Effective Stability of Near-Rectilinear Halo Orbits in the Earth-Moon System

Ninth International Meeting on Celestial Mechanics (CELMEC IX)

Luke T. Peterson[1] & Joan Gimeno[2]

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[1] Department of Aerospace Engineering & Engineering Mechanics, UT Austin

[2] Department de Matemàtiques i Informàtica, Universitat de Barcelona

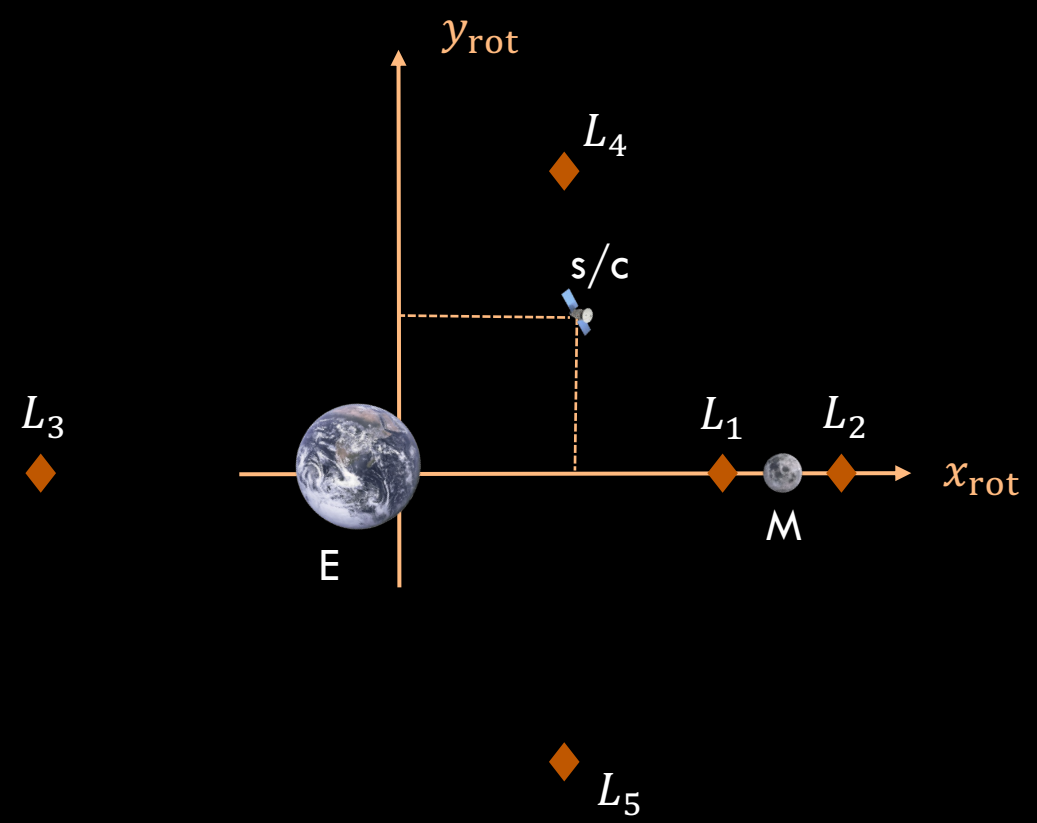


Motivation

Long-term Stability of Periodic Orbits in the Earth-Moon System

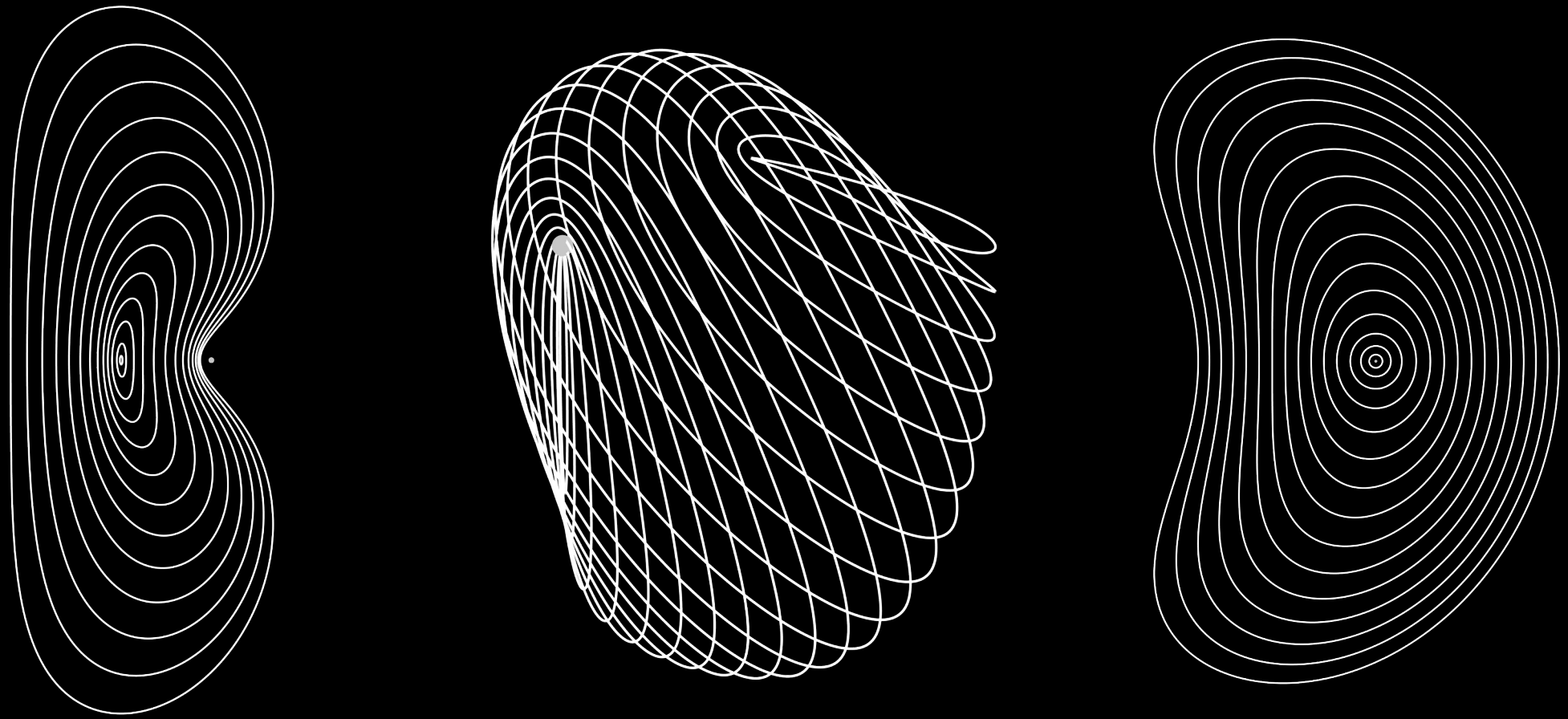
Circular Restricted 3-Body Problem (CR3BP)

- Motion of m_3 (s/c) under gravitational effects of m_1 (E) & m_2 (M)
- Assume $m_3 \ll m_1, m_2$
- m_1, m_2 in circular orbits about mutual barycenter
- Study motion of m_3 in the synodic rotating frame s.t. $m_{1,2}$ remain fixed

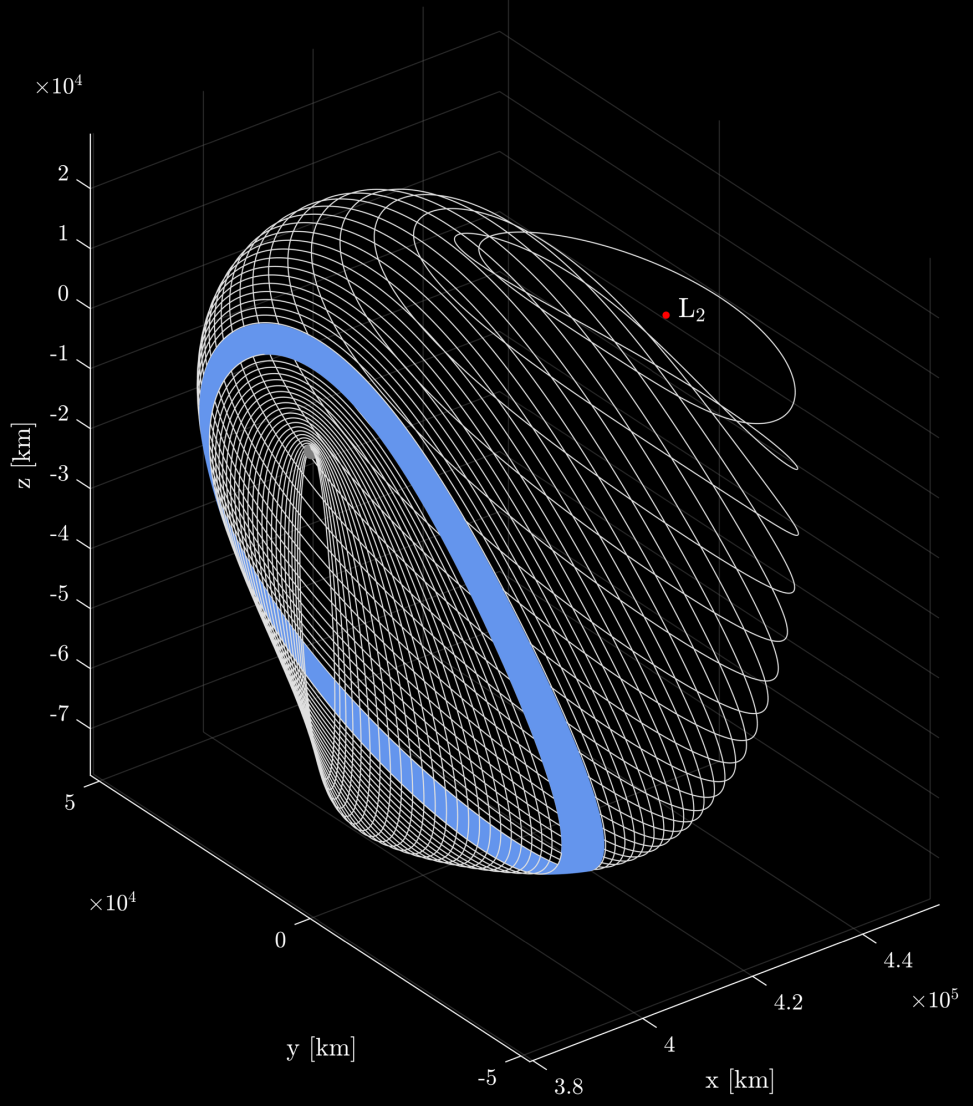


$$H_{\mathrm{CR3BP}} = \frac{1}{2}|p|^2 + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

