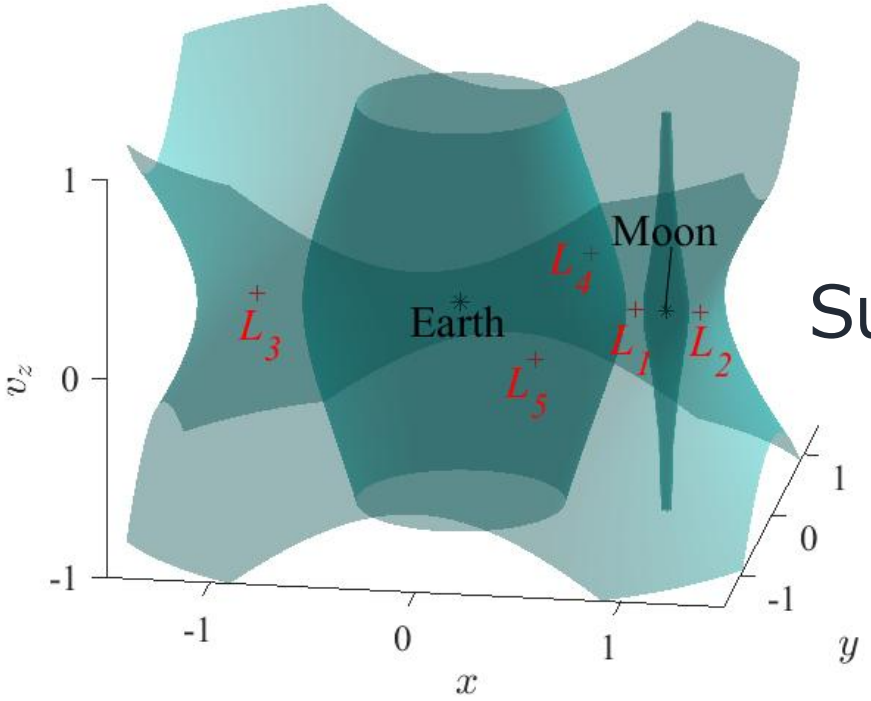
The Ninth International Meeting on Celestial Mechanics

14–18 September 2026, San Martino al Cimino, Viterbo, Italy

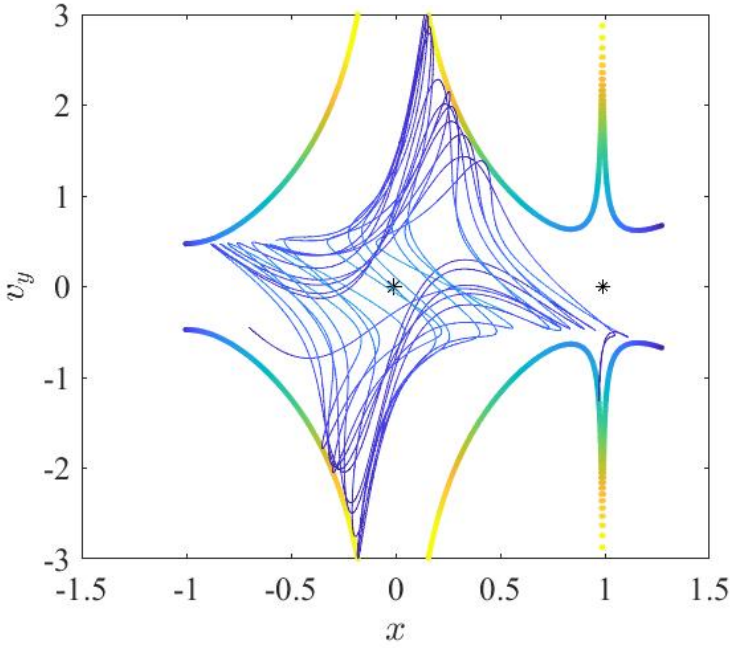
Novel Barrier Structures in the Circular Restricted Three-Body Problem



Kenta Oshima

Suwa University of Science

Dept. of Mechanical & Electrical Engineering



What I Have Today

Part 1: Barrier structure in **3D** CR3BP

Celestial Mechanics and Dynamical Astronomy (2024) 136:60
https://doi.org/10.1007/s10569-024-10230-2

ORIGINAL ARTICLE

A hidden barrier surface complementary to the zero velocity surface in the circular restricted three-body problem

Kenta Oshima[1]

Part 2: Barrier structure in **2D** CR3BP

Celestial Mechanics and Dynamical Astronomy (2025) 137:29
https://doi.org/10.1007/s10569-025-10260-4

ORIGINAL ARTICLE

On the concept of conditional barrier structures in the planar circular restricted three-body problem

Kenta Oshima[1]

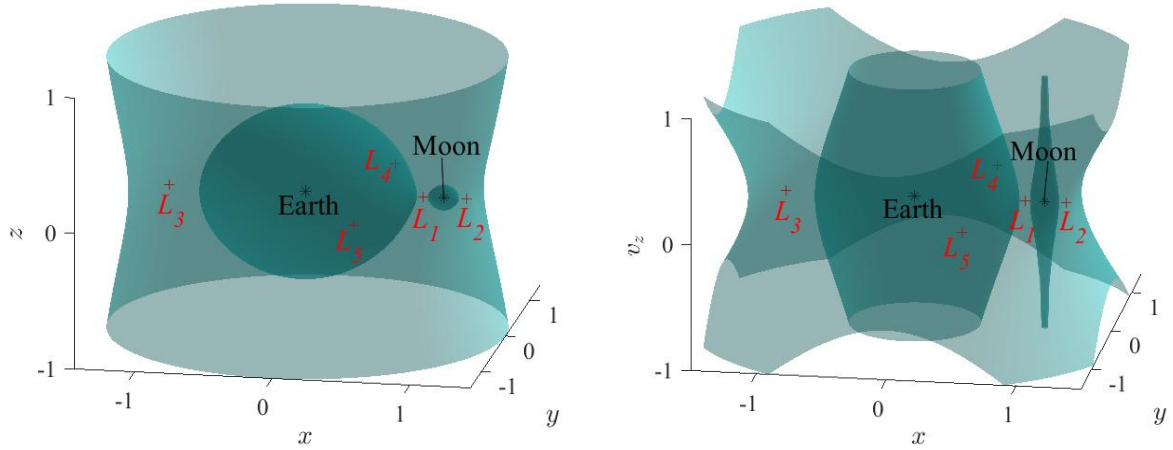
Part 1: Barrier Structure in 3D CR3BP

1. Background

2. Global Minimization of Energy Function

Zero-velocity surface **Zero-$v_x v_y z$ surface**

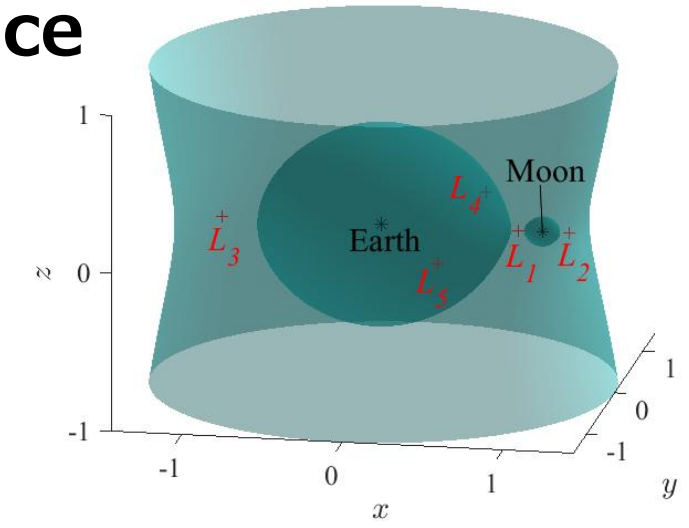
3. Comparative Roles of ZVS and ZvxvyzS



Energy Function $E(\boldsymbol{r}, \boldsymbol{v}) = \frac{1}{2}|\boldsymbol{v}|^2 + \bar{U}(\boldsymbol{r})$

Since $|\boldsymbol{v}|^2 > 0$ holds when the motion is energetically possible,

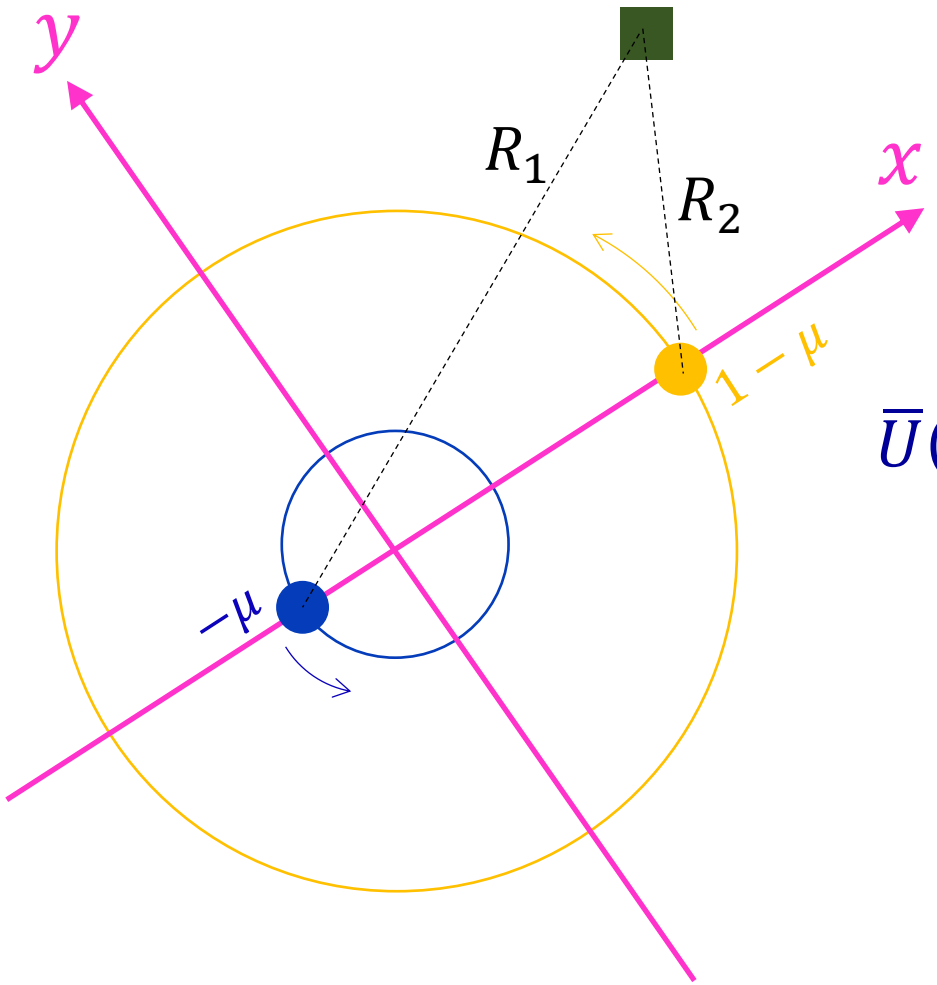
- $E^* - \bar{U}(\boldsymbol{r}) > 0$: **A region of possible motion (Hill's region)**
- $E^* - \bar{U}(\boldsymbol{r}) < 0$: **A region of impossible motion (forbidden region)**
- $E^* - \bar{U}(\boldsymbol{r}) = 0$: **A barrier structure** that separates the regions of possible and impossible motion in **position space**
  - **Zero velocity curve (ZVC)** in 2-DOF system
  - **Zero velocity surface (ZVS)** in 3-DOF system



* Energy level: $E = E^*$

Our Central Idea

$$E(\boldsymbol{r},\boldsymbol{v}) = \frac{1}{2}|\boldsymbol{v}|^2 + \boxed{\bar{U}(\boldsymbol{r})}$$



Why not examine the potential??

The effective potential in the rotating frame:

$$\bar{U}(x,y,z) = -\frac{1}{2}(x^2+y^2) - \frac{1-\mu}{R_1(x,y,z)} - \frac{\mu}{R_2(x,y,z)}$$

- Not so simple as the kinetic energy term
- **A new analysis method** is sought

Part 1: Barrier Structure in 3D CR3BP

1. Background

2. Global Minimization of Energy Function

3. Comparative Roles of ZVS and ZvxvyzS

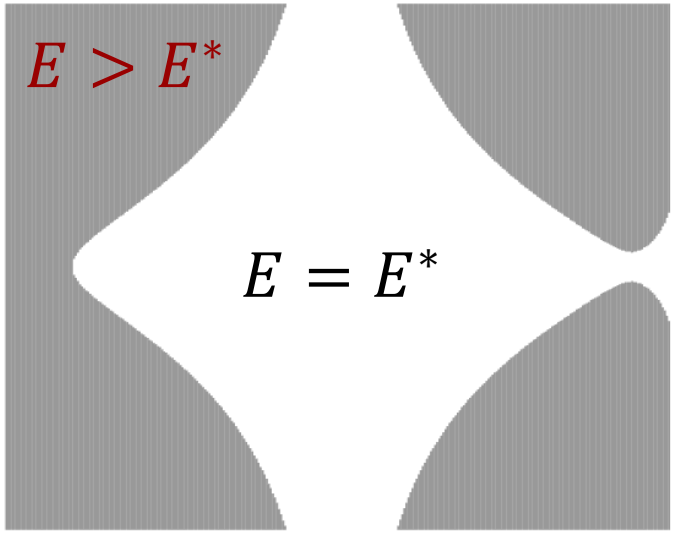
Global Energy Minimization Property

Phase-space variables : $x_1, x_2, x_3, x_4, x_5, x_6$ (3-DOF autonomous system)

We define **the forbidden region** at an energy level E^* in $x_1x_2x_3$-space s.t.

The states in the region result in $E > E^*$ whatever real values x_4, x_5, x_6 take.

*Then, states in the region cannot be defined at the energy level E^* (energetically forbidden!)



Global Energy Minimization Property

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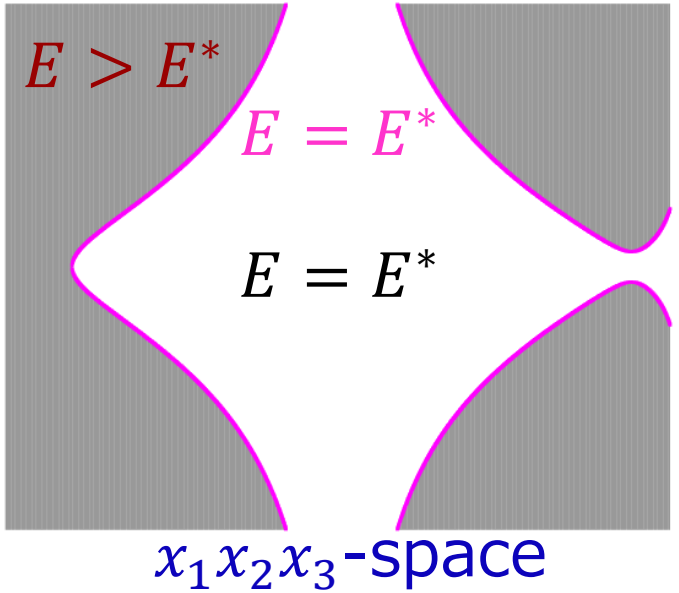
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The states in the region result in $E > E^*$ whatever real values x_4, x_5, x_6 take.

*Then, states in the region cannot be defined at the energy level E^* (energetically forbidden!)

On its **boundary** that separates the regions of possible and impossible motion in $x_1x_2x_3$-space,

a set of x_4, x_5, x_6 that can realize $E = E^*$ shall **globally minimize the energy function**

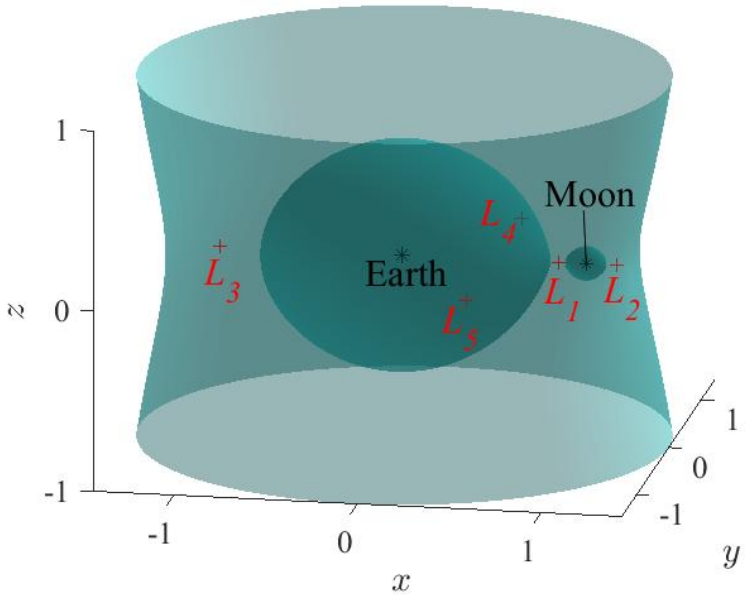


ZVS (Zero-Velocity Surface)

$$\frac{\partial E}{\partial v_x} = v_x, \qquad \frac{\partial^2 E}{\partial {v_x}^2} = 1$$

$$\frac{\partial E}{\partial v_y} = v_y, \qquad \frac{\partial^2 E}{\partial {v_y}^2} = 1$$

$$\frac{\partial E}{\partial v_z} = v_z, \qquad \frac{\partial^2 E}{\partial {v_z}^2} = 1$$

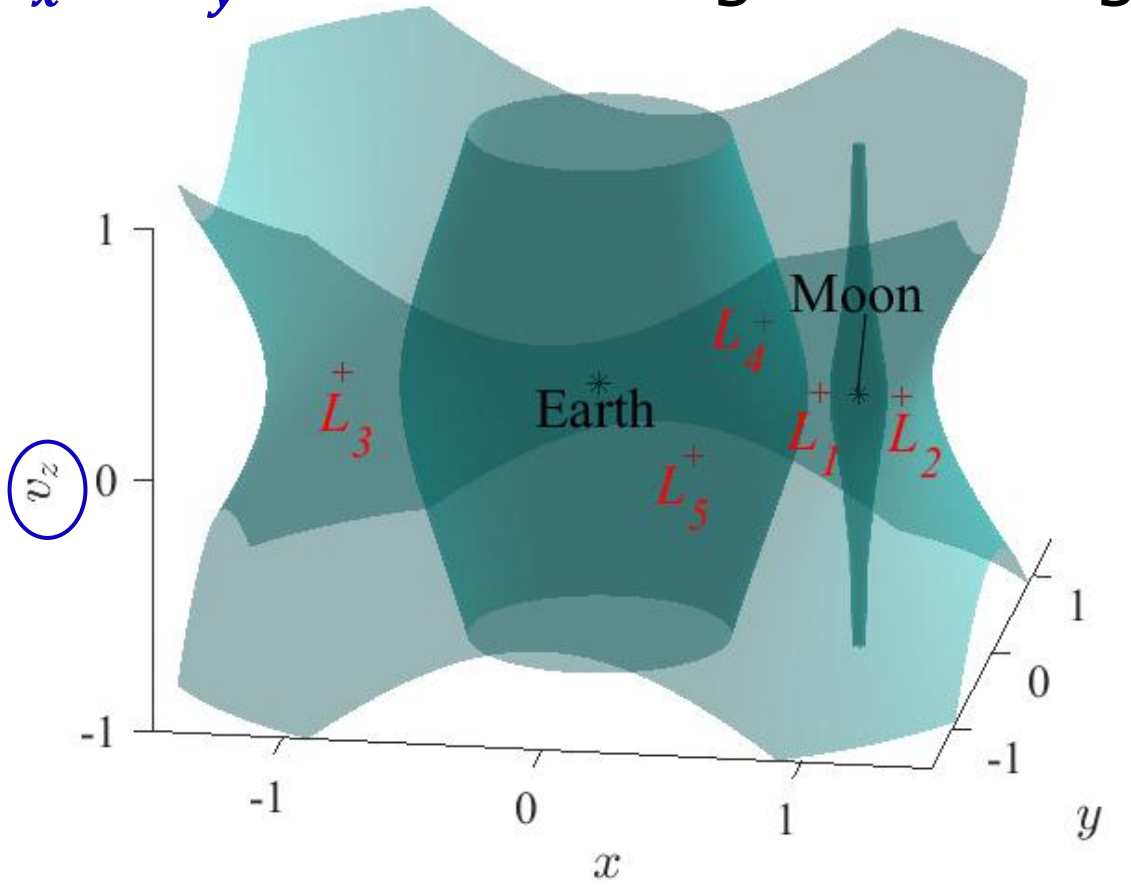


$v_x = v_y = v_z = 0$ globally minimizes the energy function

$$E = \frac{1}{2}\left({v_x}^2 + {v_y}^2 + {v_z}^2\right) + \bar{U}(x, y, z), \qquad \bar{U} = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1(x, y, z)} - \frac{\mu}{r_2(x, y, z)}$$

ZvxvyzS (Zero–$v_x v_y z$ Surface)

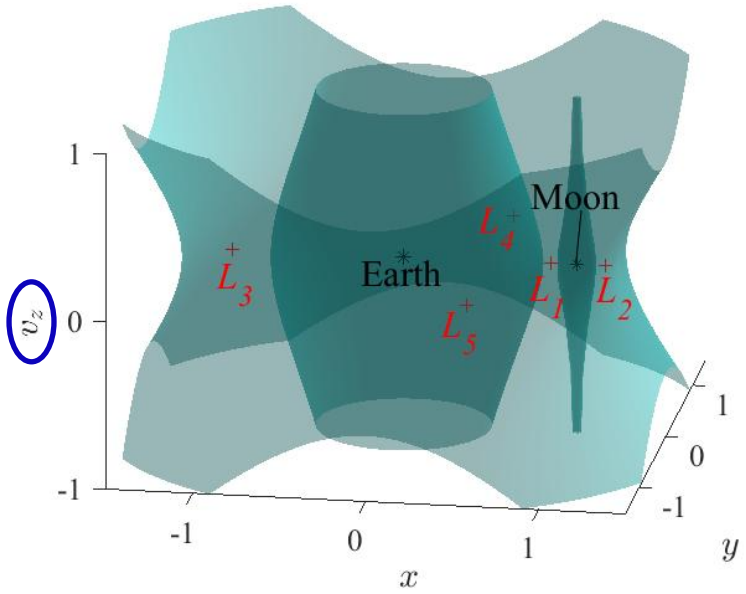
A surface in the xyv_z-space generated by a set of points satisfying $v_x = v_y = z = 0$ at a given energy level



