University of Strathclyde Glasgow

Aerospace Centre of Excellence

A Physics Informed Fourier-Chebyshev Generator of Trajectories in the Restricted Three-Body Dynamics

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FUNDAMENTALS

Dynamical systems, solutions, and challenges

The Circular Restricted Three-Body Problem
Generalities

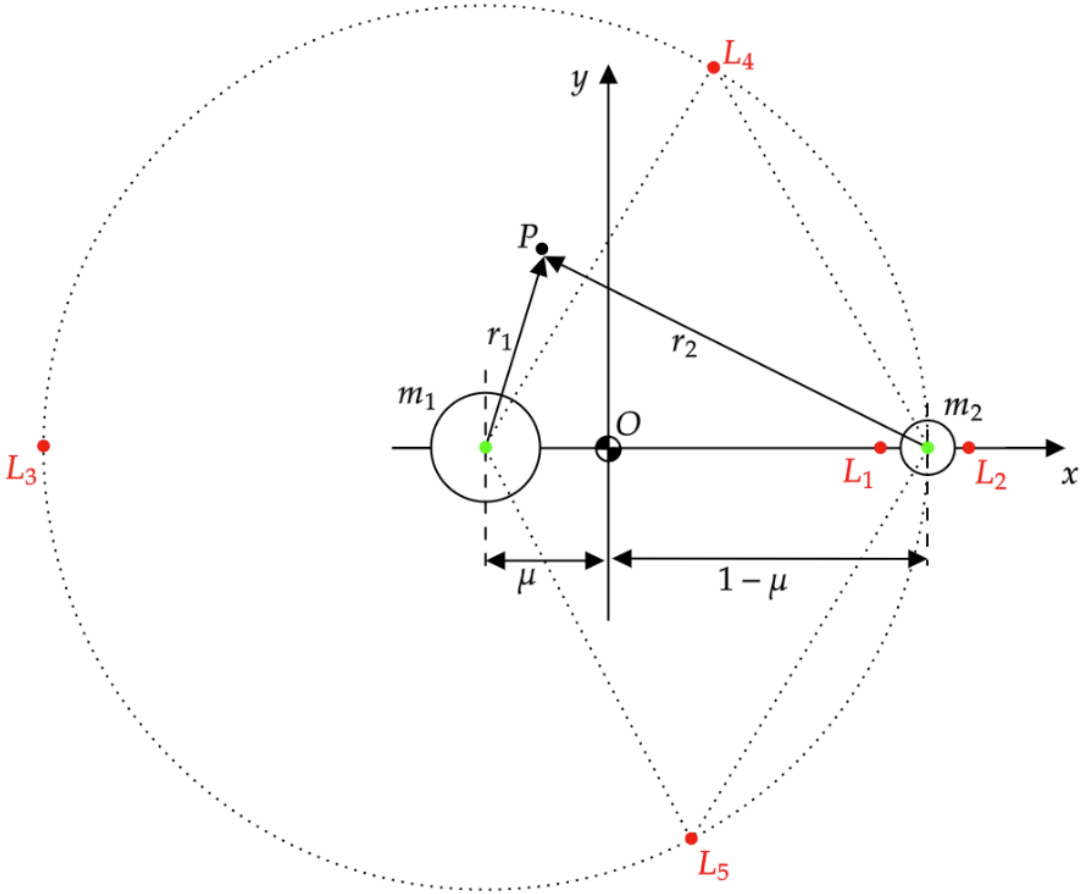
$$\ddot{x} = 2\dot{y} + U_x$$
$$\ddot{y} = -2\dot{x} + U_y$$
$$\ddot{z} = U_z.$$