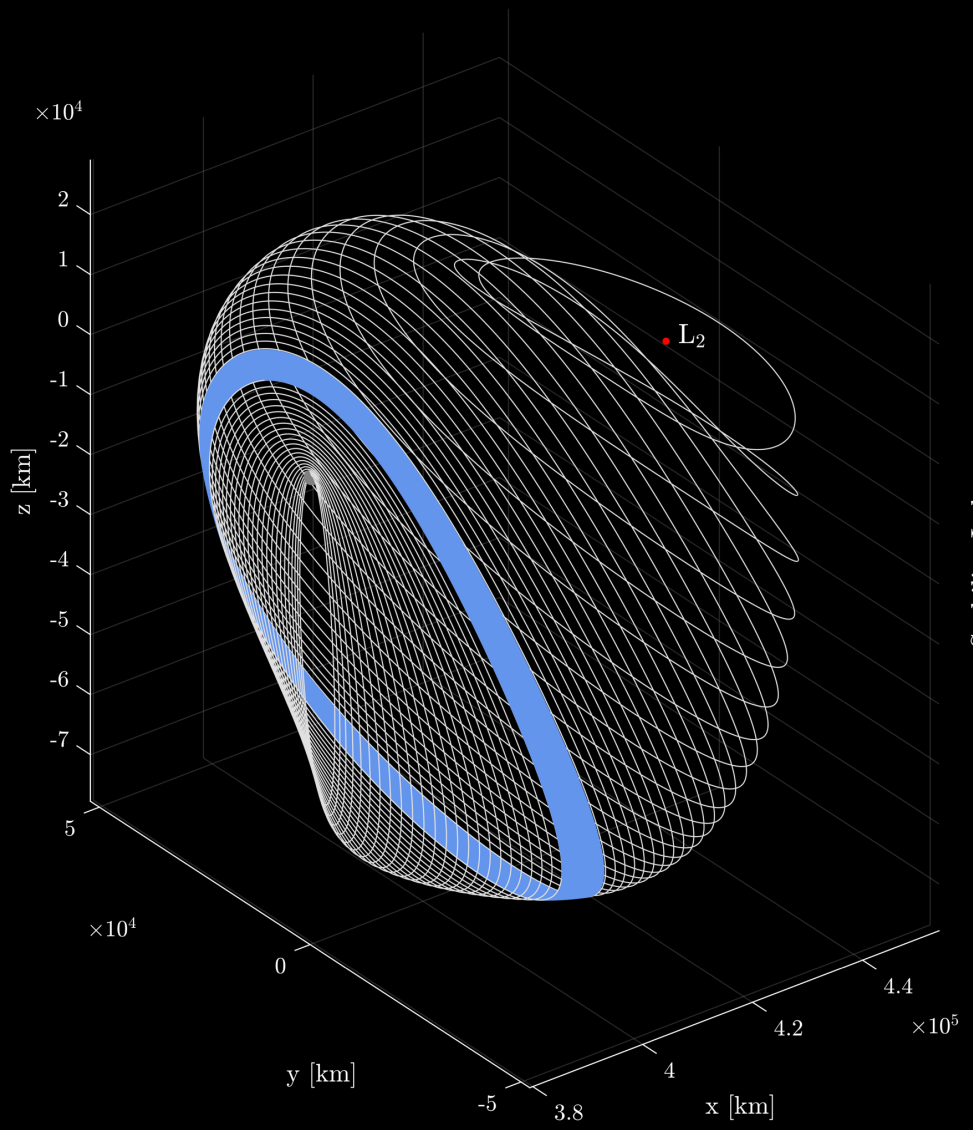
Families of Periodic Orbits

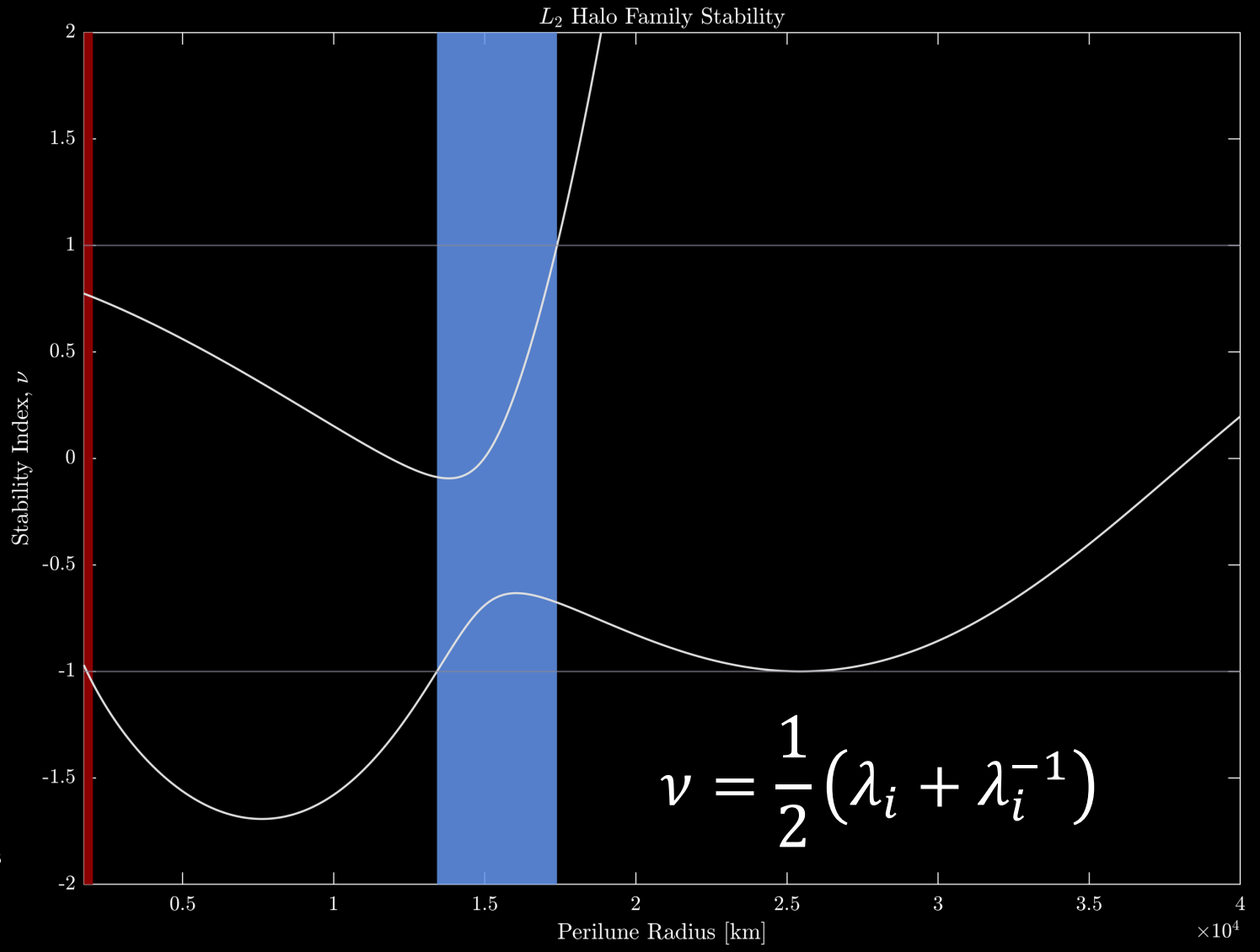


Earth-Moon L_2 Halo Orbits



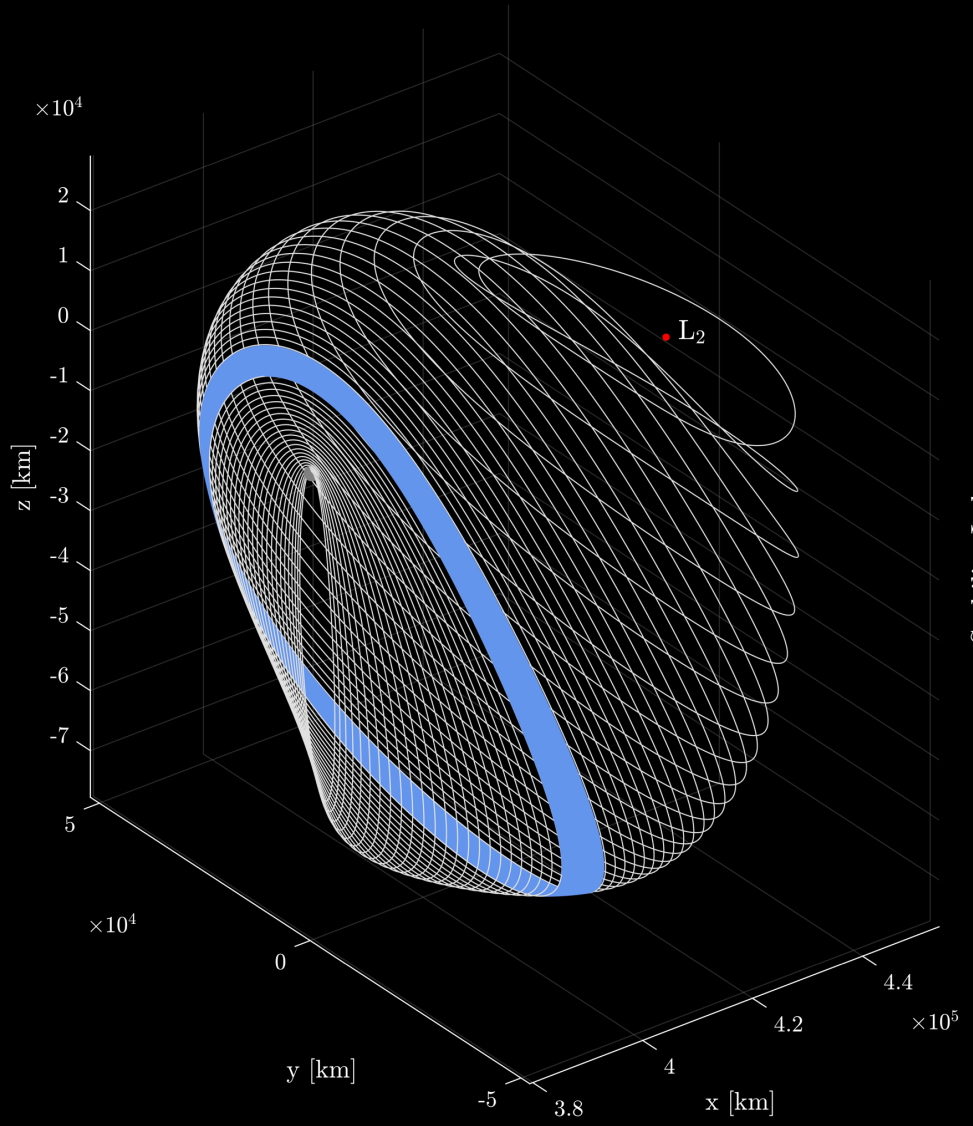
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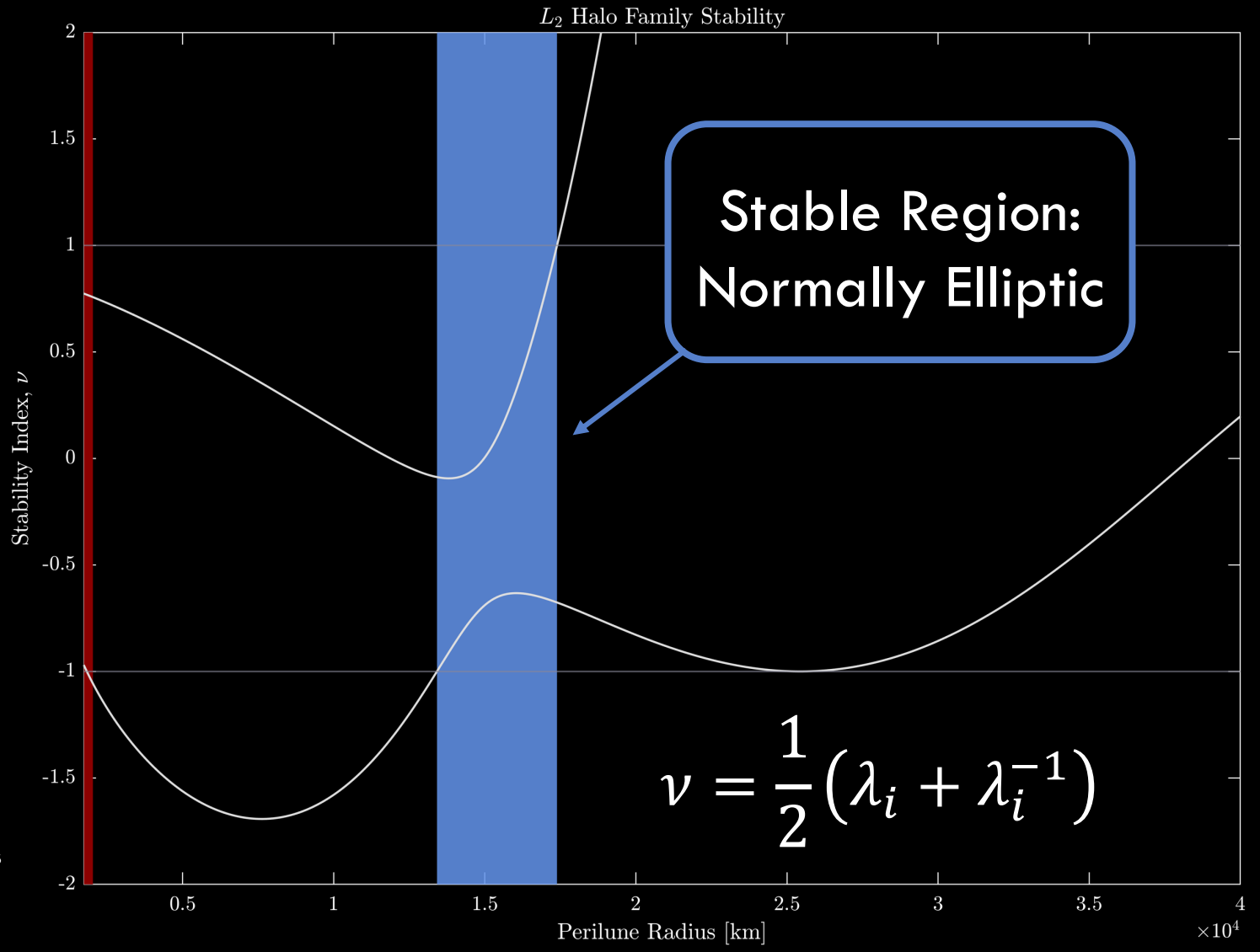




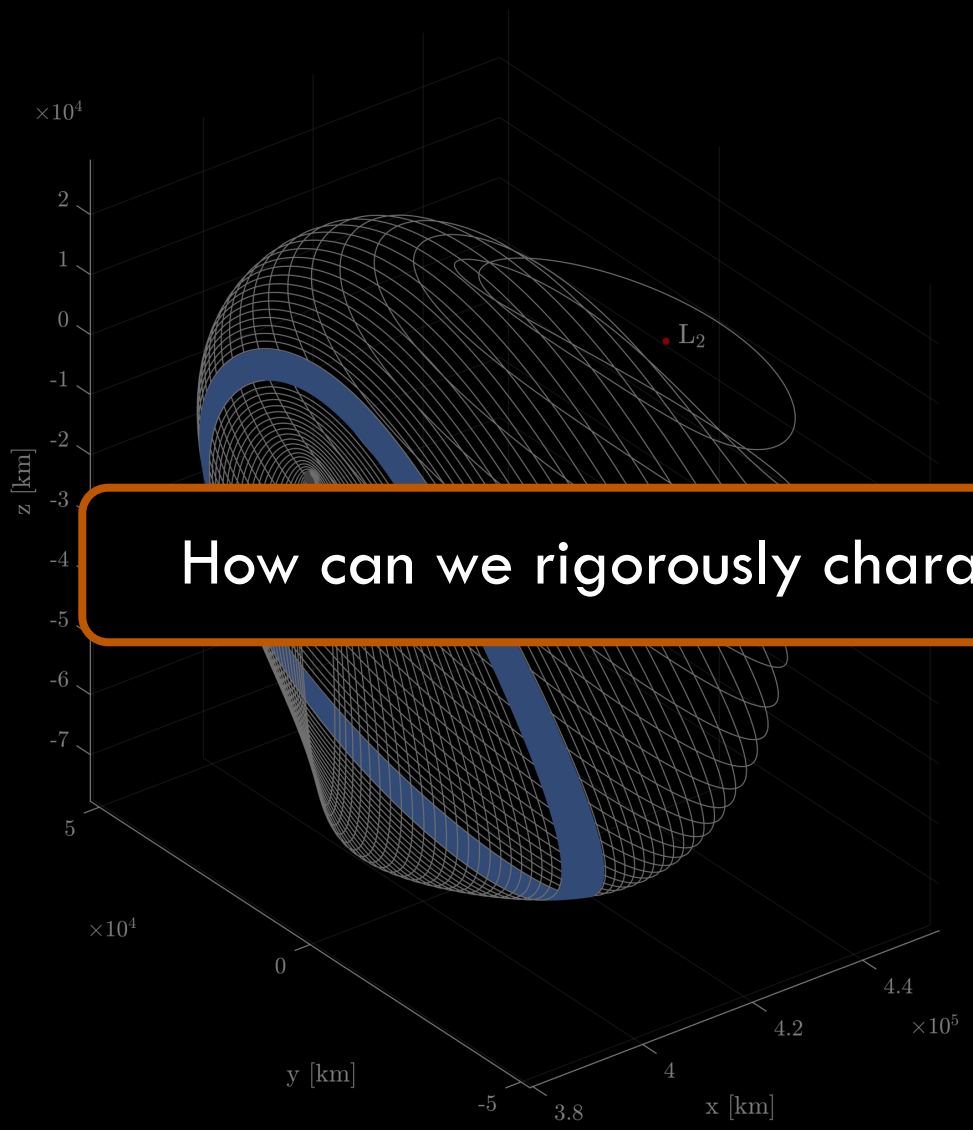
$$\nu = \frac{1}{2}\left(\lambda_i + \lambda_i^{-1}\right)$$

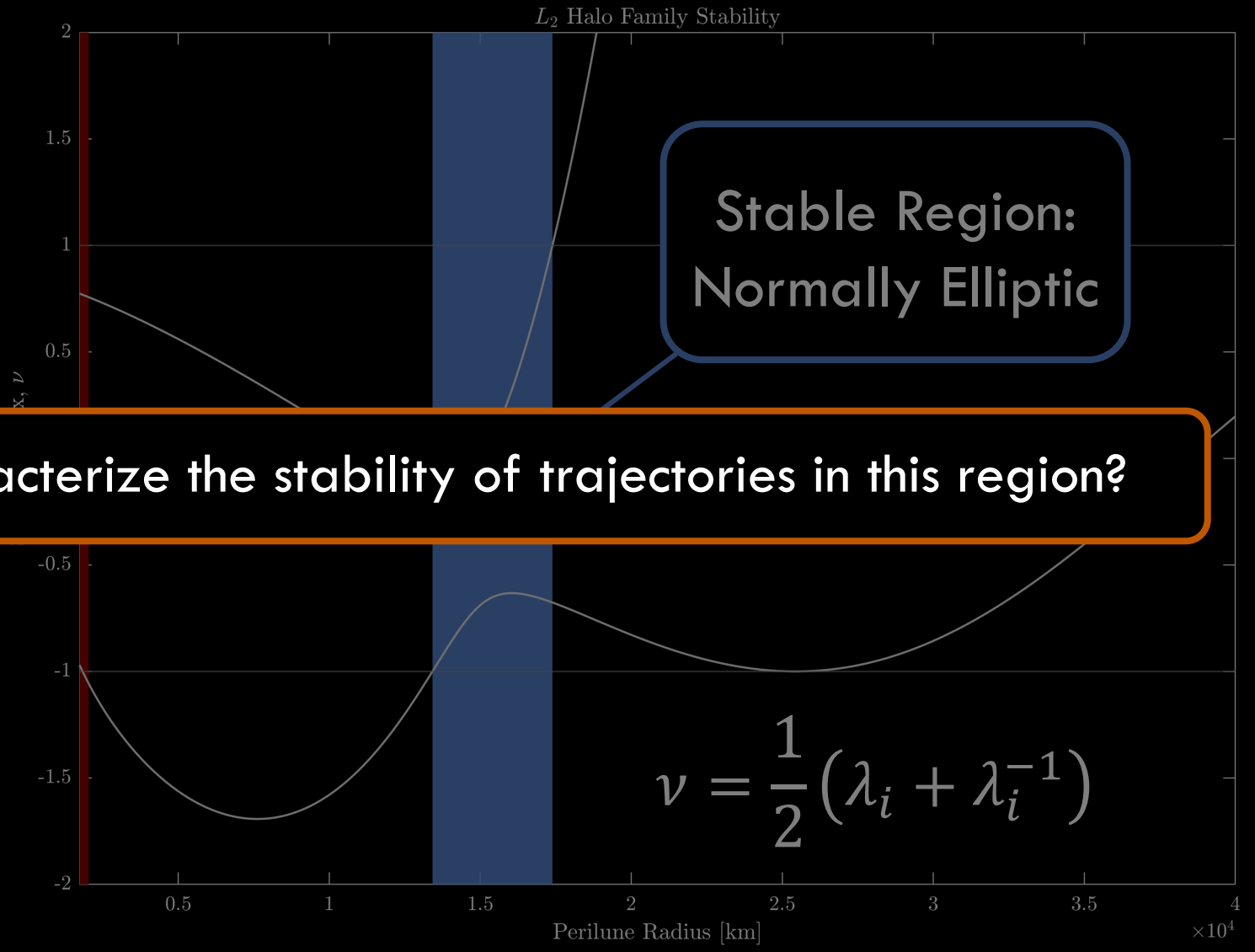
Earth-Moon L_2 Halo Orbits





Earth-Moon L_2 Halo Orbits





Stable Region:
Normally Elliptic

How can we rigorously characterize the stability of trajectories in this region?