ZvxvyzS (Zero–$v_x v_y z$ Surface)

$$\frac{\partial E}{\partial v_x} = v_x, \qquad \frac{\partial^2 E}{\partial {v_x}^2} = 1 \qquad \frac{\partial E}{\partial v_y} = v_y, \qquad \frac{\partial^2 E}{\partial {v_y}^2} = 1$$

$$\frac{\partial E}{\partial z} = z\left(\underbrace{\frac{1-\mu}{{R_1}^3} + \frac{\mu}{{R_2}^3}}_{>0}\right), \qquad \frac{\partial^2 E}{\partial z^2} = \underbrace{\frac{1-\mu}{{R_1}^3} + \frac{\mu}{{R_2}^3}}_{>0} - 3z^2\left(\frac{1-\mu}{{R_1}^5} + \frac{\mu}{{R_2}^5}\right)$$



$v_x = v_y = z = 0$ globally minimizes the energy function

$$E = \frac{1}{2}\left({v_x}^2 + {v_y}^2 + {v_z}^2\right) + \bar{U}(x, y, z), \qquad \bar{U} = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1(x, y, z)} - \frac{\mu}{r_2(x, y, z)}$$

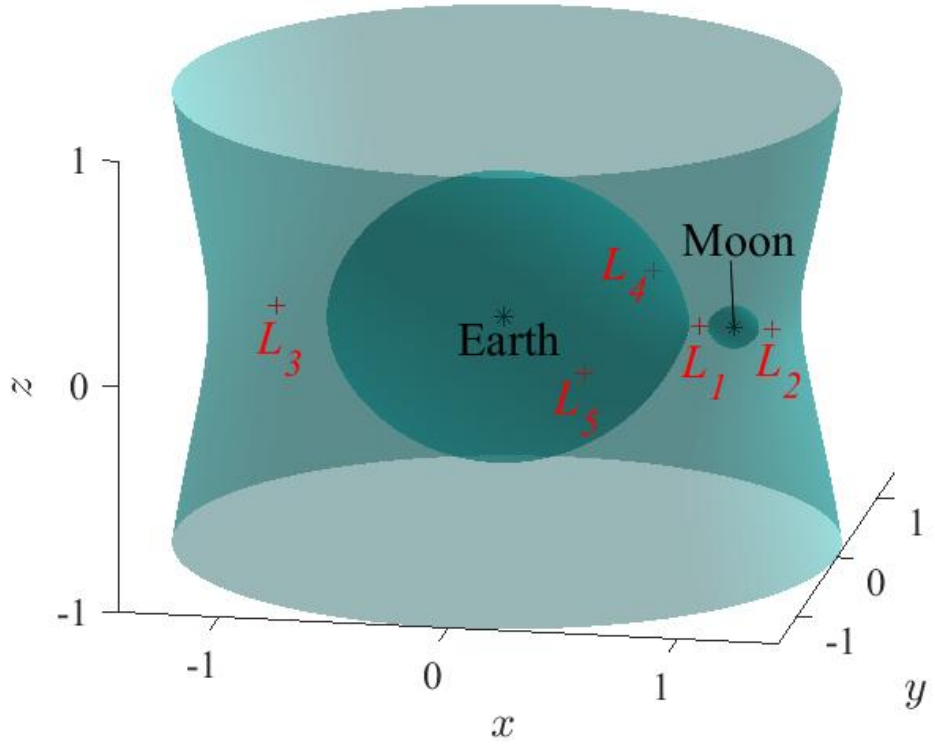
Part 1: Barrier Structure in 3D CR3BP

1. Background
2. Global Minimization of Energy Function
3. **Comparative Roles of ZVS and ZvxvyzS**

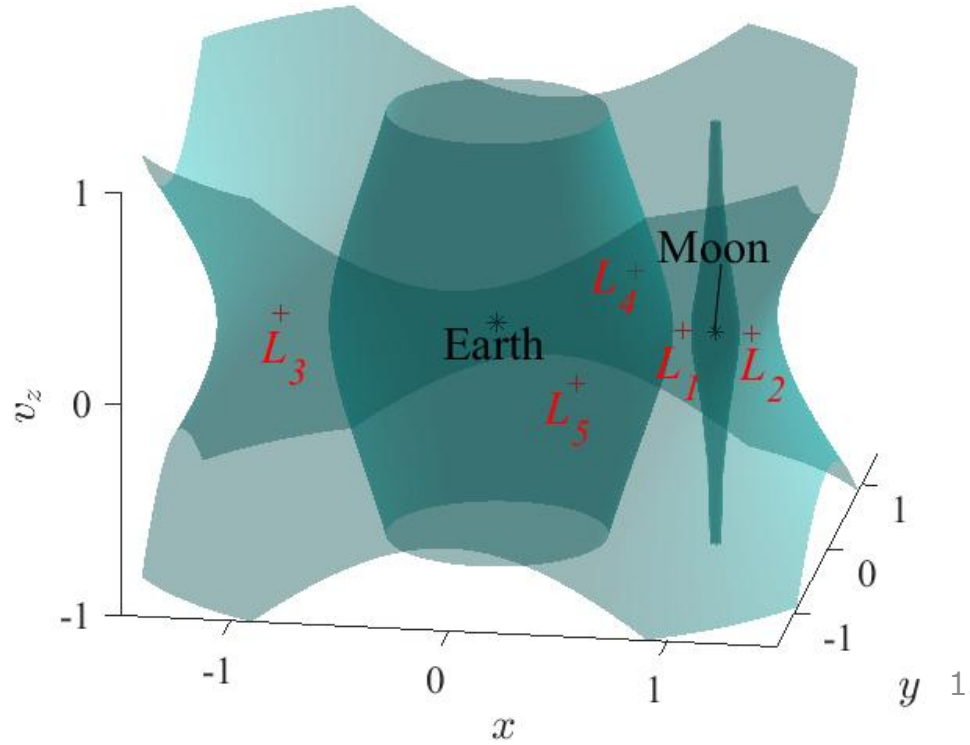
Structure Comparison

- Similarly to the ZVS, the ZvxvyzS divides the region of possible motion
- Distinctively from the ZVS, **the ZvxvyzS extends to infinity w.r.t.** $|v_z|$ near the gravitational singularities

ZVS



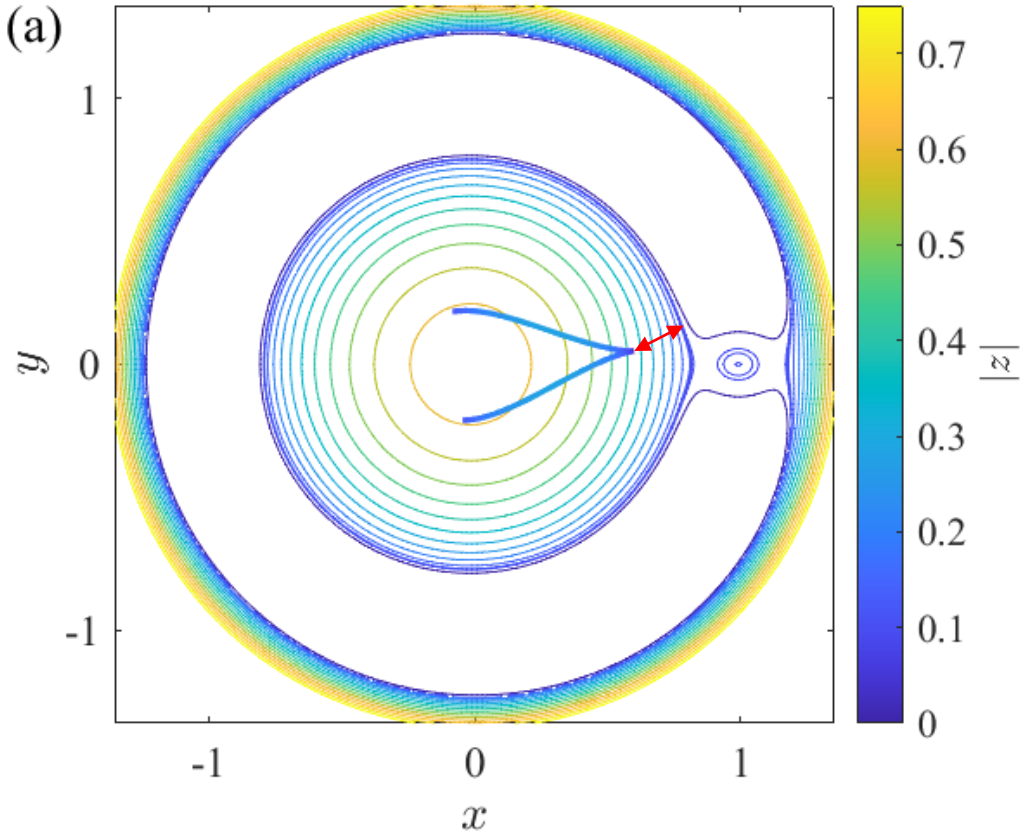
ZvxvyzS



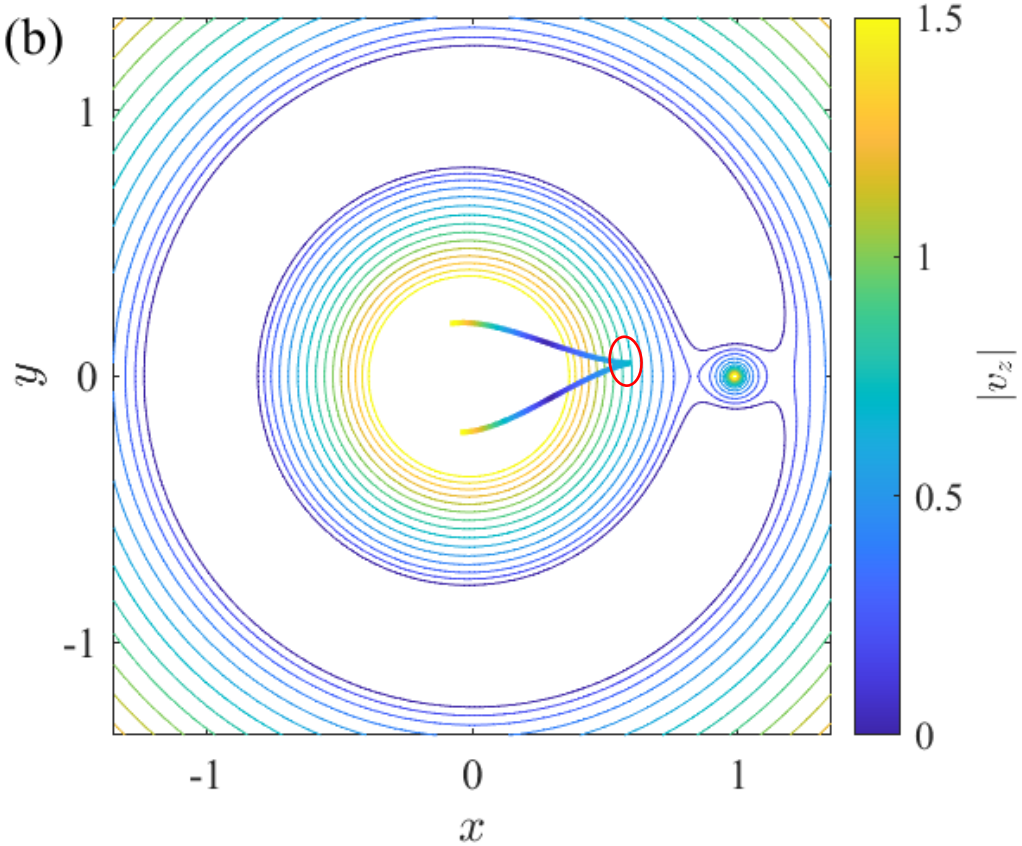
Out-of-Plane Dynamics

	$\lvert z\rvert$	$\lvert v_z\rvert$
ZVS	**Maximum**	**Minimum (=0)**
ZvxvyzS	**Minimum (=0)**	**Maximum**

ZVS



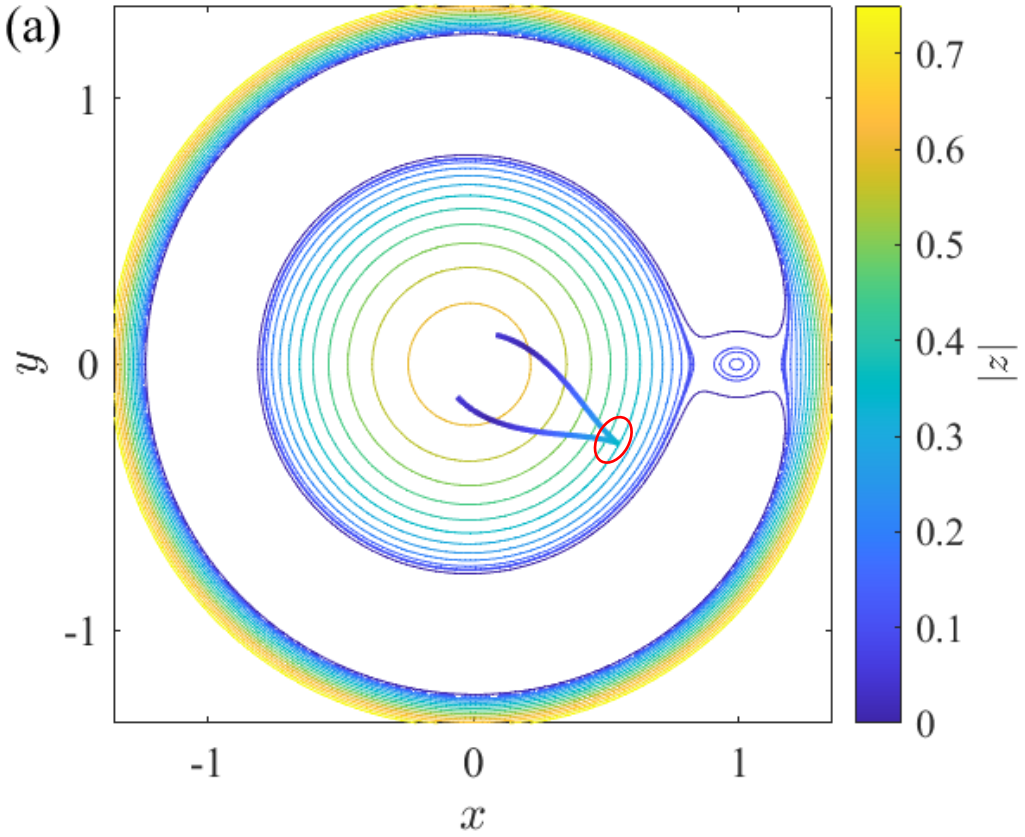
ZvxvyzS



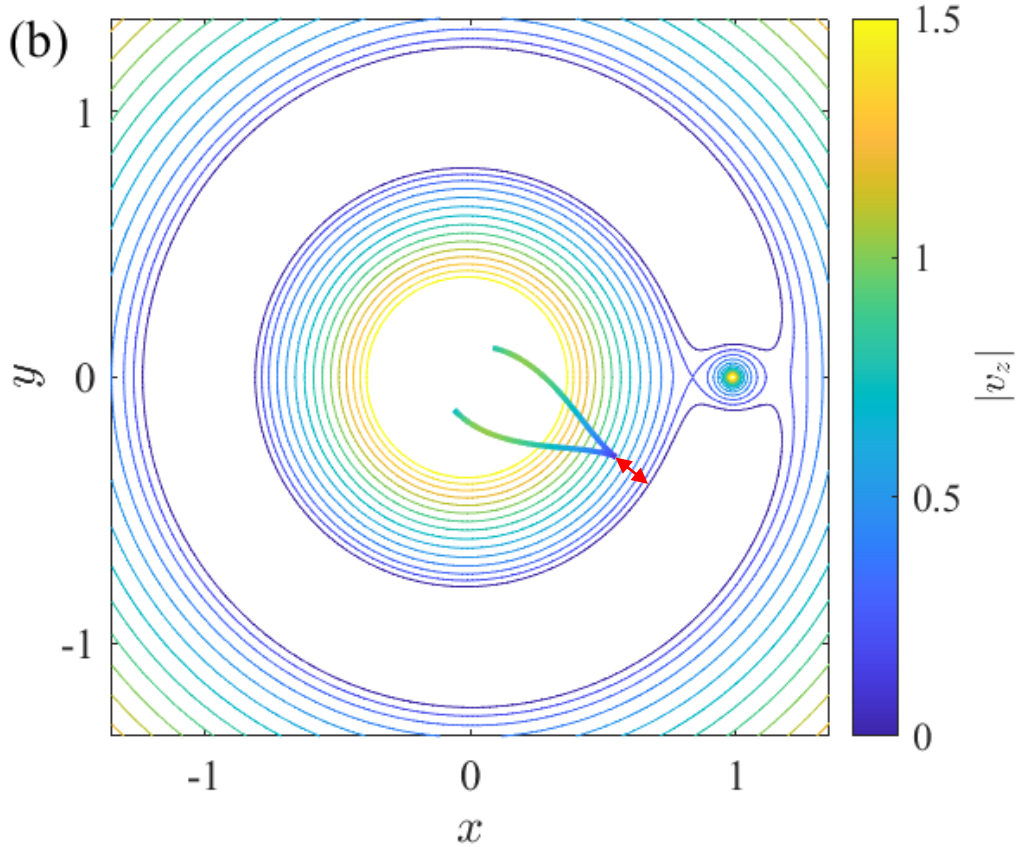
Out-of-Plane Dynamics

	$\lvert z\rvert$	$\lvert v_z\rvert$
ZVS	**Maximum**	**Minimum (=0)**
ZvxvyzS	**Minimum (=0)**	**Maximum**

ZVS



ZvxvyzS



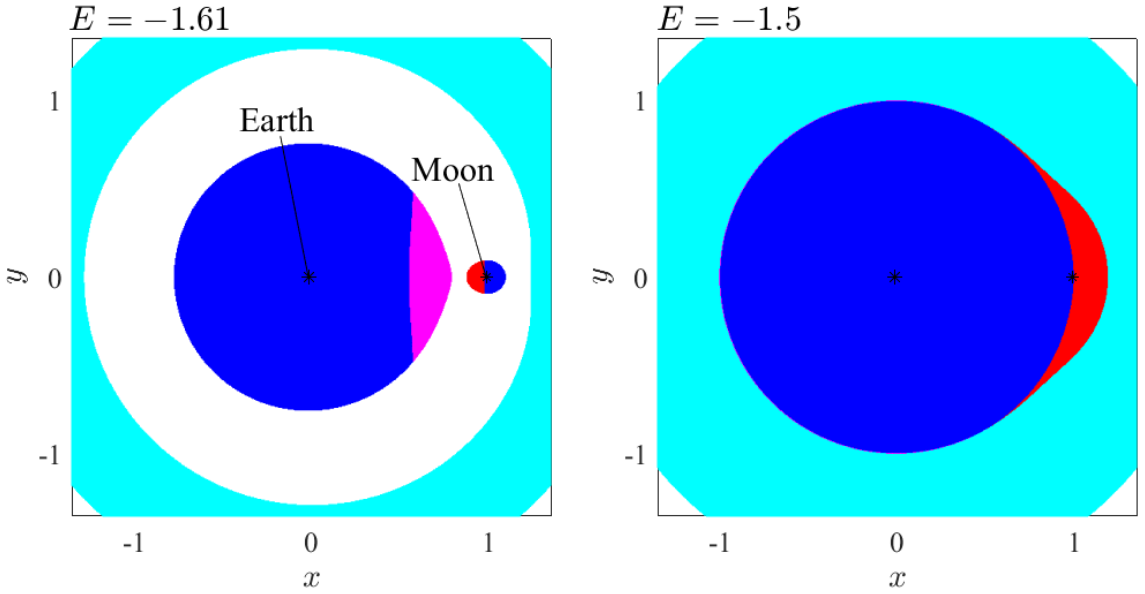
Apsidal States

Apogee & apolune
Apogee & perilune
Perigee & apolune
Perigee & perilune

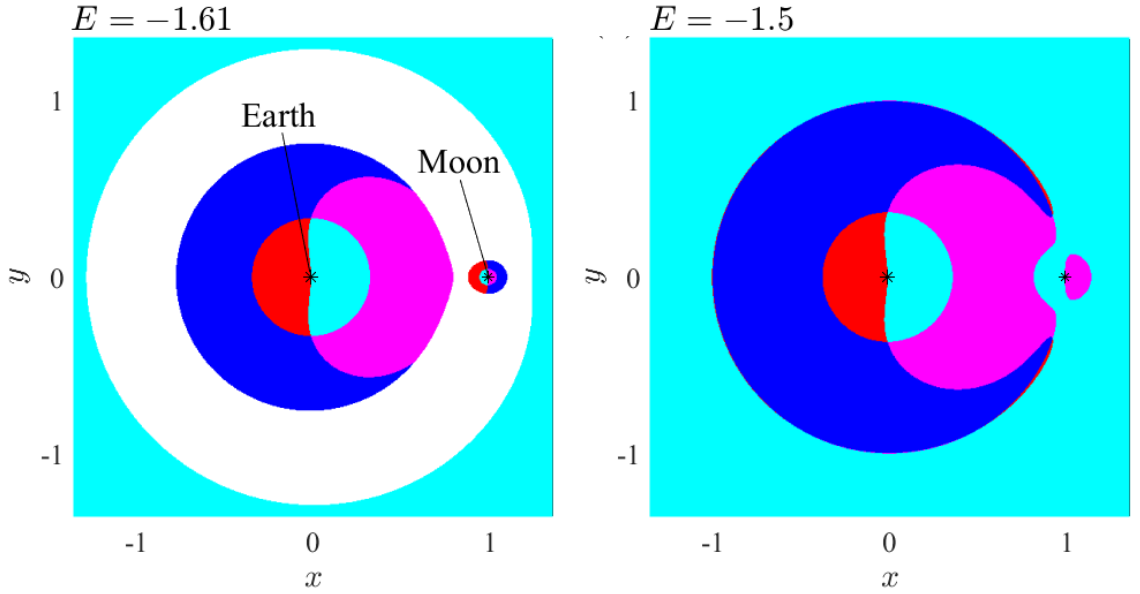
*** States on ZVS and ZvxvyzS are at apsis with respect to Earth and the Moon**