$$U(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$$

$$K(\dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

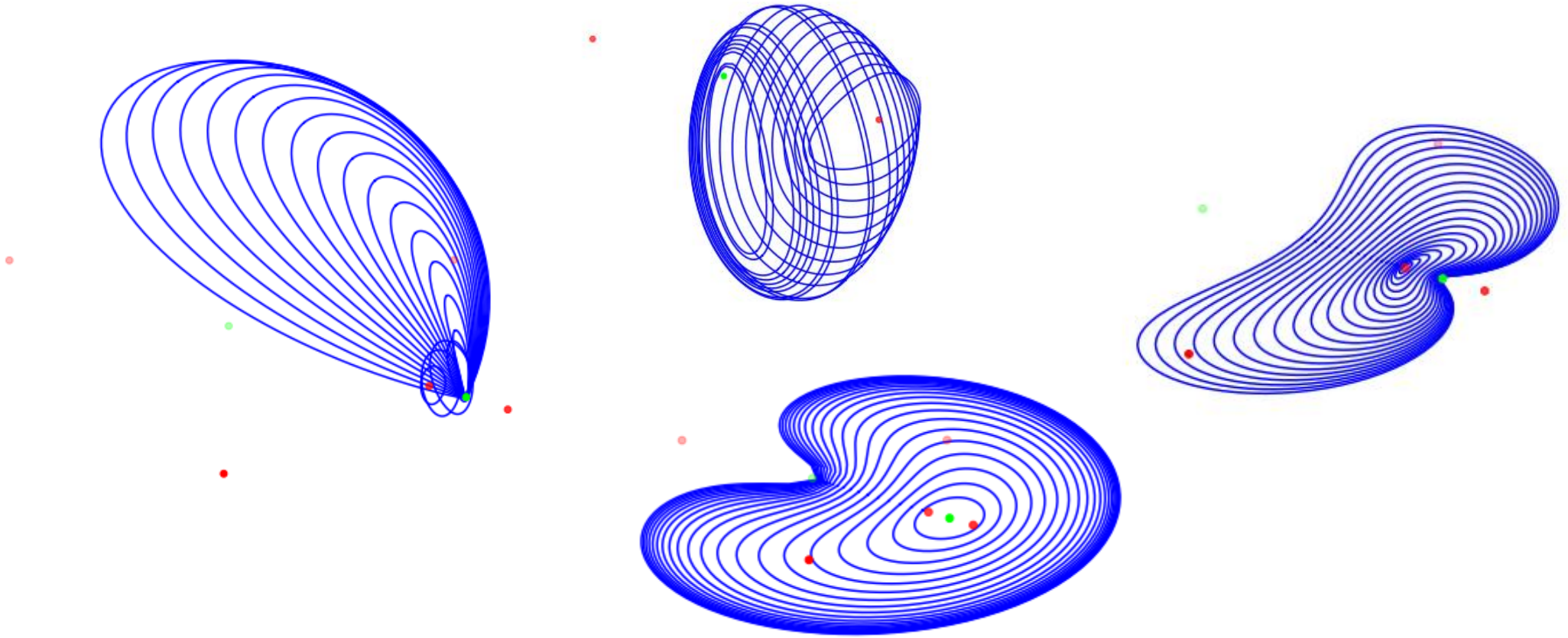
$$L(\mathbf{X}) = K + U + x\dot{y} + y\dot{x} - \frac{1}{2}\mu(1-\mu)$$