$$\nu = \frac{1}{2}\left(\lambda_i + \lambda_i^{-1}\right)$$

Long-term Confinement in Hamiltonian Systems

$$H(q,p) = H_0(I) + \epsilon H_1(q,p)$$

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Perpetual Stability

KAM (Kolmogorov-Arnold-Moser) Theory:

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- Survival of sets of positive measure covered by invariant tori (for $\epsilon > 0$ small)

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Effective Stability

Nekhoroshev Theory (1977):

- Bound the drift of trajectories over exponentially long but finite times

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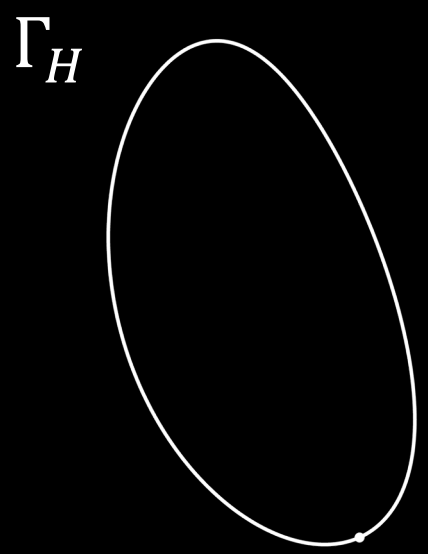
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Dynamics near periodic orbits can be constructed as *nearly integrable*

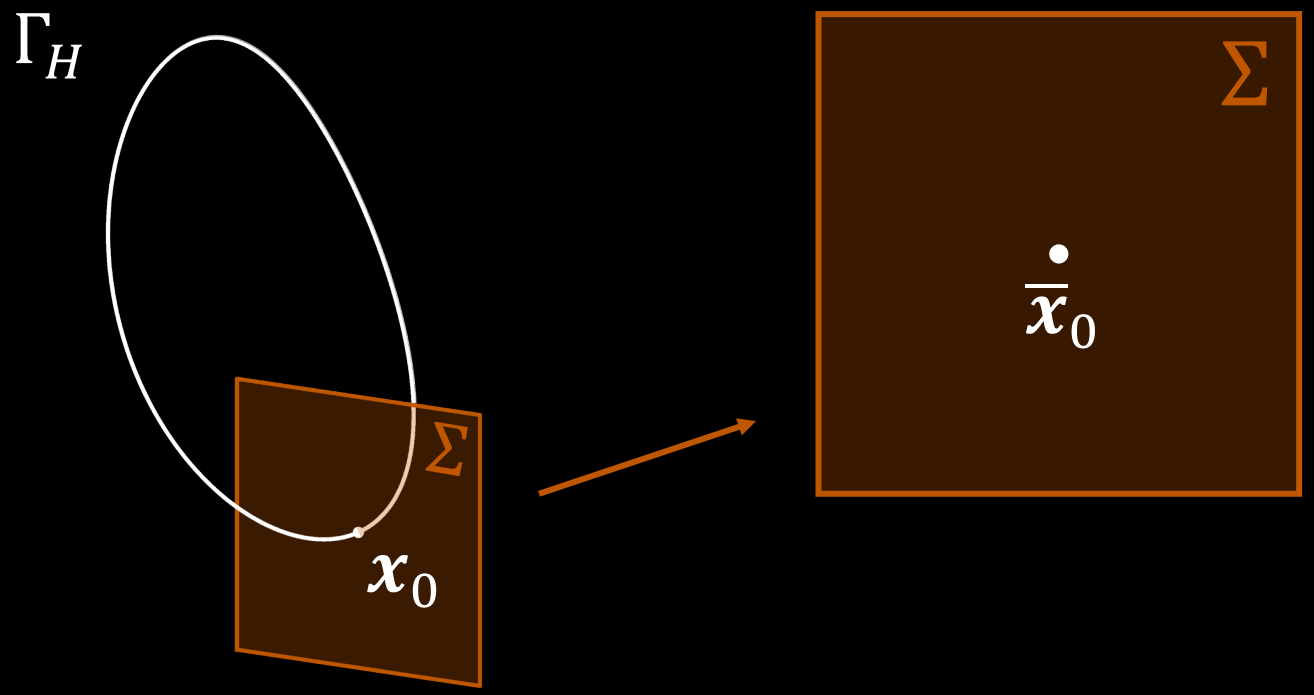
Normal Form Construction

For Fixed Points of Poincaré Maps via Jet Transport

Generating Map Dynamics via Jet Transport

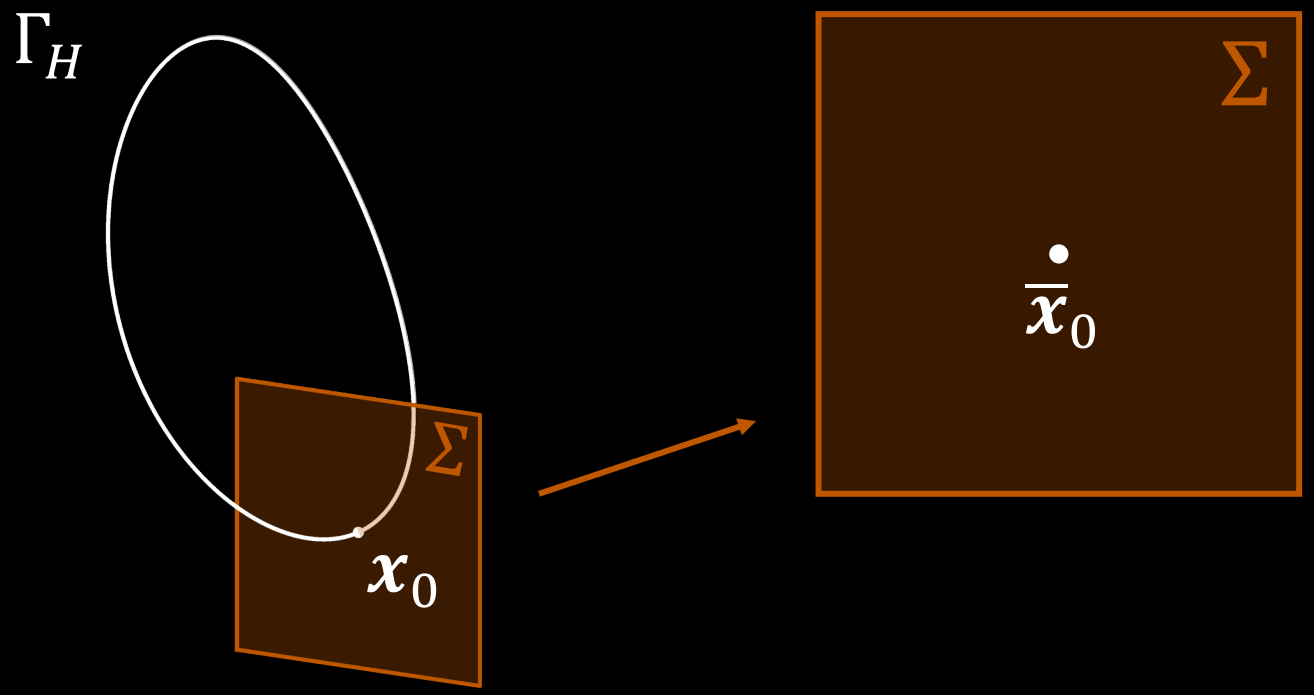


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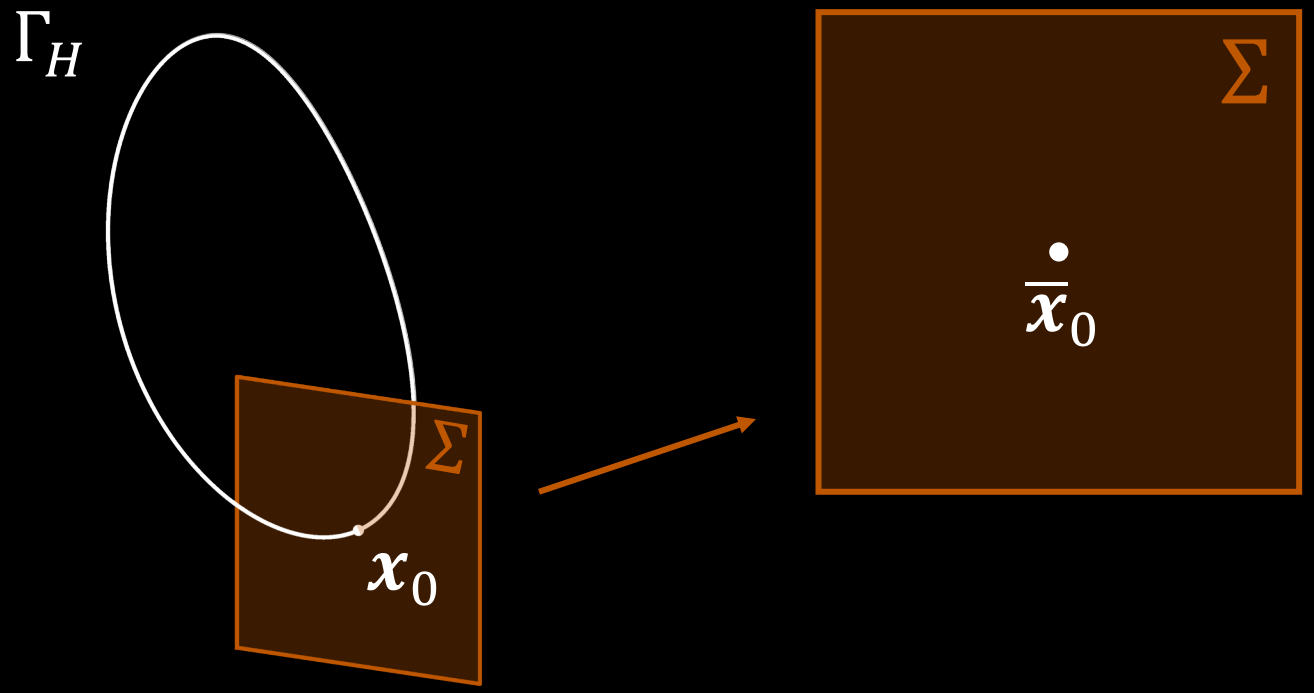


Generating Map Dynamics via Jet Transport

$P(\bar{x}_0 + \xi) =$ Poincaré return map



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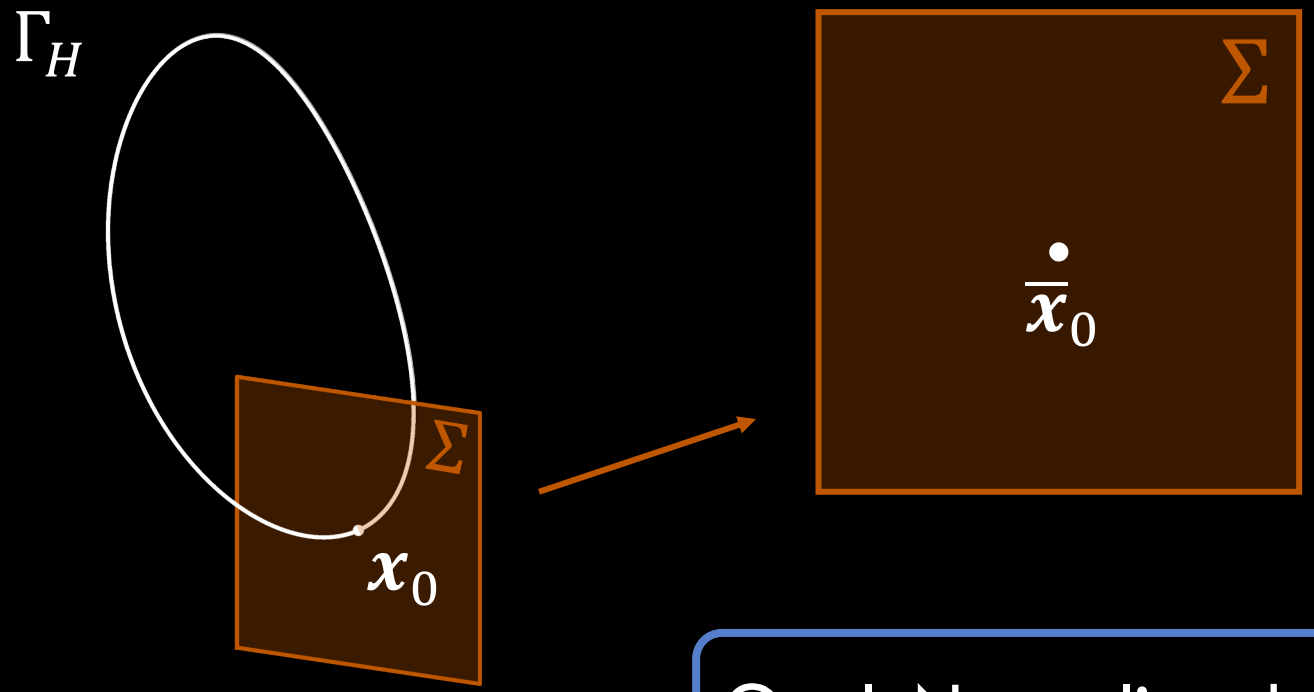
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Jet transport used to generate expansion

$$P(\overline{\boldsymbol{x}}_0 + \xi) = G(\xi)$$

$$= \sum_{k \geq 0} \sum_{|j|=k} G_j^k \xi^j$$

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Goal: Normalize dynamics

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Normalization via Near-Identity Transformations

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Linear Normalization

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Linear Normalization

Diagonalize linear system:

$$\Lambda = V^{-1} G^{1} V = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1})$$

Normal block of Monodromy matrix $\Phi(T)$

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$$\chi_k(\xi) = \sum_{|j|=k} b_j \xi^j, \qquad b_{j,i} = \frac{R_{j,i}^{k,(k-1)}}{\lambda_i - \lambda^j}$$

Solution to a homological equation!

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$$(\lambda_i - \lambda^j) b_{j,i} = R_{j,i}^{k,(k-1)} - F_{j,i}^{k,(k)}$$

$F_{j,i}^{k,(k)} \to 0$ (unless unavoidable resonance)

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Diophantine: $|\lambda_i - \lambda^j| \geq \frac{\gamma}{k^\tau}$ for $|j| = k$

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$$z' = F^{(N)}(z, \bar{z}), \qquad z_i' = z_i \exp\left(\mathrm{i}\Omega_i^{(N)}(r)\right)$$

$\Omega_i^{(N)}(r) =$ nonlinear frequencies $\qquad r_i^2 = z_i \bar{z}_i =$ orbital radii $=$ const.

