Marchal's family: Inclined co-orbitals in the three-body problem

Alexandre Prieur

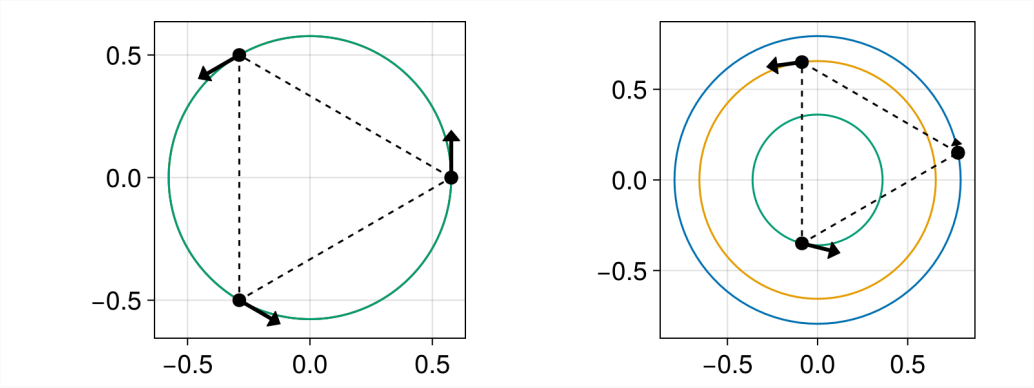
With Philippe Robutel & Jacques Fejoz

14/09/2026



Introduction

The Lagrange relative equilibrium



Lagrange relative equilibrium of the 3BP

Elliptic if

$$\frac{m_0 * m_1 + m_0 * m_2 + m_1 * m_2}{(m_0 + m_1 + m_2)^2} > \frac{1}{27}$$

$$\begin{cases} (1-\varepsilon, \varepsilon, 0) \rightarrow \varepsilon_G \approx 0.0385 \\ (1-2\varepsilon, \varepsilon, \varepsilon) \rightarrow \varepsilon_G \approx 0.0191 \end{cases}$$

The Lagrange relative equilibrium – Lyapunov families

Lyapunov families emerge from the elliptic directions of Lagrange

3 planar Lyapunov families, one orthogonal to the orbital plane : the vertical family

The vertical family – equal masses

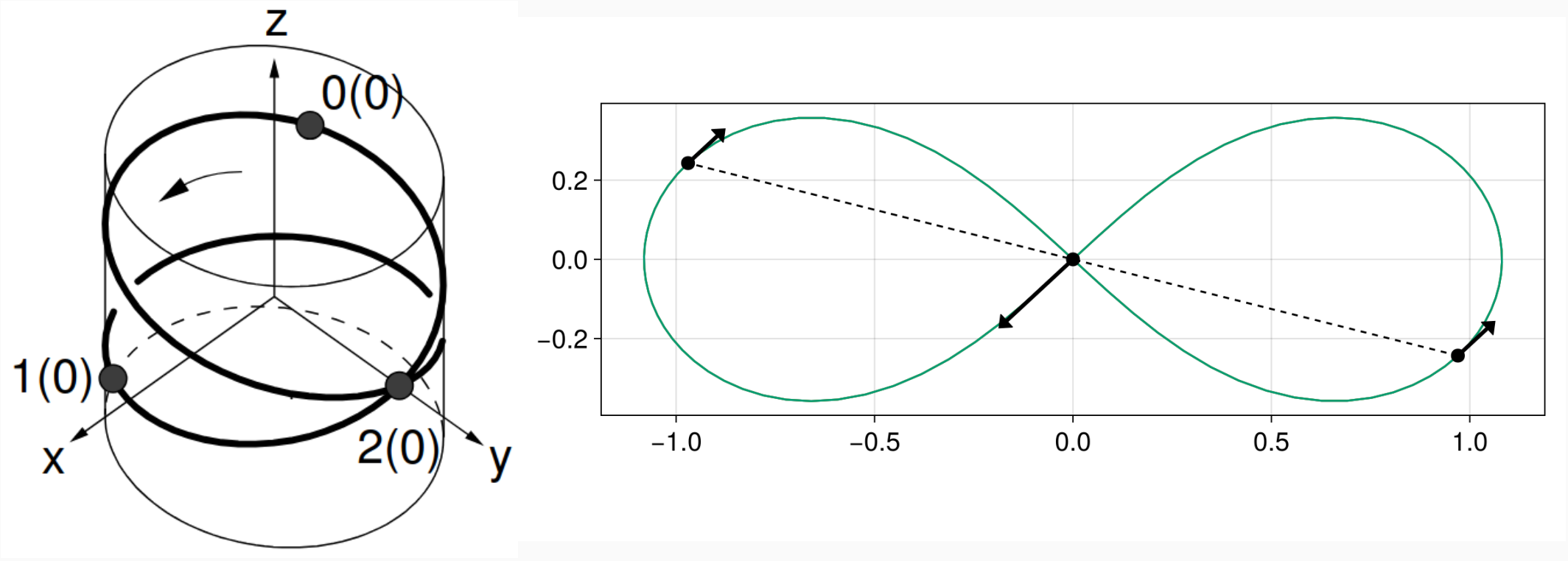
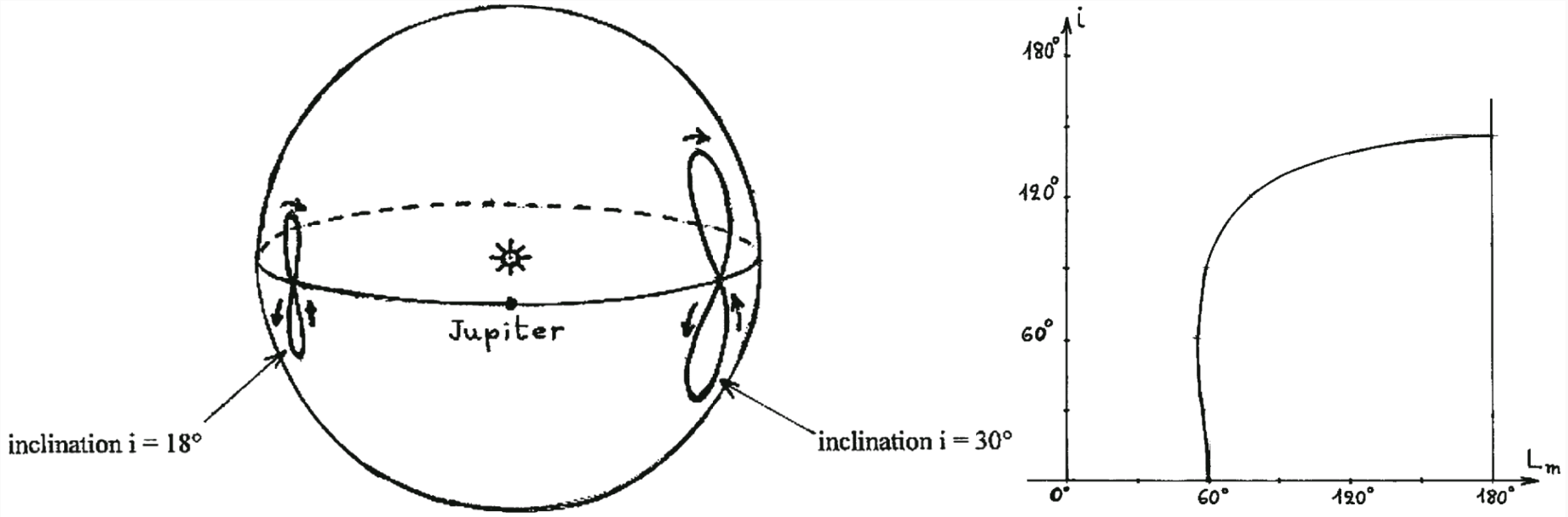


Figure 2: Left: (Chenciner and Fejoz, 2008, fig 3), an orbit of P_{12} in a rotating frame. Right: the Eight.

Link conjectured by Marchal in 2000, proved in (Calleja et al., 2024)

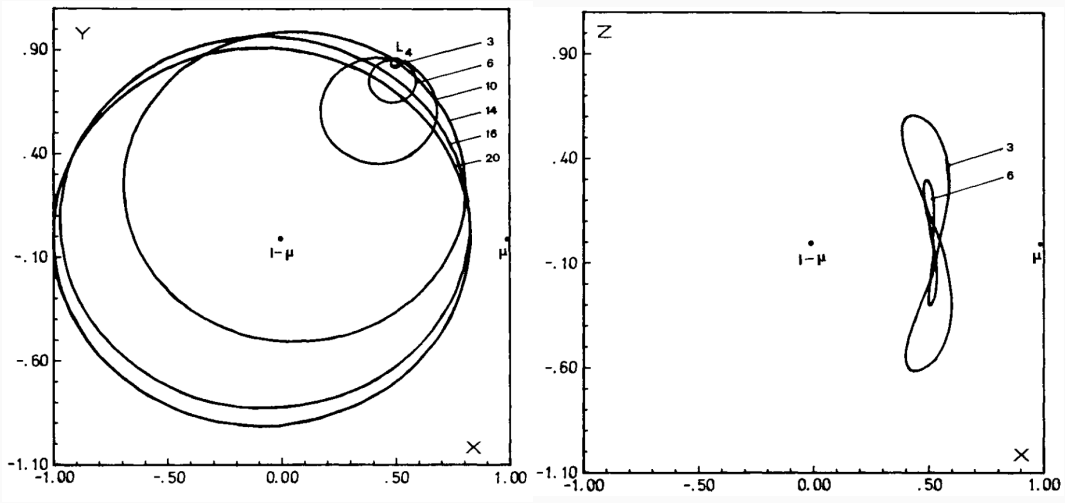
The vertical family – small masses



- (Marchal, 2009) – $\mathcal{VF}_L$ in the average restricted problem
  - Linearly stable in the directions he considers
- (Leleu, 2016) – numerical hints of a stable region in the full planetary problem