$$\mathcal{A}(T) = \int_0^T L(\mathbf{X}(t))\, dt$$



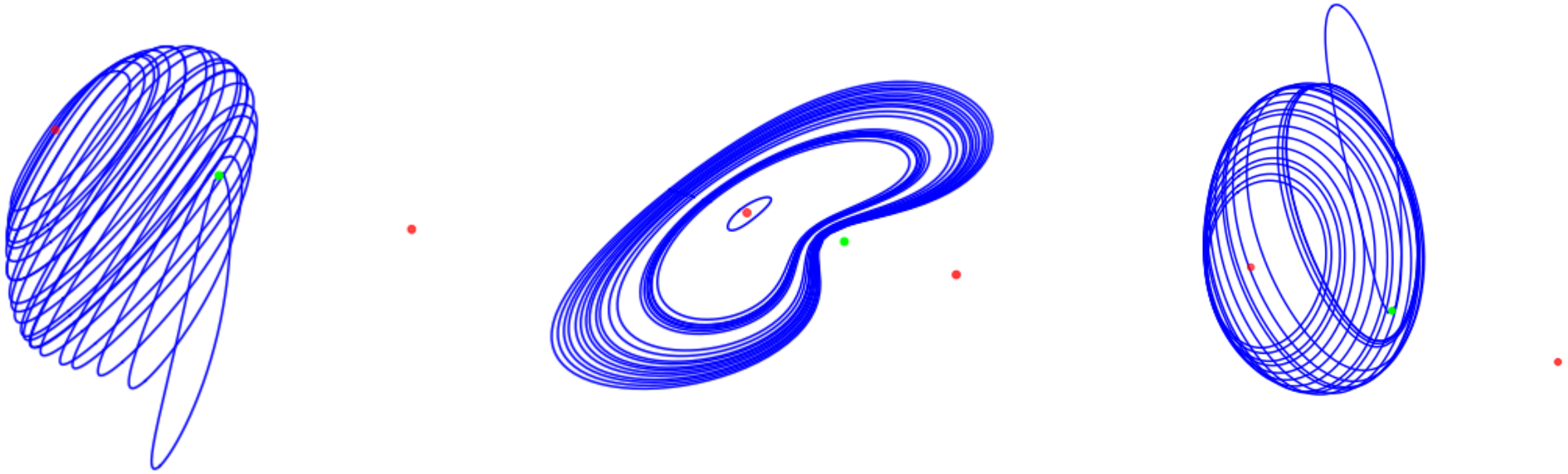
The Circular Restricted Three-Body Problem
Solutions

- 7 families of Periodic Orbits in the Earth-Moon $L_1 - L_2$ region
  Halo, Lyapunov, DROs



The Circular Restricted Three-Body Problem
Solutions

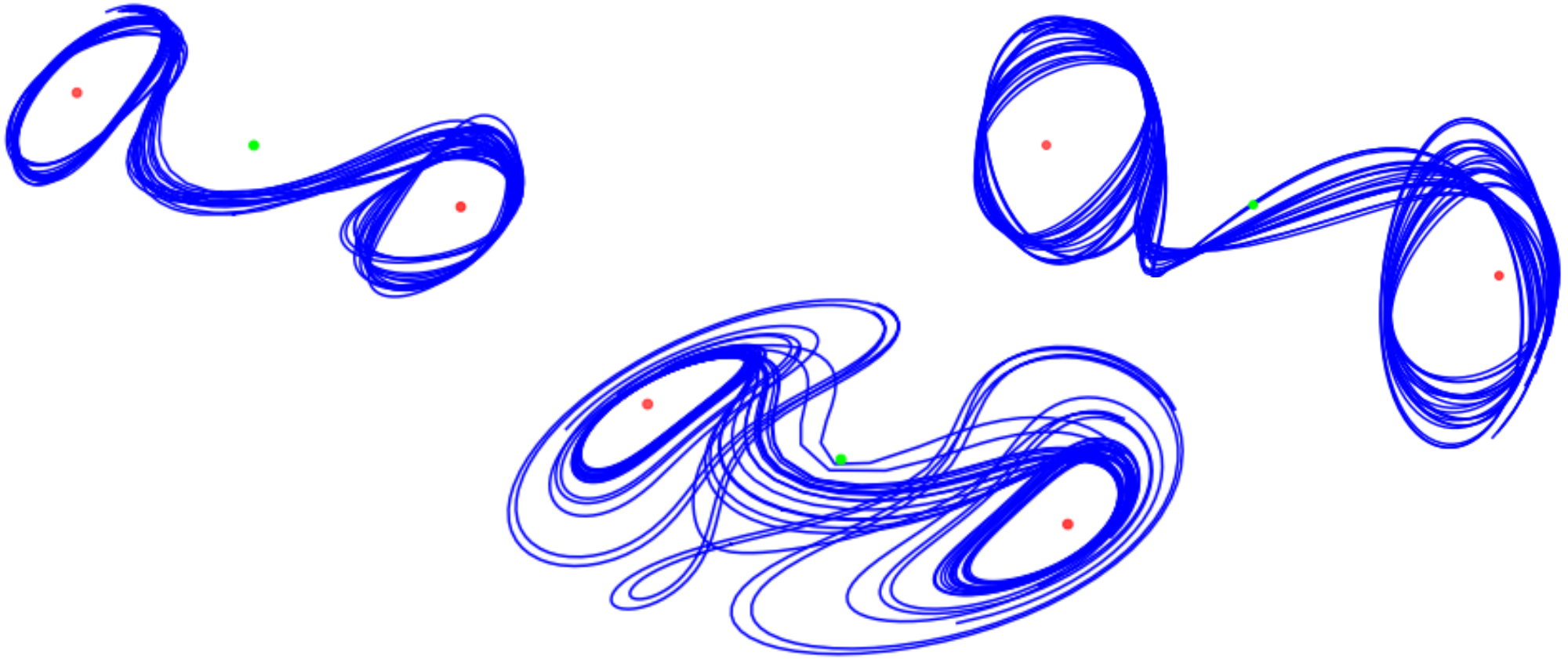
- 7 families of Periodic Orbits in the Earth-Moon $L_1 - L_2$ region
  Halo, Lyapunov, DROs
- 7 families of Quasi Periodic Orbits associated to the Periodic Orbits
  GMOS correction from the perturbed-centre dynamics
  Olikara and Scheeres, 2012