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$$R^{(N)}(\xi) \propto \mathcal{O}(|\xi|^{N+1})$$

Encapsulates all non-integrable terms…

Exclusively drives drift away from fixed point

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Bound $\| R^{(N)} \|_a$ over a local domain → Establish Nekhoroshev escape times

Proof Sketch

Classical Analytical Bounds

Cauchy Estimates

Estimate size of coefficients of the initial map:

$$C_k := \sum_{|j|=k} |G_j^k| \leq \frac{\tilde{C}}{\tilde{\rho}^k}$$

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Coefficient Growth due to Normalization

Iterative Lemma:

$$\epsilon_N := \| R^{(N)} \|_{\rho_N} \leq \epsilon_2 \left(\frac{2N^\tau}{\gamma}\right)^N$$

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Truncated Remainder Bound

Let $a < \frac{\rho}{2}$ be a target physical confinement radius.
Then:

$$\| R^{(N)} \|_a \leq \epsilon_2 \left(\frac{2a}{\rho}\right) (AaN^\tau)^N, \qquad A = \frac{4}{\rho\gamma}$$

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Optimal Normalization Order

Factorial growth of coefficients will overtake geometric decay of high-order polynomials.

Stop normalization at the optimized order to minimize size of the remainder:

$$N_{\text{opt}} := \text{argmin}_N \, (AaN^\tau)^N = \left\lfloor \frac{1}{e} \left(\frac{1}{Aa}\right)^{\frac{1}{\tau}} \right\rfloor$$

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Radial Drift

Let $(z, \bar{z})$ be the complexified NF coordinates. For each $i = 1, \ldots, n$, define the orbital radius: $r_i^2 = z_i \bar{z}_i$

The one-step drift in radius is bounded by:

$$|\Delta r_i| \leq \| R^{(N_{\text{opt}})} \|_a \leq \epsilon_{\text{opt}}$$

Main Results

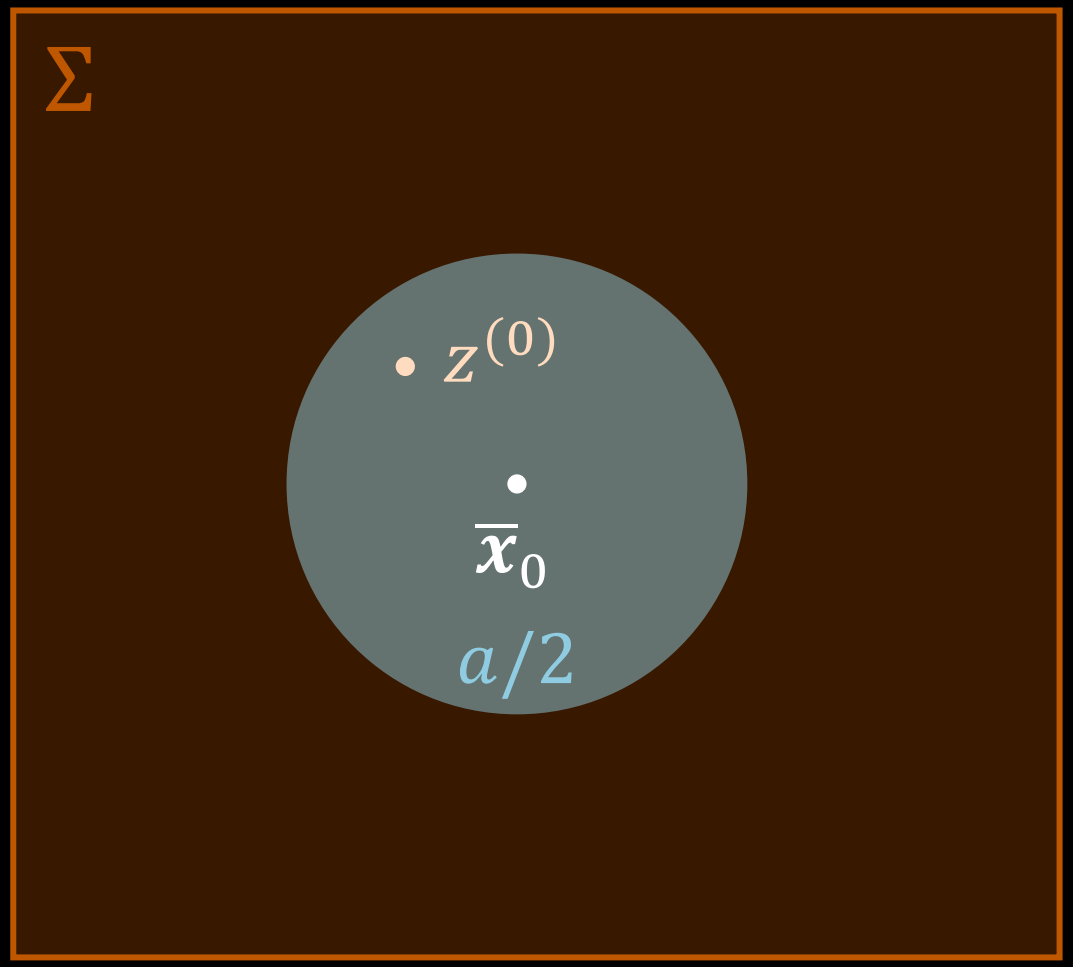
Stability Time + Confinement Region

Theorem: Effective Stability Time

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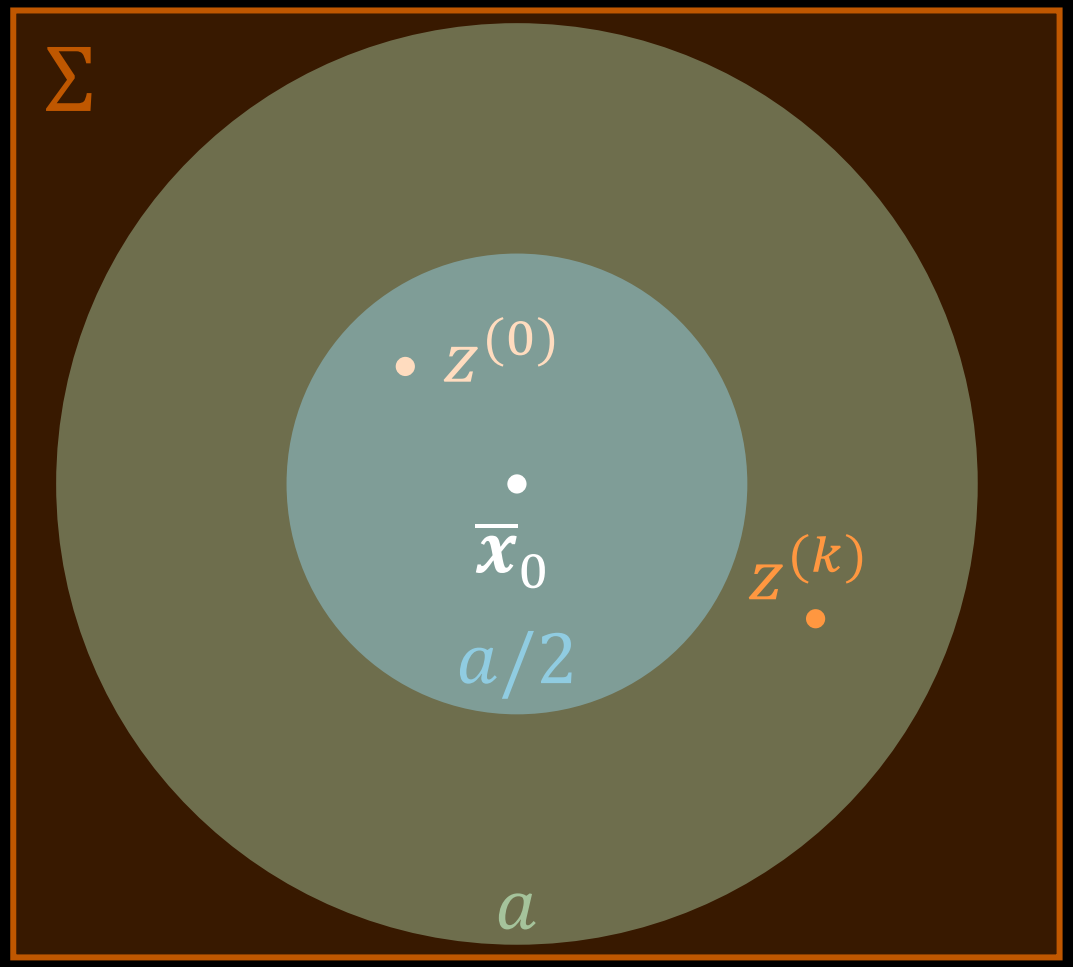
Theorem: Effective Stability Time



If:

$$z^{(0)} \in \mathcal{D}_{a/2}, \qquad \underbrace{a}_{\text{confinement radius}} \leq a_0 := \frac{1}{A(2e)^{\tau}}$$

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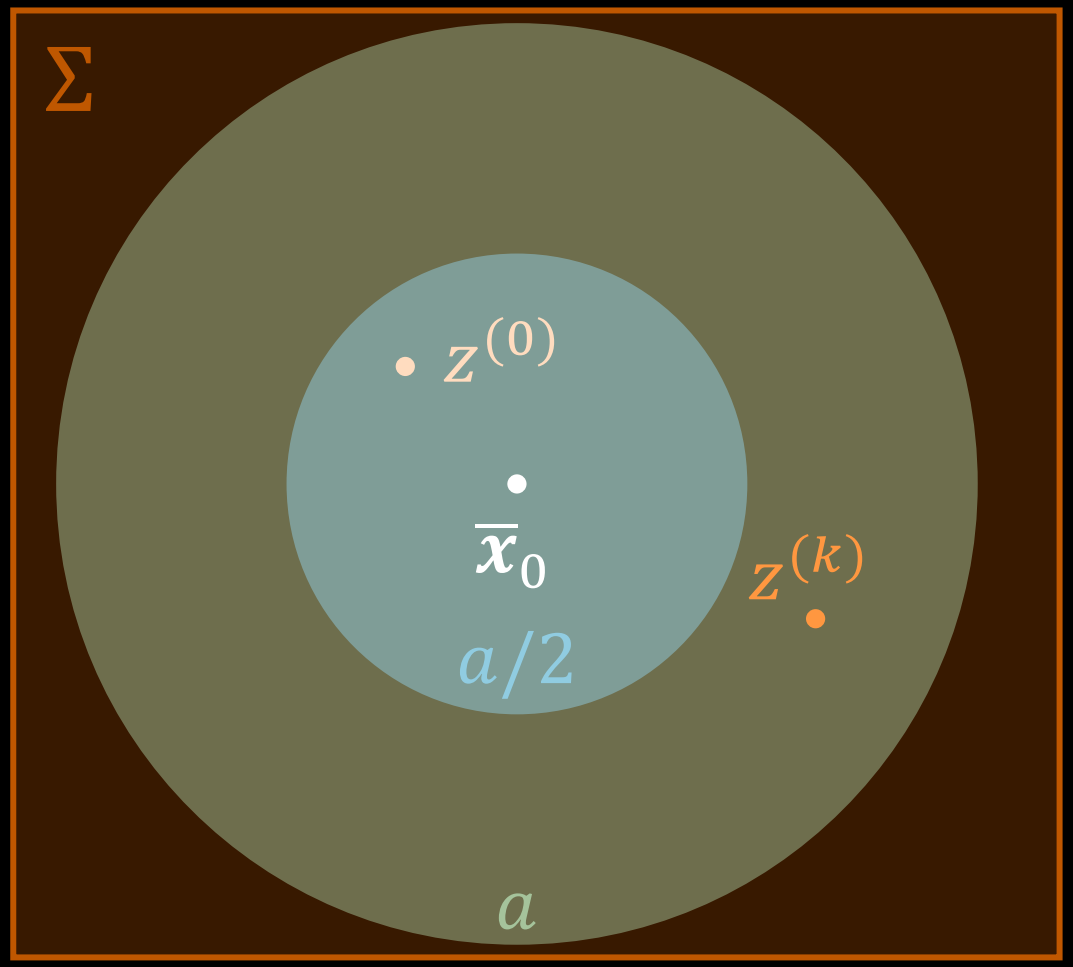
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Theorem: Effective Stability Time



$A = 4/\rho\gamma$ & $C = \rho/4\epsilon_2$ positive constants

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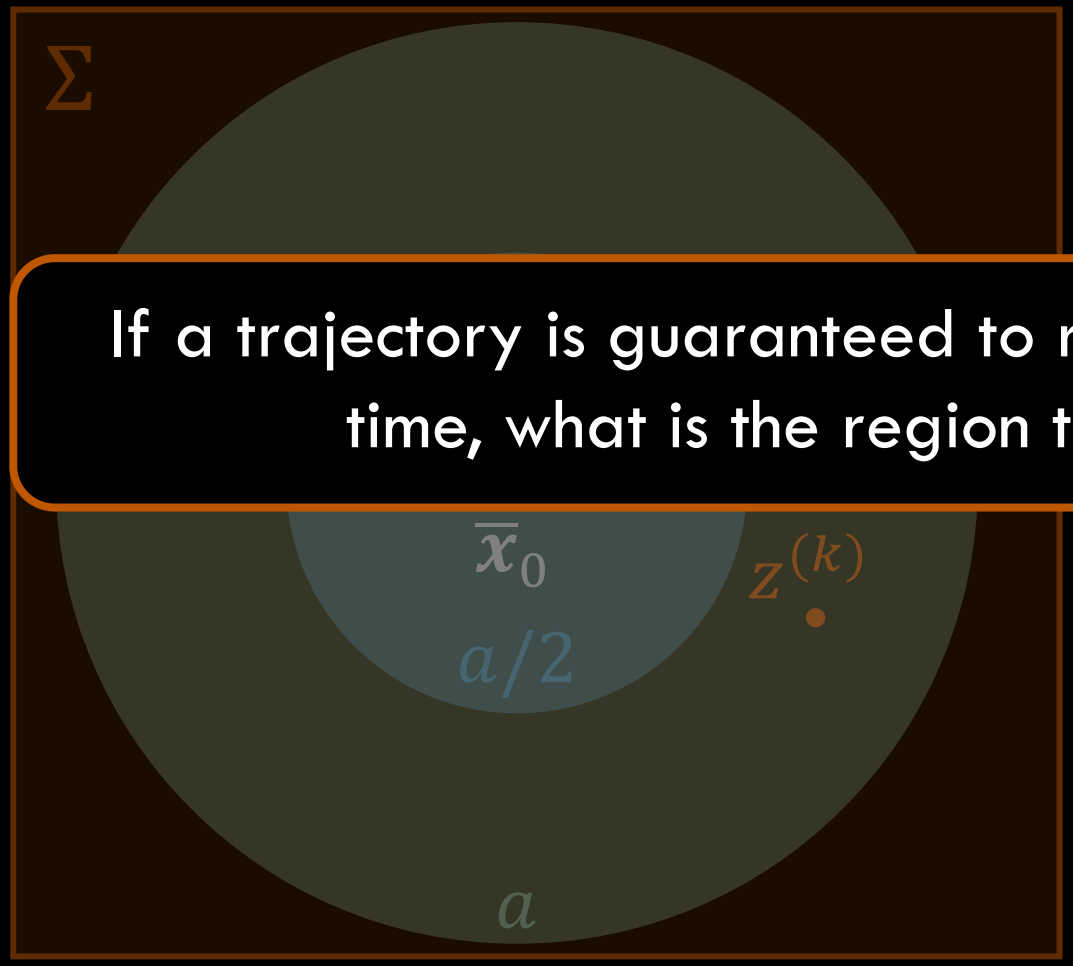
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$$\underbrace{T_{\text{eff}}(a) = C \exp\left(\tau \left\lfloor \frac{1}{e}\left(\frac{1}{Aa}\right)^{\frac{1}{\tau}} \right\rfloor\right)}_{\text{Effective stability time}}$$

Theorem: Effective Stability Time



If:

$$z^{(0)} \in \mathcal{D}_{a/2}, \qquad a \leq a_0 := \frac{1}{A(2e)^{\tau}}$$

If a trajectory is guaranteed to remain stable for an exponentially long time, what is the region that contains it during that time?

$$z^{(k)} \in \mathcal{D}_a, \qquad \forall k \leq T_{\text{eff}}(a)$$

$$T_{\text{eff}}(a) = C \exp\left(\tau \left\lfloor \frac{1}{e}\left(\frac{1}{Aa}\right)^{\frac{1}{\tau}} \right\rfloor\right)$$

Effective stability time

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Corollary: Confinement Region

Let $T > C$ be a prescribed minimum effective stability time.

