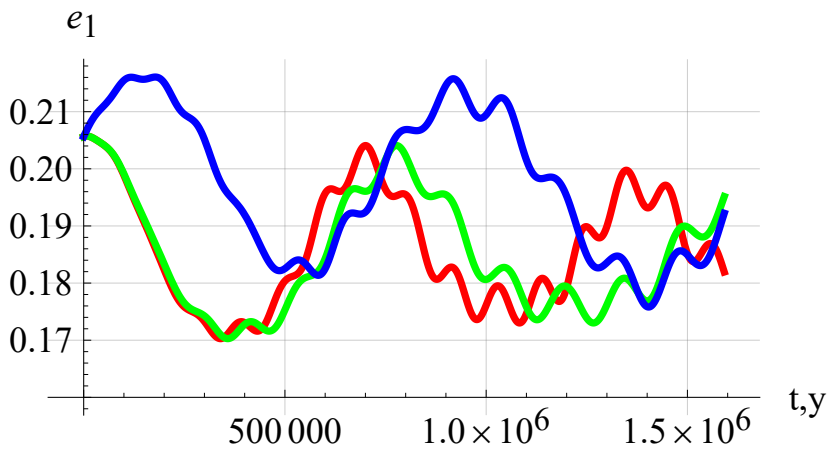
$a_1 = 0.387099$ AU, $a_2 = 0.723332$ AU, $a_3 = 1$; $a_4 = 1.523662$; $a_5 = 5.203363$ AU; $a_6 = 9.537070$ AU ;
$e_1 = 0.205631$; $e_2 = 0.006773$; $e_3 = 0.01671$; $e_4 = 0.093412$; $e_5 = 0.048393$; $e_6 = 0.054151$;
$i_1 = 7.00487$; $i_2 = 3.39471$; $i_3 = 0.00005$; $i_4 = 1.85061$; $i_5 = 1.30530$; $i_6 = 2.48446$;
$\Omega_1 = 48.33167$; $\Omega_2 = 76.68069$; $\Omega_3 = 348.7394$; $\Omega_4 = 49.5785$; $\Omega_5 = 100.5562$; $\Omega_6 = 113.71504$;
$\omega_1 = 29.1248$; $\omega_2 = 54.8523$; $\omega_3 = 114.208$; $\omega_4 = 286.462$; $\omega_5 = 274.198$; $\omega_6 = 338.717$.

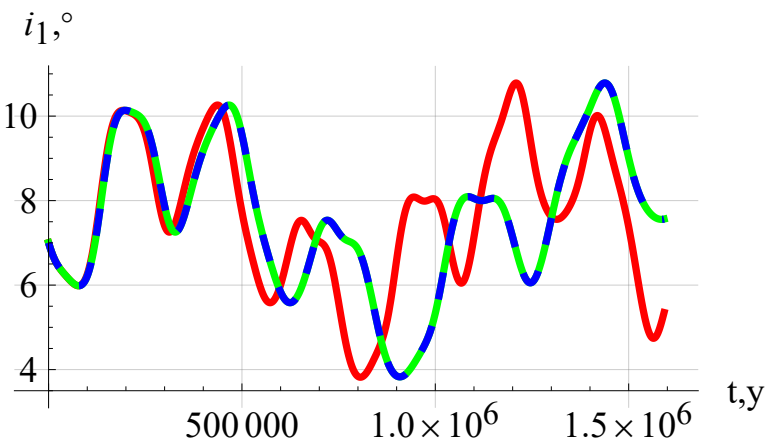
Time t will be measured in the dimensionless units $t^* = n_3 t$, where $n_3 = \sqrt{G(m_{00} + m_{30})/a_3^3}$. The distances will be measured in the dimensional units $r^* = r/a_3$. Dimensionless mass: $m_{10}^* = m_{10}/m_{00}$. The mass of the Sun decreases according to the Eddington-Jeans law:

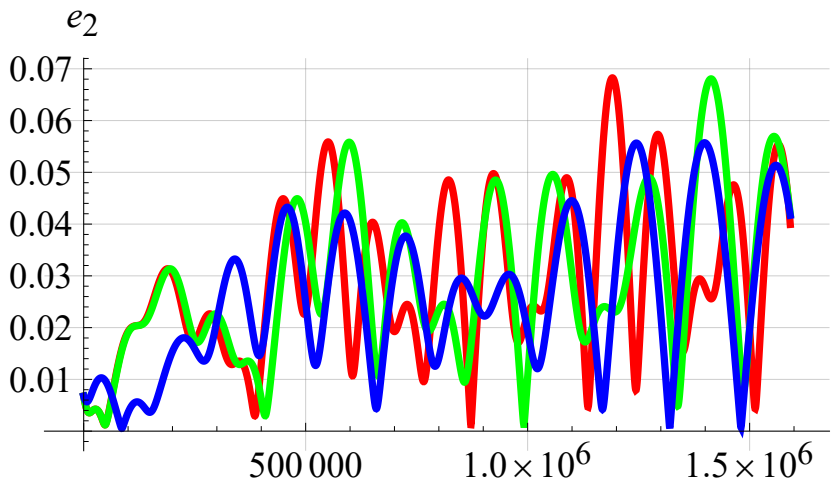
$$\frac{dm_0^*}{dt} = -\beta_0 m_0^{*n_0};$$