- The ZVS bounds the motion at apoapsis around the bodies
- Distinctively from the ZVS, **the ZvxvyzS bounds the motion at periapsis near the bodies**

ZVS

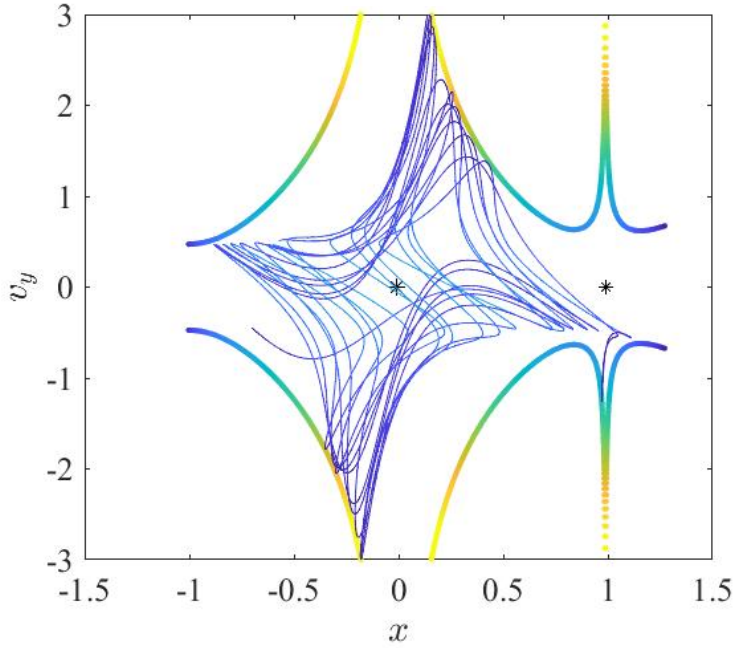


ZvxvyzS



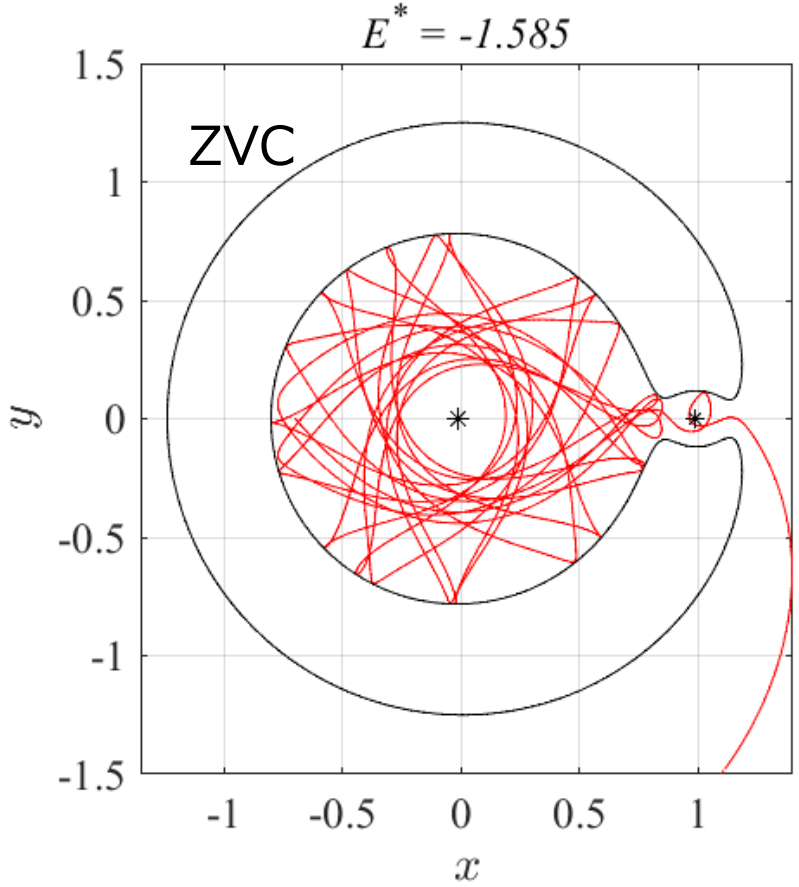
Part 2: Barrier Structure in 2D CR3BP

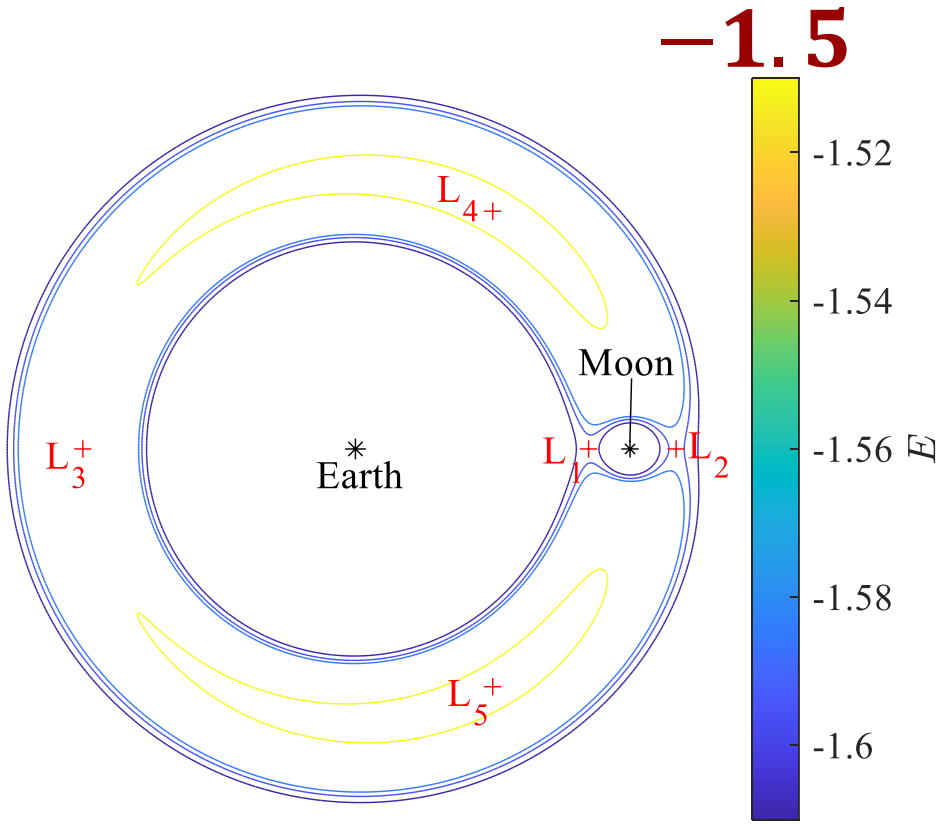
1. **Background**
2. Local Minimization of Energy Function
3. Conditional Barrier Structure



Zero-Velocity Curve (ZVC)

- ZVC places **bounds on the motion** in 2-DOF systems
- ZVC **disappears at high energy** ($E^* > -1.5$ in planar CR3BP)

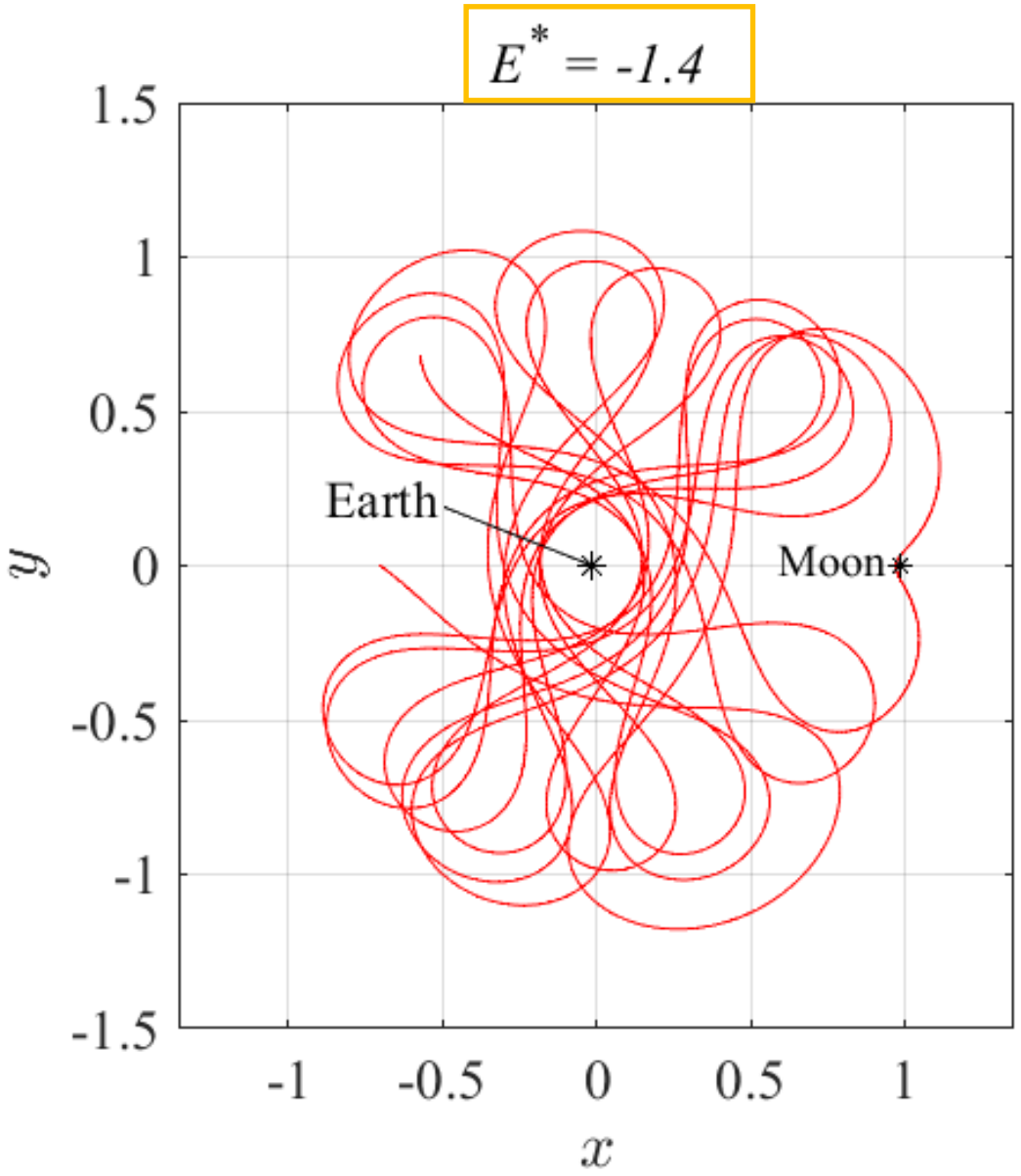




Motivation

- Although ZVC ceases to exist, **high-energy trajectories** cannot freely move
- They appear to be **still regulated** by certain structures

Unveiled barrier structure in the well-studied planar CR3BP??



Global Energy Minimization in 2D CR3BP

- $v_x = 0$ and $v_y = 0$ globally minimize E
- $x = \pm\infty$ and $y = \pm\infty$ globally minimize E
- $R_1(x, y) = 0$ and $R_2(x, y) = 0$ globally minimize E (infinite velocity)

$$E = \frac{1}{2}\left({v_x}^2 + {v_y}^2\right) + \bar{U}(x, y), \qquad \bar{U} = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{R_1(x, y)} - \frac{\mu}{R_2(x, y)}$$

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Barrier structures that are determined from **finite values** are of interest



- Only **ZVC** can be obtained from the global energy minimization
- There can be x **and** y **that locally minimize** E

Part 2: Barrier Structure in 2D CR3BP

1. Background

2. Local Minimization of Energy Function

3. Conditional Barrier Structure

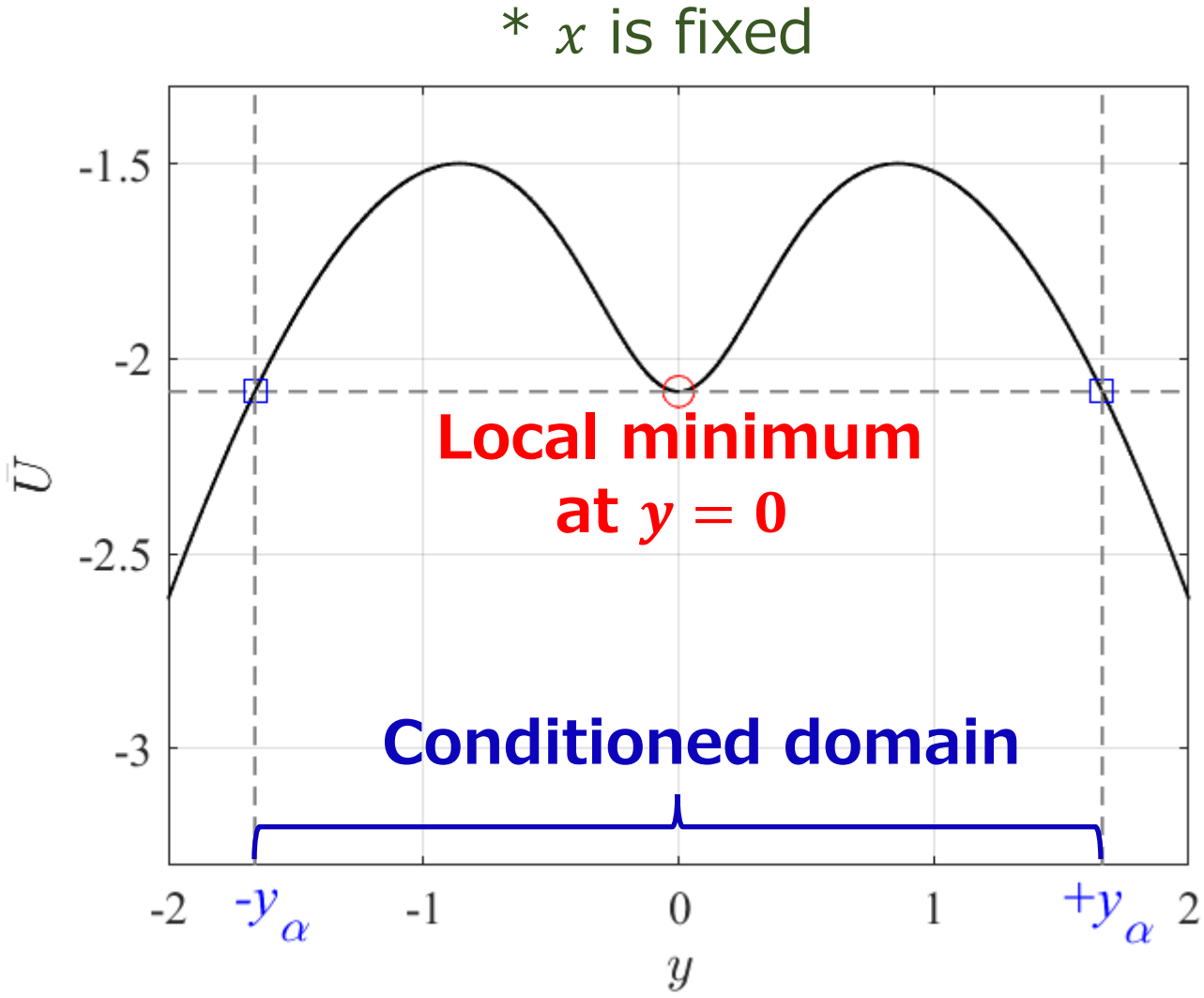
Local Minimum and Conditioned Domain

$y = 0$ in the right figure
globally minimizes $\bar{U}$ (and E)

within **the conditioned domain**

$$(-y_\alpha, +y_\alpha) := \{y \in \mathbb{R} \mid -y_\alpha < y < +y_\alpha\}$$

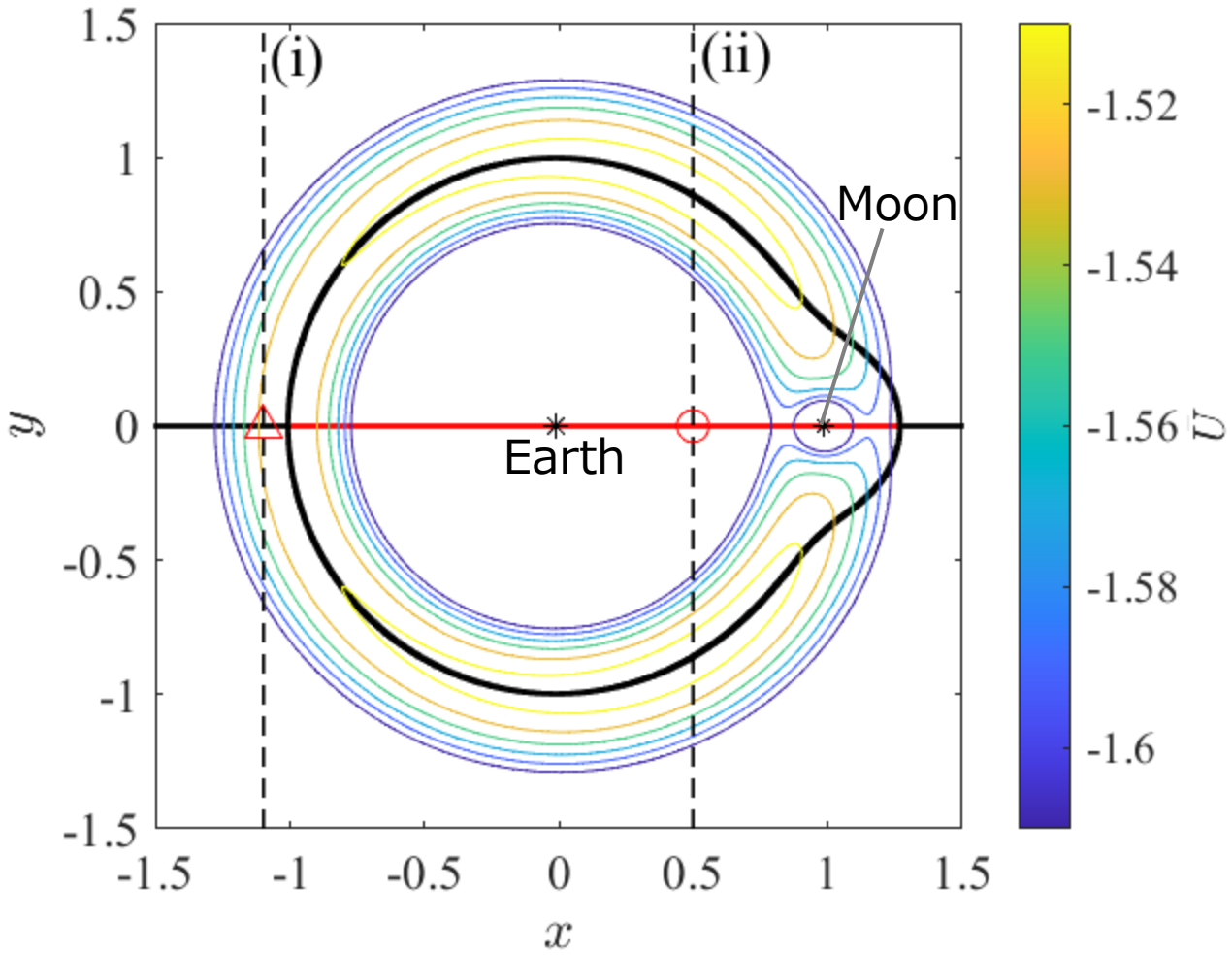
The corresponding structure is expected to **conditionally play the role of a barrier**

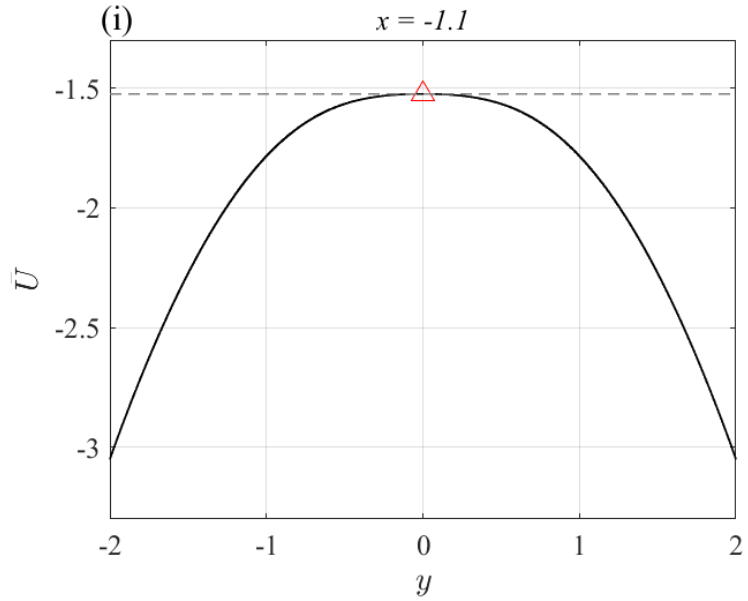


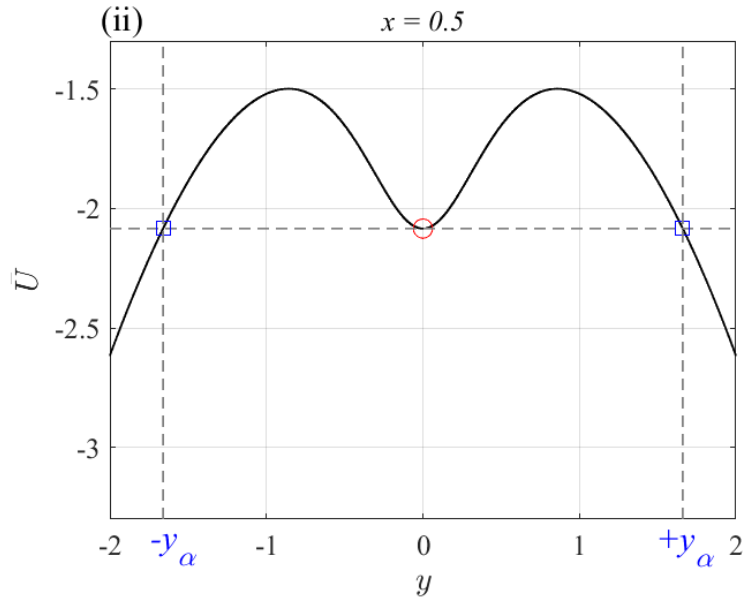
Local Energy Minimization in 2D CR3BP

Energy minimizing & maximizing conditions **for y**

Red: $\frac{\partial E}{\partial y} = 0$ & $\frac{\partial^2 E}{\partial y^2} > 0$ **Black:** $\frac{\partial E}{\partial y} = 0$ & $\frac{\partial^2 E}{\partial y^2} < 0$