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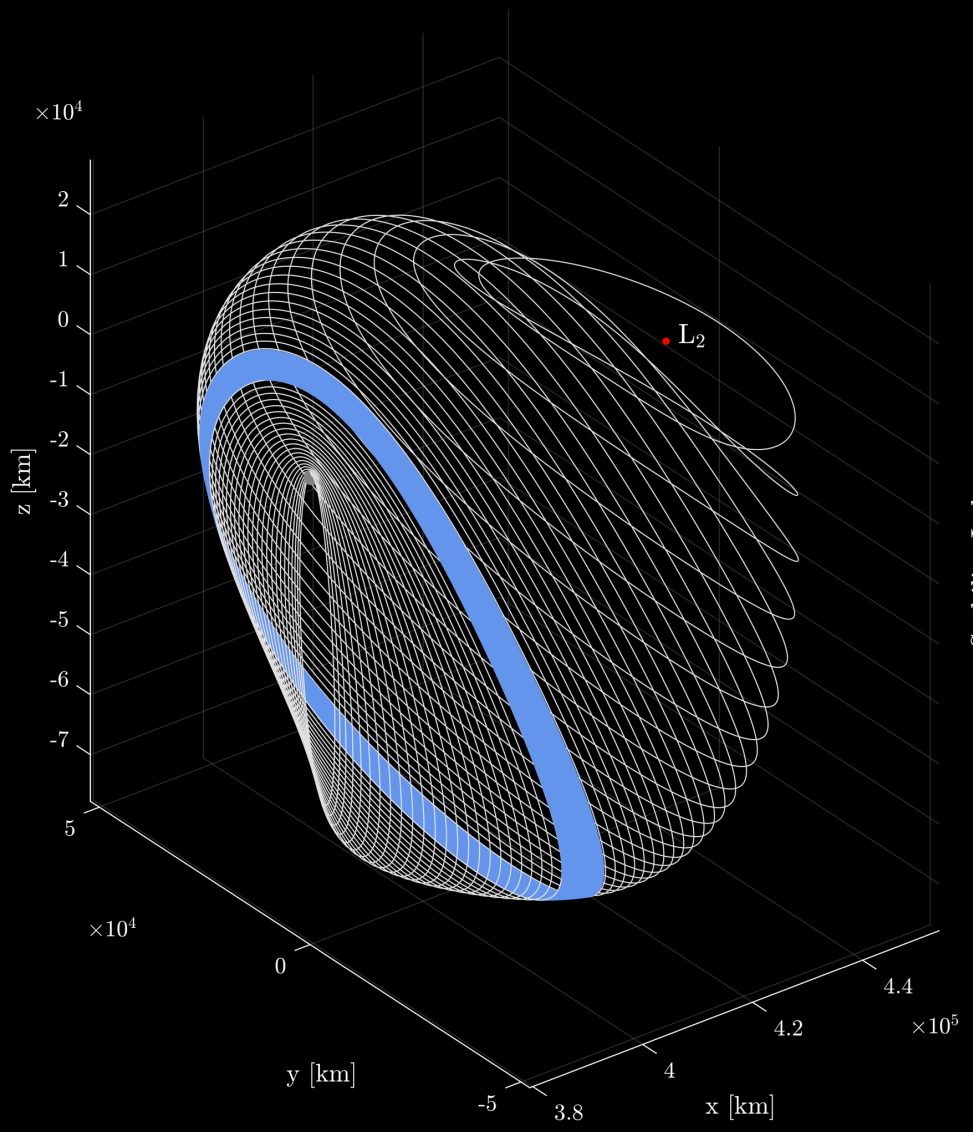
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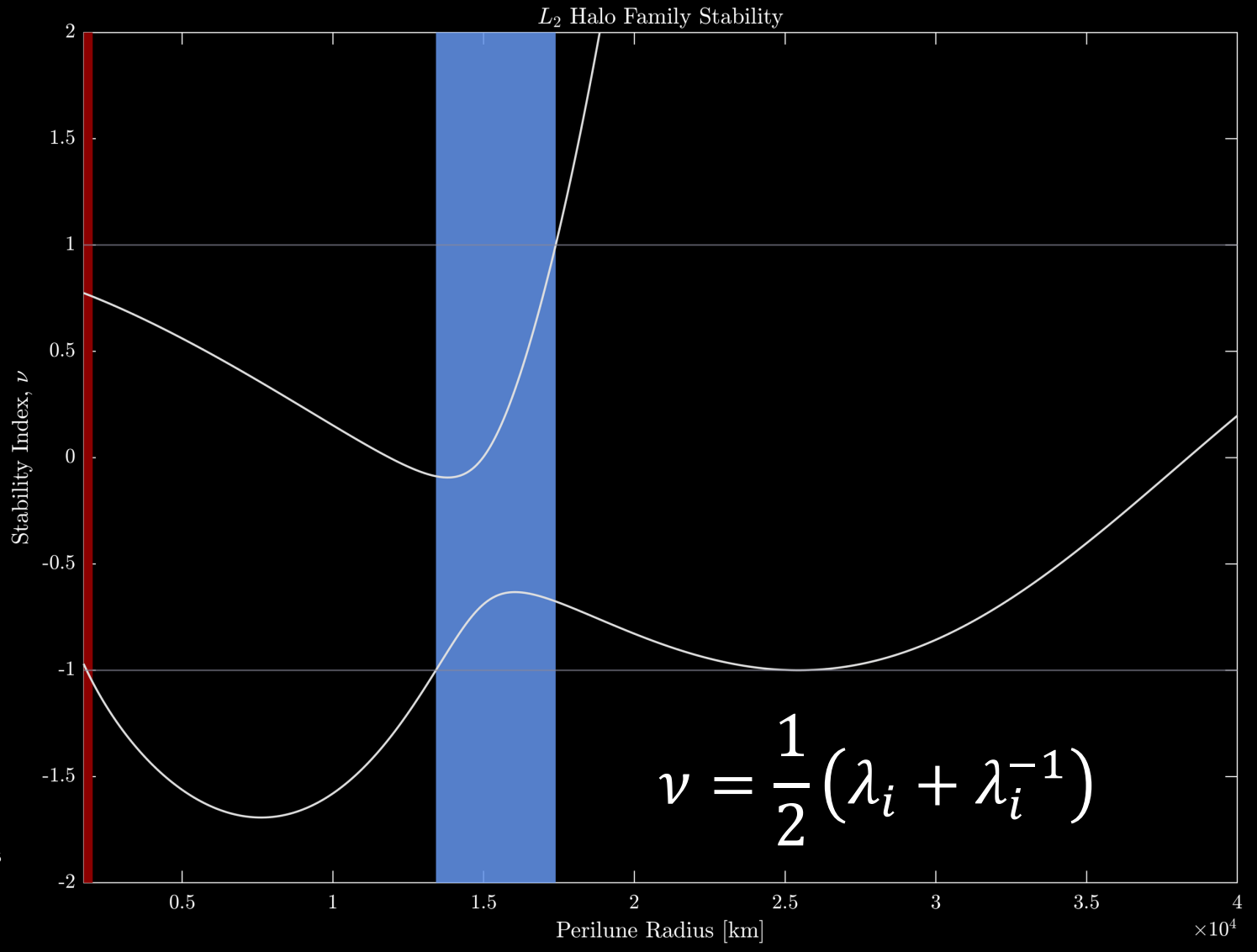
To guarantee confinement of the discrete trajectory $\forall k \leq T$, the confinement radius a must be bounded by:

$$a(T) \leq \frac{1}{Ae^{\tau}\left(1 + \frac{1}{\tau}\ln\left(\frac{T}{C}\right)\right)^{\tau}}$$

where $A = \frac{4}{\rho\gamma}$ and $C = \frac{\rho}{4\epsilon_2}$.