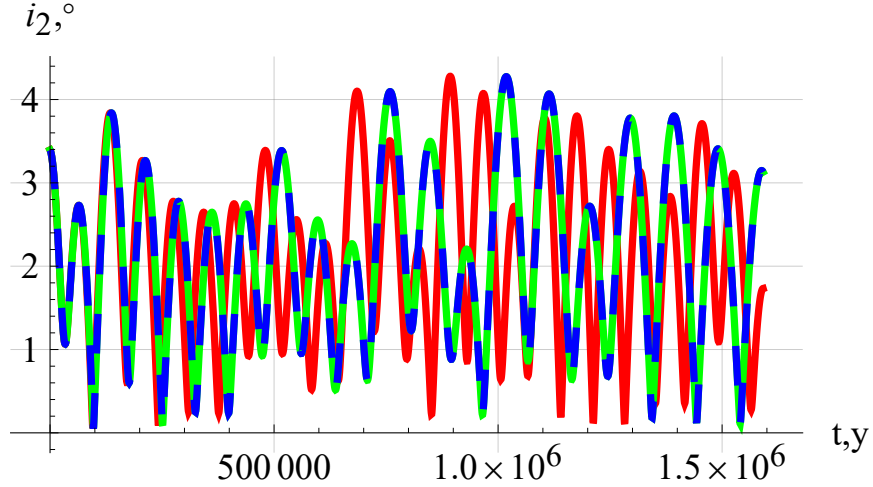
where $n_0 = 3$, $\beta_0 = 5 \cdot 10^{-8}$. Dimensionless relative velocity of the particles leaving the Sun is $V_0^* = 10$.

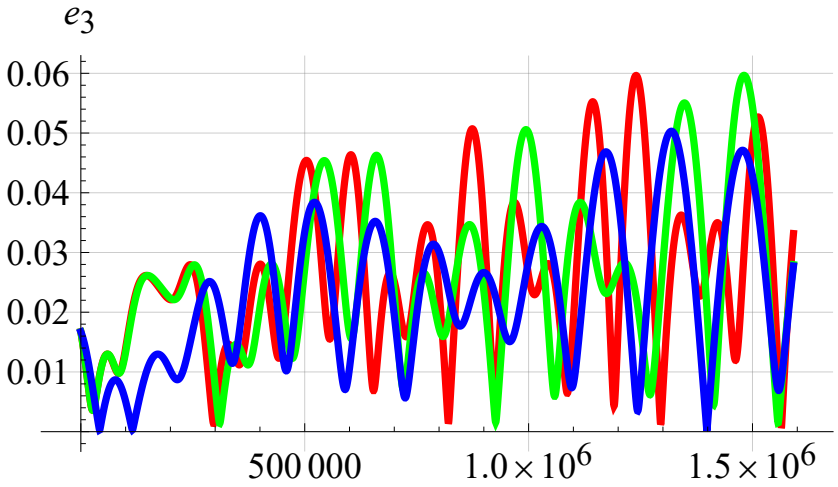
Notations: constant masses - red curve, isotropic mass change - green curve, anisotropic mass change - blue curve.