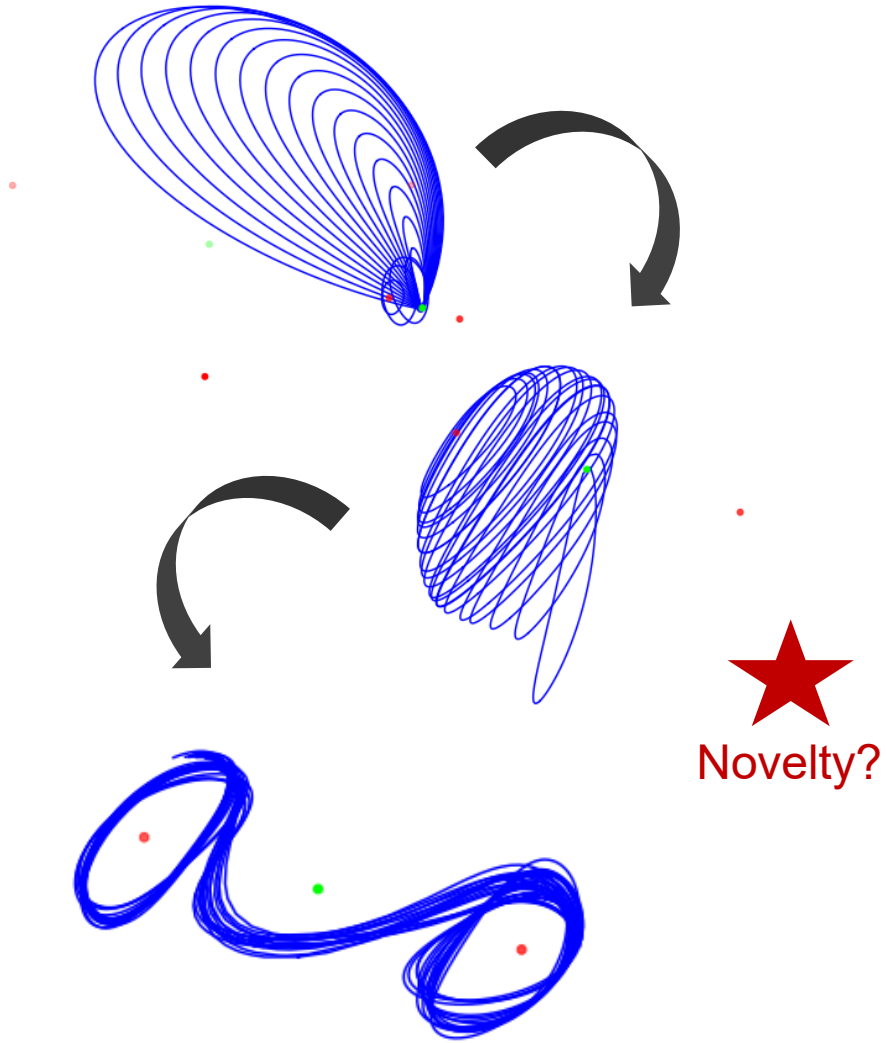
Note: only one sample of the resulting manifold is retained per QPO.