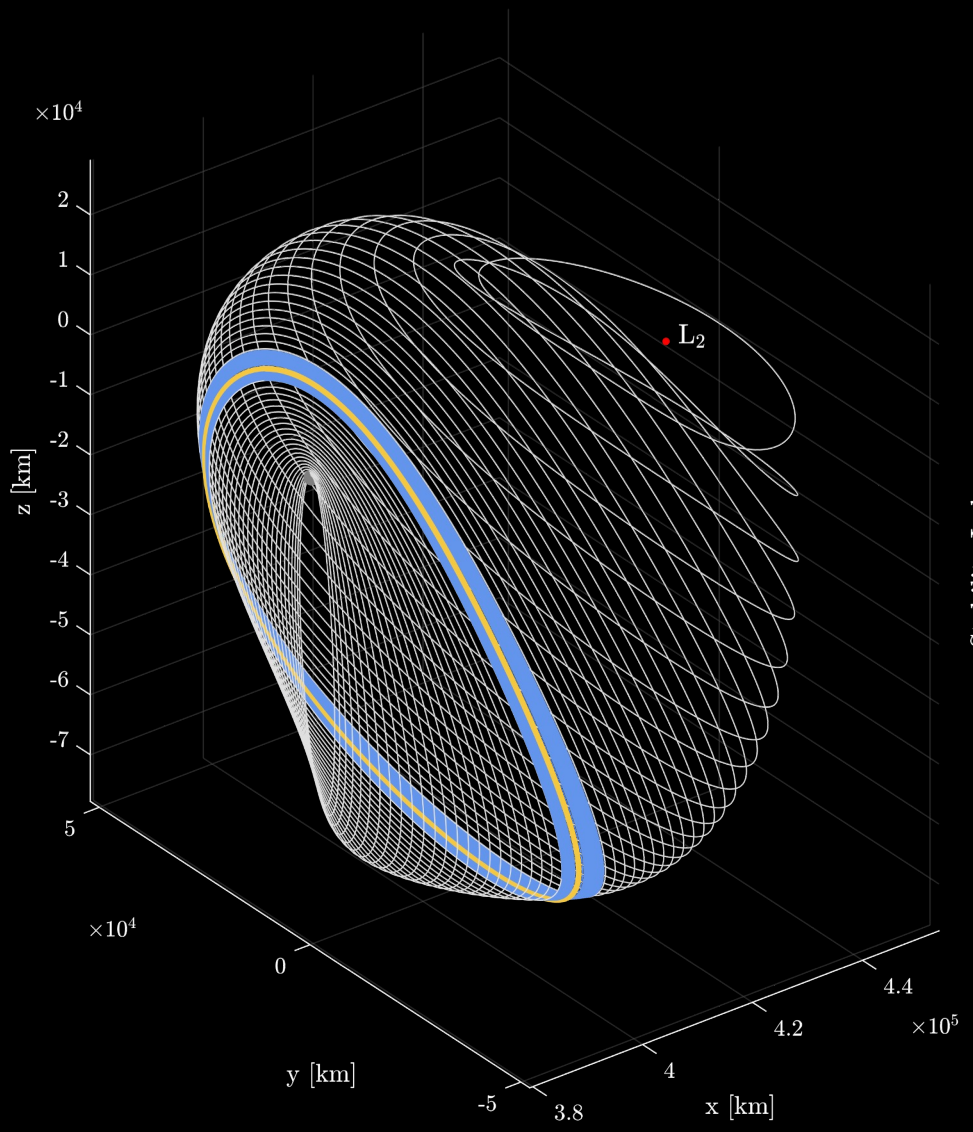
Earth-Moon L_2 Halo Orbits

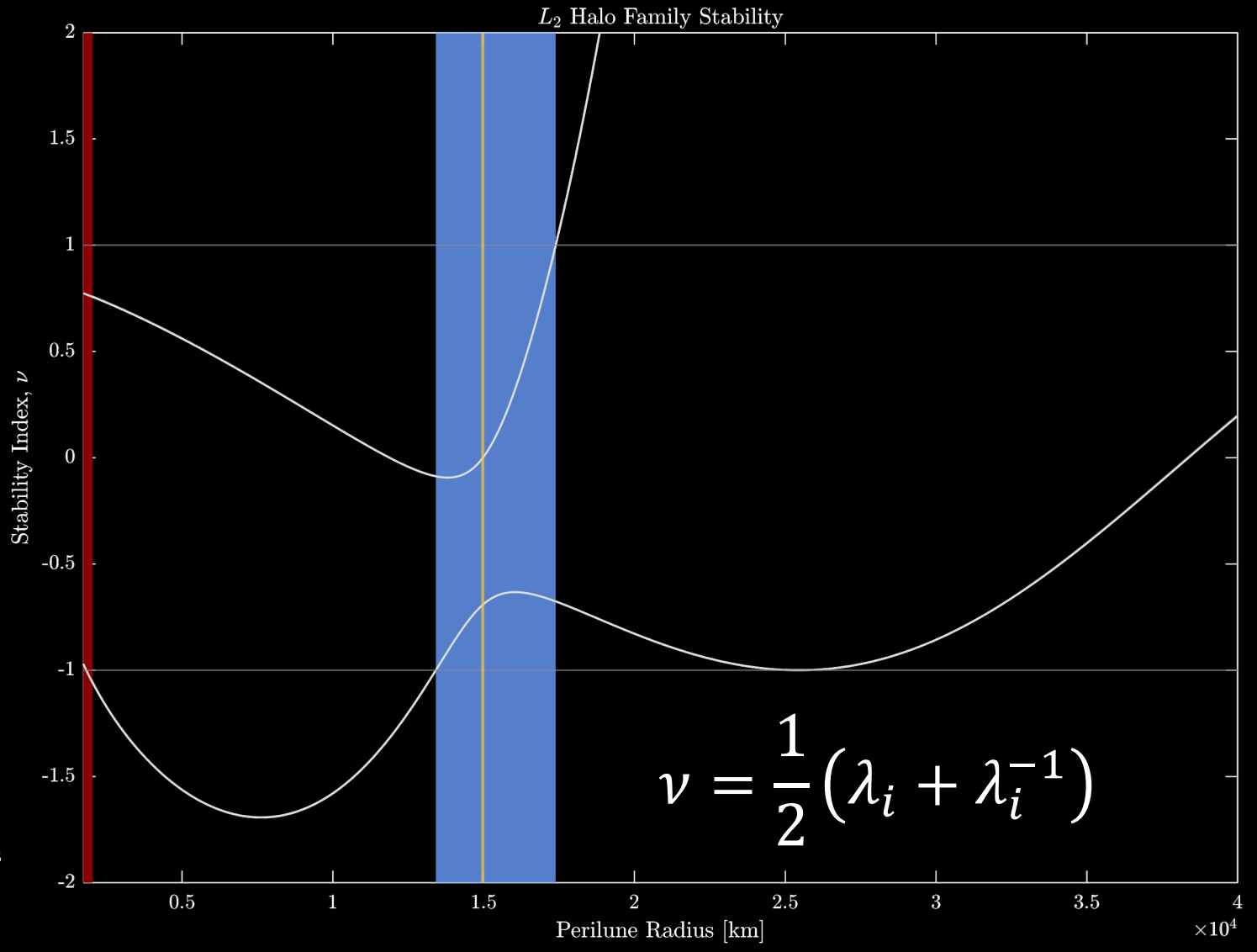




$$\nu = \frac{1}{2}\left(\lambda_i + \lambda_i^{-1}\right)$$

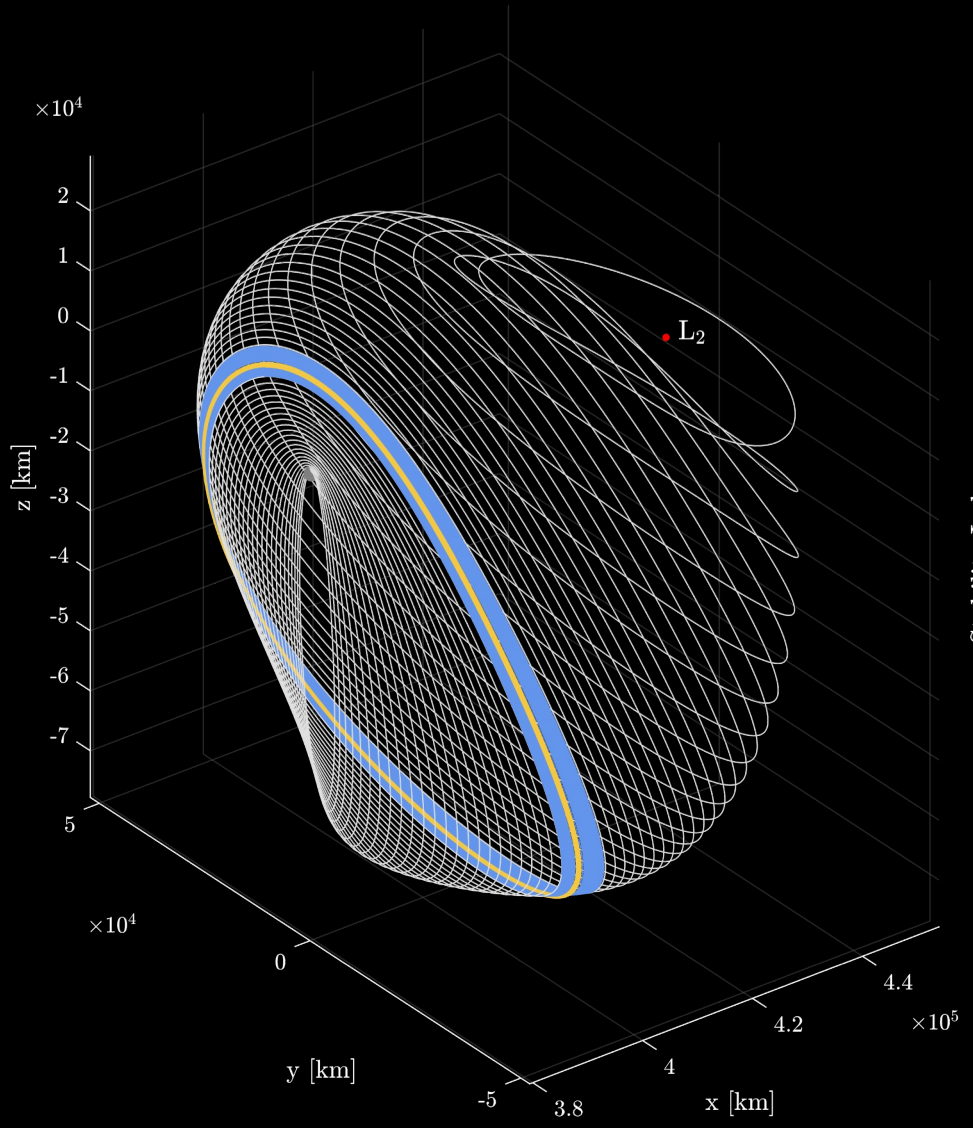
Earth-Moon L_2 Halo Orbits

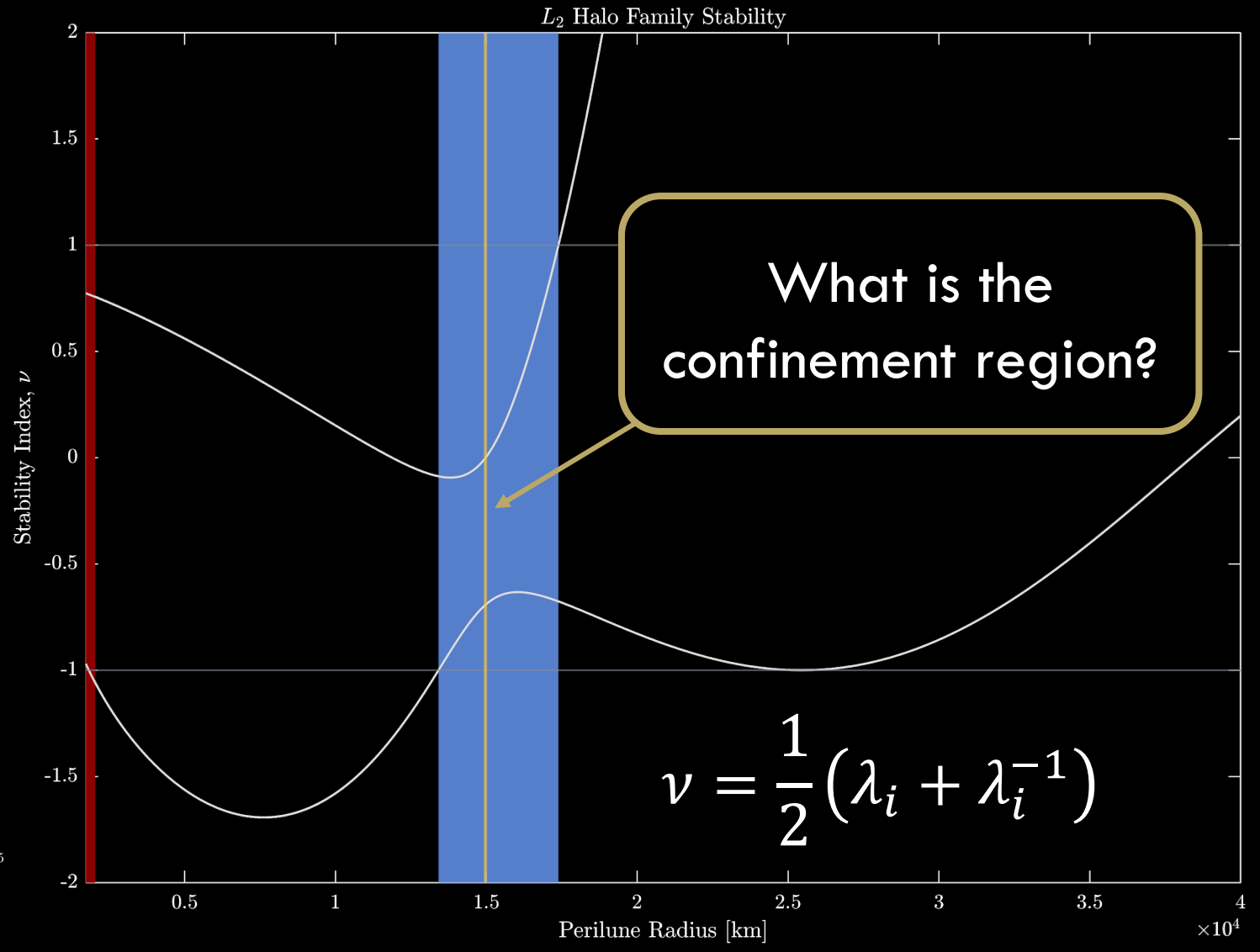




$$\nu = \frac{1}{2}\left(\lambda_i + \lambda_i^{-1}\right)$$

Earth-Moon L_2 Halo Orbits

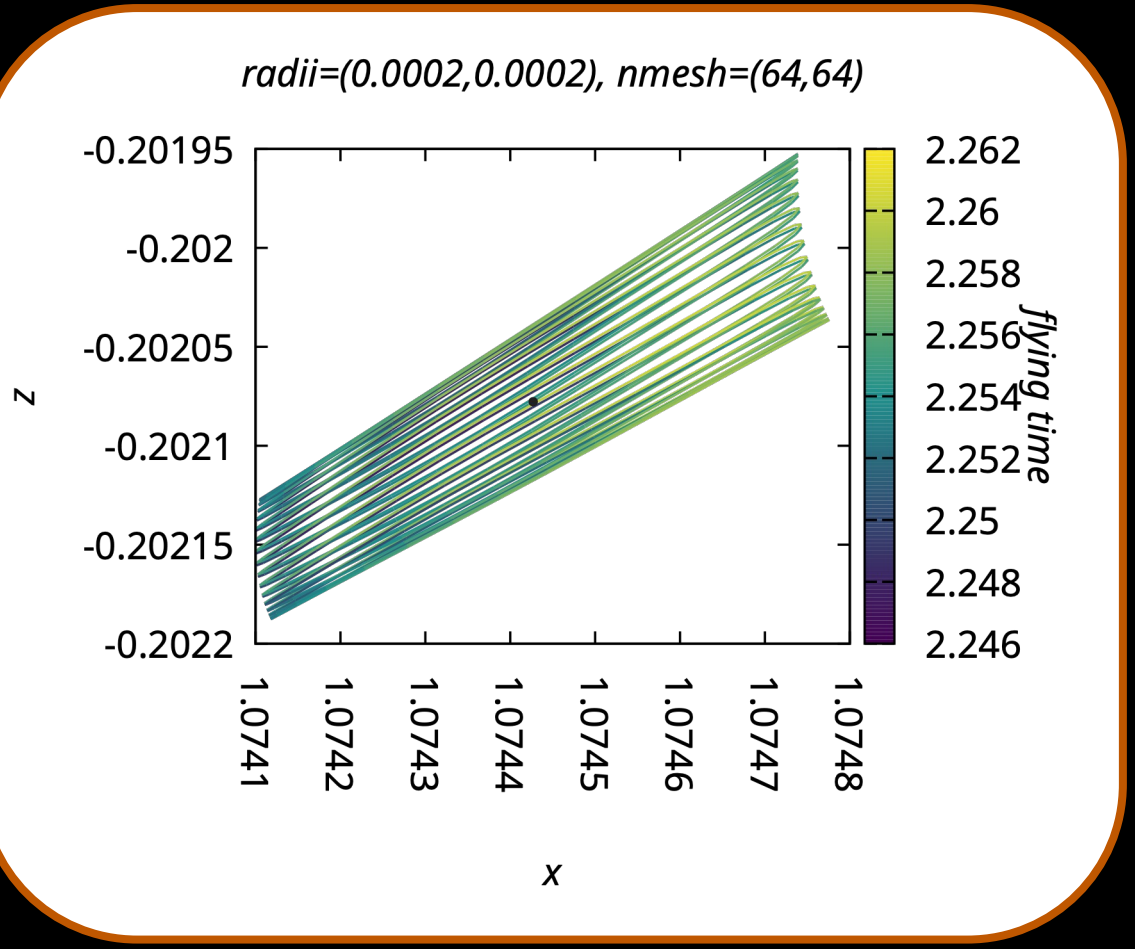




What is the confinement region?

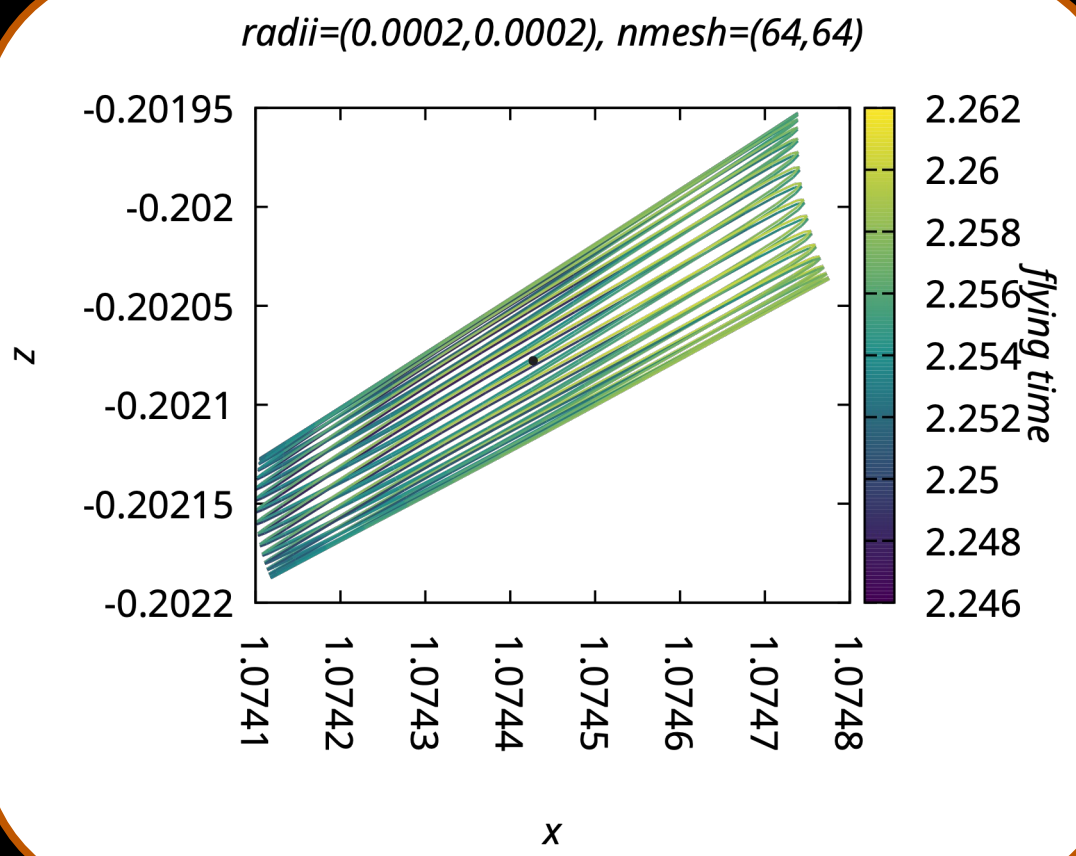
$$\nu = \frac{1}{2}\left(\lambda_i + \lambda_i^{-1}\right)$$

$T = 2\times10^{10}$ years



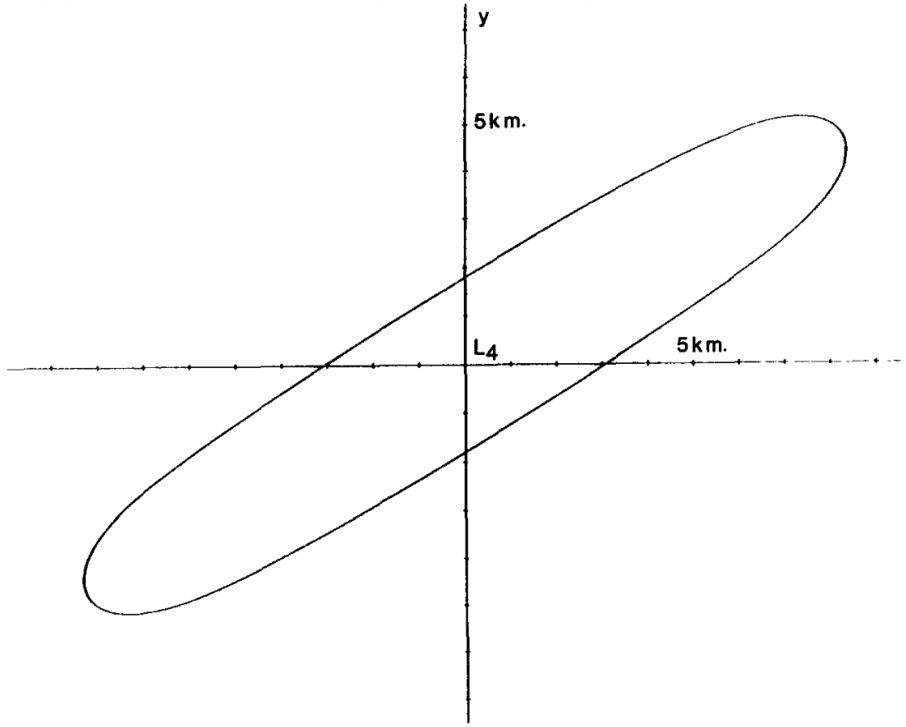
$a(T) \approx 38\,\mathrm{km}$

$T = 2\times10^{10}$ years



$a(T) \approx 38\,\text{km}$

Comparison: Sun-Jupiter L_4



Giorgilli, Delshams, Fontich, Galgani, Simó (1987)

$a(T) \approx \mathcal{O}(1\,\text{km}) \rightarrow \mathcal{O}(10^8\,\text{km})$

Historical Perspective: Sun-Jupiter L_4

Historical Perspective: Sun-Jupiter L_4

Effective Stability for a Hamiltonian System
near an Elliptic Equilibrium Point, with
an Application to the Restricted Three Body Problem

ANTONIO GIORGILLI

Dipartimento di Fisica dell'Università, Via Celoria 16, Milano, Italy

AMADEO DELSHAMS

Departament de Matemàtica Aplicada i Análisi,
Universitat de Barcelona, Barcelona, Spain

ERNEST FONTICH

Departament de Matemàtica Aplicada, ETS EIB,
Universitat Politècnica de Barcelona, Barcelona, Spain

LUIGI GALGANI

Dipartimento di Matematica dell'Università, Via Saldini 50, Milano, Italy

AND

CARLES SIMÓ

Departament de Matemàtica Aplicada i Análisi,
Universitat de Barcelona, Barcelona, Spain

Received September 10, 1987

1987

$O(1\,\text{km})$

Historical Perspective: Sun-Jupiter L_4

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AMADEO DELSHAMS

Departament de Matemàtica Aplicada i Anàlisi,
Universitat de Barcelona, Barcelona, Spain

ERNEST FONTICH

Departament de Matemàtica Aplicada, ETSEIB,
Universitat Politècnica de Barcelona, Barcelona, Spain

LUIGI GALGANI

Dipartimento di Matematica dell'Università, Via Saldini 50, Milano, Italy

AND

CARLES SIMÓ

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$\mathcal{O}(1\,\text{km})$

1987

ESTIMATES OF THE ERROR IN NORMAL FORMS OF HAMILTONIAN SYSTEMS. APPLICATIONS TO EFFECTIVE STABILITY AND EXAMPLES

Carles Simó
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona, Gran via 585,
Barcelona 08007, Spain.