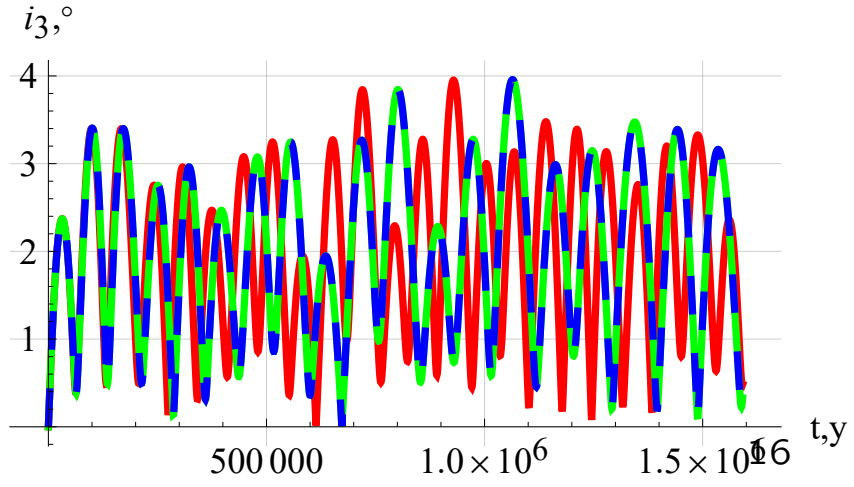

e_1
0.21
0.20
0.19
0.18
0.17
500 000
1.0×10^6
1.5×10^6
t,y

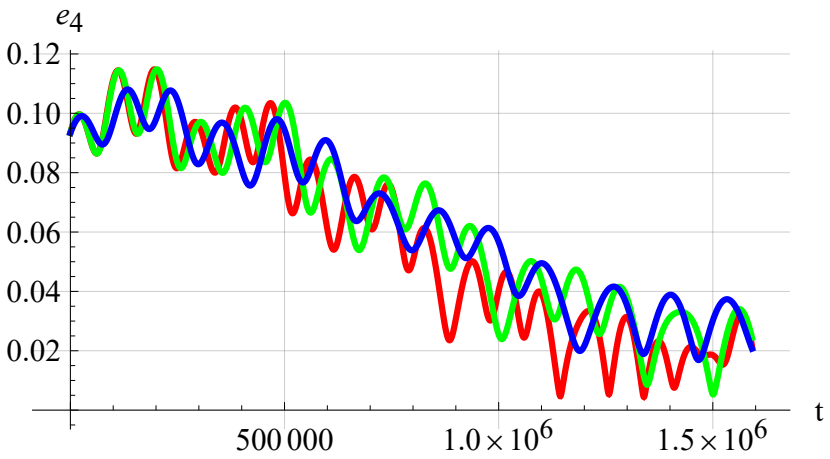

i_1,°
10
8
6
4
500 000
1.0×10^6
1.5×10^6
t,y

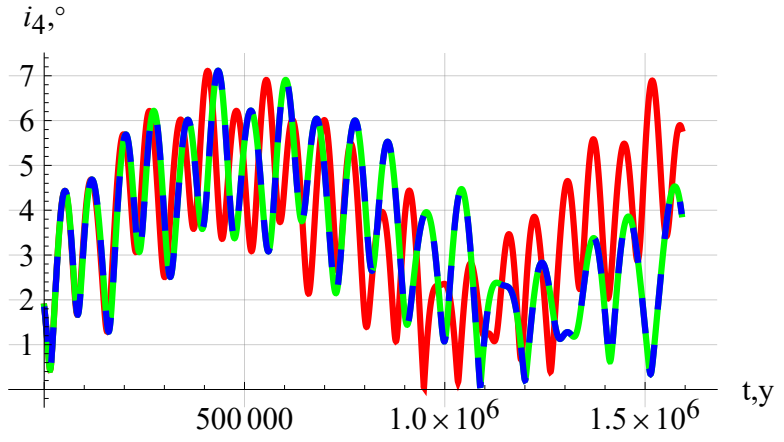

e_2
0.07
0.06
0.05
0.04
0.03
0.02
0.01
500 000
1.0×10^6
1.5×10^6
t,y

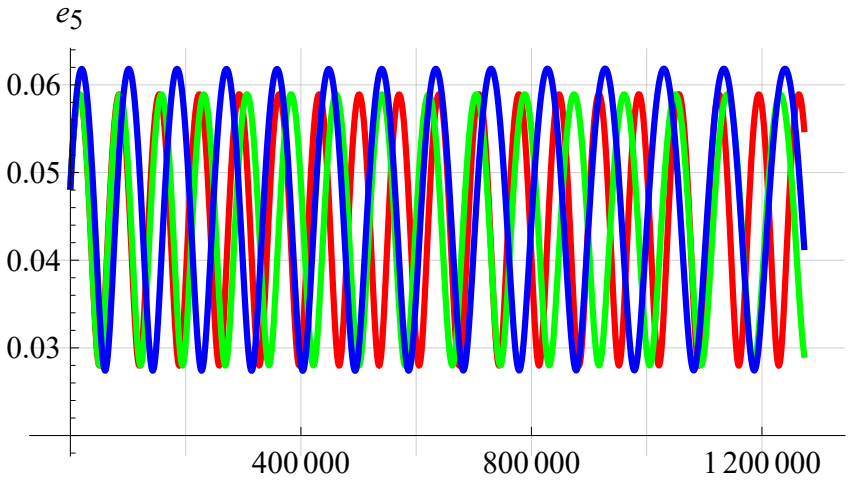

i_2,°
4
3
2
1
500 000
1.0×10^6
1.5×10^6
t,y