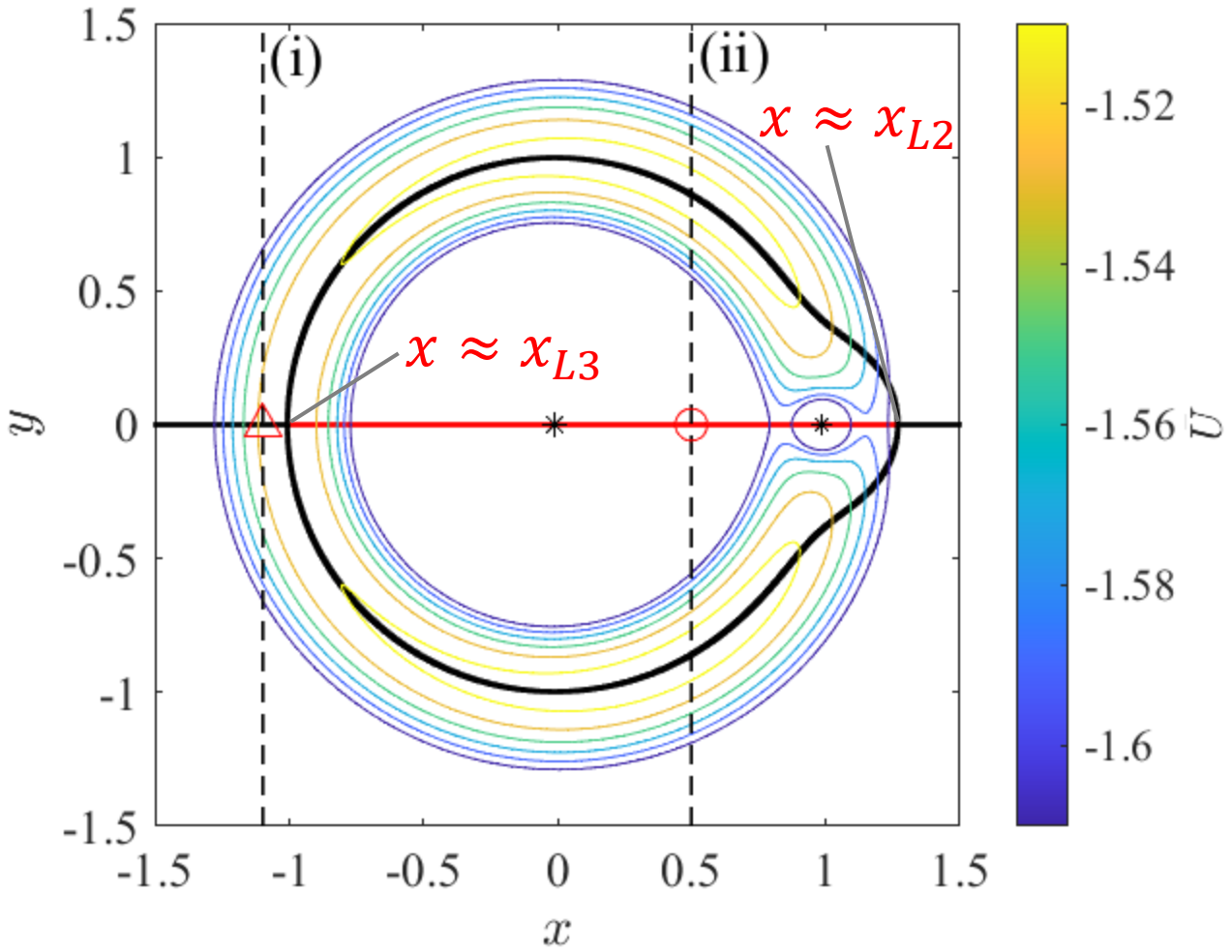




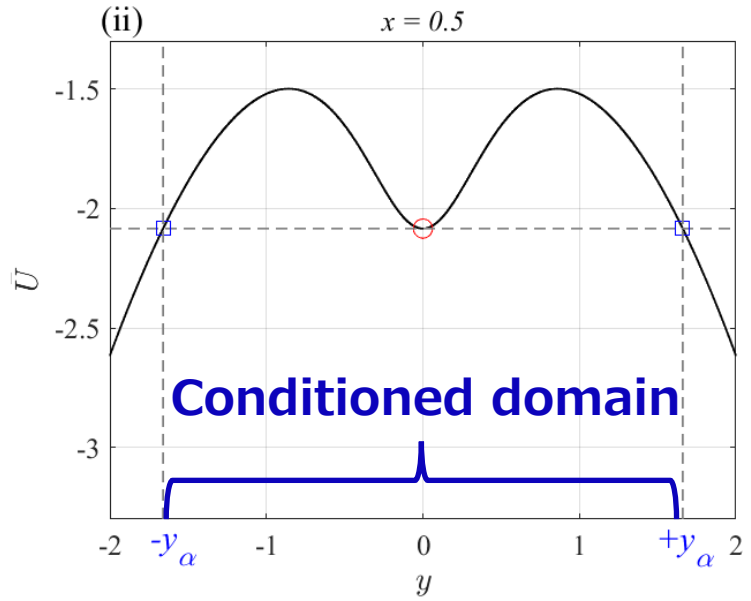
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- ✓ $y_{min} = 0$ locally minimizes E with x limited by the extent of red line
- ✓ $\bar{U}(x, y) = \bar{U}(x, -y)$ due to symmetry
- ✓ y_α is a positive real solution of $\bar{U}(x, y_\alpha) - \bar{U}(x, 0) = 0$



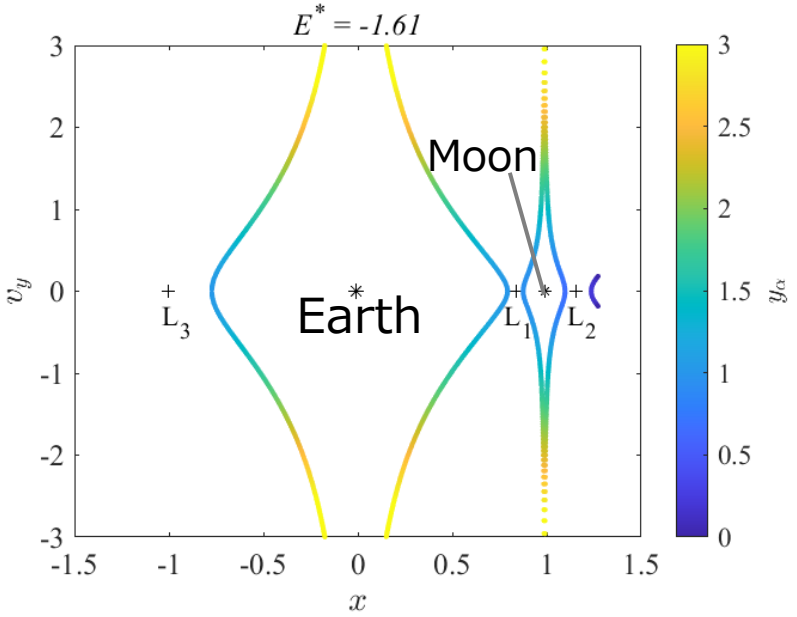
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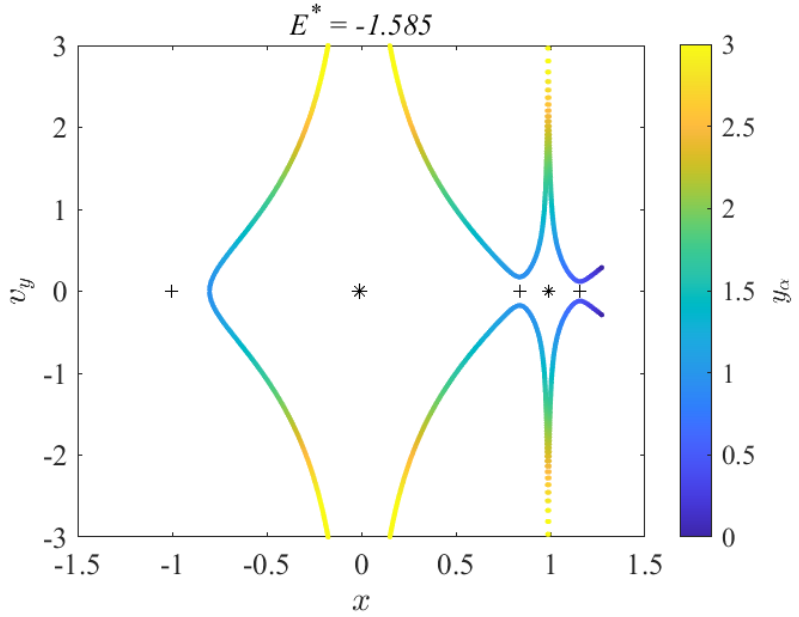
1. Background
2. Local Minimization of Energy Function
3. **Conditional Barrier Structure**

* Other planes are also available

Conditional Barrier in xv_y-plane

- The states on the curve (conditional barrier) satisfy $y = v_x = 0$ with $\frac{\partial^2 E}{\partial y^2} > 0$
- The color denotes the conditioned domain $(-y_\alpha, +y_\alpha) := \{y \in \mathbb{R} \mid -y_\alpha < y < +y_\alpha\}$

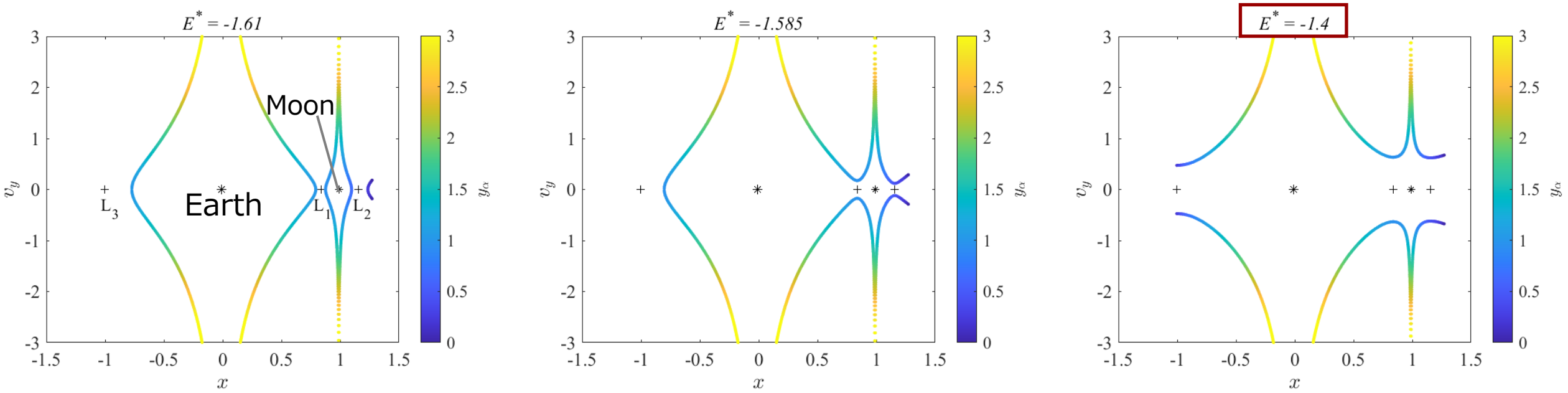




* Other planes are also available

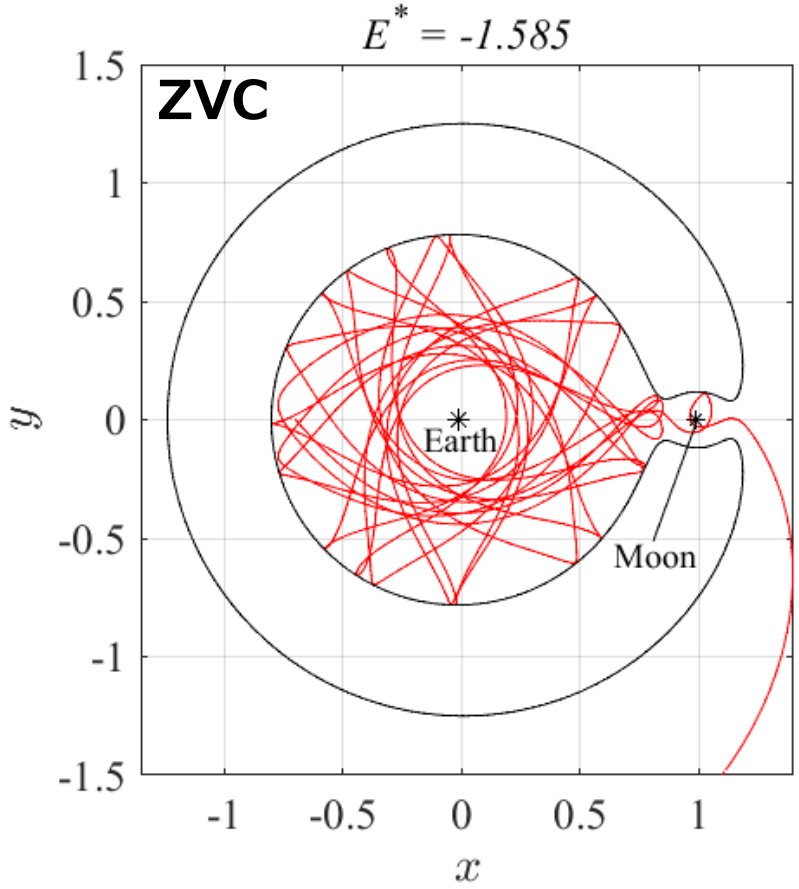
Conditional Barrier in xv_y-plane

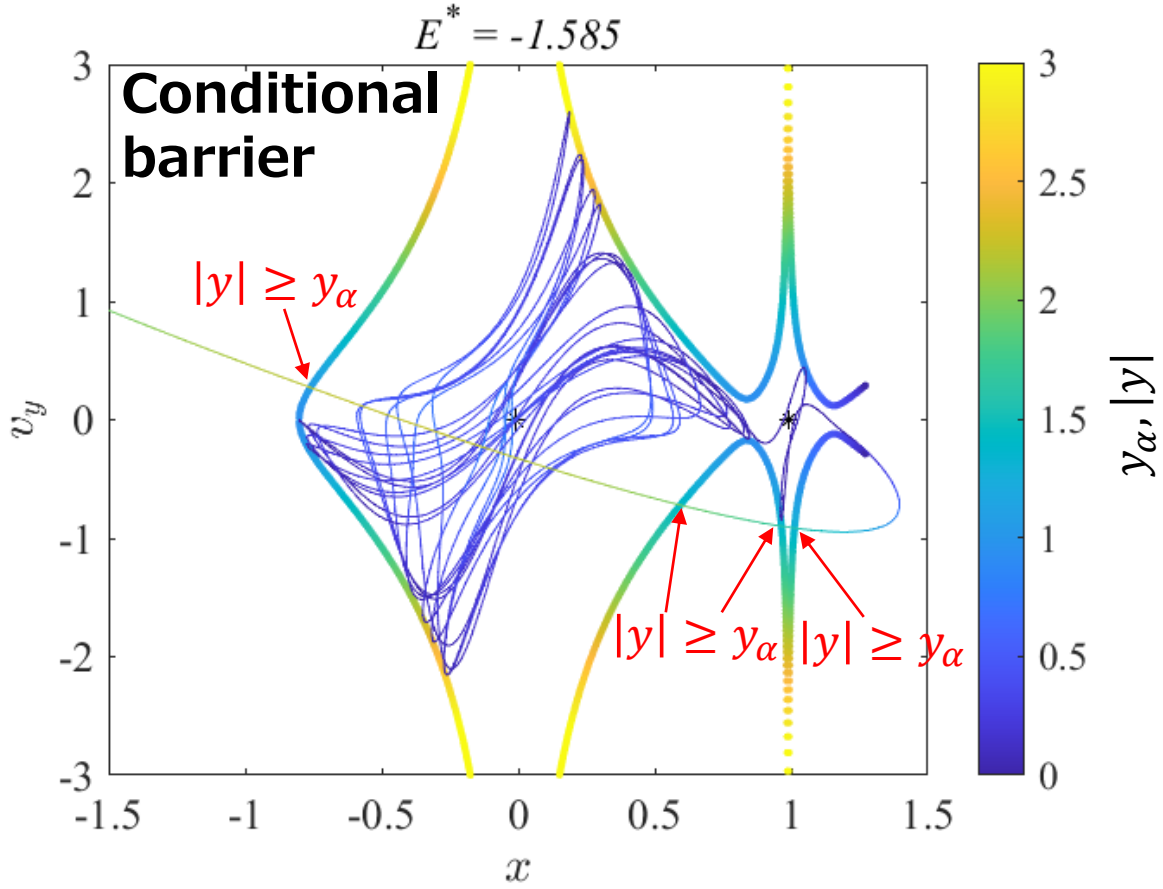
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- The color denotes the conditioned domain $(-y_\alpha, +y_\alpha) := \{y \in \mathbb{R} \mid -y_\alpha < y < +y_\alpha\}$
- The conditional barrier **robustly exists at $E^* > -1.5$**
- Higher energies contribute to greater values of $|v_y|$ on the conditional barrier



Low-Energy Case (ZVC Exists)

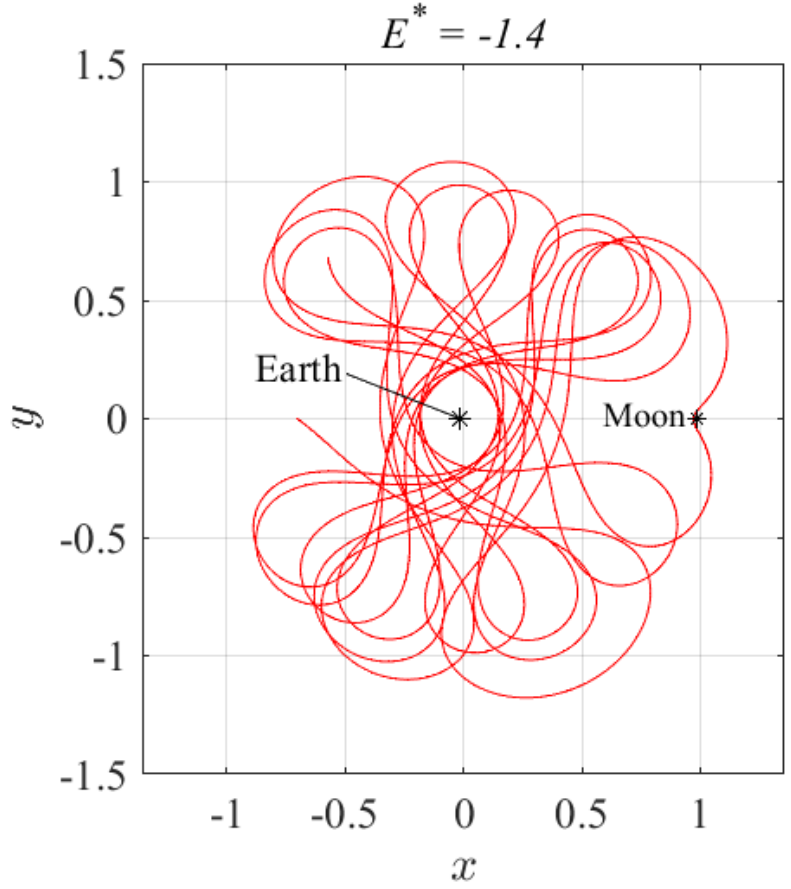
- Both ZVC and conditional barrier **place bounds on the motion**
- Trajectories can cross over conditional barrier when $|y| \geq y_\alpha$ holds
  * ZVC does not bound the corresponding segment as well

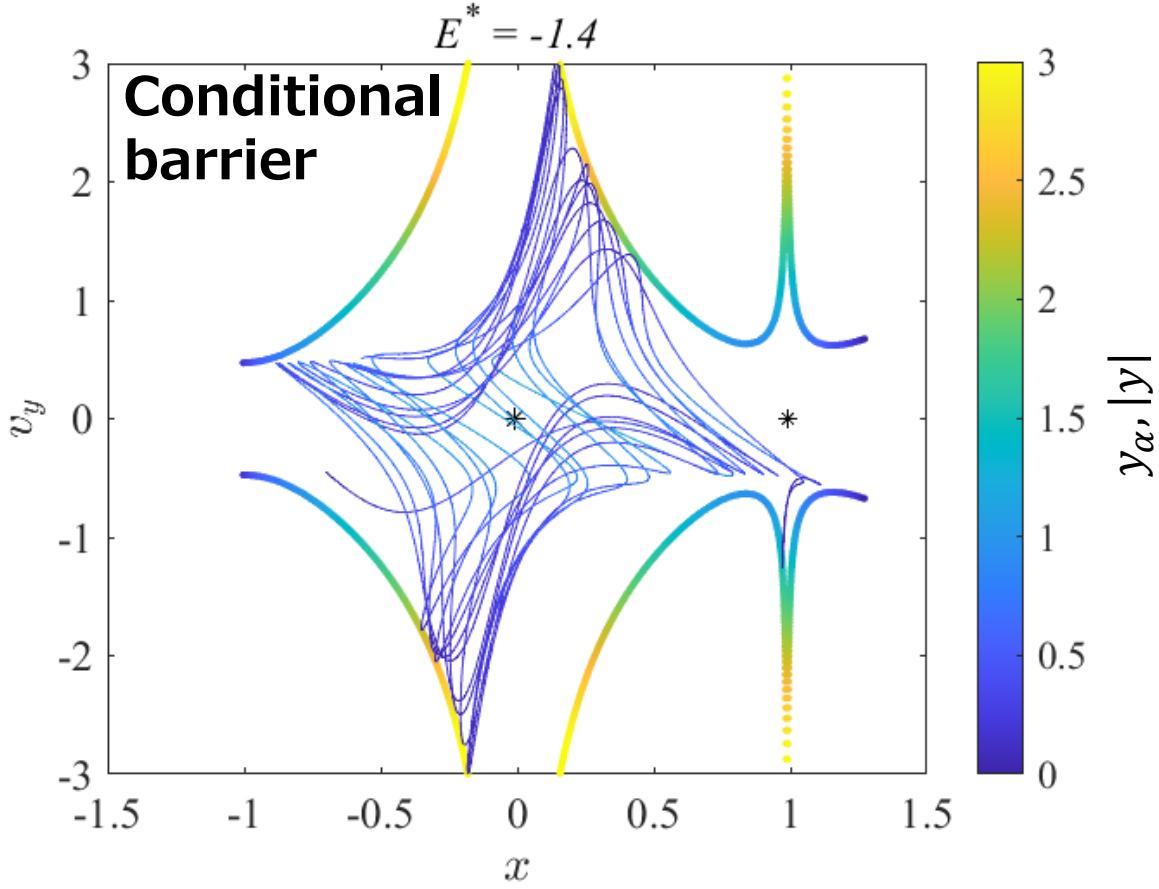




High-Energy Case (ZVC Disappears)

- The conditional barrier **still places bounds on the motion**
- Although the ability as a barrier is conditional, it is **effective around the bodies where three-body effect is substantial** (e.g., cis-lunar space)





Conclusions

- **A new barrier surface coined as the ZvxvyzS** that separates the regions of possible and impossible motion in 3D CR3BP

- ZVS and ZvxvyzS are **equally important and complement each other**

- The concept of **conditional barrier** in 2D CR3BP

- Conditional barrier **robustly exists and regulates the motion** in both **low- and high-energy** regimes

Thank you for your attention!