9,842 km

1988

Historical Perspective: Sun-Jupiter L_4

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Dipartimento di Fisica dell'Università, Via Celoria 16, Milano, Italy

AMADEO DELSHAMS

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain

ERNEST FONTICH

Departament de Matemàtica Aplicada, ETSEIB, Universitat Politècnica de Barcelona, Barcelona, Spain

LUIGI GALGANI

Dipartimento di Matematica dell'Università, Via Saldini 50, Milano, Italy

AND

CARLES SIMÓ

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain

Received September 10, 1987

$O(1\,\text{km})$

1987

ON THE STABILITY OF THE LAGRANGIAN POINTS IN THE SPATIAL RESTRICTED PROBLEM OF THREE BODIES

ALESSANDRA CELLETTI

Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, 67100 Coppito-L'AQUILA (Italy)

and

ANTONIO GIORGILLI

Dipartimento di Matematica, Via Saldini 50, 20133-MILANO (Italy)

(Received: 30 July, 1990; accepted: 8 February, 1991)

14 473 km

1990

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Carles Simó
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona, Gran via 585,
Barcelona 08007, Spain.

9 842 km

1988

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Dipartimento di Fisica dell'Università, Via Celoria 16, Milano, Italy

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Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain

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$O(1\text{ km})$

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9 842 km

1988

On the stability of the Trojan asteroids*

Antonio Giorgilli[1] **and Charalambos Skokos**[2]

[1] Dipartimento di Matematica dell' Università di Milano, Via Saldini 50, I-20133 Milano, Italy
[2] Department of Physics, University of Athens, Panepistimiopolis, GR-15784 Zografos, Athens, Greece

Received 8 February 1996 / Accepted 11 March 1996

$O(10^8\text{ km}) \approx 0.127\ d(S,J)$

1996

Historical Perspective: Sun-Jupiter L_4

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AMADEO DELSHAMS

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain

ERNEST FONTICH

Departament de Matemàtica Aplicada, ETSEIB, Universitat Politècnica de Barcelona, Barcelona, Spain

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Dipartimento di Matematica dell'Università, Via Saldini 50, Milano, Italy

AND

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$\mathcal{O}(1\text{ km})$

1987

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14 473 km

1990



≈ 51 cm

1997

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$\mathcal{O}(10^8\text{ km}) \approx 0.127\, d(S,J)$

1996

Historical Perspective: Sun-Jupiter L_4

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AMADEO DELSHAMS

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain

ERNEST FONTICH

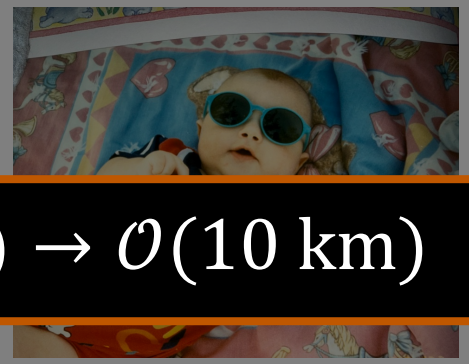
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1997

Scaling from $\mathrm{d}(S,J)$ to $\mathrm{d}(E,M)$: $\mathcal{O}(10\,000\ \mathrm{km}) \rightarrow \mathcal{O}(10\ \mathrm{km})$

Our result: ≈ 38 km

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9 842 km

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$\mathcal{O}(10^8\ \mathrm{km}) \approx 0.127\ \mathrm{d}(S,J)$

1996

Thank you



Effective Stability of Near-Rectilinear Halo Orbits in the Earth-Moon System

Ninth International Meeting on Celestial Mechanics (CELMEC IX)

Luke T. Peterson[1] & Joan Gimeno[2]

15 September 2026

[1] Department of Aerospace Engineering & Engineering Mechanics, UT Austin

[2] Department de Matemàtiques i Informàtica, Universitat de Barcelona



Back-up Slides

Just in case

Cauchy Estimates

Cauchy estimates of the initial map:

$$C_k := \sum_{|j|=k} |G_j^k| \leq \frac{\tilde{C}}{\tilde{\rho}^k}$$

Goal: Estimate size of Taylor coefficients of the initial Poincaré map

Classical technique (see Papa Rudin) to give optimal bounds on derivatives of a holomorphic function.

Proof For analytic maps on $\mathcal{D}_\rho$, the standard Cauchy estimate gives:

$$|G_j^k| \leqslant \frac{1}{\rho^k} \sup_{|s|_\infty < \rho} |G(s)|. \tag{25}$$

Let M be the constant

$$M := \sup_{|s|_\infty < \rho} |G(s)|, \tag{26}$$

so that the previous inequality is written more simply as:

$$|G_j^k| \leqslant \frac{M}{\rho^k}. \tag{27}$$

Now, we bound the C_k by

$$C_k \leqslant \sum_{|j|=k} \frac{M}{\rho^k} = \frac{M}{\rho^k}\binom{n+k-1}{k}, \tag{28}$$

where the binomial coefficient comes from the number of j such that $|j| = k$ in dimension n.

To bound the binomial coefficient, consider

$$\binom{n+k-1}{k} = \frac{(k+n-1)!}{k!(n-1)!} = \frac{1}{(n-1)!}\prod_{m=1}^{n-1}(k+m). \tag{29}$$

For all $k \geqslant 1$ and all $1 \leqslant m \leqslant n-1$,

$$k + m \leqslant k + (n-1), \tag{30}$$

so that

$$\prod_{m=1}^{n-1}(k+m) \leqslant (k+n-1)^{n-1}. \tag{31}$$

Since $k \geqslant 1$, we have that $(n-1) \leqslant (n-1)k$ and thus (adding k to both sides) $k+n-1 \leqslant nk$. Substituting this into the previous bound:

$$\binom{n+k-1}{k} \leqslant \frac{n^{n-1}}{(n-1)!}k^{n-1} =: b \cdot k^{n-1}. \tag{32}$$

Thus, substituting the bound on the binomial coefficient, we obtain the new bound for C_k as:

$$C_k \leqslant \frac{M \cdot b}{\rho^k}k^{n-1}. \tag{33}$$

Next, we need to remove the polynomial growth of the k^{n-1}, a common Nekhoroshev trick. Call $C = M \cdot b$, so that we have

$$C_k \leqslant \frac{C}{\rho^k}k^{n-1}. \tag{34}$$

We want a bound of the form:

$$C_k \leqslant \frac{\tilde{C}}{\rho^k}, \tag{35}$$

with explicit constants. The trick will be to replace k^{n-1} by some exponential in k and slightly shrinking the analyticity radius. Hence, we find a constant for "exponential domination." Fix $\epsilon > 0$. Consider the function for $x > 0$:

$$f(x) := x^{n-1}e^{-\epsilon x}. \tag{36}$$

This function f is smooth and has a global maximum since $f \to 0$ as $x \to \infty$. Taking its derivative, we get

$$f'(x) = x^{n-1}e^{-\epsilon x}\left((n-1) - \epsilon x\right). \tag{37}$$

So, the critical point is: $x_* = (n-1)/\epsilon$. And the maximum value occurs at x_* and is

$$f(x_*) = \left(\frac{n-1}{\epsilon e}\right)^{n-1} e^{\epsilon x}. \tag{38}$$

The discrete version for integers $k \geqslant 1$ gives the bound

$$k^{n-1} \leqslant \left(\frac{n-1}{\epsilon e}\right)^{n-1} e^{\epsilon k}. \tag{39}$$

Hence, we use this bound on the polynomial growth to obtain a uniform bound on the C_k via:

$$C_k \leqslant \frac{C}{R^k} \leqslant \frac{C}{R^k}\left(\frac{n-1}{\epsilon e}\right)^{n-1} e^{\epsilon k}. \tag{40}$$

Denote by $\tilde{C}$ the new constant term

$$\tilde{C} := C \cdot (n - 1/\epsilon e)^{n-1}. \tag{41}$$

Convert the remaining exponential into a slightly smaller radius

$$\frac{1}{\rho^k}e^{\epsilon k} = \frac{1}{(\rho e^{-\epsilon})^k}. \tag{42}$$

Define $\tilde{\rho} := \rho e^{-\epsilon} < \rho$, and note that $\tilde{\rho} \to \rho$ as $\epsilon \to 0$. Finally, we obtain the desired Cauchy estimate:

$$C_k \leqslant \frac{\tilde{C}}{\tilde{\rho}^k}. \tag{43}$$

□

Coefficient Growth due to Normalization

Iterative Lemma:

$$\epsilon_N := \| R^{(N)} \|_{\rho_N} \leq \epsilon_2 \left(\frac{2N^\tau}{\gamma} \right)^N$$

Lemma 2 (Coordinate Transformation) *Let $\rho > 0$ be the radius of the polydisc $\mathcal{D}_\rho \subset \mathbb{C}^n$. Let $G^{k,(k-1)}(s)$ be the homogeneous polynomial of exact degree k representing the un-normalized terms after $k-1$ steps. Assume the linear frequencies satisfy the Diophantine condition $|\lambda_i - \lambda^j| \geqslant \frac{\gamma}{k^\tau}$ for $\tau \geqslant 2$ and for all $|j| = k$.*

Let $\delta = \frac{\rho}{6N}$ be the domain shrinking factor. Then, the homogeneous polynomial $\chi_k(s)$ of degree k that solves the homological equation satisfies:

1. *Function bound: $\|\chi_k\|_\rho \leqslant \frac{k^\tau}{\gamma} \|R^{k,(k-1)}\|_\rho$;*
2. *Derivative bound (Cauchy estimate): $\|\mathrm{D}\chi_k\|_{\rho-\delta} \leqslant \frac{k^{\tau+1}}{\rho \cdot \gamma} \|R^{k,(k-1)}\|_\rho$.*

Moreover, if the un-normalized terms are small enough to satisfy the threshold condition

$$\frac{k^\tau}{\delta\gamma} \|R^{k,(k-1)}\|_\rho \leqslant \frac{1}{4}, \tag{46}$$

then the coordinate transformation $c_k(s) = s - \chi_k(s)$ satisfies the following diffeomorphism guarantees:

3. *Forward Map: c_k is a biholomorphism onto its image, mapping $\mathcal{D}_{\rho-\delta} \to \mathcal{D}_\rho$;*
4. *Inverse Map: The inverse transformation c_k^{-1} is rigorously defined as an analytic mapping $c_k^{-1} : \mathcal{D}_{\rho-2\delta} \to \mathcal{D}_{\rho-\delta}$.*

Lemma 3 (Composition and Remainder Bound) *Assume the hypotheses of Lemma 2. Furthermore, assume the intermediate map $G^{(k-1)}$ satisfies the domain bound $\|G^{>2,(k-1)}\|_{\rho-2\delta} \leqslant \delta/2$. Then, the newly composed map after the k-th normalization step, $G^{(k)} = c_k^{-1} \circ G^{(k-1)} \circ c_k$ satisfies:*

1. *Domain of Analyticity (Topological Validity): The full map $G^{(k)}$ is a well-defined analytic function on the shrunken polydisc $\mathcal{D}_{\rho-3\delta}$, and it maps this domain strictly into $\mathcal{D}_\rho$.*
2. *Recursive Remainder Bound: There exists a universal constant $C > 0$ such that the norm of the new remainder is bounded by:*

$$\begin{aligned} \|R^{(k)}\|_{\rho_k} \leqslant \|R^{>k,(k-1)}\|_{\rho_{k-1}} + \frac{k^\tau}{\gamma} \|R^{k,(k-1)}\|_{\rho_{k-1}} \\ + \|R^{k,(k-1)}\|^2_{\rho_{k-1}} \left[\frac{k^{\tau+1}}{\rho_{k-1}\gamma} + \frac{k^{2\tau}}{\gamma^2} \left(\frac{1}{\delta} + \frac{k}{\rho_{k-1}} \right) \right] \end{aligned} \tag{73}$$

Lemma 4 (Iterative Lemma) *Assume the hypotheses of Lemma 2 and 3 hold for all steps $k \leqslant N$. Define the quantities:*

$$\rho_k := \rho_{k-1} - 3\delta, \qquad \delta = \frac{\rho}{6N}, \tag{110}$$

$$\epsilon_k = \|R^{(k)}\|_{\rho_k} \tag{111}$$

Define the linear small divisor and quadratic penalties by

$$L := \frac{N^\tau}{\gamma}, \qquad Q := \frac{2N^{\tau+1}}{\rho\gamma} + \frac{8N^{2\tau+1}}{\rho\gamma^2}, \tag{112}$$

respectively. Suppose the initial perturbation ϵ_2 satisfies the strict threshold condition

$$\epsilon_2 \leqslant \frac{L}{Q(1+2L)^{N-2}}. \tag{113}$$

Then, for all $2 \leqslant k \leqslant N$, the quadratic remainder terms remain strictly bounded by the linear terms, $Q\epsilon_{k-1} \leqslant L$, and the sequence of remainders ϵ_k satisfies the recursive bound:

$$\epsilon_k \leqslant \epsilon_{k-1}(1+2L) \leqslant \epsilon_{k-1} \left(\frac{2N^\tau}{\gamma} \right). \tag{114}$$

Consequently, unrolling this recursion yields the factorial-type bound:

$$\epsilon_N \leqslant \epsilon_2 \left(\frac{2N^\tau}{\gamma} \right)^N. \tag{115}$$

Truncated Remainder Bound

Let $a < \frac{\rho}{2}$ be a target physical confinement radius.
Then:

$$\| R^{(N)} \|_a \leq \epsilon_2 \left(\frac{2a}{\rho}\right)(AaN^\tau)^N, \qquad A = \frac{4}{\rho\gamma}$$

Goal: Translate coefficient bound into geometric bound on the actual remainder function evaluated over a specific domain.