A historical note: forgetting previous work

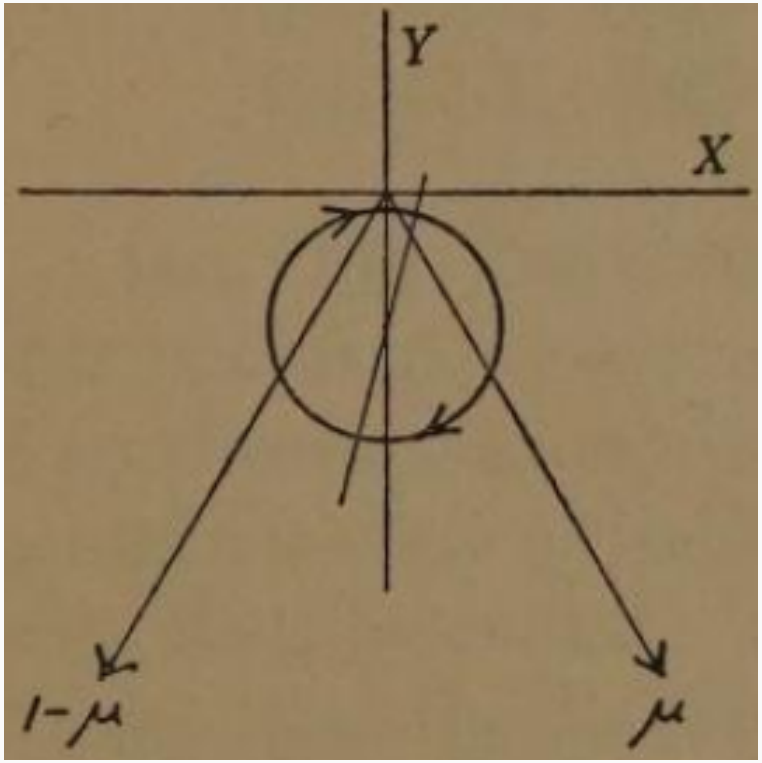
(Marchal, 2009) results are in (Zagouras, 1985)

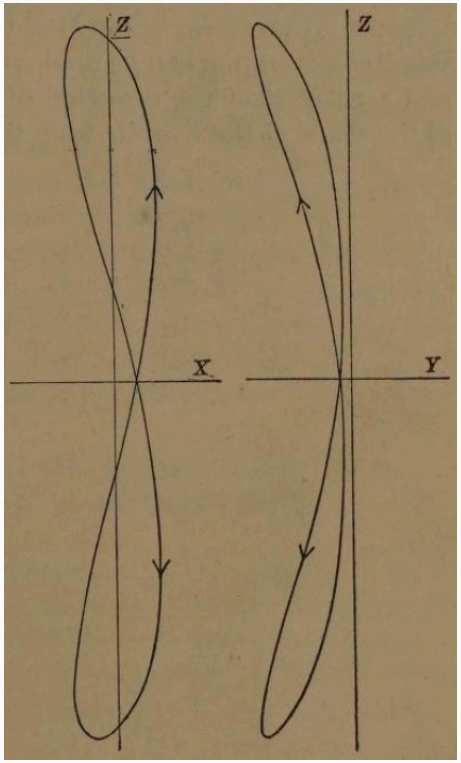


But (Zagouras, 1985)'s method is described in "Zagouras, C.G.: 1982, Post doctoral dissertation, University of Patras, Greece."

A historical note: forgetting previous work

... And (Zagouras, 1985) is an extension of (Moulton, 1920), chap IX (by Buck)!





Marchal's generalized conjecture

These periodic orbits belong to a three parameter family of similar periodic orbits of the general three body problem, two parameters being the mass ratios of the three bodies and the third a generalization of the inclination i. [...] The one parameter family P_{12} of (Marchal, 2001) is another limit case of this three parameter family: the case of three equal masses.

— (Marchal, 2009)

Questions

Q1: Does the family persist in the non-restricted/non-average problem?
Q2: What is the stability of the orbits along this family?
Q3: Does it have an impact on the global stability of the co-orbital region?
Q4: What is the link between the family and P_{12} ?

$\mathcal{VF}_L$ in the average circular planetary problem

Frame of reference

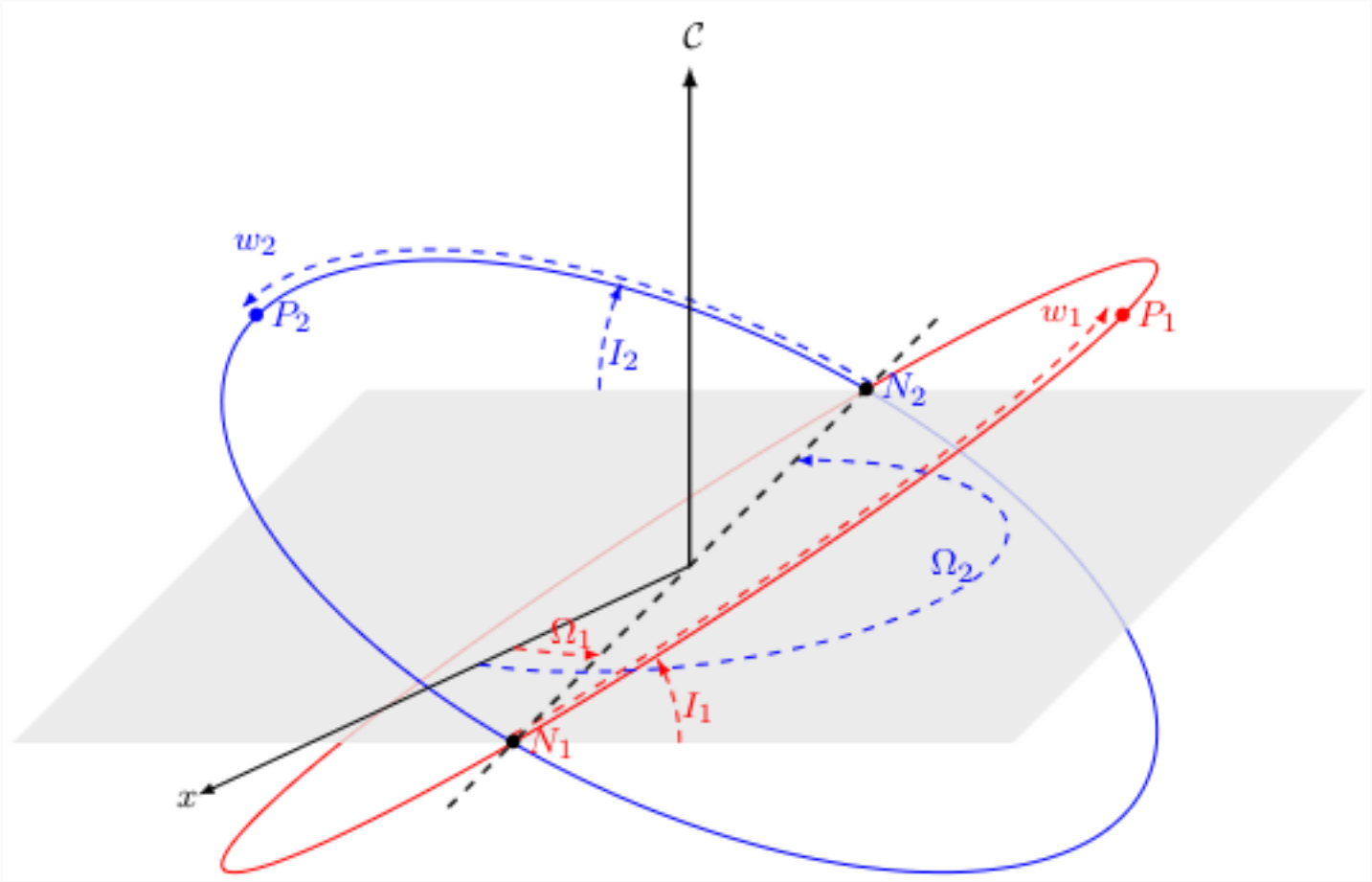


Figure 6: Canonical heliocentric : (Robutel, 2015) heliocentric positions, barycentric momenta; $(Oxy) \perp \boldsymbol{C}$

Co-orbital motion in the planetary case

Complex Poincaré-like coordinates

Planetary problem

$$\left(1 - \varepsilon\left(m_1 + m_2\right), \varepsilon m_1, \varepsilon m_2\right), \varepsilon \ll 1$$

ε-neighborhood of co-orbital motions

$$a_1, a_2 = a^\star + \mathcal{O}(\varepsilon)$$

Averaging over fast angle ζ_2

The submanifold of eccentricity 0 is invariant under the flow: circular problem

A one-parameter family: $\mathcal{VF}_L$

$$\bar{F}(Z_1, \zeta_1, C) = \alpha Z_1^2 + \varepsilon\beta F_1(\zeta_1, C)$$

For a fixed C_0, one-parameter family of equilibrium points satisfying

$$Z_1 = 0, \quad \zeta_1 = \zeta(C_0), \quad \frac{\partial F_1}{\partial \zeta_1}(\zeta(C_0), C_0) = 0$$