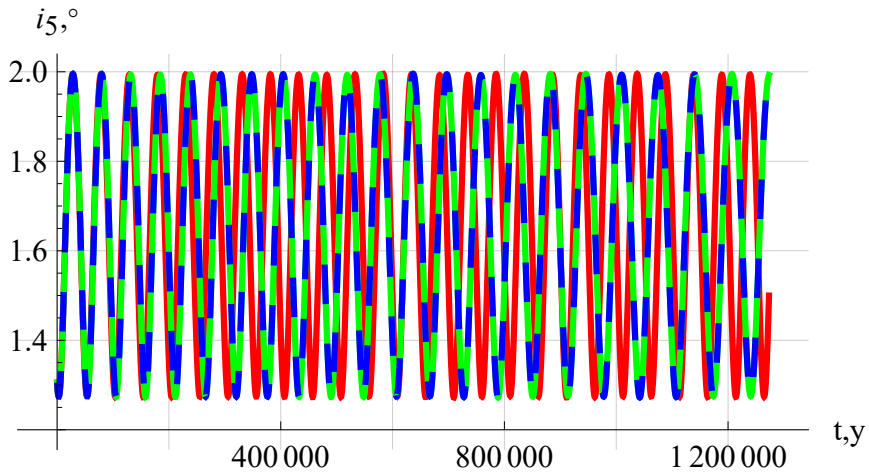

e_3
0.06
0.05
0.04
0.03
0.02
0.01
500 000
1.0×10^6
1.5×10^6
t

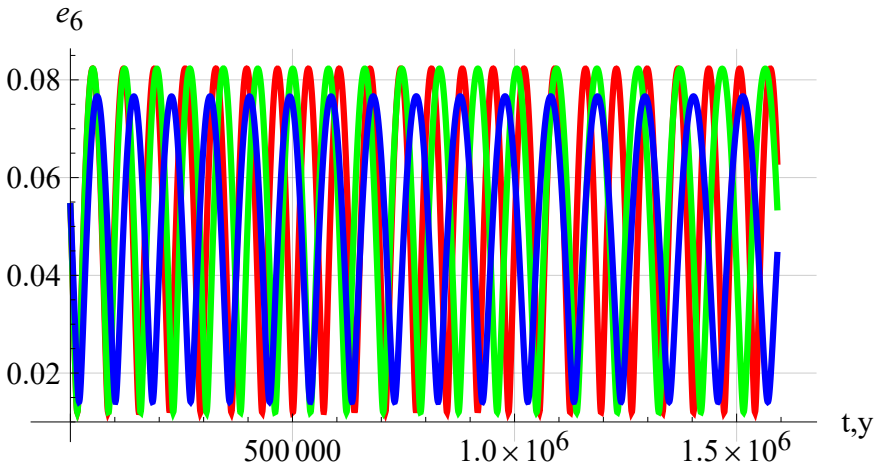

i_3,°
4
3
2
1
500 000
1.0×10^6
1.5×10^6
t,y

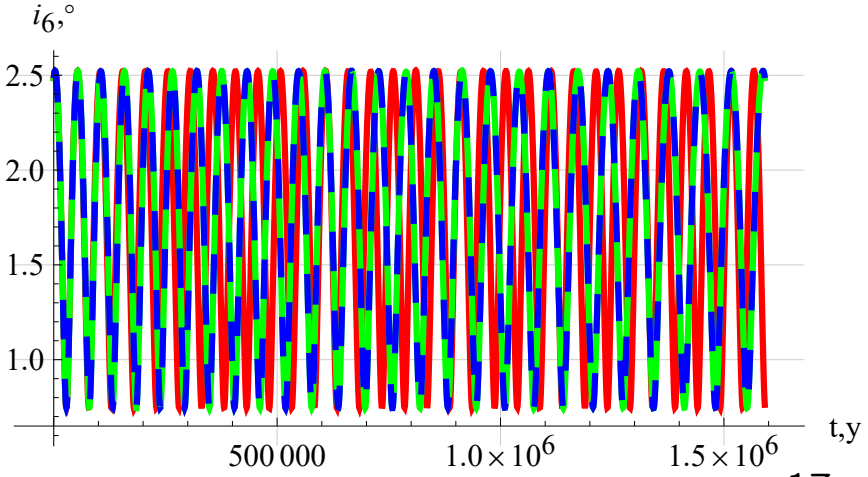

e_4
0.12
0.10
0.08
0.06
0.04
0.02
500 000
1.0×10^6
1.5×10^6
t

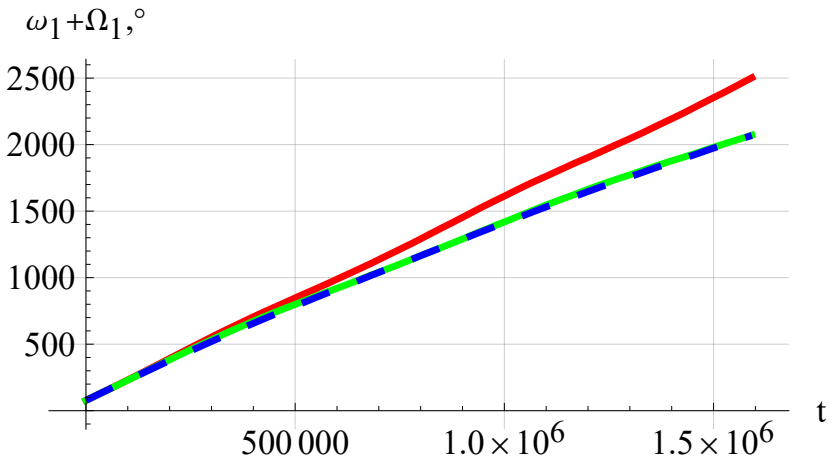

i_4,°
7
6
5
4
3
2
1
500 000
1.0×10^6
1.5×10^6
t,y