Questions?

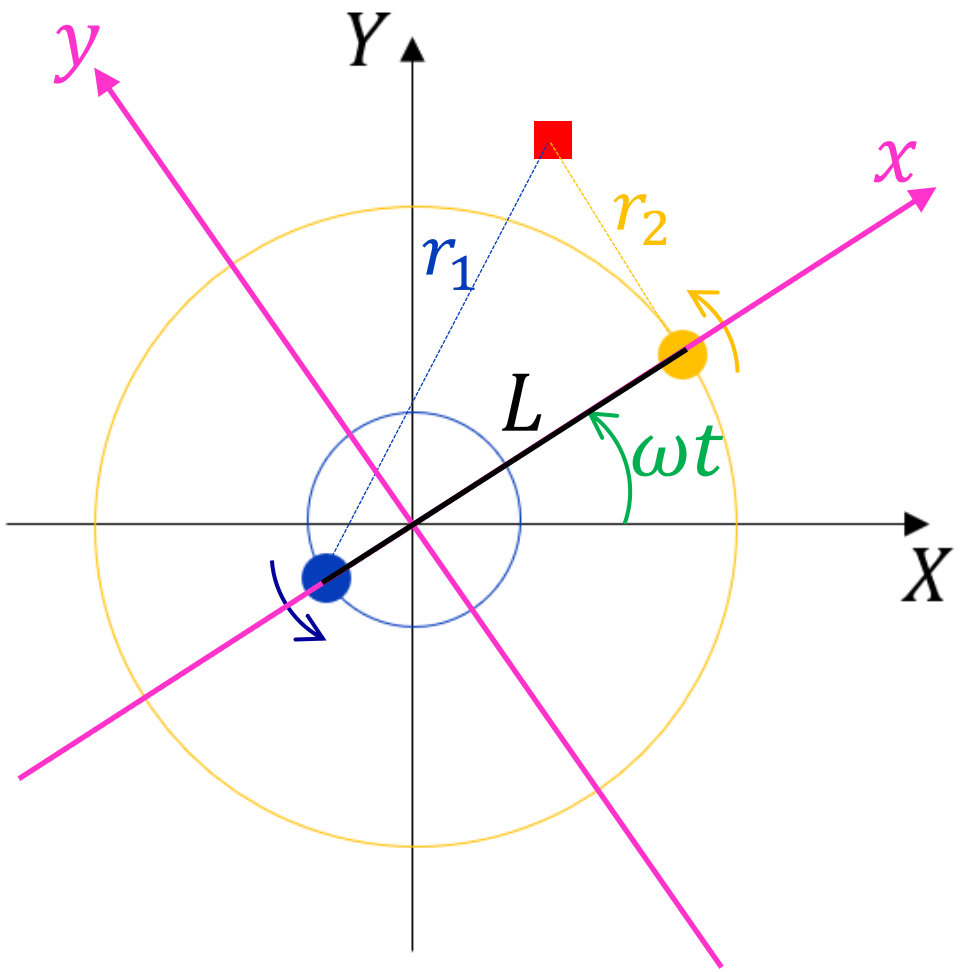
E-mail: oshima_kenta@rs.sus.ac.jp

Web: https://researchmap.jp/espresso



Rotating Frame

A rotating frame that corotates with P_1 & P_2 around the Z-axis



- The xy-plane is the orbital plane of P_1 & P_2
- The x-axis lies along the line from P_1 to P_2
- The z-axis coincides with the Z-axis
- The y-axis completes a right-hand system
- P_1 is **fixed** at $[-\mu, 0, 0]^T$
- P_2 is **fixed** at $[1-\mu, 0, 0]^T$

⇨ **Explicit dependence on time can be eliminated**

Part 1

Necessary & Sufficient Conditions

- $X = [x_1, x_2, x_3, x_4, x_5, x_6]$ is on the boundary of the forbidden region in $x_1x_2x_3$-space
  $\Rightarrow$ X satisfies $E = E^*$ and x_4, x_5, x_6 globally minimizes E (already described)

- X satisfies $E = E^*$ and x_4, x_5, x_6 globally minimizes E
  $\Rightarrow$ X is on the boundary of the forbidden region in $x_1x_2x_3$-space

- If X is in the forbidden region, $E > E^*$ by the following definition, so contradicted

  The states in the forbidden region in $x_1x_2x_3$-space *result in* $E > E^*$ *whatever real values* x_4, x_5, x_6 *take.*

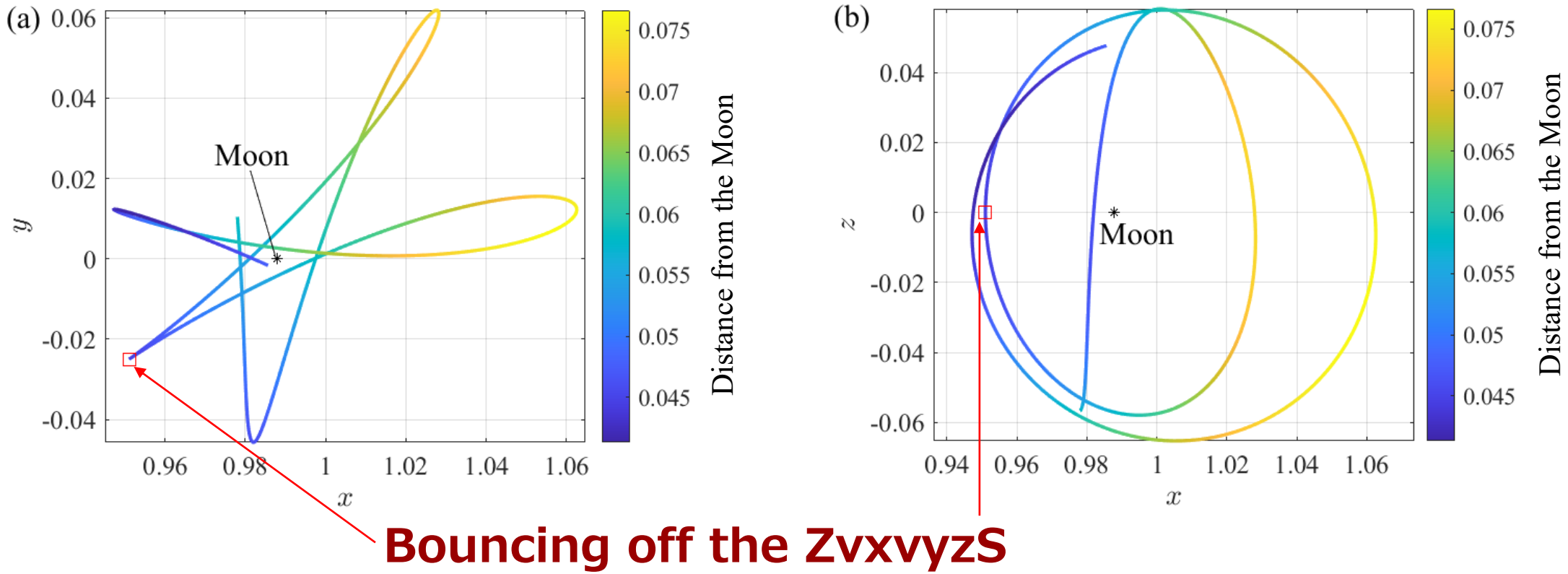
- If X is in the region of possible motion, by the following definition, global minimization of E leads to $E < E^*$, so contradicted

  The states in the region of possible motion in $x_1x_2x_3$-space *can result in* $E < E^*$ *with real values of* x_4, x_5, x_6.

- Therefore, X must be on the boundary

Periapsis on ZvxvyzS

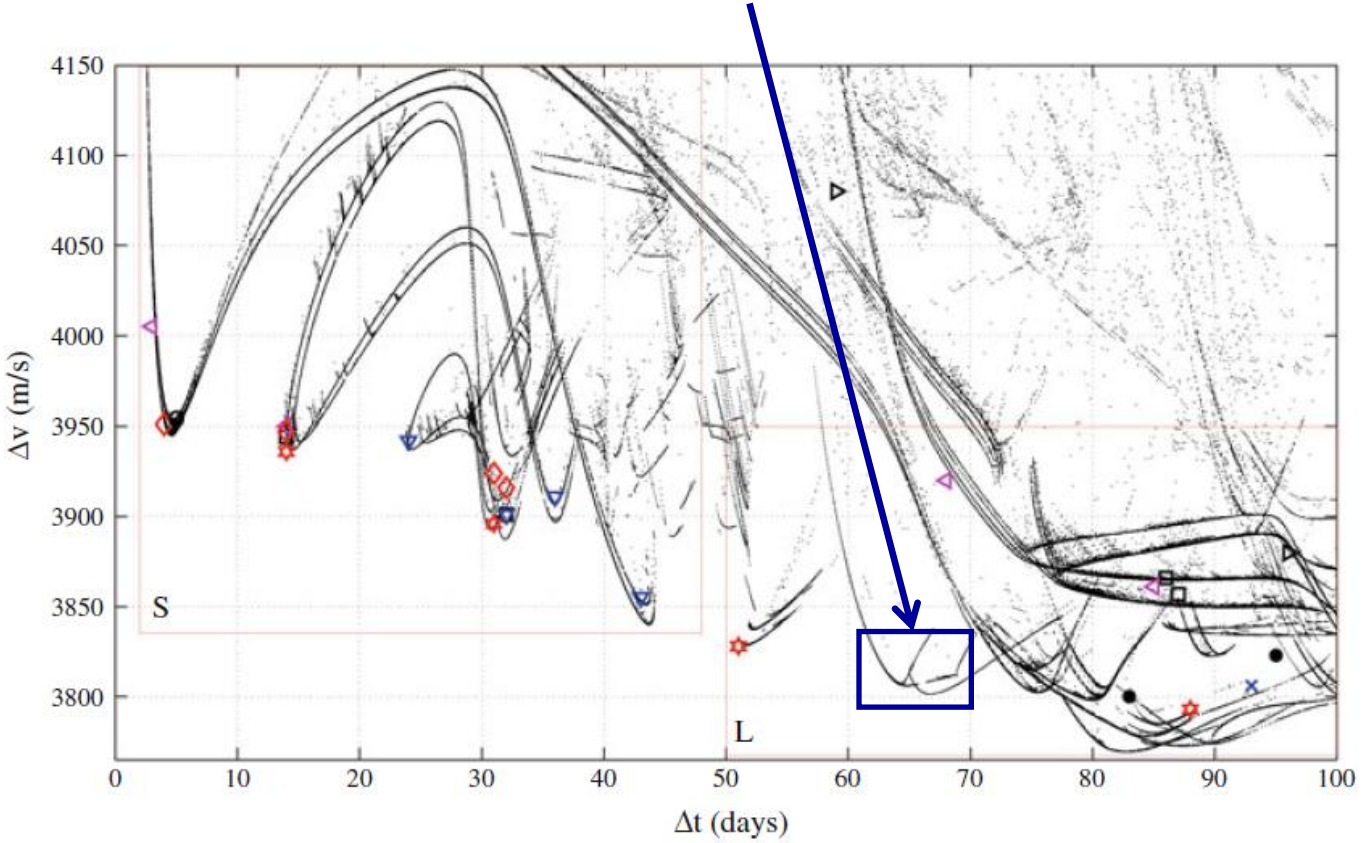
- **Nearer to the bodies, larger $|v_z|$** produces greater out-of-plane oscillation that makes the states on the ZvxvyzS **at $z = 0$ periapsis**



Application: Medium-Energy Transfer [*]

Global search for 2-impulse transfer [**]

An unexplored unique family was found



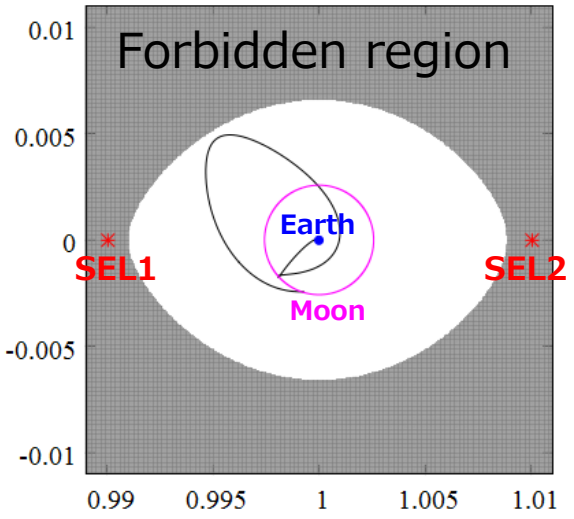
Inherent difficulties in applying the manifold-based method

- $E < E_{L1}$ at apogee

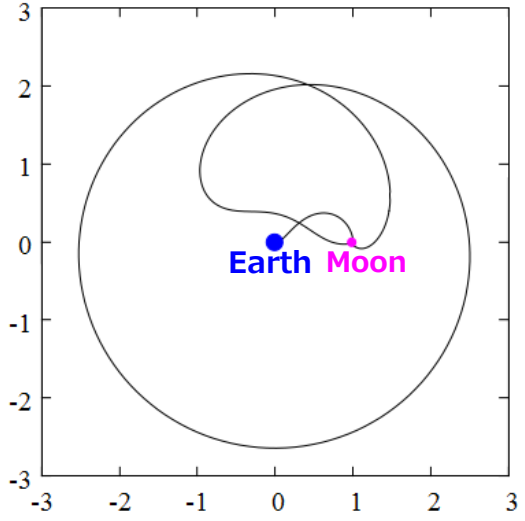
⇛ **No SEL1/L2 manifold tube!!**

- $E > -1.5$ at lunar capture

⇛ **High-energy manifold tube??**



Sun-Earth rotating frame



Earth-Moon rotating frame

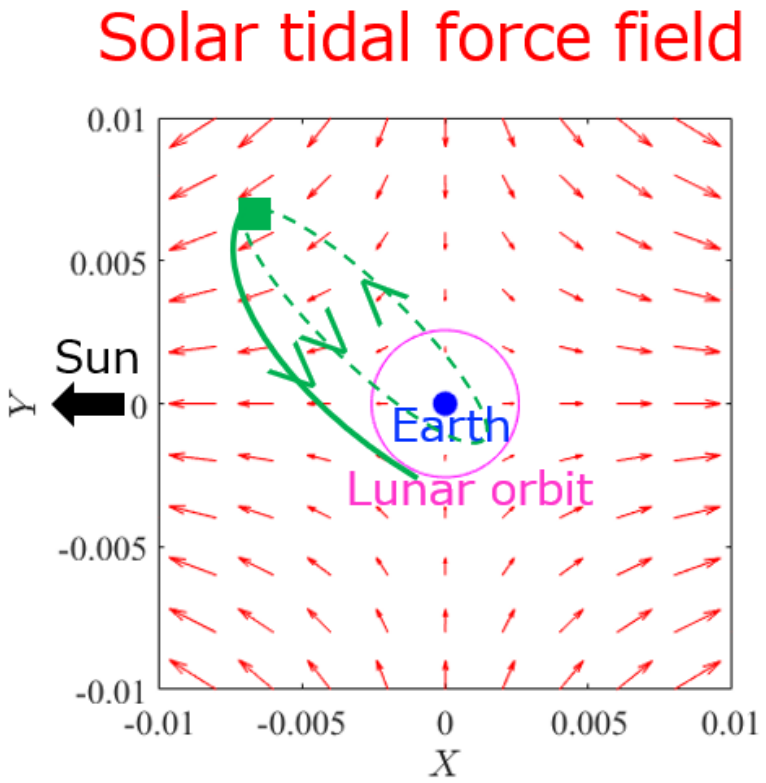
[*] Oshima, K., Topputo, F., Campagnola, S., Yanao, T.: Analysis of medium-energy transfers to the Moon. CMDA (2017).

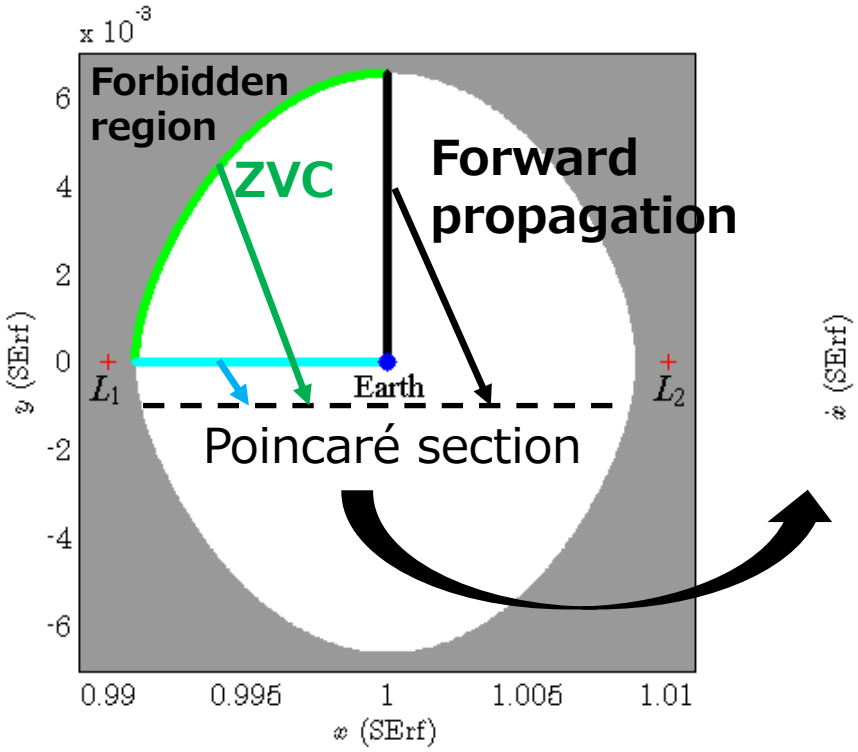
[**] Topputo, F.: On optimal two-impulse Earth-Moon transfers in a four-body model. CMDA (2013).