F_1 independant of masses! $\Rightarrow$ Study valid for all small masses

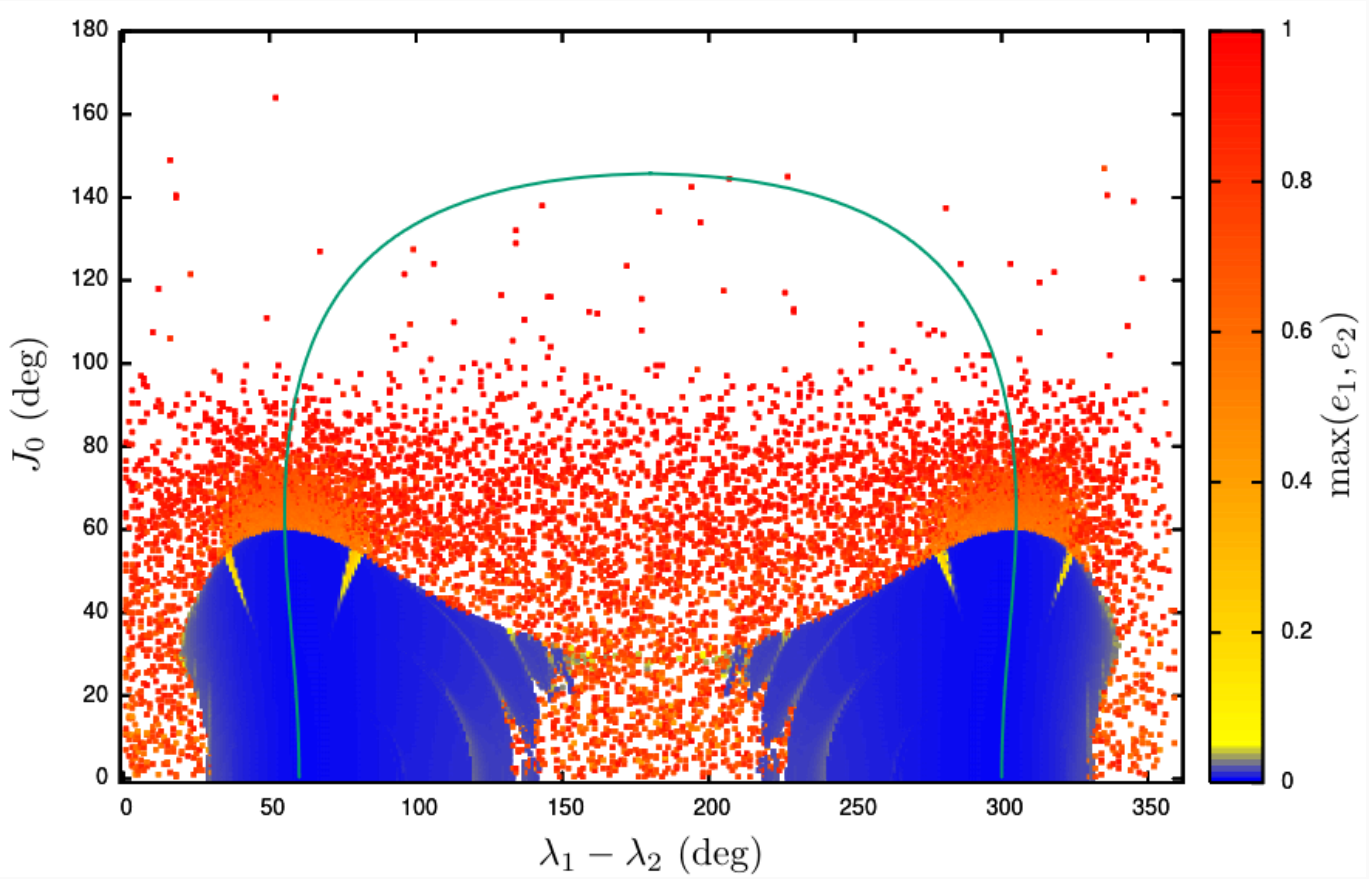
Next ?

- Analytical approximation
- Or numerical search

What can we say?

Analytical: 20^{th} degree expansion in $\sin\left(\frac{J_{\max}}{2}\right)$

Stability: $\mathcal{VF}_L$ is stable in the average **circular** planetary problem



Numerical search in the full problem

A new set of coordinates

- Numerical integration: need coordinates with explicit expression of the Hamiltonian/vector field
- The Jacobi reduction should be explicit in these coordinates

Hill coordinates

$$\begin{cases} R_j & r_j \\ G_j & w_j \\ G_j \cos\left(I_j\right) & \Omega_j \end{cases}$$

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How to search for periodic orbits?

Looking for zeros of

$$f : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^n \\ (\boldsymbol{x}; C) & \mapsto \Phi_T\left(\boldsymbol{x}; C\right) - \mathrm{x} \end{cases}$$

But the zeros aren't locally unique!

Solution: add sections

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But the zeros aren't locally unique!

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How to search for periodic orbits?

Looking for zeros of

$$f : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^{n+m} \\ (\boldsymbol{x}; C) & \mapsto \begin{pmatrix} \Phi_T(\boldsymbol{x};C) - \boldsymbol{x} \\ \sigma(\boldsymbol{x};C) \end{pmatrix} \end{cases}$$

But the zeros aren't locally unique!

Solution: add sections

How to search for periodic orbits?

In our case:

$$\begin{cases} \sigma_1(\boldsymbol{x}; C) = w_1 + w_2 \\ \sigma_2(\boldsymbol{x}; C) = \mathrm{J} - J_0 \end{cases}$$ Selecting a point on the orbit

Focusing on the case of two equal masses: $(1 - 2\varepsilon, \varepsilon, \varepsilon)$

Rootfinding algorithm: multi-dimensional Newton-Raphson method

Numerical integration with `taylor` software (Jorba and Zou, 2005) via `TaylorInterface.jl` (in julia (Bezanson et al., 2017))

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Small masses: $\varepsilon = 10^{-3}$

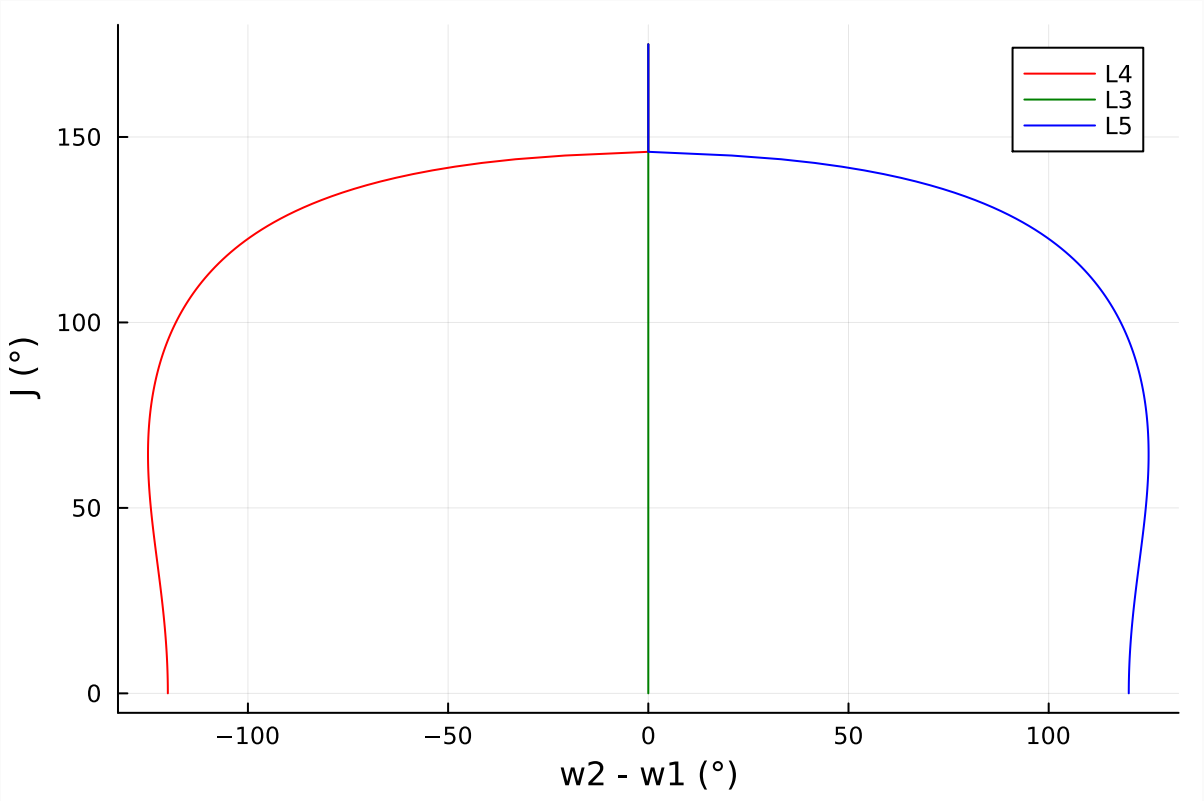
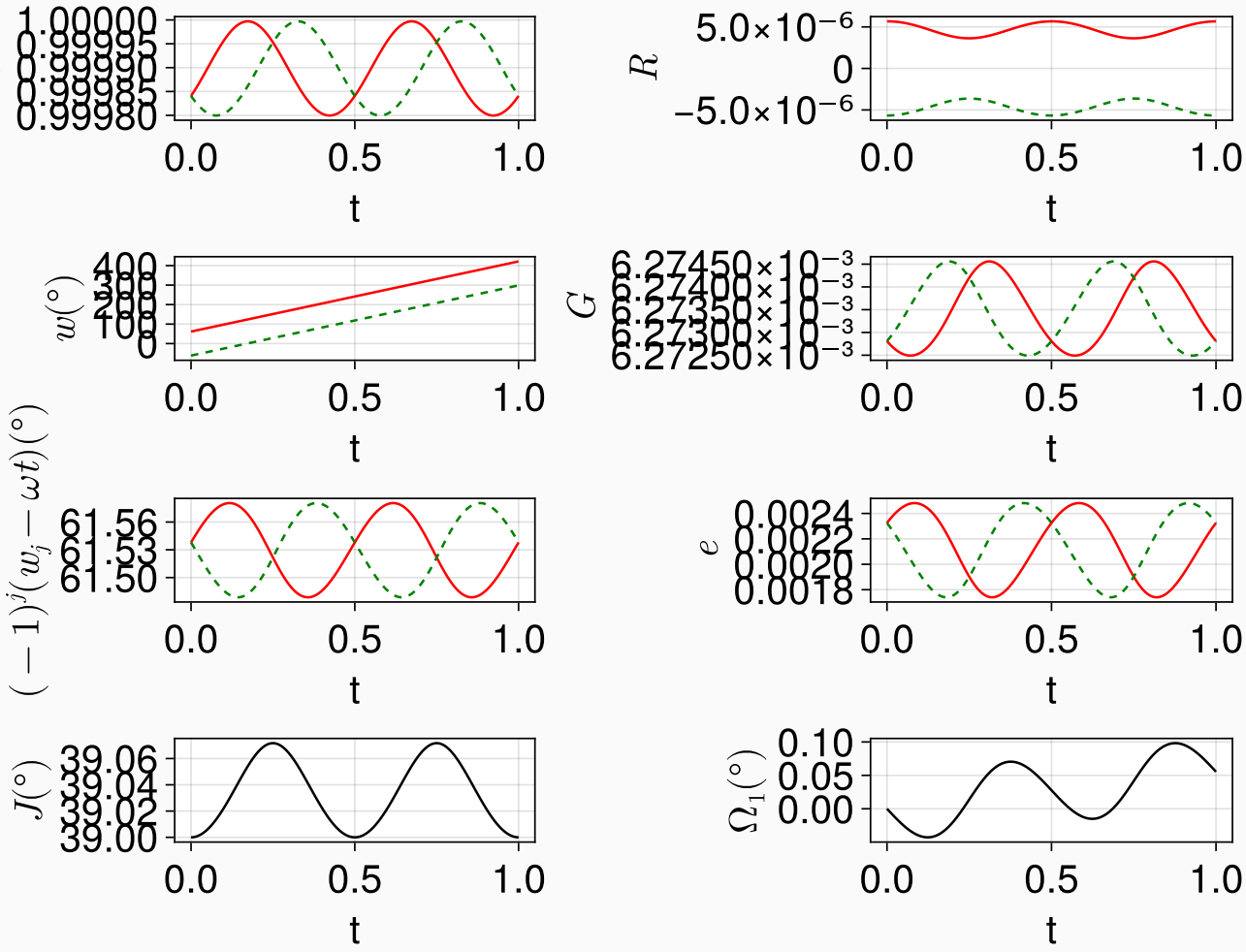