The Circular Restricted Three-Body Problem
Solutions

- 7 families of Periodic Orbits in the Earth-Moon $L_1 - L_2$ region
- 7 families Quasi Periodic Orbits associated to the Periodic Orbits
- 10 families of Heteroclinic Connections between Periodic Orbits (forward and backward)



The Circular Restricted Three-Body Problem
Our research



- **Systematically explore** the state of solutions
- **Generate** physical orbits
- Generate ***novel*** solutions

A former contribution on Periodic Orbits:

Litteri, W., Gil, A.F., **Vasile**, M., et al.
Generation of Periodic Orbits in the Restricted Three-body Problem with a Variational Autoencoder.
Celest Mech Dyn Astron 138, 25 (2026).

ORBIT GENERATION
A Generative AI approach

Orbit Generation Framework
A unified DFT-Chebyshev representation

Fourier Transform and Chebyshev decomposition of the residual.

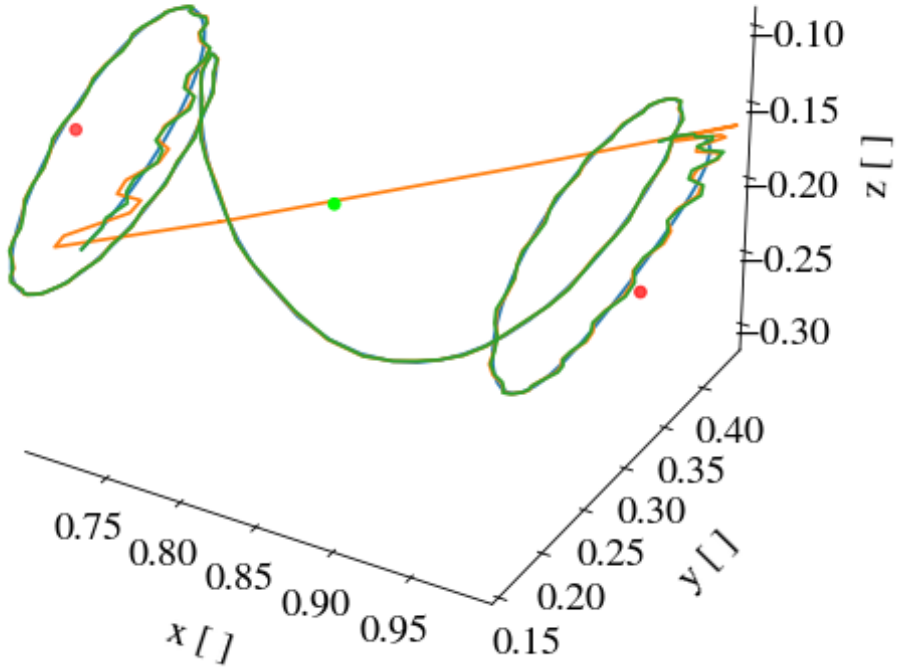
$$\mathbf{X}(t) \approx \tilde{\mathbf{X}}_{\mathrm{FFT}}(t) + \tilde{\mathbf{r}}_{\mathrm{Cheb}}(t) = \tilde{\mathbf{X}}_{\mathrm{FFT-Cheb}}(t)$$

$$\mathbf{r}(t) = \mathbf{X}(t) - \tilde{\mathbf{X}}_{\mathrm{FFT}}(t)$$

Compact representation for both periodic and non periodic trajectories.

$$n_{\mathrm{DFT}} = 50, n_{\mathrm{Cheb}} = 50,$$
$$d_v = 2 + 3(2n_{\mathrm{FFT}} + n_{\mathrm{Cheb}}) = 452.$$

Note: On 200 nodes, the trajectory parametrisation is of dimension 1202 (1200 + Period + Energy).