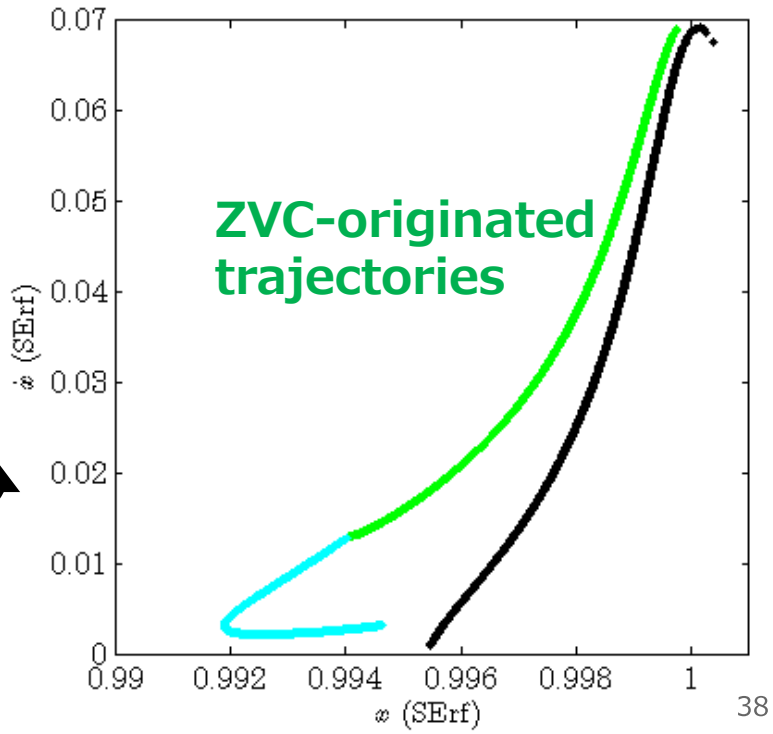
Application: Medium-Energy Transfer

Instead of using **SEL1/L2 manifold tubes** to find a Sun-perturbed arc, we generated **trajectories bouncing off near ZVC in an appropriate quadrant (2nd or 4th) around Earth**

⇒ **Perigee altitude increases & relative velocity wrt Moon decreases**



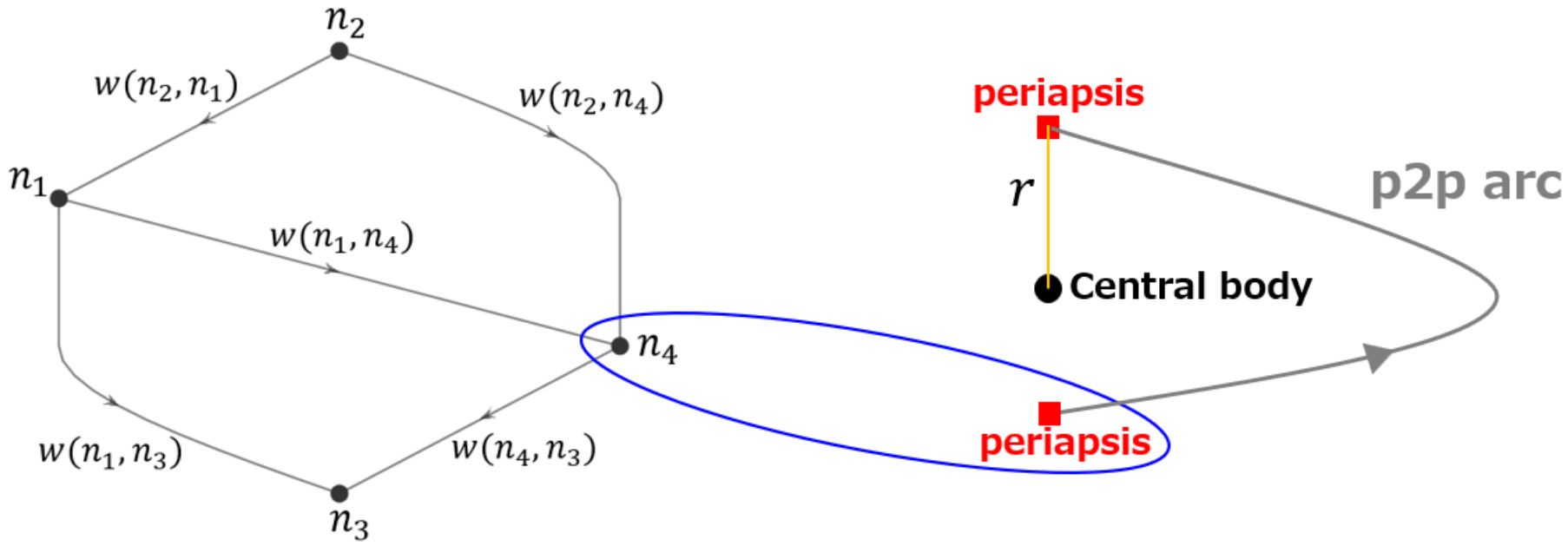




Application: Graph-Based Trajectory Design [*]

Initial **periapsis states** of **periapsis-to-periapsis (p2p) arcs** were adopted as **the graph nodes**

An apsis condition $\dot{r} = 0$ leads to **5-dim.** search in 3D CR3BP



[*] Oshima, K.: A graph-based framework of low-energy transfer design. Acta Astronaut. (2025).

Application: Graph-Based Trajectory Design

7 X **3-dim. search** was carried out to generate **p2p arcs**

- Cannot cover all the possible p2p arcs
- Prioritized the feasibility of computing the p2p arcs
- **Provided finer resolution on pivotal dynamical objects**

Rotating-frame coordinates

$$v_x = v_y = v_z = 0 \quad (1)$$

$$v_x = v_y = z = 0 \quad (2)$$

$$v_x = y = v_z = 0 \quad (3)$$

$$v_x = y = z = 0 \quad (4)$$

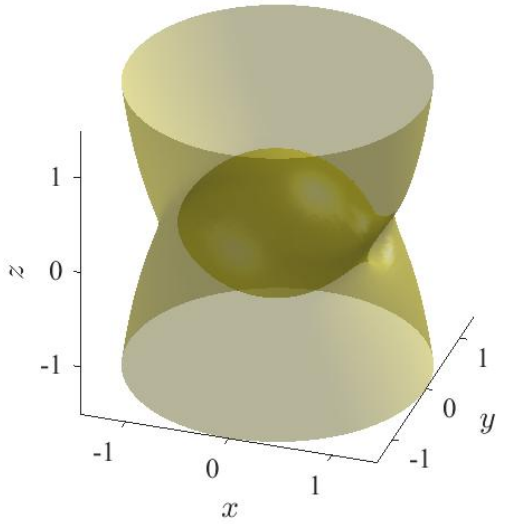
$$x - x^* = v_y = v_z = 0 \quad (5)$$

$$x - x^* = v_y = z = 0 \quad (6)$$

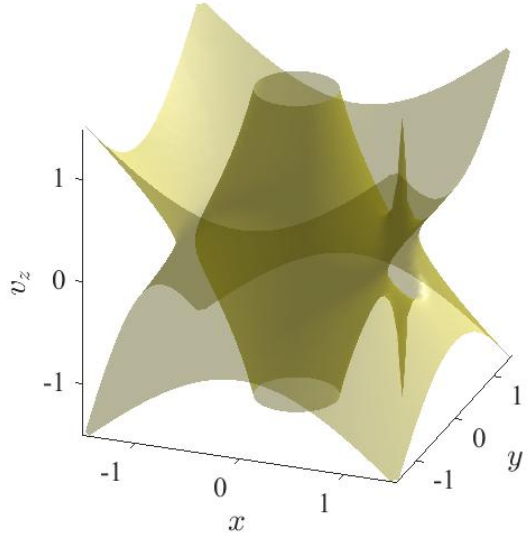
$$x - x^* = y = v_z = 0 \quad (7)$$

x^*: Abscissa of a body

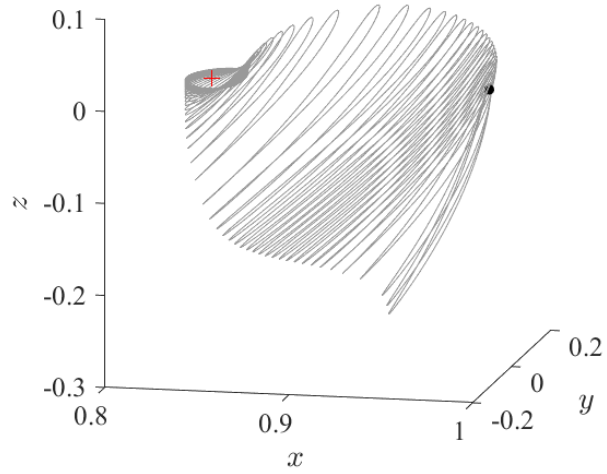
(1) Zero-velocity surface (ZVS)



(2) Novel barrier surface (ZvxvyzS)



(3) xz-planar symmetric orbits



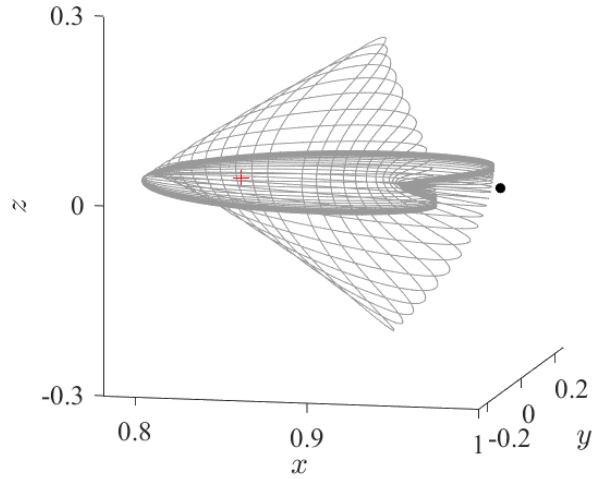
$$v_x = v_y = v_z = 0 \quad (1)$$

$$v_x = v_y = z = 0 \quad (2)$$

$$v_x = y = v_z = 0 \quad (3)$$

$$v_x = y = z = 0 \quad (4)$$

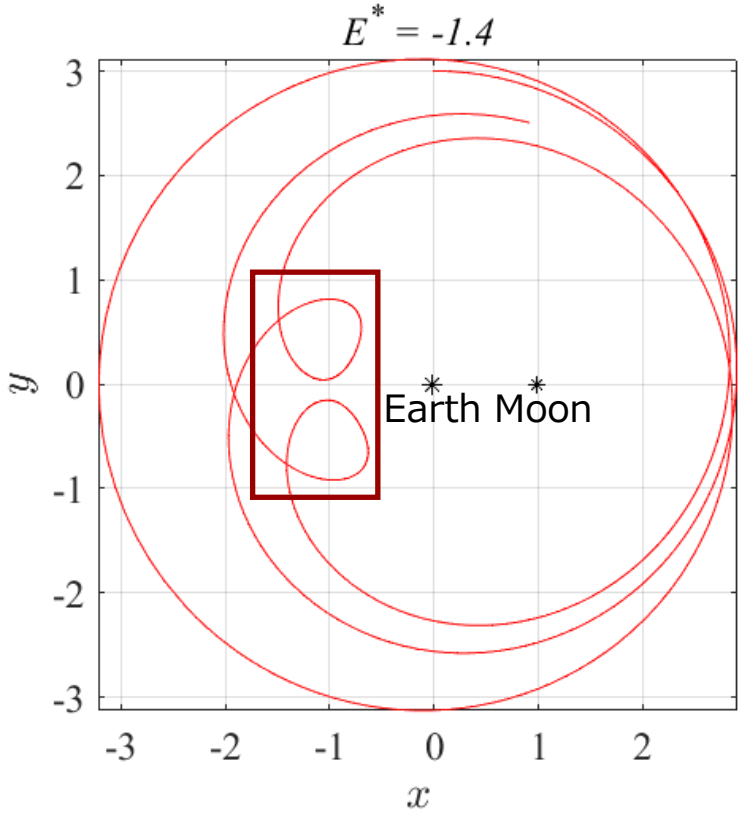
(4) Axi-symmetric orbits

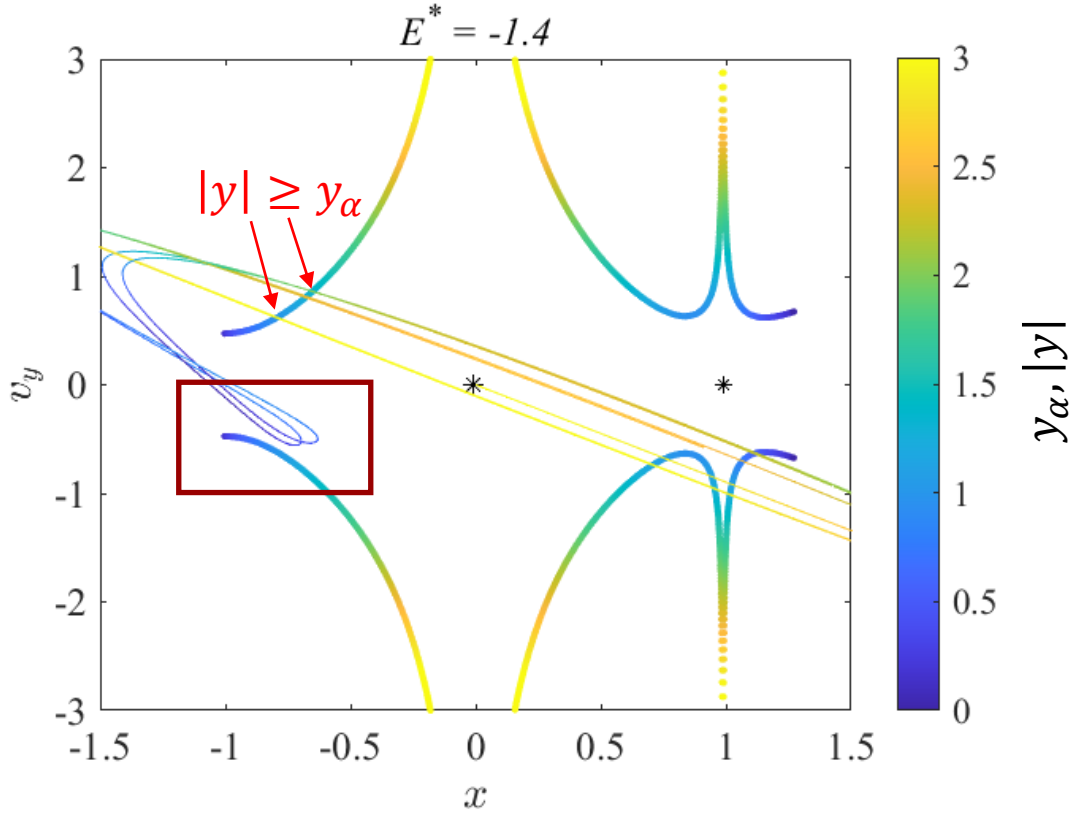


Part 2

Large-Orbit Case (ZVC Disappears)

- Conditional barrier is not very effective for a large orbit that $|y| \geq y_\alpha$ often holds
- Still, the conditional barrier becomes effective for the segment that **three-body effect is substantial**





Why Does It Work?

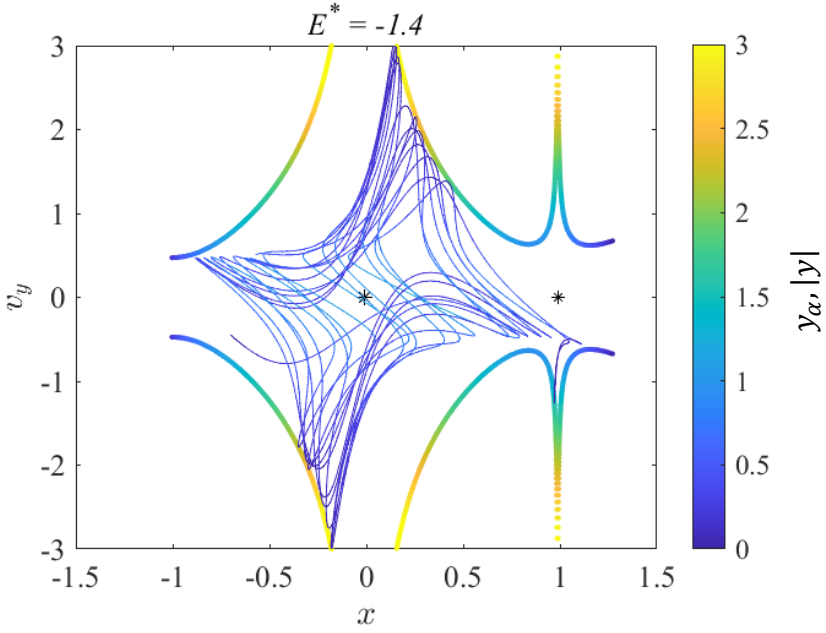
Assume a trajectory **reaches the conditional barrier** in xv_y-plane **with** $|y| < y_\alpha$

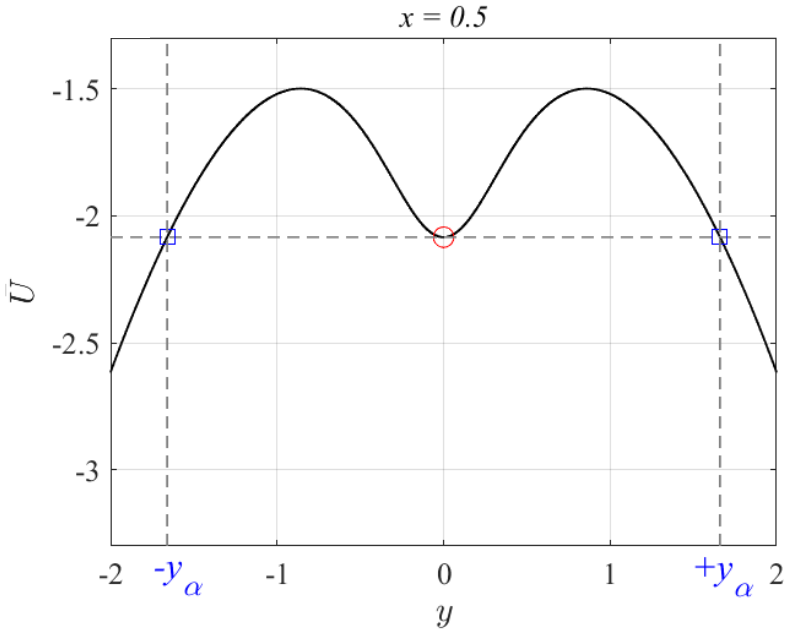
At any point (x, v_y) on the conditional barrier,
the state at intersection must be $y = v_x = 0$

* Otherwise $E > E^*$ due to **the energy minimizing condition**

Substituting $y = v_x = 0$ into EoM yields $\frac{dx}{dt} = \frac{dv_y}{dt} = 0$,
indicating **it redirects the motion in** xv_y**-plane and the trajectory cannot cross over it!**

* **Similarly to ZVC** in xy-plane ($\frac{dx}{dt} = \frac{dy}{dt} = 0$)



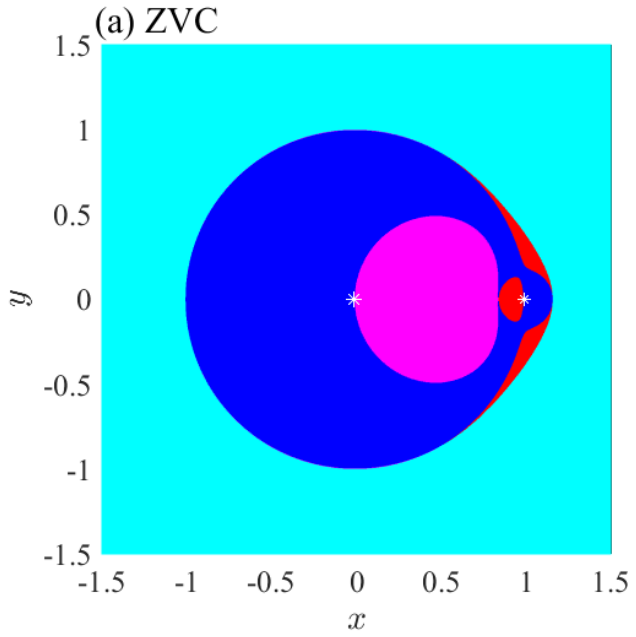


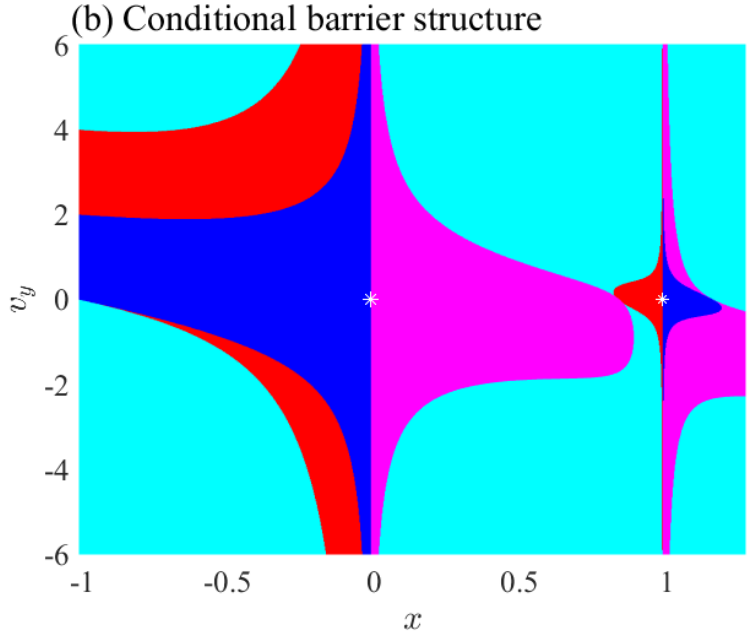
Apsidal States

Apogee & apolune
Apogee & perilune
Perigee & apolune
Perigee & perilune

*** States on ZVC and Conditional barrier are at apsis wrt Earth and Moon**

- The ZVC bounds the motion at apogee or apolune around the bodies
- Distinctively from the ZVC, **the conditional barrier bounds the motion at periapsis near the bodies** with relatively large $|v_y|$