Figure 8: $\mathcal{VF}_L$ from L_4 and L_5, $\mathcal{VF}_E$ from L_3

Coordinates along trajectory at $\varepsilon = 10^{-3}$

Coordinates at $J = 39°$



Trajectory examples

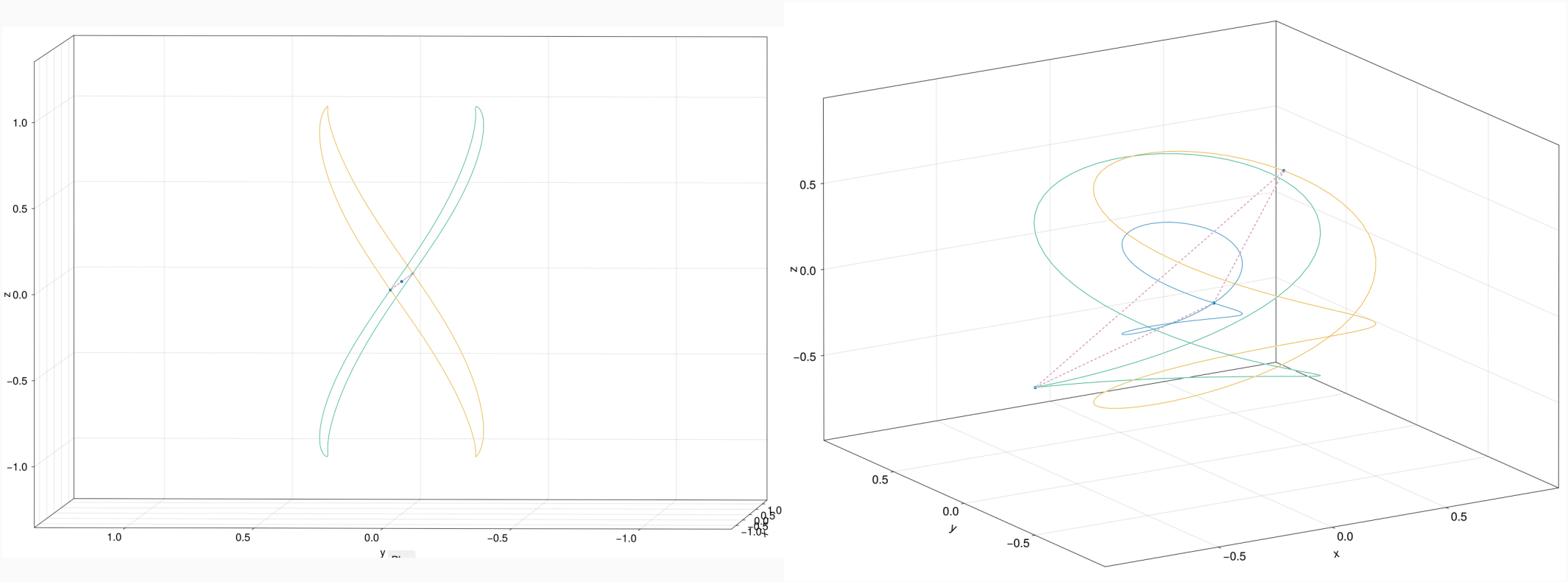


Figure 10: $\mathcal{VF}_L$ orbits in the draconic (left) and uniformly rotating (right) plane, for $\varepsilon = 0.21, J = 106°$.