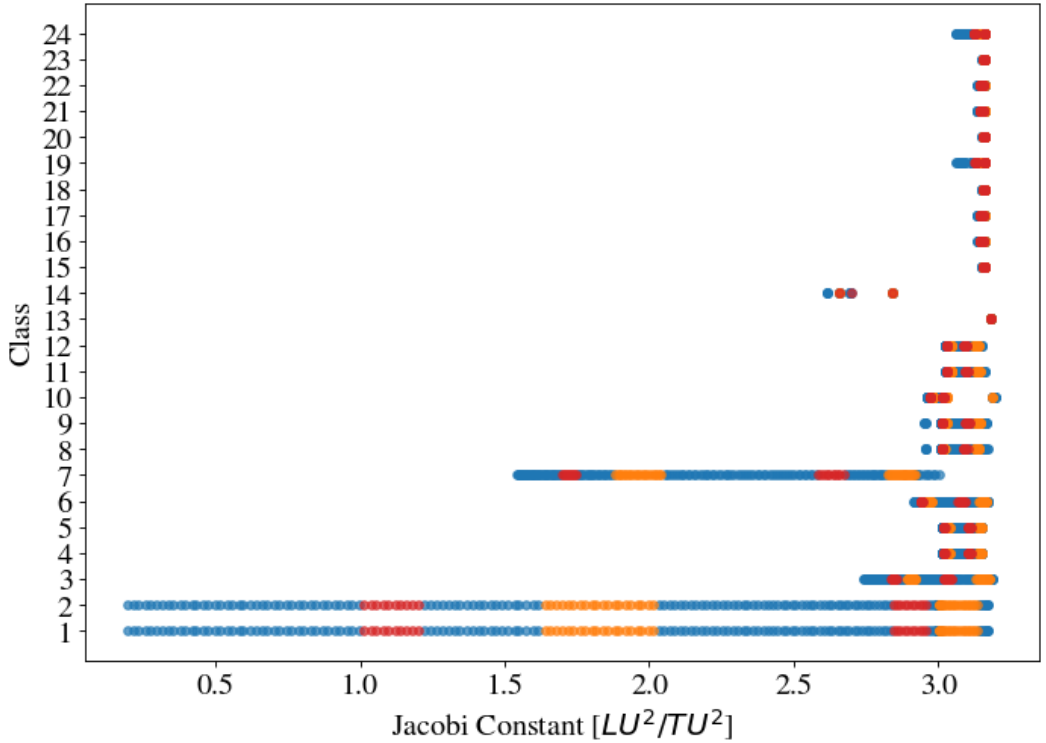


Comparison between the **reconstructed trajectory** using the mixed Fourier-Chebyshev representation (in green) **and the reference trajectory** (blue) for the worst reconstructed sample by **FFT-only** (orange) in the dataset.

Orbit Generation Framework
Dataset

Type of subset		Number of samples
Training	The model is optimised on these objects	$n_{\text{train}} = 3038$
Validation	Training assessment (e.g., detect overfitting)	$n_{\text{val}} = 868$
Test	Model assessment	$n_{\text{test}} = 432$

- 24 families of orbits
- Features space of $d_v = 452$
- Subset partitioning by **energy bands**



Subset partitioning strategy:
training, **validation**, **test** bands.

Orbit Generation Framework
Normalising Flow Models

Learn an invertible and differentiable transformation $f : \mathbb{R}^{d_v} \rightarrow \mathbb{R}^{d_v}$

Forward process: $\mathbf{w} = f(\mathbf{v})$

Inverse process: $\mathbf{v} = f^{-1}(\mathbf{w}) = g(\mathbf{w})$

Real Non-Volume Preserving Normalising Flow (**Real-NVP**)
Dinh et al., 2017

$\mathbf{w}$ is the *latent variable*

Training function:

$$\mathcal{L}_{\text{NLL}} = -\frac{1}{n_{\text{train}}} \sum_{k=1}^{n_{\text{train}}} \log p\left(\mathbf{v}_k\right)$$

$$\log p(\mathbf{v}) = \log p(\mathbf{w}) + \log \left|\mathbf{J}(f(\mathbf{v}))\right|$$