Final component before optimizing truncation order (in the usual way).

Lemma 5 (Truncated Remainder Bound) *Let the assumptions of Lemma 4 hold. Let $a > 0$ be a target physical radius such that $a < \rho/2$. Then, the remainder $R^{(N)}$ after N normalization steps, evaluated on the strictly smaller polydisc $\mathcal{D}_a$, satisfies the exact algebraic bound:*

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2a}{\rho}\right)(AaN^\tau)^N, \qquad A = \frac{4}{\rho\gamma}. \tag{130}$$

Proof By the geometric definition of domain shrinkage, the radius of analyticity after N normalization steps is:

$$\rho_N = \rho - \sum_{k=1}^{N} 3\delta = \rho - 3N\left(\frac{\rho}{6N}\right) = \rho - \frac{\rho}{2} = \frac{\rho}{2} \tag{131}$$

Thus, by Lemma 4, the remainder $R^{(N)}$ is a well-defined analytic function on the polydisc $\mathcal{D}_{\rho/2}$ and its norm is bounded by:

$$\|R^{(N)}\|_{\rho/2} = \epsilon_N \leqslant \epsilon_2 \left(\frac{2N^\tau}{\gamma}\right)^N \tag{132}$$

Because N steps of the normal form procedure have been completed, all non-resonant terms up to and including degree N have been eliminated from the remainder. Consequently, the Taylor series expansion of $R^{(N)}$ around the origin begins with terms of at least degree $N+1$.

By the general Schwarz Lemma (or standard Cauchy bounds), the norm of such a function on a smaller interior domain $\mathcal{D}_a$ (where $a < \rho_N$) scales strictly by the ratio of the radii to the power of the lowest non-vanishing degree. Therefore, projecting from ρ_N down to a, we obtain:

$$\|R^{(N)}\|_a \leqslant \|R^{(N)}\|_{\rho_N} \left(\frac{a}{\rho_N}\right)^{N+1}. \tag{133}$$

Substituting $\rho_N = \rho/2$ yields the spatial decay factor:

$$\|R^{(N)}\|_a \leqslant \epsilon_N \left(\frac{2a}{\rho}\right)^{N+1}. \tag{134}$$

Finally, inserting the recursive bound from Lemma 4, we obtain:

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2N^\tau}{\gamma}\right)^N \left(\frac{2a}{\rho}\right)^{N+1}. \tag{135}$$

Factoring out one power of $(2r\rho)$ aligns the exponents:

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2a}{\rho}\right)\left(\frac{4aN^\tau}{\rho\gamma}\right)^N. \tag{136}$$

Finally, defining the geometric constant $A = \frac{4}{\rho\gamma}$ completes the proof. □

Optimal Normalization Order

Factorial growth of coefficients will overtake geometric decay of high-order polynomials.

Halt normalization at the optimized order to minimize magnitude of the remainder:

$$N_{\text{opt}} := \text{argmin}_N \, (AaN^\tau)^N = \left\lfloor \frac{1}{e} \left(\frac{1}{Aa} \right)^{\frac{1}{\tau}} \right\rfloor$$

Proposition 6 (Optimal Truncation) *Assume the remainder after N normalization steps satisfies the bound:*

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2a}{\rho} \right) (AaN^\tau)^N, \qquad A = \frac{4}{\rho\gamma} \tag{137}$$

with $\epsilon_2, \rho, \gamma, \tau > 0$ constants independent of N and a. For a fixed physical radius a, satisfying the strict algebraic condition

$$a \leqslant \frac{1}{A(2e)^\tau} \tag{138}$$

the optimal number of normalization steps, $N_{opt} \geqslant 2$, that minimizes the remainder bound is:

$$N_{opt} = \left\lfloor \frac{1}{e} \left(\frac{1}{Aa} \right)^{\frac{1}{\tau}} \right\rfloor. \tag{139}$$

Evaluated at N_{opt}, the remainder is exponentially small with respect to a, satisfying

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2a}{\rho} \right) \exp(-\tau N_{opt}). \tag{140}$$

Proof To prove the upper bound of the remainder, we isolate the factor that depends on the normalization order N. Let

$$g(N) = (AaN^\tau)^N, \tag{141}$$

where N here is thought of as a continuous (i.e., not discrete) variable. Since the pre-factor $\epsilon_2(2a/\rho) > 0$ and is independent of N, it suffices to minimize g. Moreover, since $g(N) > 0$ for $N \geqslant 1$, the minimum of $g(N)$ coincides with the minimum of $\ln g(N)$, so we will minimize the latter:

$$\ln g(N) = N \ln(AaN^\tau) = N \ln(Aa) + \tau N \ln N \tag{142}$$

Computing the derivative with respect to N, we get:

$$\frac{d}{dN} \ln g(N) = \ln(AaN^\tau) + \tau \tag{143}$$

Setting the derivative to zero yields

$$\ln(AaN^\tau) = -\tau, \tag{144}$$

and exponentiating gives

$$AaN^\tau = e^{-\tau}. \tag{145}$$

Solving for N, we obtain:

$$N = \frac{1}{e} \left(\frac{1}{Aa} \right)^{\frac{1}{\tau}} \tag{146}$$

Note that this is a minimum by

$$\frac{d^2}{dN^2} \ln g(N) = \frac{\tau}{N} > 0. \tag{147}$$

Because the normal form procedure requires a natural number of steps, we select the optimal discrete order by taking the floor

$$N_{\text{opt}} = \left\lfloor \frac{1}{e} \left(\frac{1}{Aa} \right)^{\frac{1}{\tau}} \right\rfloor. \tag{148}$$

Let us show now that $N_{\text{opt}} \geqslant 2$. If $a \leqslant \frac{1}{A(2e)^\tau}$, we can simply plug in this value of a to the expression for N_{opt} to find

$$N_{\text{opt}} \geqslant \left\lfloor \frac{1}{e} \left(\frac{1}{A\left(\frac{1}{A(2e)^\tau}\right)} \right)^{\frac{1}{\tau}} \right\rfloor = \left\lfloor \frac{1}{e} (2e)^{\tau \cdot \frac{1}{\tau}} \right\rfloor = 2. \tag{149}$$

To compute the magnitude of the remainder at this optimal order, we substitute the exact optimal relationship back into the continuous function $g(N)$:

$$g(N_{\text{opt}}) = (AaN_{\text{opt}})^{N_{\text{opt}}} \leqslant (e^{-\tau})^{N_{\text{opt}}} = \exp(-\tau N_{\text{opt}}). \tag{150}$$

Multiplying by the N-independent pre-factor, the total optimal remainder bound is

$$\|R^{(N)}\|_a \leqslant \epsilon_2 \left(\frac{2a}{\rho} \right) \exp(-\tau N_{\text{opt}}). \tag{151}$$

This completes the proof. □

Radial Drift

Let $(z, \overline{z})$ be the complexified NF coordinates. For each $i = 1, \ldots, n$, define the orbital radius: $r_i^2 = z_i \overline{z}_i$

The one-step drift in radius is bounded by:

$$|\Delta r_i| \leq \| R^{(N_{\text{opt}})} \|_a \leq \epsilon_{\text{opt}}$$

Proposition 7 (Radial Drift Bound) *Let $(z, \bar{z})$ be the complexified normal form coordinates in the local polydisc $\mathcal{D}_r$, where $z_i = x_i + \mathrm{i} y_i$. For each degree of freedom, $j = 1, \ldots, n$, we define the orbital radius in the i-th phase plane as $r_i^2 = z_i \bar{z}_i$. Under the discrete Poincaré map $G^{N_{opt}}$, the one-step drift in the radius $\Delta r_i := r_i^{(1)} - r_i^{(0)}$ is strictly bounded by the exponentially small remainder. Specifically, for any point in $\mathcal{D}_a$:*

$$|\Delta r_i| \leqslant \|R^{(N_{opt})}\|_a \tag{152}$$

where $\|R^{(N_{opt})}\|_r \leqslant \epsilon_{N_{opt}}$, the exponentially small upper bound derived in Proposition 5.

Proof In normalized coordinates, the Poincaré map decomposes into an integrable twist map, $F^{(N_{\text{opt}})}$, and the truncated remainder, $R^{(N_{\text{opt}})}$, where the i-th element of the integrable twist map is given by:

$$F_i^{(N_{\text{opt}})}(z^{(0)}, \bar{z}^{(0)}) = z_i^{(0)} \exp[\mathrm{i}\omega_i(r)]. \tag{153}$$

Consequently, the twist map is an exact isometry on the radii, hence preserving the radius

$$|F_i^{(N_{\text{opt}})}| = |z_i^{(0)}| = r_i^{(0)}. \tag{154}$$

Applying the map, the updated i-th coordinate is

$$z_i^{(1)} = F_i^{(N_{\text{opt}})}(z^{(0)}, \bar{z}^{(0)}) + R_i^{(N_{\text{opt}})}(z^{(0)}, \bar{z}^{(0)}). \tag{155}$$

To bound the drift of the radius, $r_i^{(1)} = |z_i^{(1)}|$, we just apply the reverse triangle inequality:

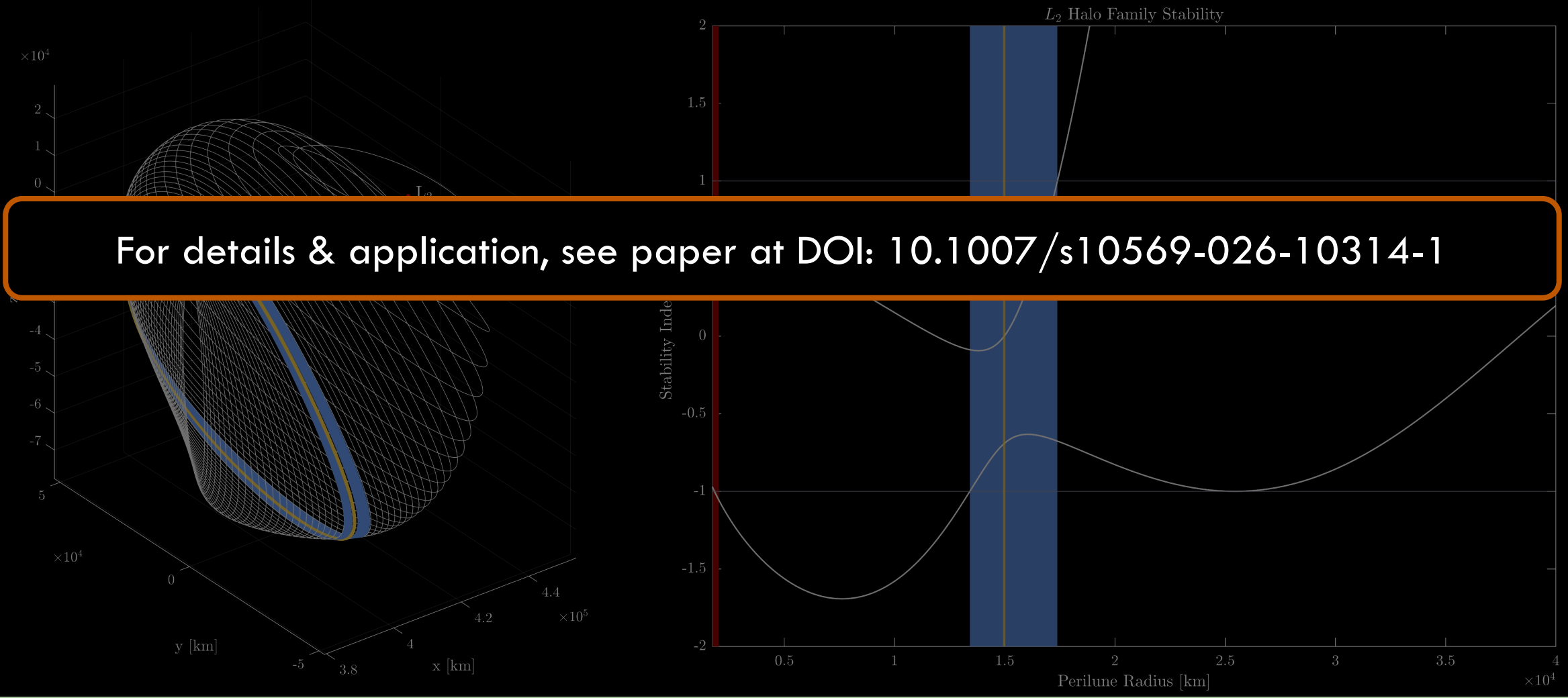
$$|r_i^{(1)} - r_i^{(0)}| = \left| |z_i^{(1)}| - |F_i^{(N_{\text{opt}})}| \right| \leqslant \left| z_i^{(1)} - F_i^{(N_{\text{opt}})} \right| = |R_i^{(N_{\text{opt}})}|. \tag{156}$$

Taking the supremum over the domain $\mathcal{D}_a$ immediately yields the result:

$$|\Delta r_i| \leqslant \|R_i^{(N_{\text{opt}})}\|_a. \tag{157}$$

as desired. □

Example: Earth-Moon L_2 NRHO



For details & application, see paper at DOI: 10.1007/s10569-026-10314-1

Why did we pick a "bad" example?

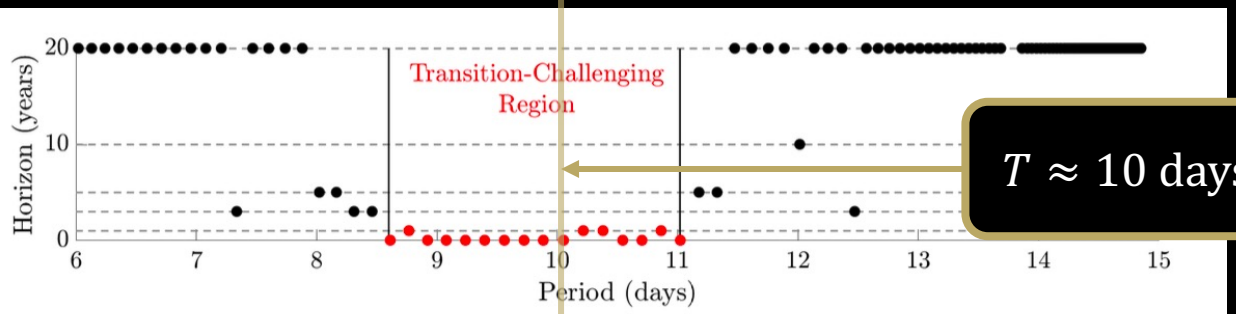


Fig. 1 ***Transition-challenging*** **region within the Earth-Moon L_2 halo family, recreated from Park and Howell, Figure 2. [11]**

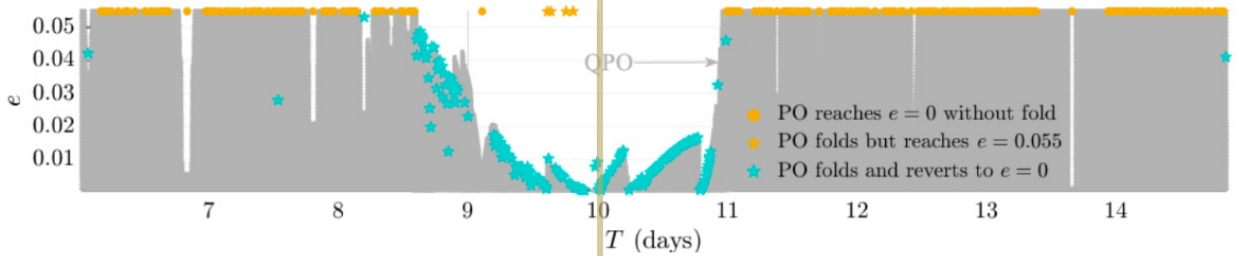


Fig. 15 Global trend across the Earth-Moon L_2 halo family

$T \approx 10$ days