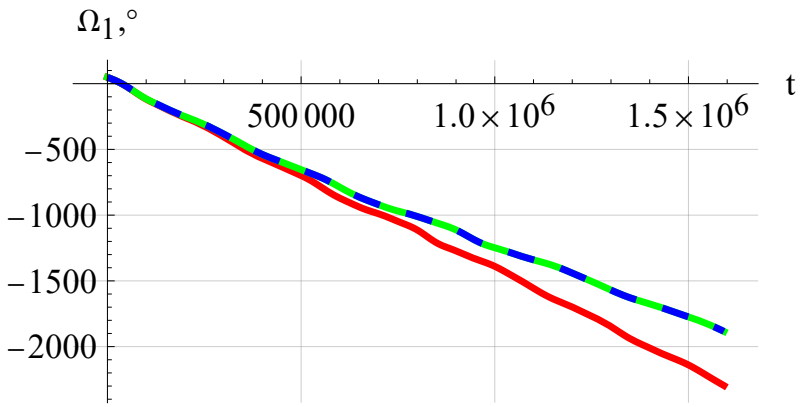

e_5
0.06
0.05
0.04
0.03
400 000
800 000
1 200 000
t,y

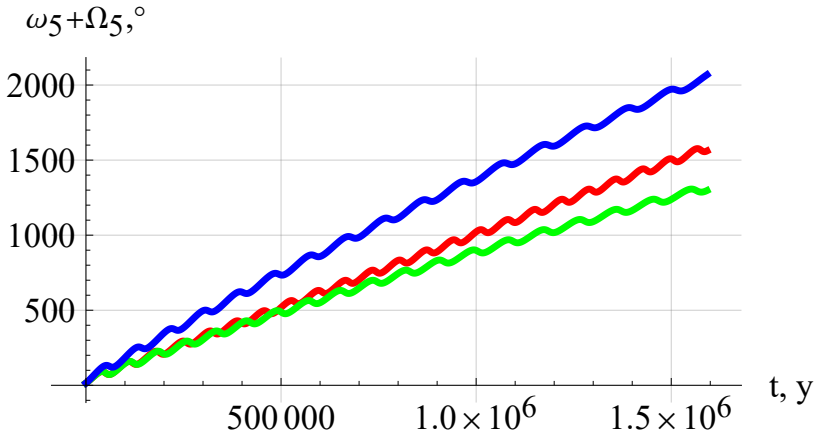

i_5,°
2.0
1.8
1.6
1.4
400 000
800 000
1 200 000
t,y

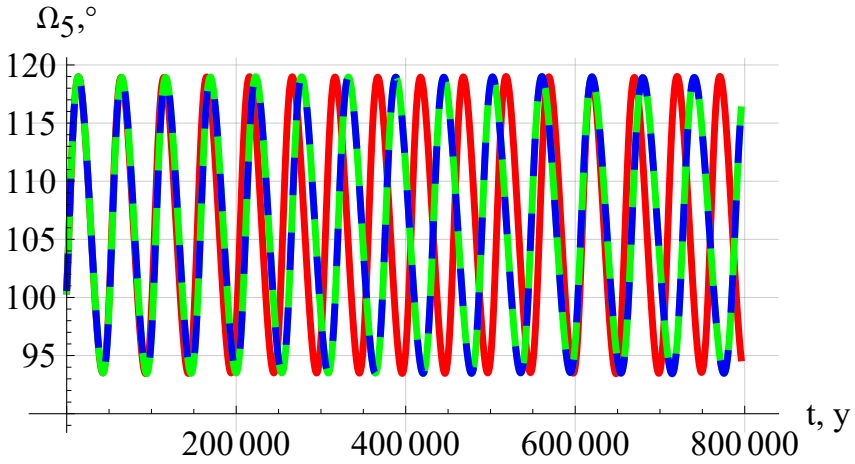

e_6
0.08
0.06
0.04
0.02
500 000
1.0×10^6
1.5×10^6
t,y

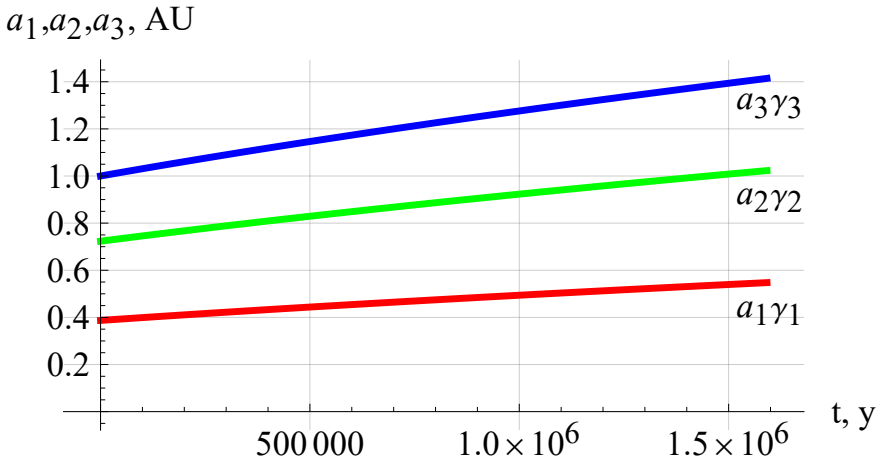

i_6,°
2.5
2.0
1.5
1.0
500 000
1.0×10^6
1.5×10^6
t,y