Trajectory example – $\mathcal{VF}_E$

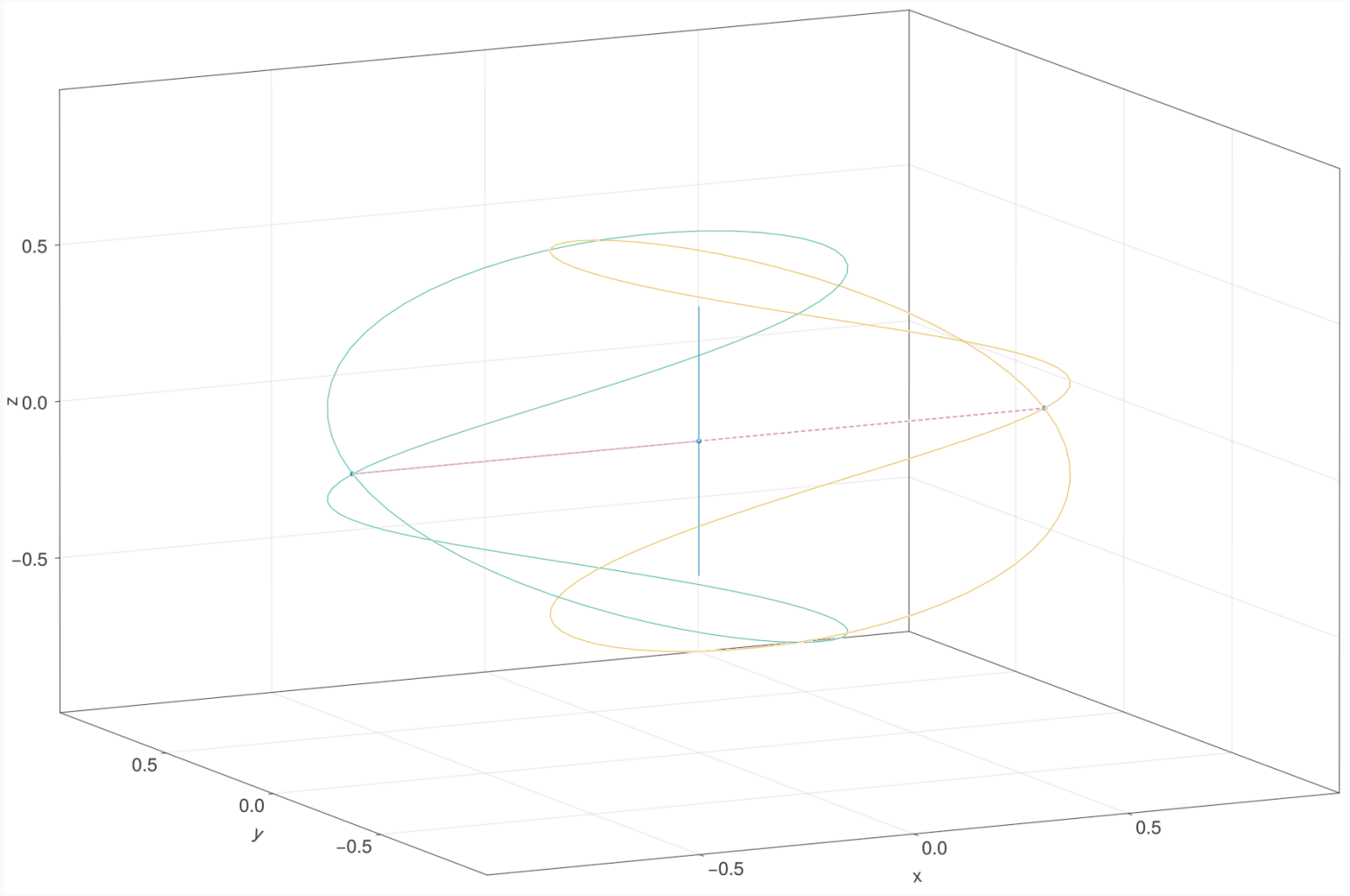
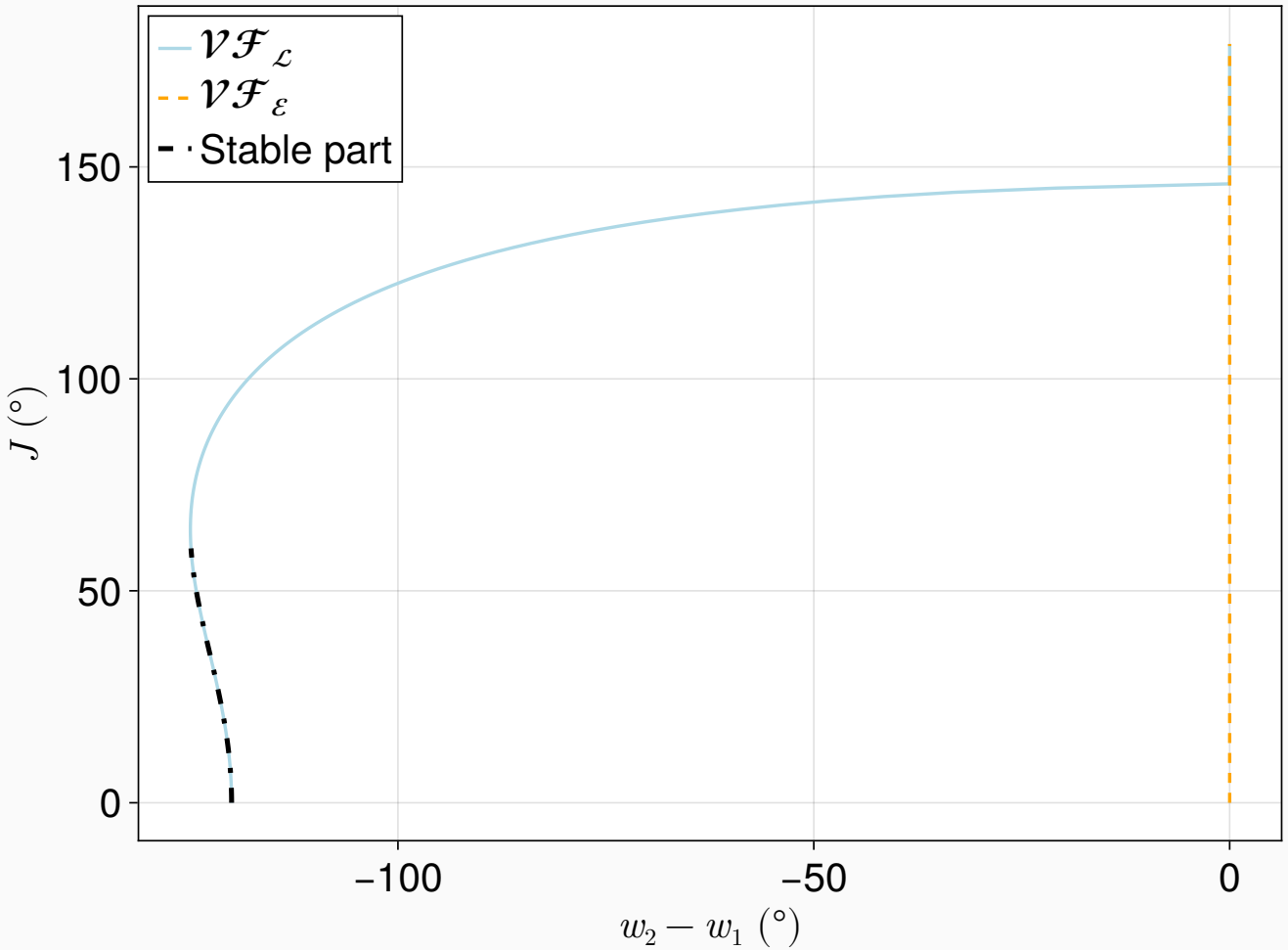


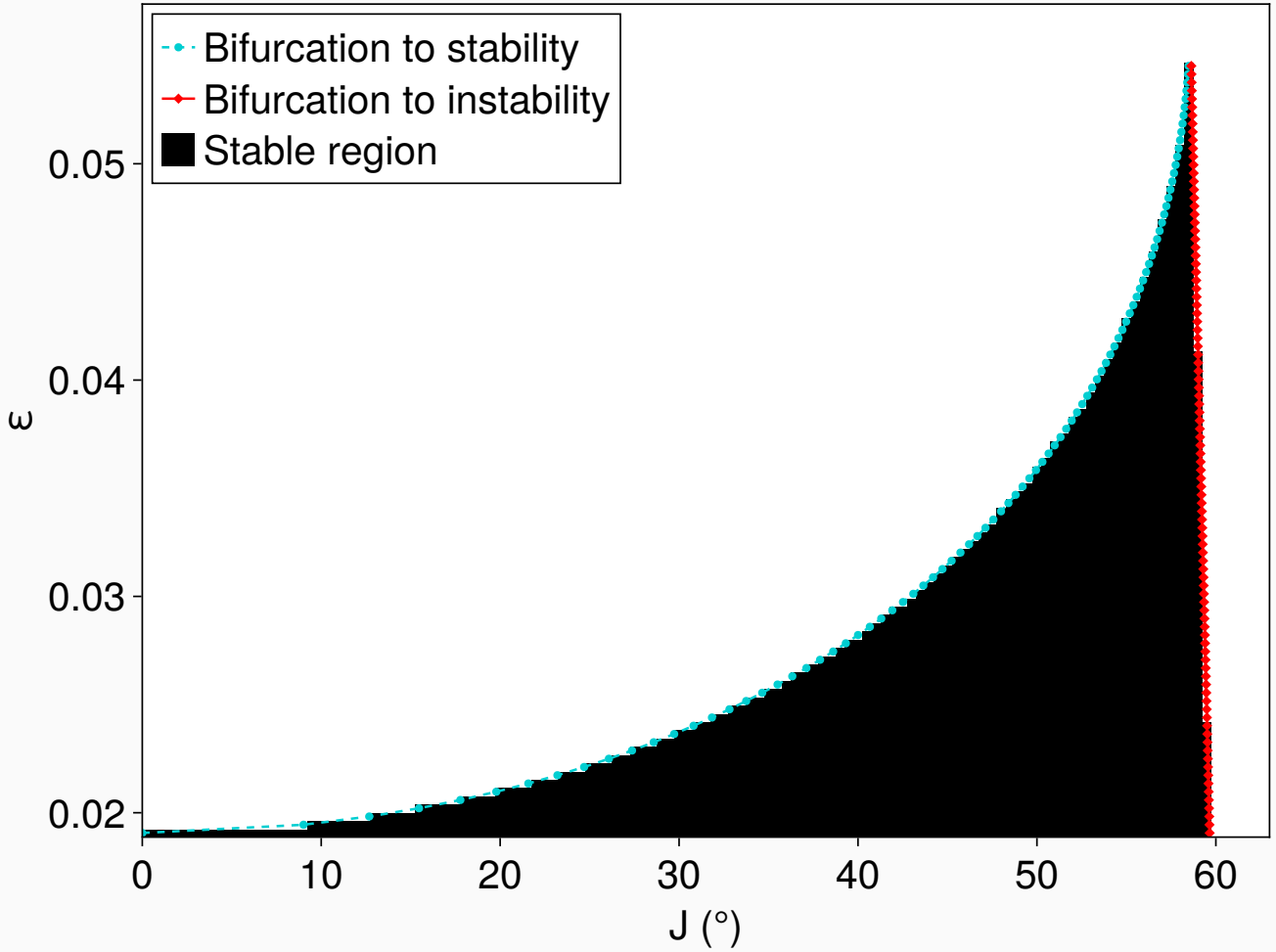
Figure 11: $\mathcal{VF}_E$ orbit at $\varepsilon = 0.21, J = 130°$.