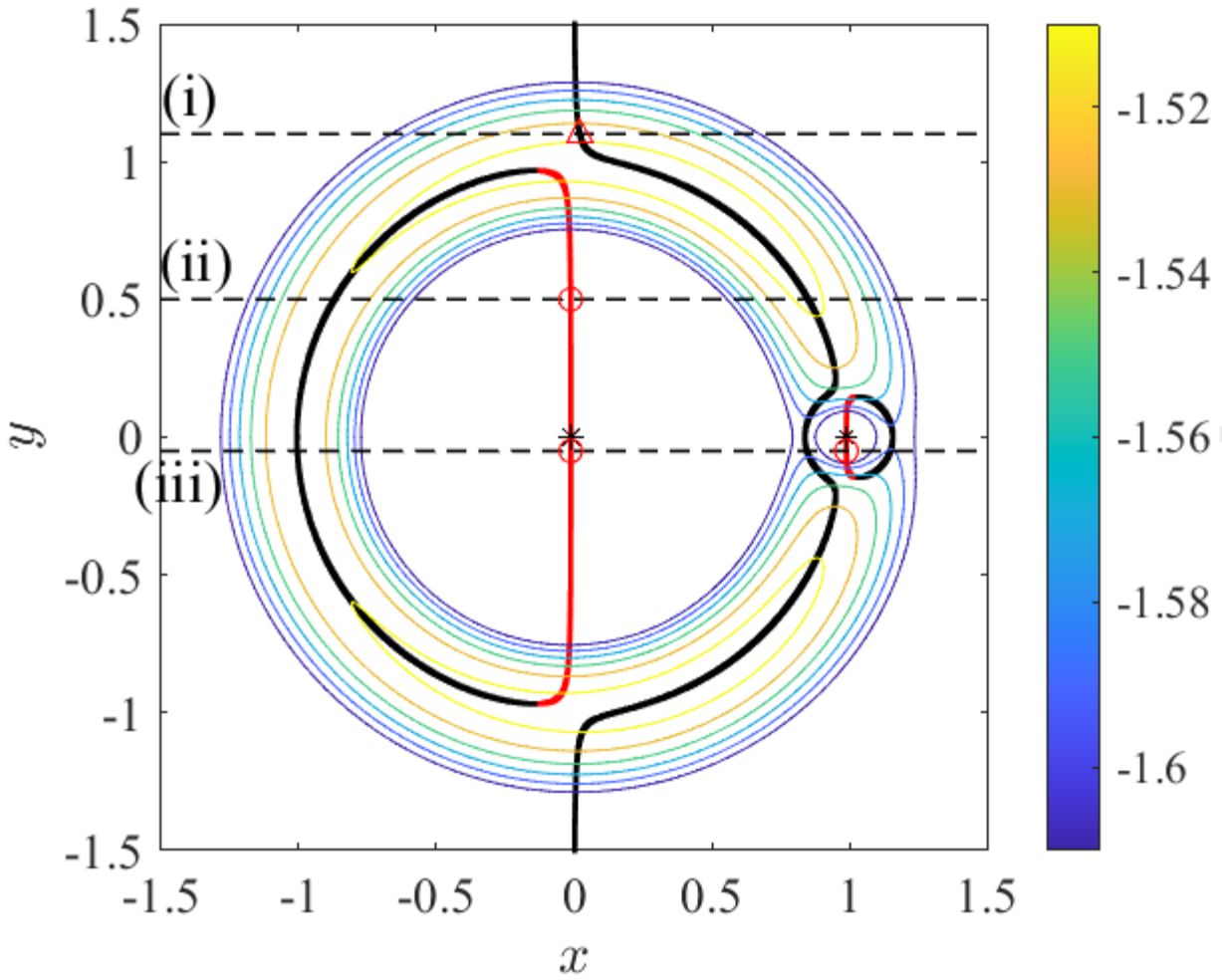


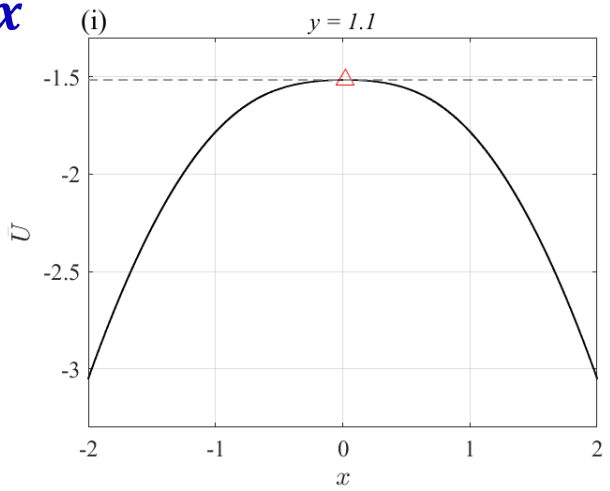
yvx-plane

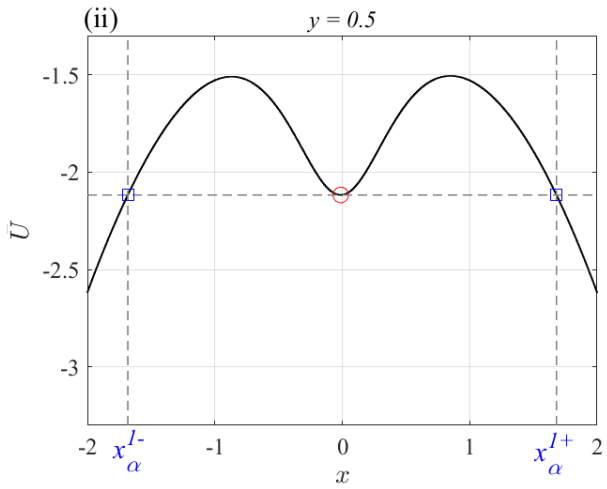
Local Energy Minimizers and Where to Find Them

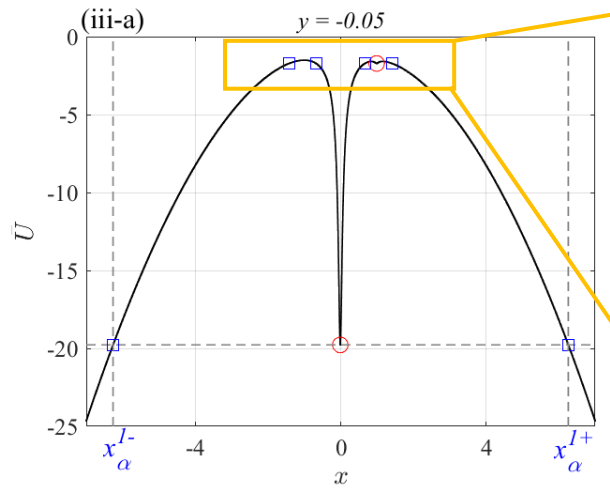
Energy minimizing & maximizing conditions **for x**

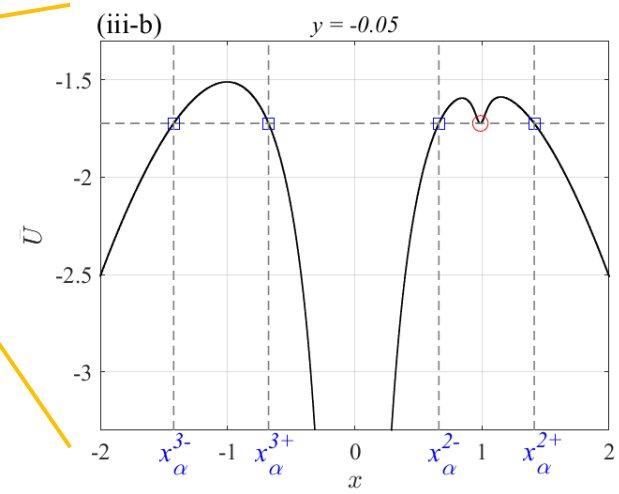
Red: $\frac{\partial E}{\partial x} = 0$ & $\frac{\partial^2 E}{\partial x^2} > 0$ **Black:** $\frac{\partial E}{\partial x} = 0$ & $\frac{\partial^2 E}{\partial x^2} < 0$







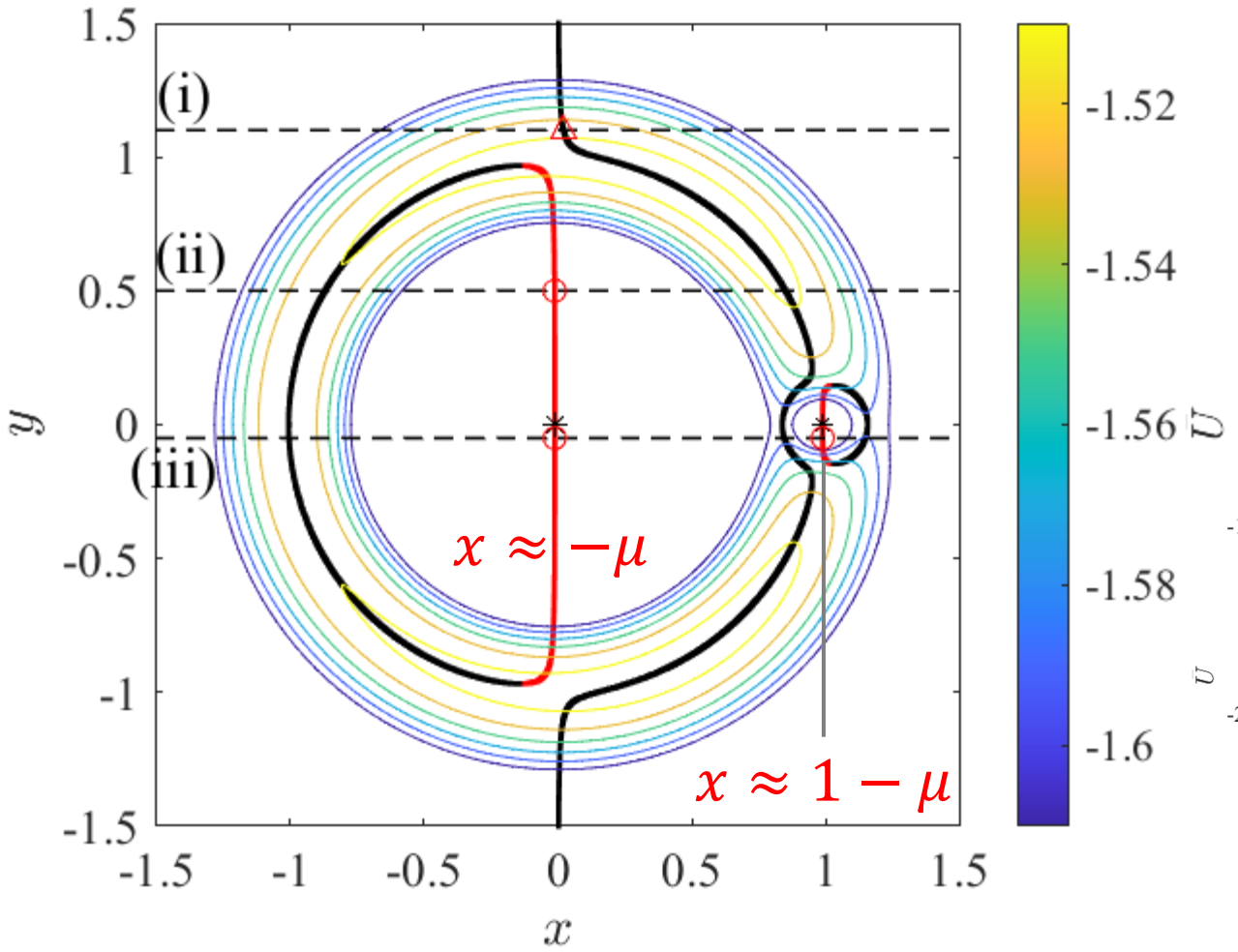




Local Energy Minimizers and Where to Find Them

Energy minimizing & maximizing conditions **for x**

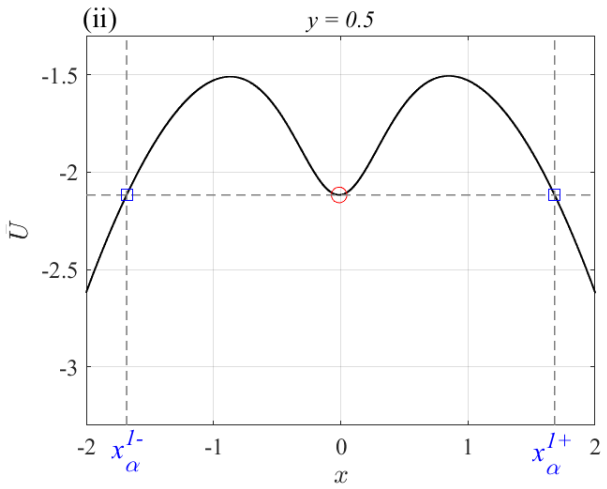
Red: $\frac{\partial E}{\partial x} = 0$ & $\frac{\partial^2 E}{\partial x^2} > 0$ **Black:** $\frac{\partial E}{\partial x} = 0$ & $\frac{\partial^2 E}{\partial x^2} < 0$

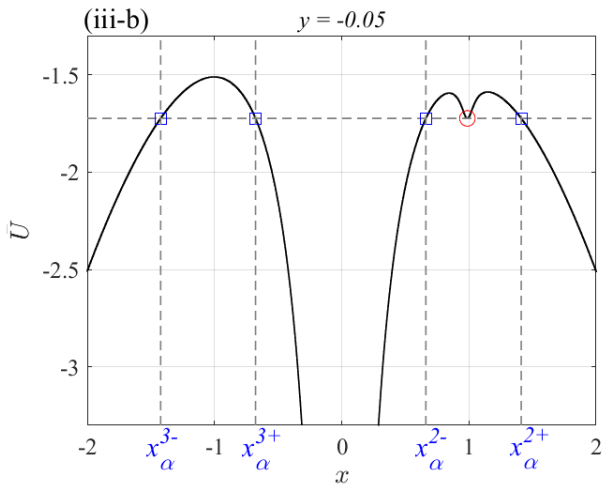


✓ $x_{min} \approx -\mu$ and $x_{min} \approx 1-\mu$ are local minimizers with y limited by the extent of red curves

✓ $\bar{U}(x,y) \neq \bar{U}(-x,y)$

✓ $x_\alpha^{j\pm}$ $(j = 1,2,3)$ are real solutions of $\bar{U}\left(x_\alpha^{j\pm}, y\right) - \bar{U}(x_{min}, y) = 0$



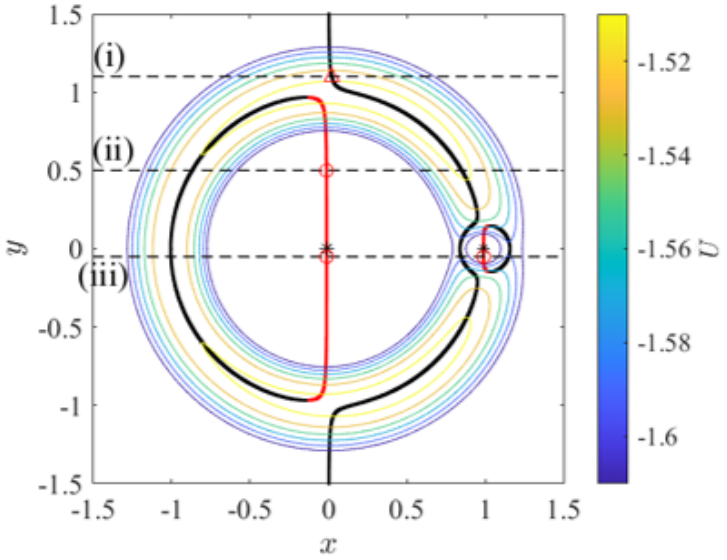


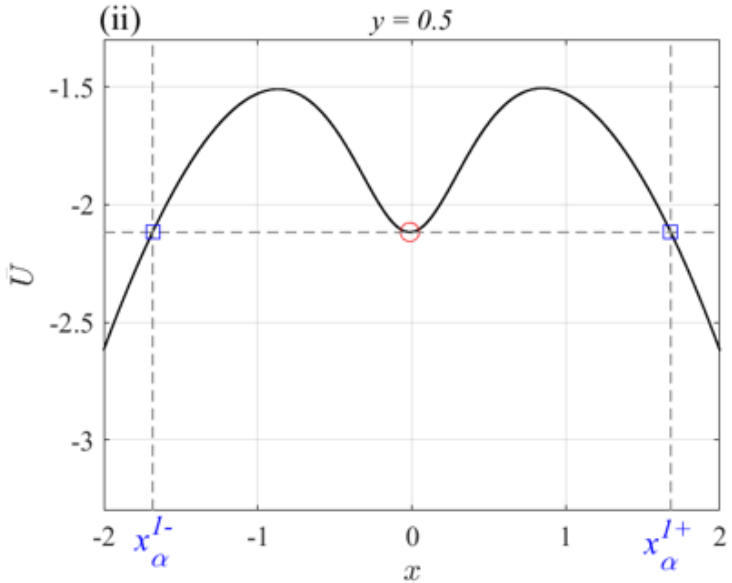
Conditional Barriers in yv_x-plane (1)

Blue: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ includes the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

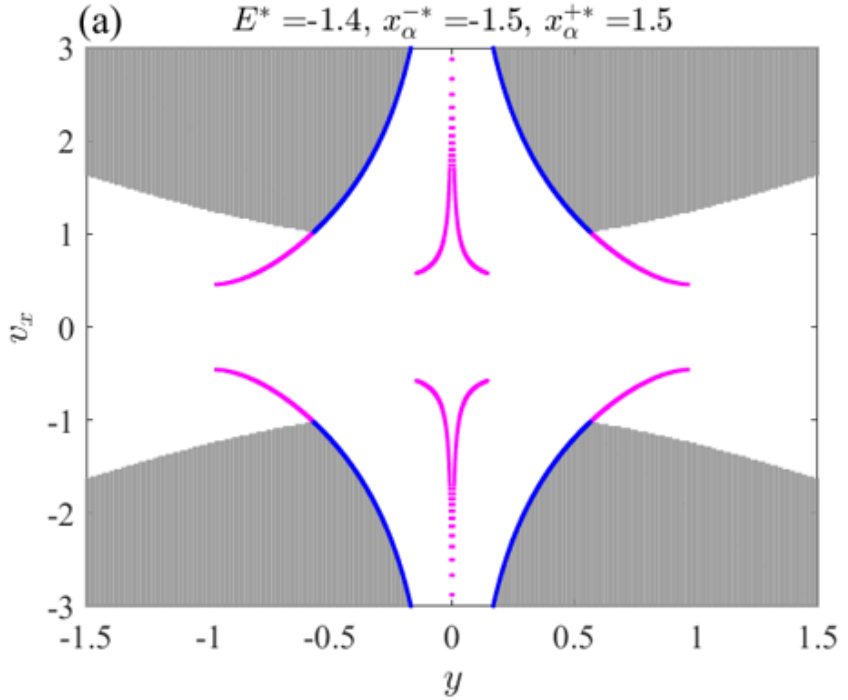
Magenta: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ does not include the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

Gray: Forbidden region within the given range $(x_\alpha^{-*}, x_\alpha^{+*})$





Outer curve: $x \approx -\mu$ Inner curve: $x \approx 1-\mu$

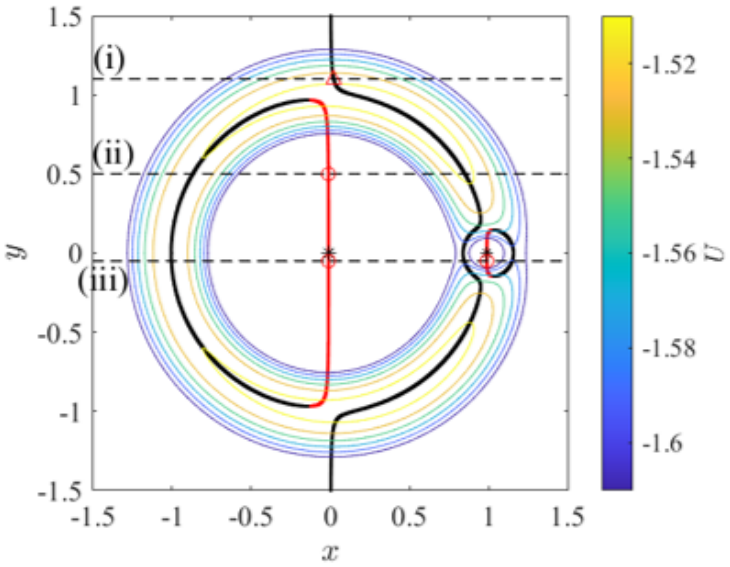


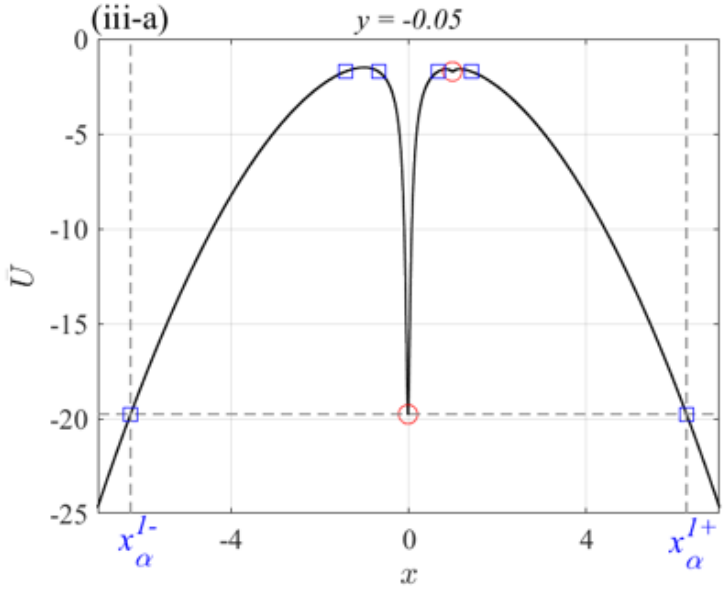
Conditional Barriers in yv_x-plane (2)

Blue: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ includes the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

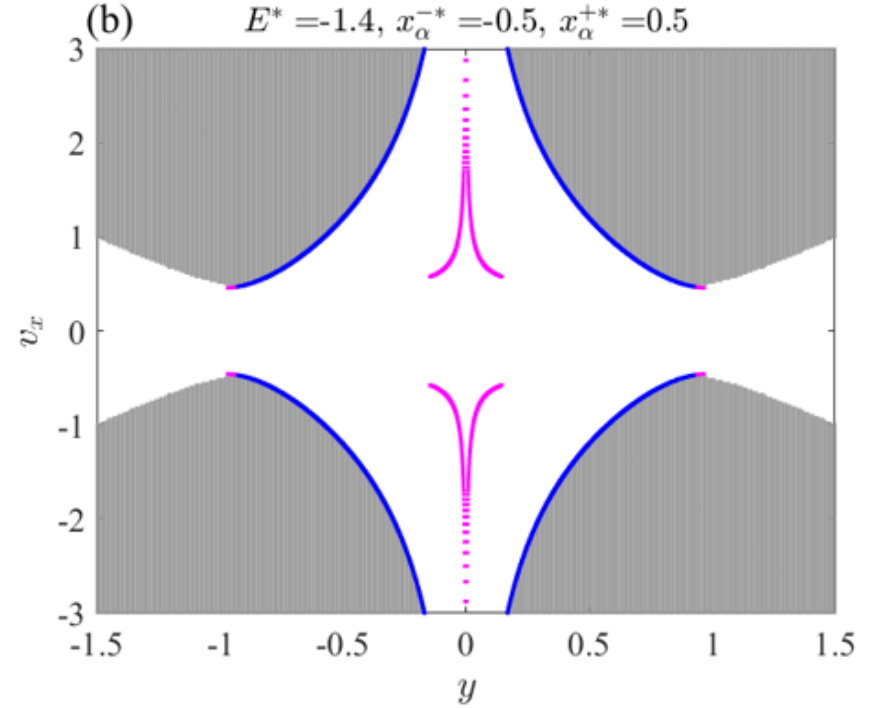
Magenta: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ does not include the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