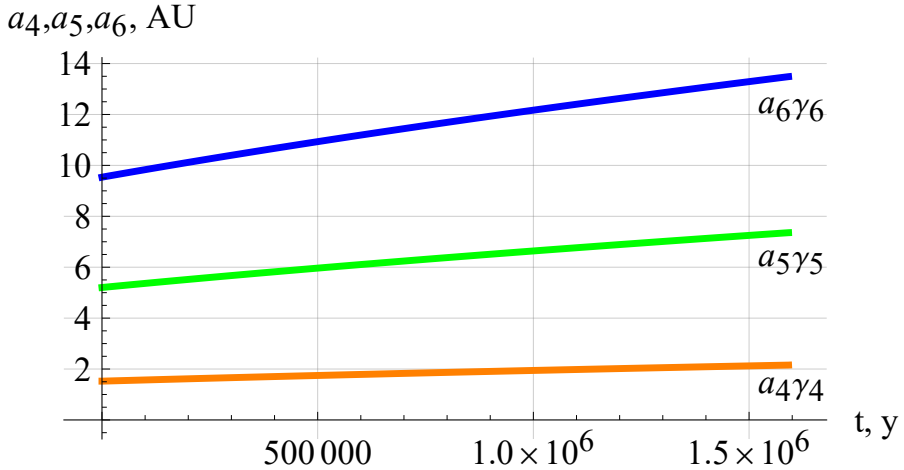

$\omega_1+\Omega_1$,°
2500
2000
1500
1000
500
500 000
1.0×10^6
1.5×10^6
t


Ω_1,°
t
500 000
1.0×10^6
1.5×10^6
−500
−1000
−1500
−2000

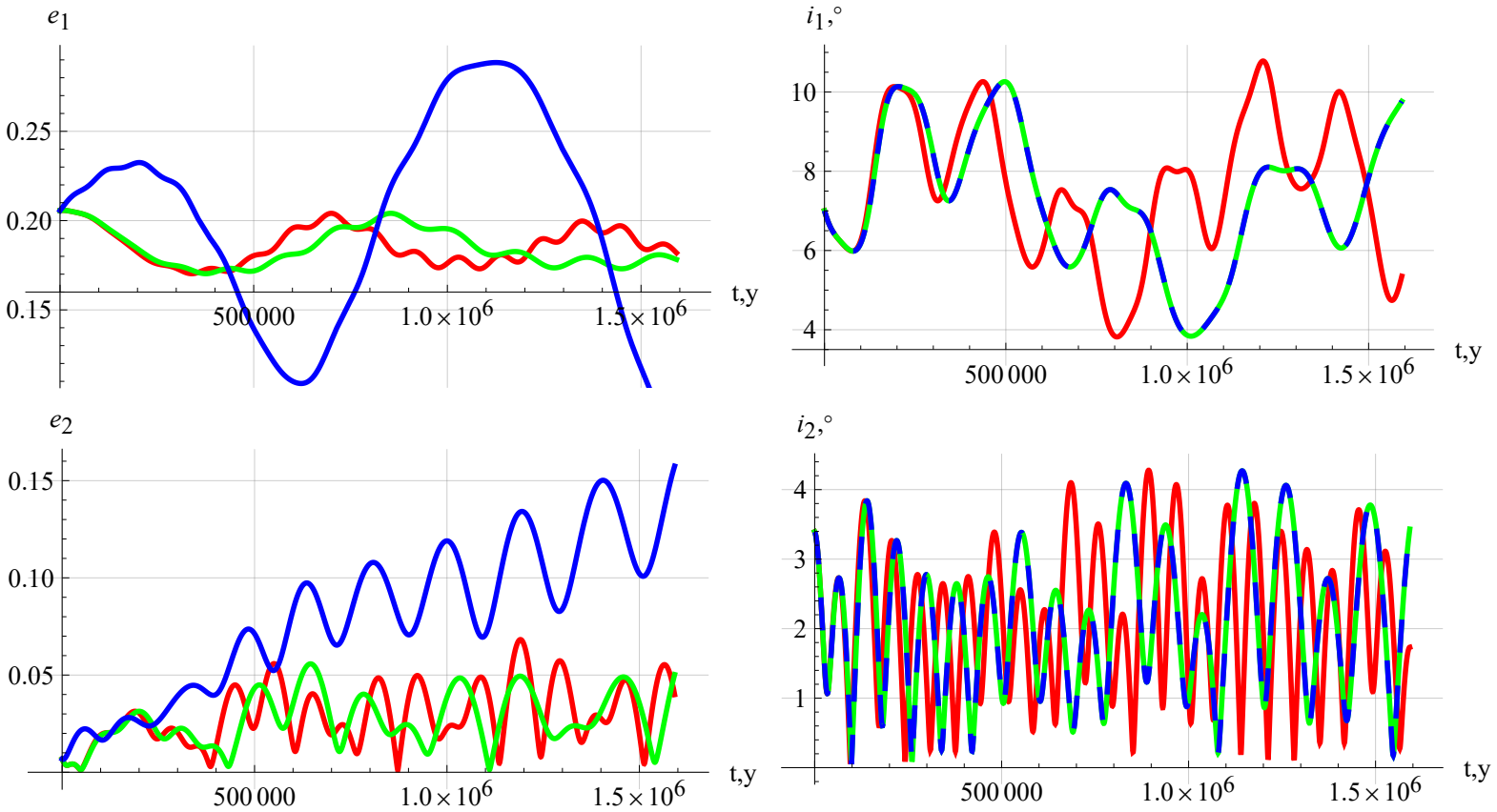

$\omega_5+\Omega_5$,°
2000
1500
1000
500
500 000
1.0×10^6
1.5×10^6
t, y

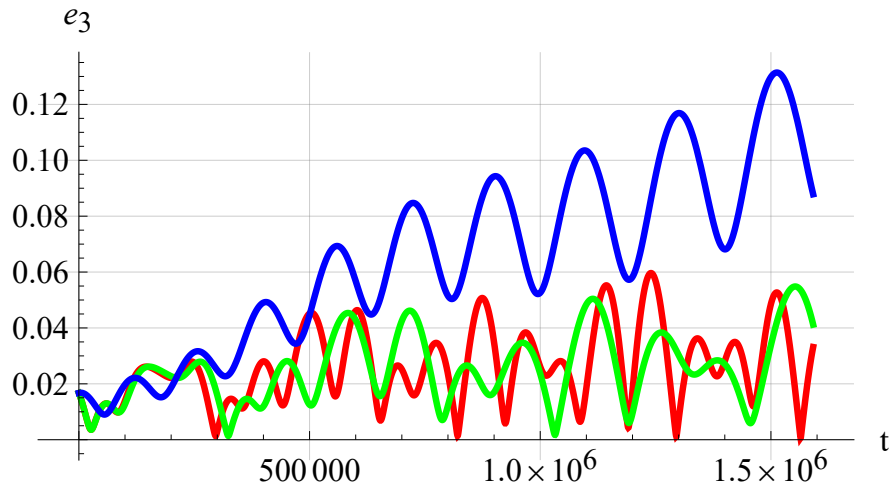

Ω_5,°
120
115
110
105
100
95
200 000
400 000
600 000
800 000
t, y