Stability

Stability at $\varepsilon = 10^{-3}$

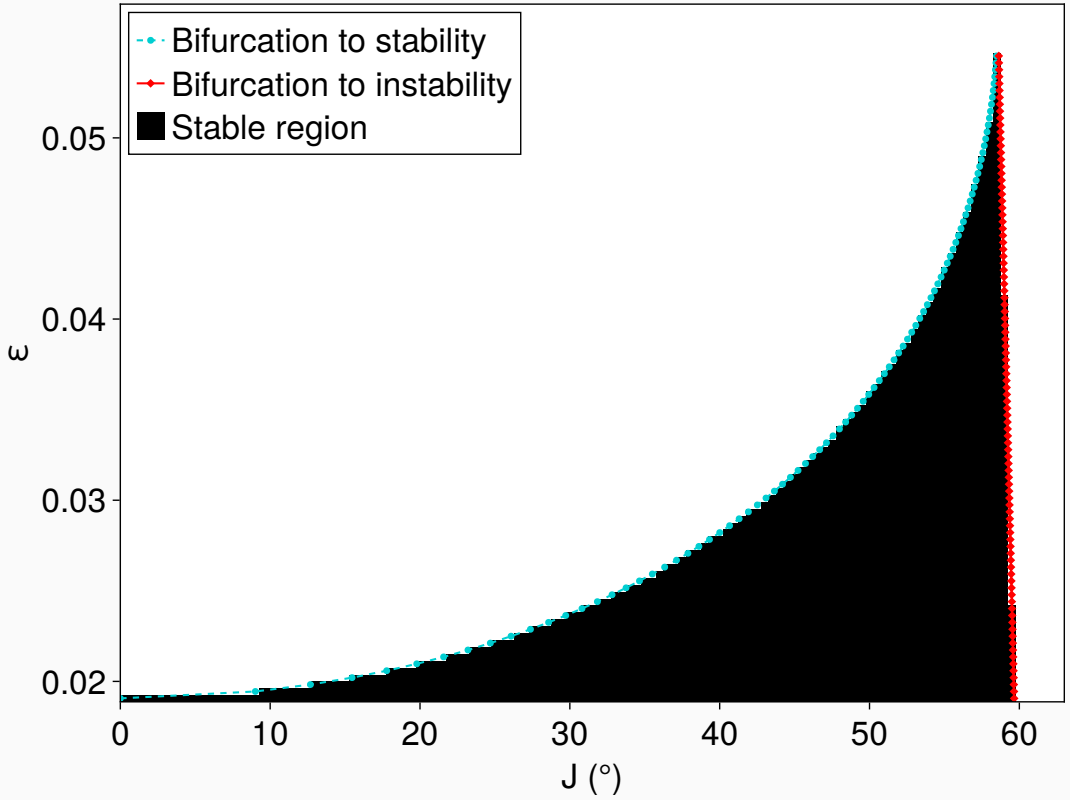


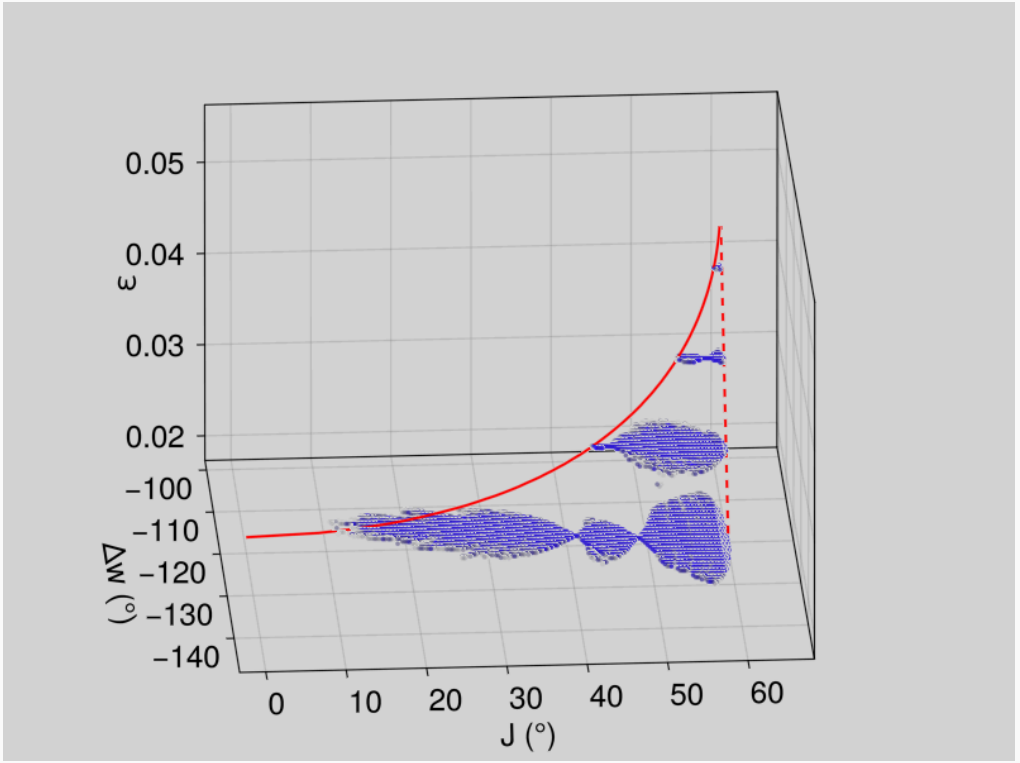
Increasing masses: above Gascheau's value



Similar to (Roberts, 2002)

Above Gascheau's value (long-term stability)





Astronomical consequences?

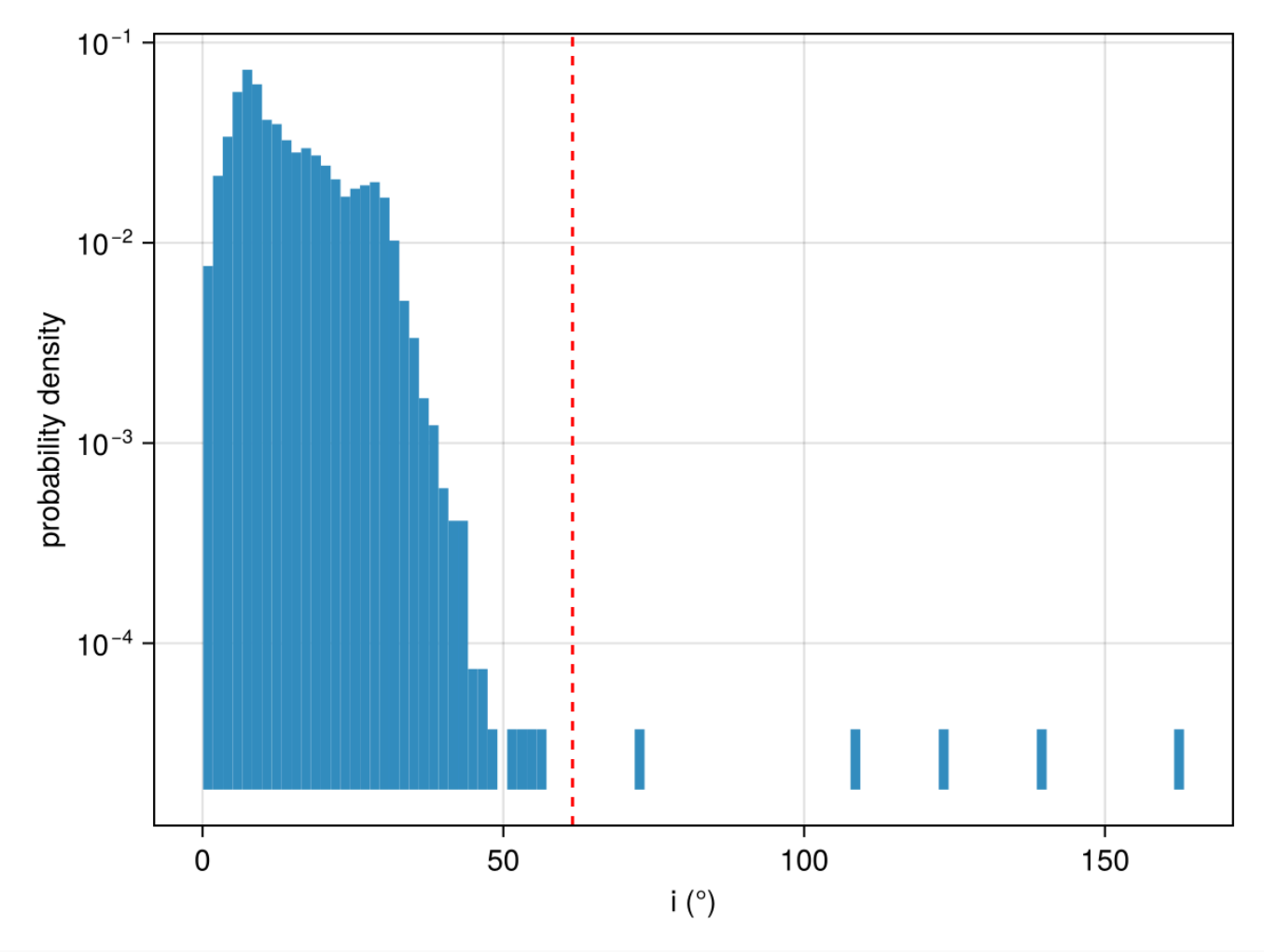


Figure 16: Inclination of Jupiter's Trojans.