Gray: Forbidden region within the given range $(x_\alpha^{-*}, x_\alpha^{+*})$





Outer curve: $x \approx -\mu$ Inner curve: $x \approx 1-\mu$

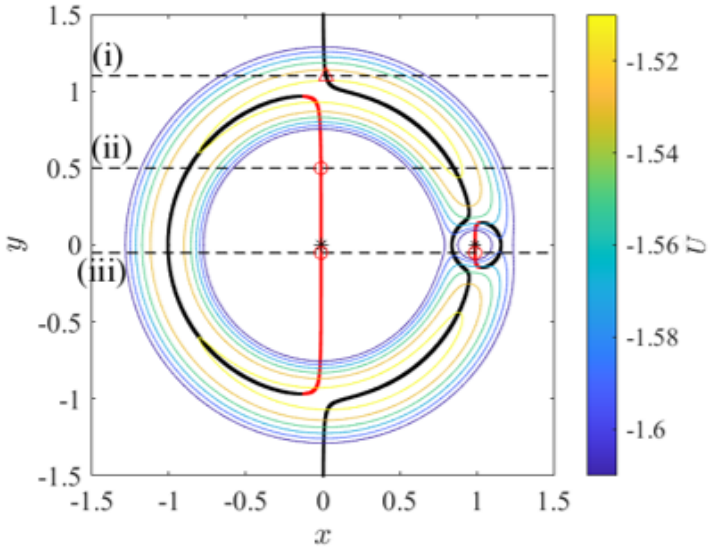


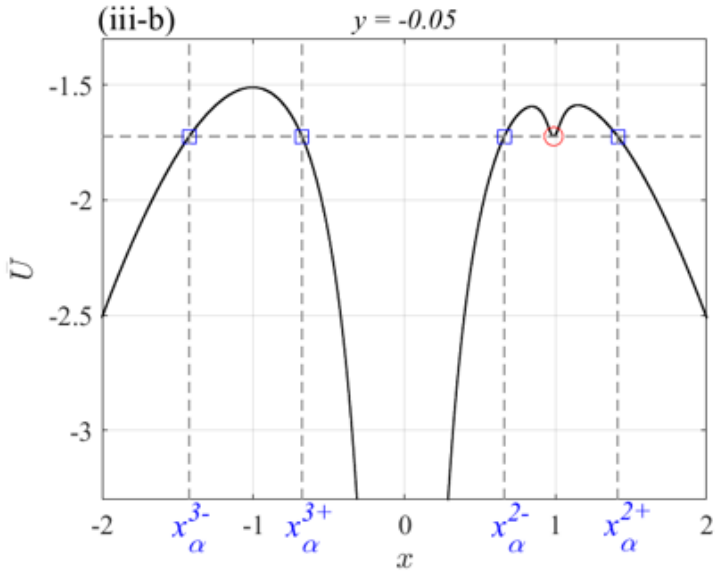
Conditional Barriers in yv_x-plane (3)

Blue: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ includes the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

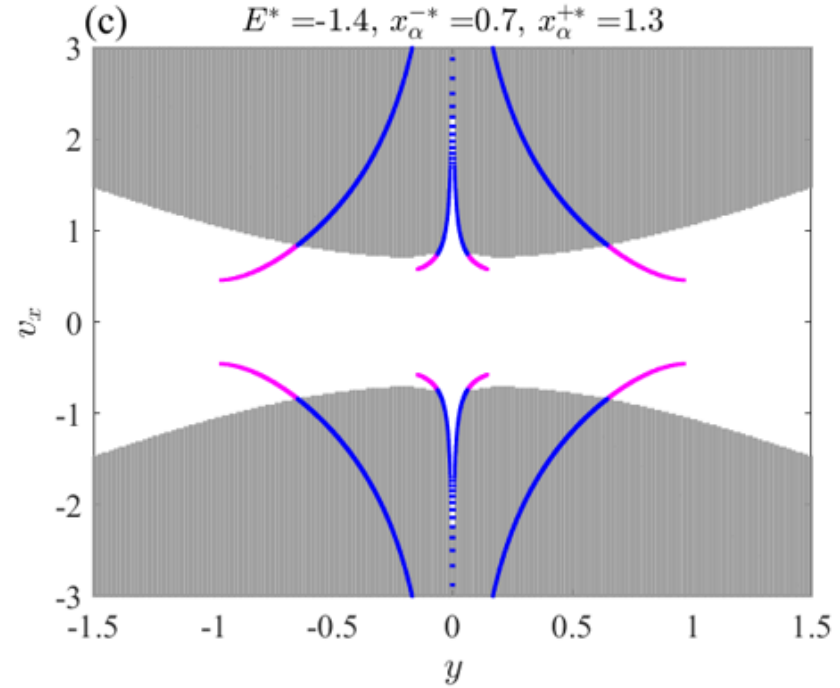
Magenta: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ does not include the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

Gray: Forbidden region within the given range $(x_\alpha^{-*}, x_\alpha^{+*})$





Outer curve: $x \approx -\mu$ Inner curve: $x \approx 1-\mu$



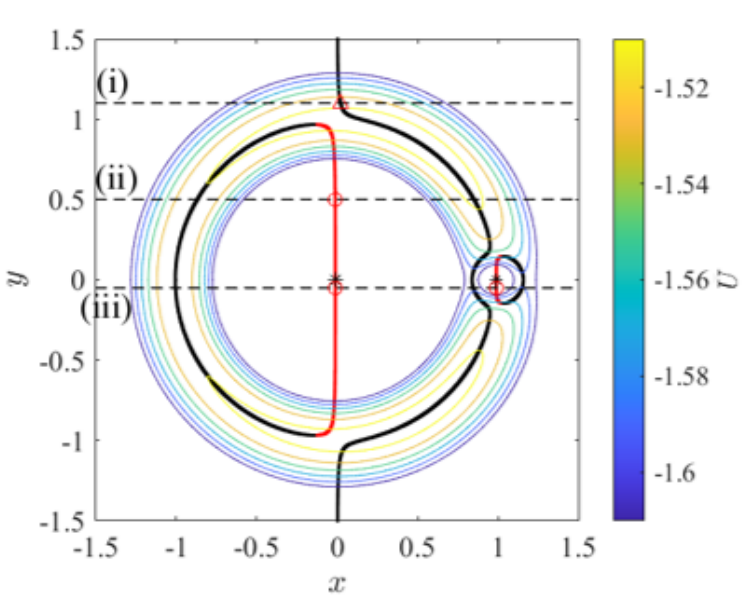
Conditional Barriers in yv_x-plane (4)

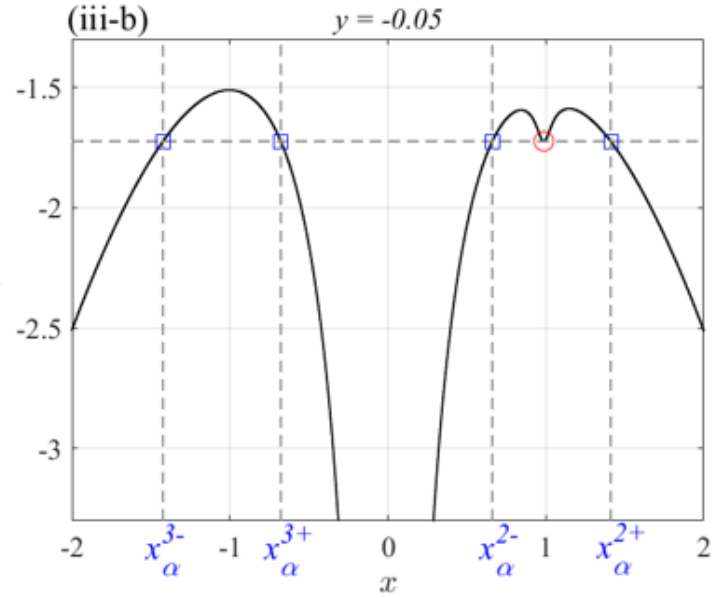
Blue: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ includes the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

Magenta: The conditioned range $\left(x_\alpha^{j-}, x_\alpha^{j+}\right)$ does not include the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

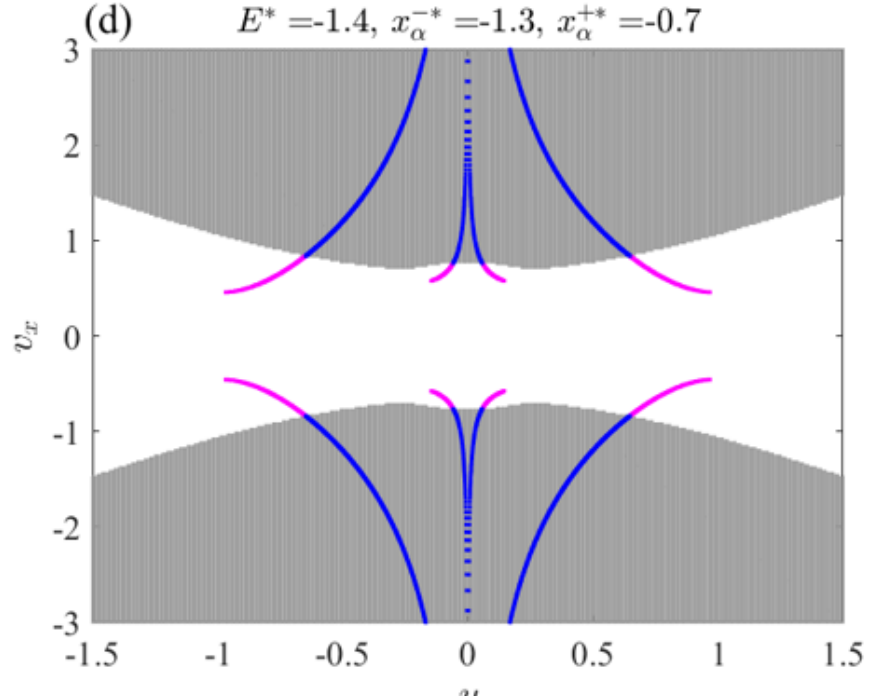
Gray: Forbidden region within the given range $(x_\alpha^{-*}, x_\alpha^{+*})$

The states with −1.3 < x < −0.7 always result in higher energies than those associated with the conditional barrier structure at x ≈ 1 − μ, thus reasoning that the conditional barrier structure is embedded in the conditional forbidden



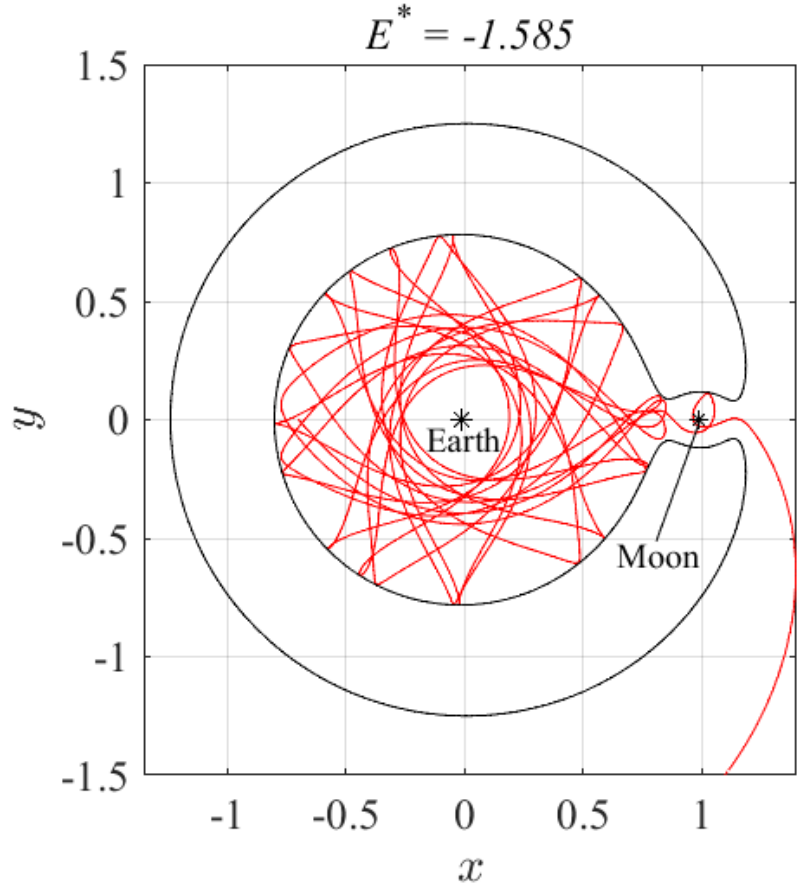


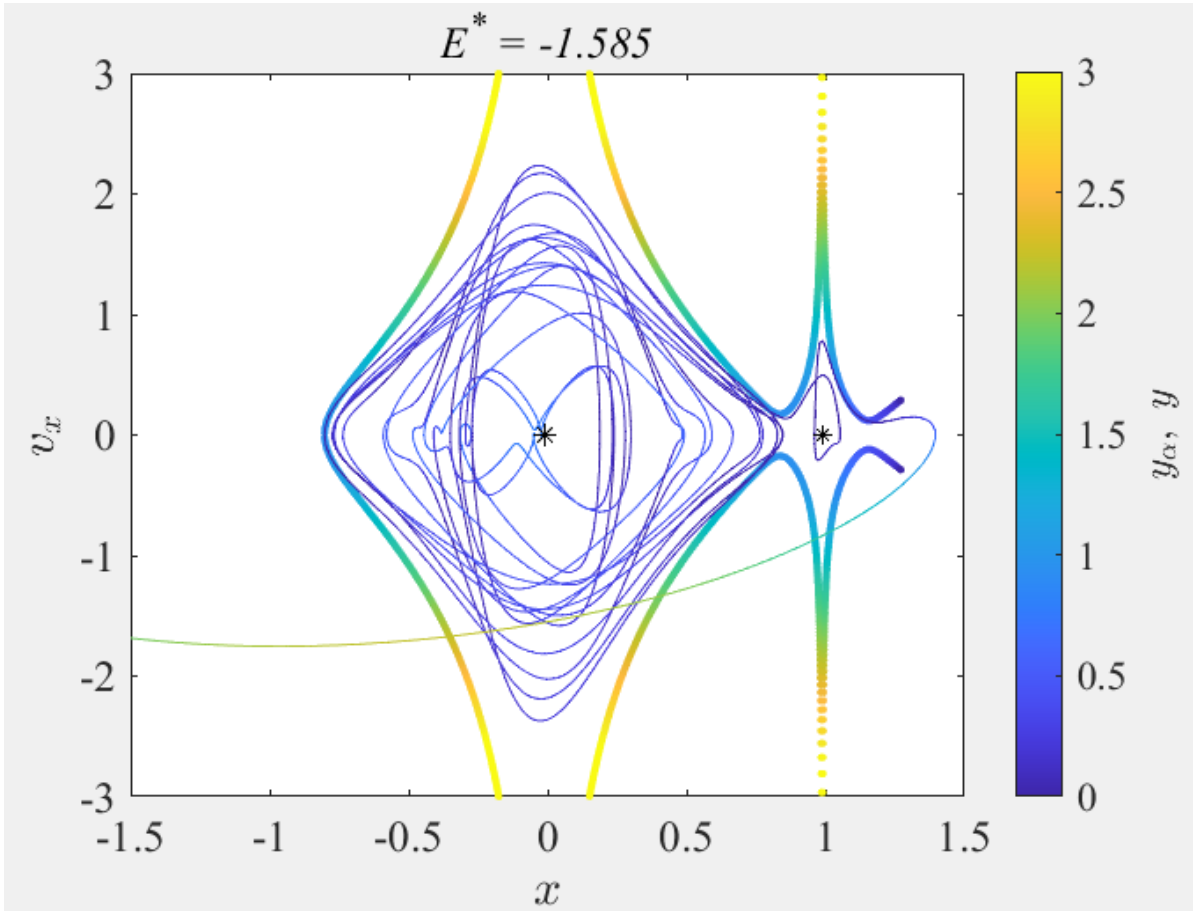
Outer curve: $x \approx -\mu$ Inner curve: $x \approx 1-\mu$



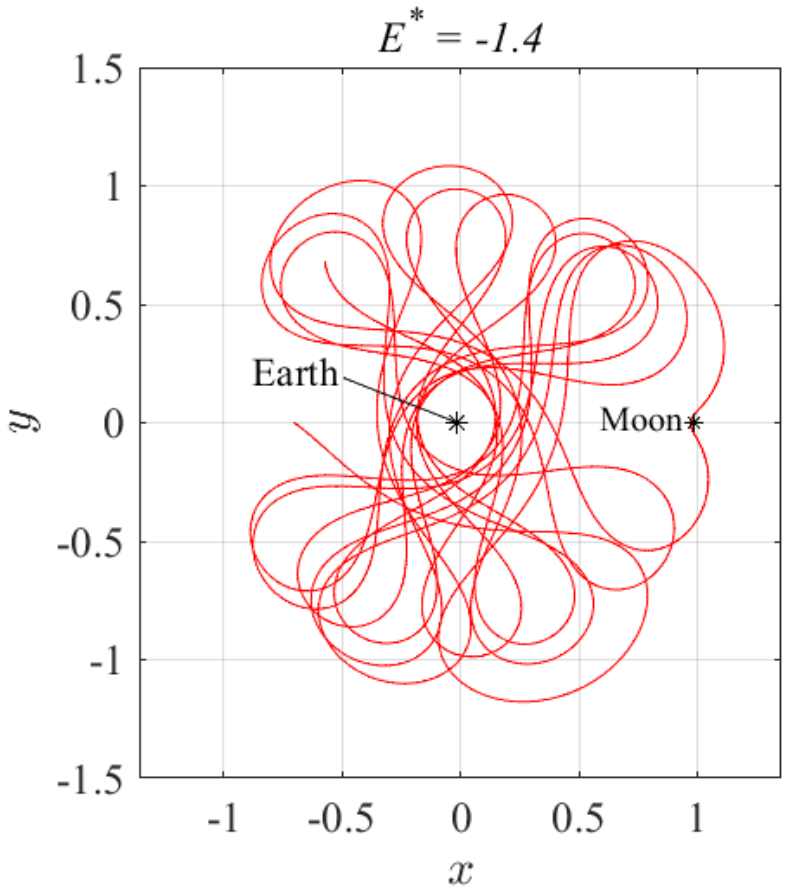
xvx-plane

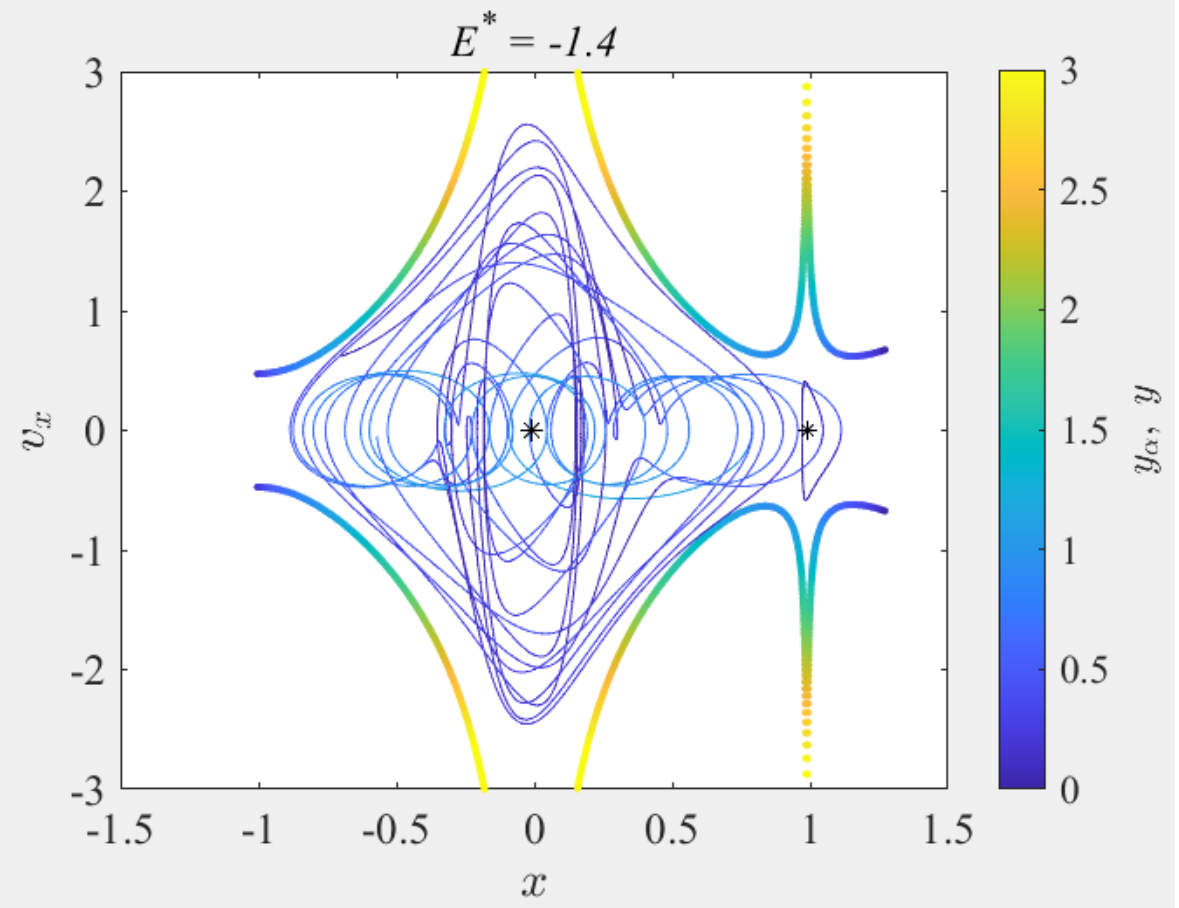
xvx-plane (1)





xvx-plane (2)


$E^* = -1.4$
Earth
Moon
y
x
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1


$E^* = -1.4$
v_x
x
y_α, y
3
2
1
0
-1
-2
-3
-1.5
-1
-0.5
0
0.5
1
1.5
2.5