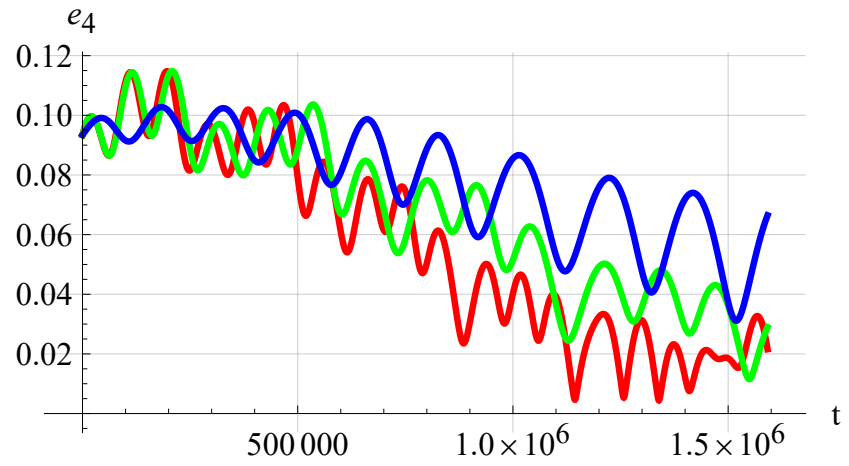

a_1,a_2,a_3, AU
1.4
1.2
1.0
0.8
0.6
0.4
0.2
$a_3\gamma_3$
$a_2\gamma_2$
$a_1\gamma_1$
500 000
1.0×10^6
1.5×10^6
t, y

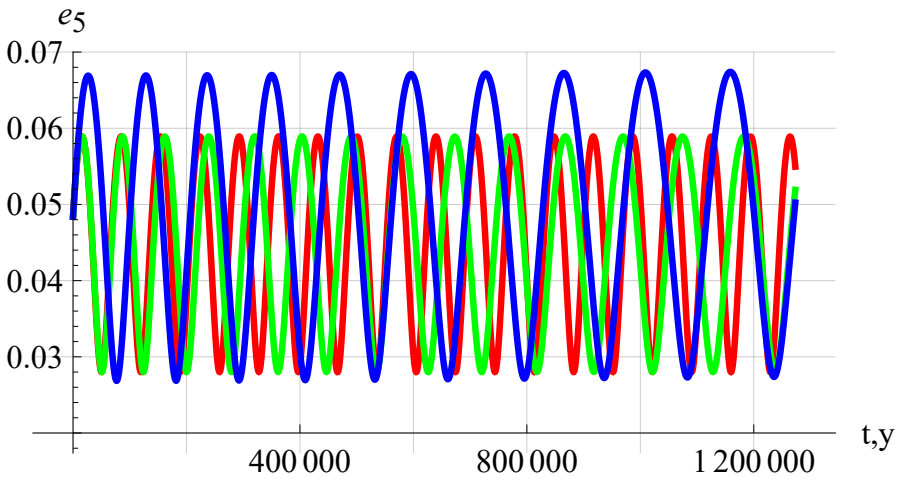

a_4,a_5,a_6, AU
14
12
10
8
6
4
2
$a_6\gamma_6$
$a_5\gamma_5$
$a_4\gamma_4$
500 000
1.0×10^6
1.5×10^6
t, y

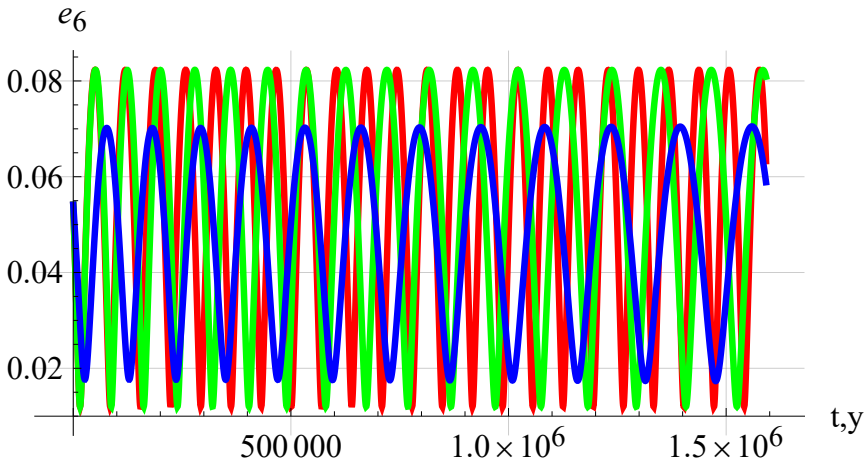
An influence of the mass loss on the orbital elements increases if the rate of the mass loss becomes two times greater: $\beta_0 = 10^{-7}$.




e_3
0.12
0.10
0.08
0.06
0.04
0.02
500 000
1.0×10^6
1.5×10^6
t


e_4
0.12
0.10
0.08
0.06
0.04
0.02
500 000
1.0×10^6
1.5×10^6
t


e_5
0.07
0.06
0.05
0.04
0.03
400 000
800 000
1 200 000
t,y


e_6
0.08
0.06
0.04
0.02
500 000
1.0×10^6
1.5×10^6
t,y

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