Towards equal masses

Symmetries

Half-time symmetry

$$\forall x \in \mathcal{VF}_L, \Phi_{\frac{T}{2}}(\boldsymbol{x}; C) = -\boldsymbol{x}$$

$$\Rightarrow f : (\boldsymbol{x}; C) \mapsto \begin{pmatrix} \Phi_{\frac{T}{2}}(\boldsymbol{x}; C) + \boldsymbol{x} \\ \sigma_1(\boldsymbol{x}; C) \\ \sigma_2(\boldsymbol{x}; C) \end{pmatrix}$$

Cuts integration time in half

Time-reversal symmetry

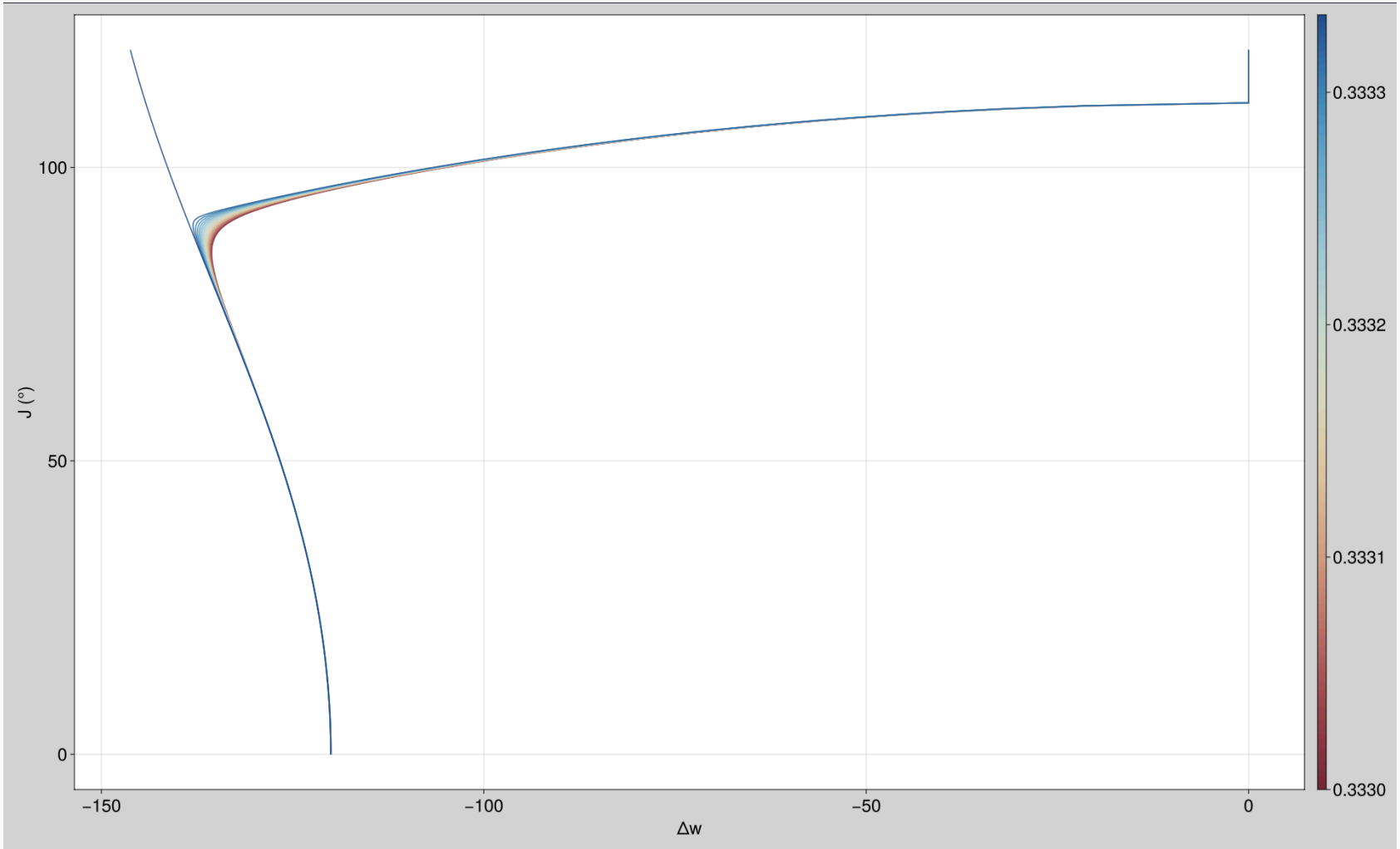
$$\forall x \in \mathcal{VF}_L, t \in \mathbb{R},$$

$$\Phi_{-t}(\boldsymbol{x}; C) = R_y(\pi) p_{1\leftrightarrow 2} \Phi_t(\boldsymbol{x}; C)$$

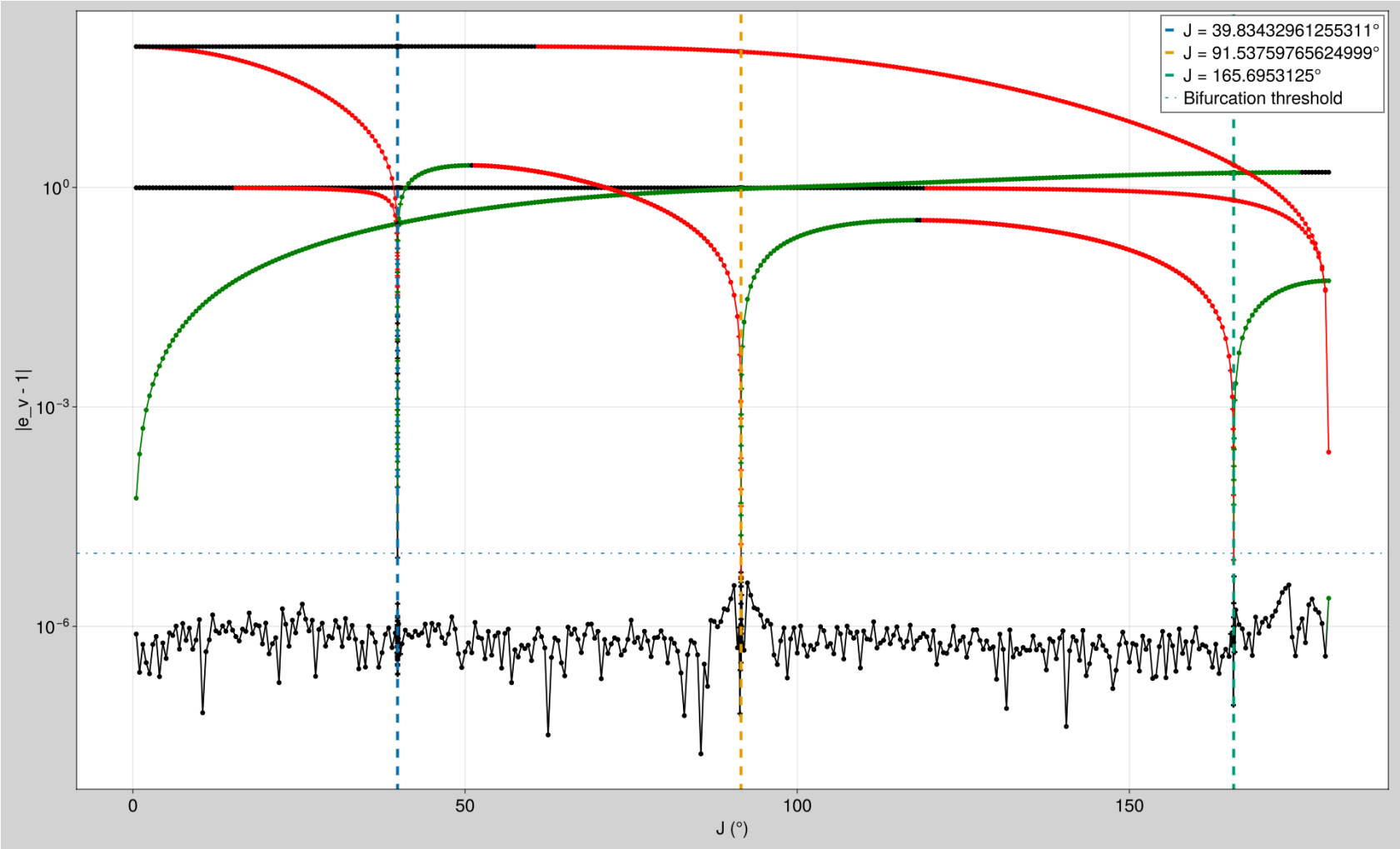
Allows to add sections

$$\sigma_3(\boldsymbol{x}; C) = \begin{pmatrix} r_1 - r_2 \\ R_1 + R_2 \\ G_1 - G_2 \end{pmatrix}$$

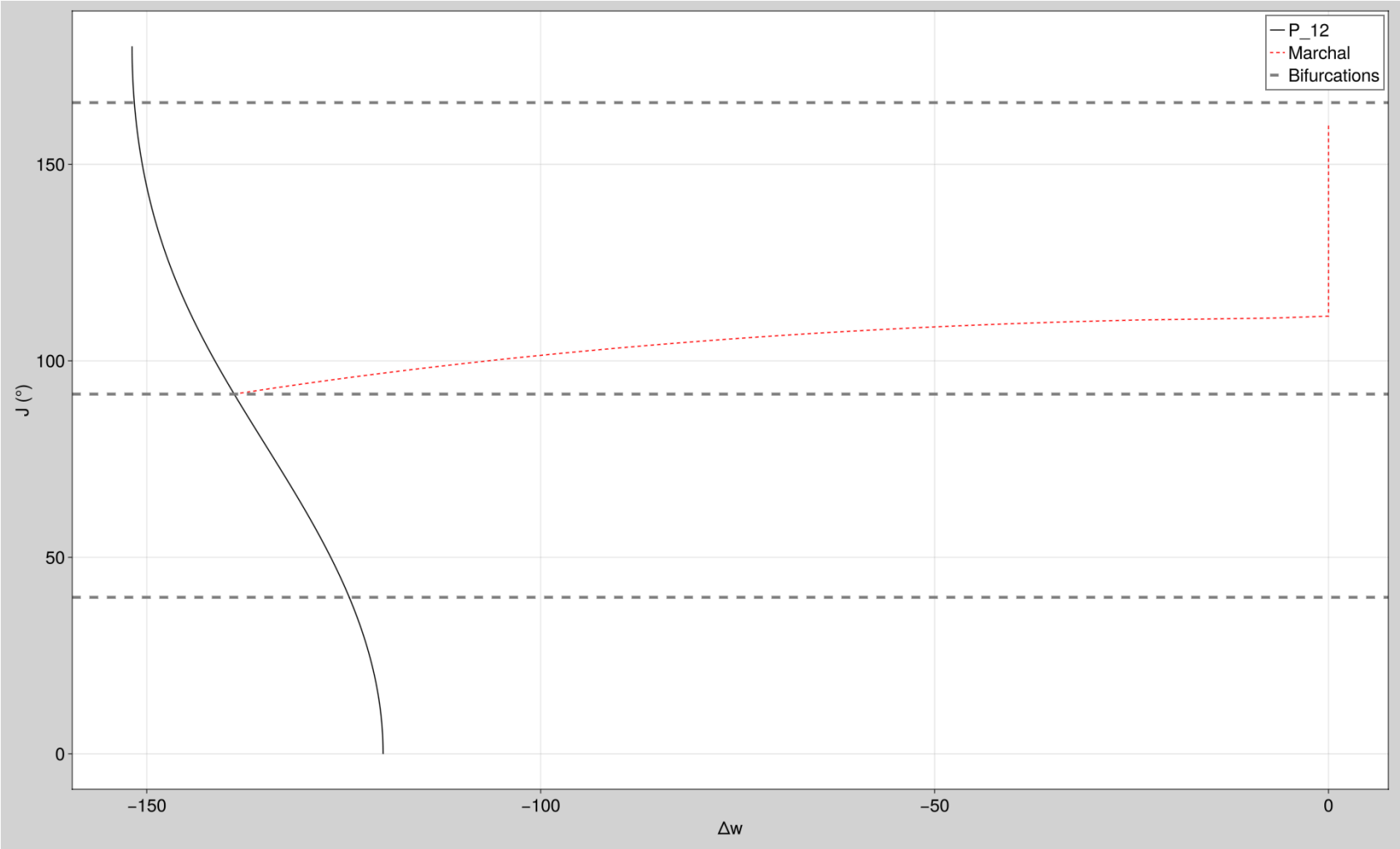
Approaching equal masses



Bifurcations at equal masses



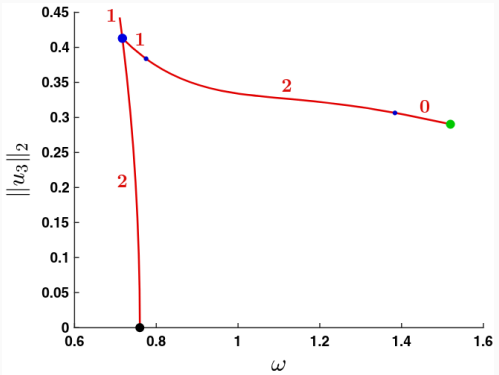
$\mathcal{VF}_L$ and P_{12}: Marchal's family



The other bifurcations



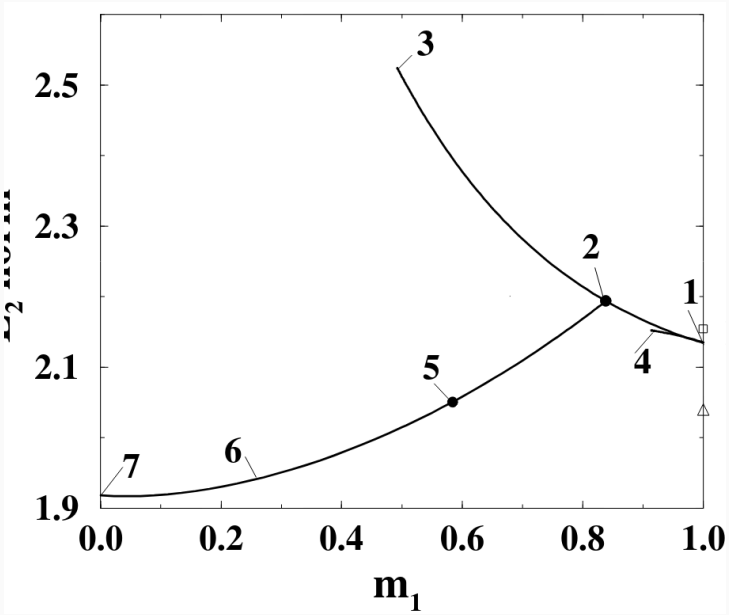
- Third bifurcation unexplored for now
- First bifurcation to a “lazy Eight”
  - Cf. (Calleja et al., 2021), and Hénot, Fejoz & Chenciner

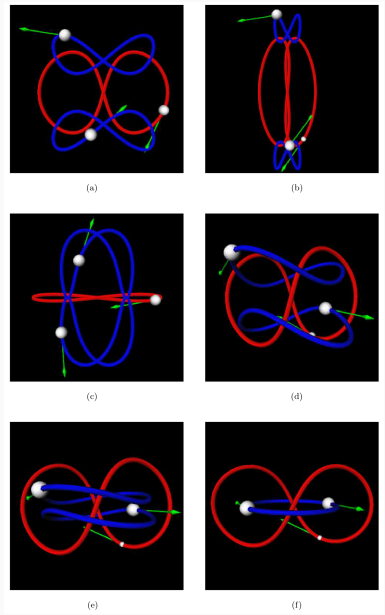


Bifurcations at non-equal masses

These bifurcations also exist at low masses!

P_{12} confirmed at almost-equal masses ($\frac{1}{3} - \varepsilon \approx 10^{-5}$)





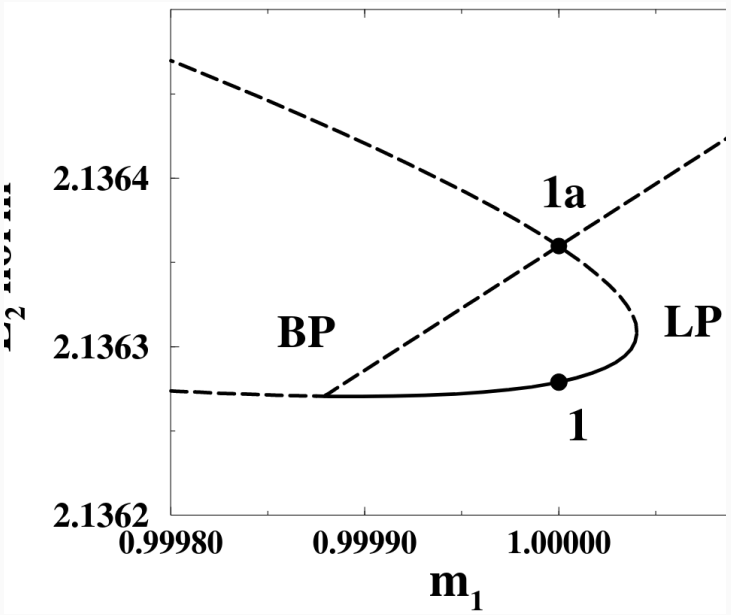


Figure 22: (Doedel et al., 2003)

Conclusion

Further work: bifurcations

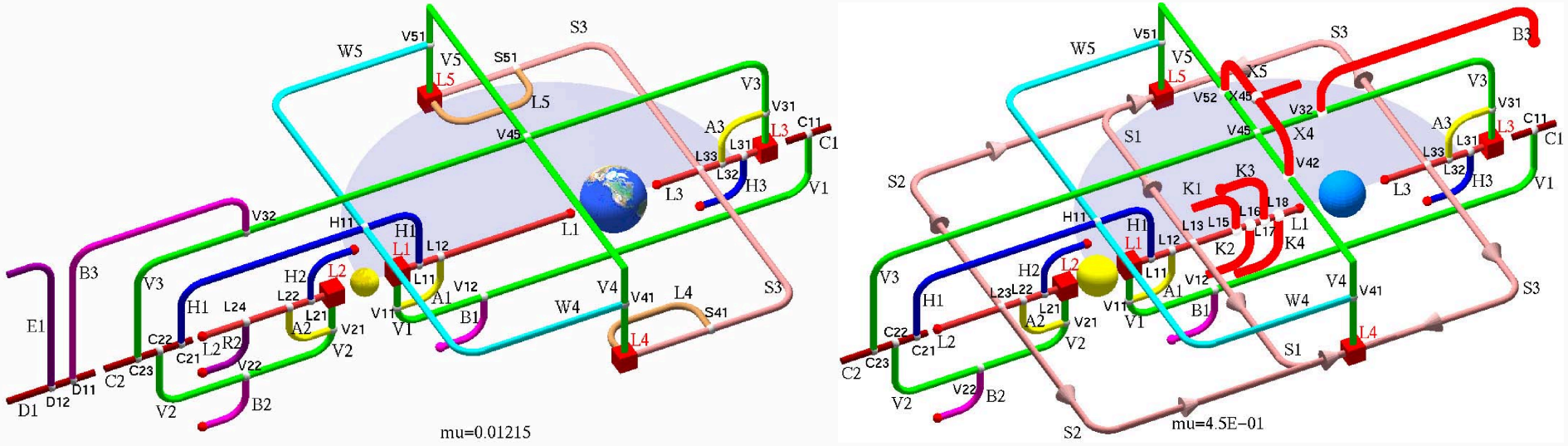


Figure 23: (Doedel et al., 2007, fig 3 and 51).

- Detailed study and following of bifurcations along $\mathcal{VF}_L$ and P_{12}
  - Continue P_{12} at non-equal masses
- Apply `TaylorInterface.jl` and our numerical toolbox to more problems

Conclusion

We showed...

- Persistence of $\mathcal{VF}_L$ in the full problem
- Numerical validation of Marchal's generalized conjecture, up to bifurcations

And we developped an open-source numerical toolbox for investigating recurring dynamics

Much remains to be done!

In memory of Àngel and Christian

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