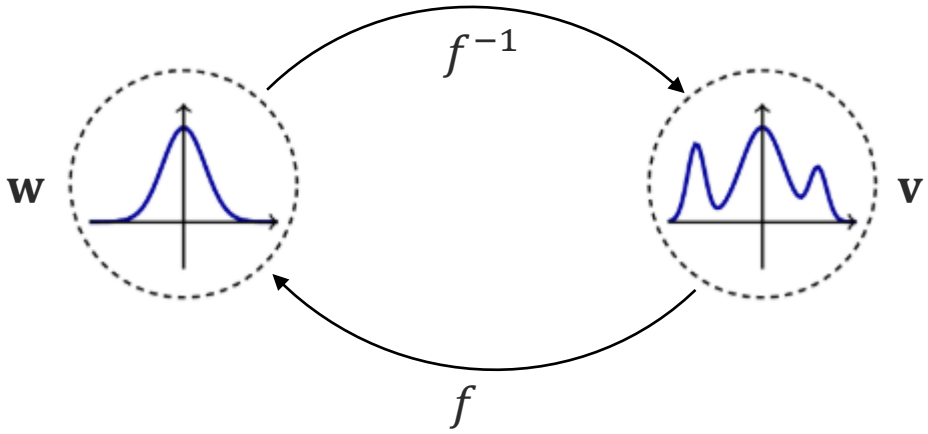
$$f = f_{n_a} \circ f_{n_a - 1} \circ \ldots \circ f_2 \circ f_1$$

$$\mathbf{y}_k = f_k(\mathbf{x}_k, \mathbf{b}_k, s_k, t_k) \quad f_k, k = 1, \ldots, n_a$$

$$\mathbf{y}_k = \mathbf{b}_k \odot \mathbf{x}_k + (1 - \mathbf{b}_k) \odot (\mathbf{x}_k \odot e^{s(\mathbf{b}_k \odot \mathbf{x}_k)} + t(\mathbf{b}_k \odot \mathbf{x}_k))$$

$n_{train} = 3038$

$n_a = 16$ (number of layers)



Orbit Generation Framework
Physics-informed training

The Gradient of the Action Functional is used as an integral metric on the generated trajectory.

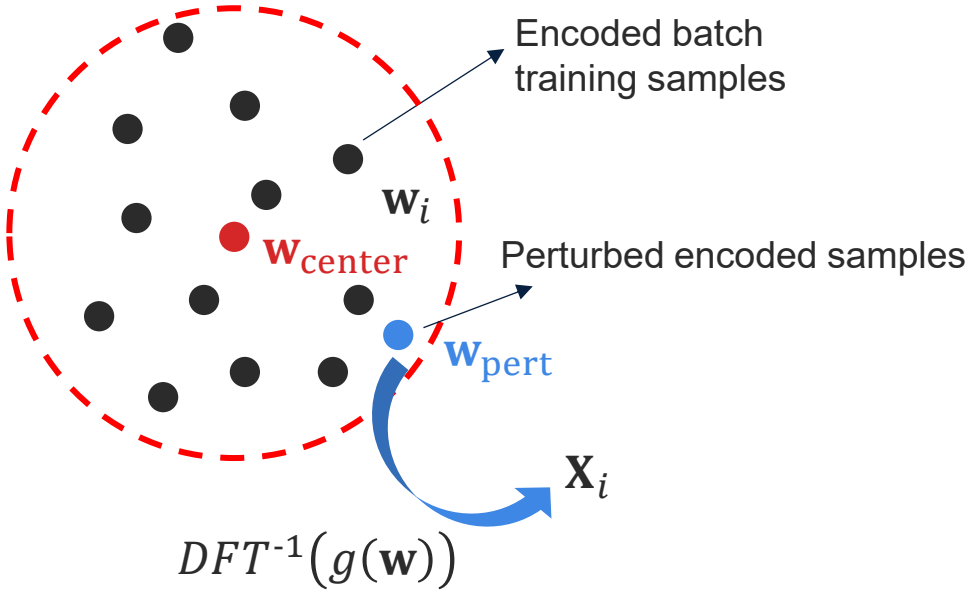
$$\bar{\mathcal{L}}_{\text{phys}} = \frac{1}{B}\sum_{i=1}^{B}\mathcal{L}_{\text{phys}}(\mathbf{X}_i);$$

$$\mathcal{L}_{\text{phys}}(\mathbf{X}_i) = \frac{1}{2N_g}\sum_{j=1}^{N_g}\left([\nabla_{\mathbf{X}_i}\mathcal{A}(\mathbf{X}_i)]_j\right)^2.$$

$$\mathcal{L} = \mathcal{L}_{\text{NLL}} + \lambda_{\text{phys}}\bar{\mathcal{L}}^{q}_{\text{phys}}$$

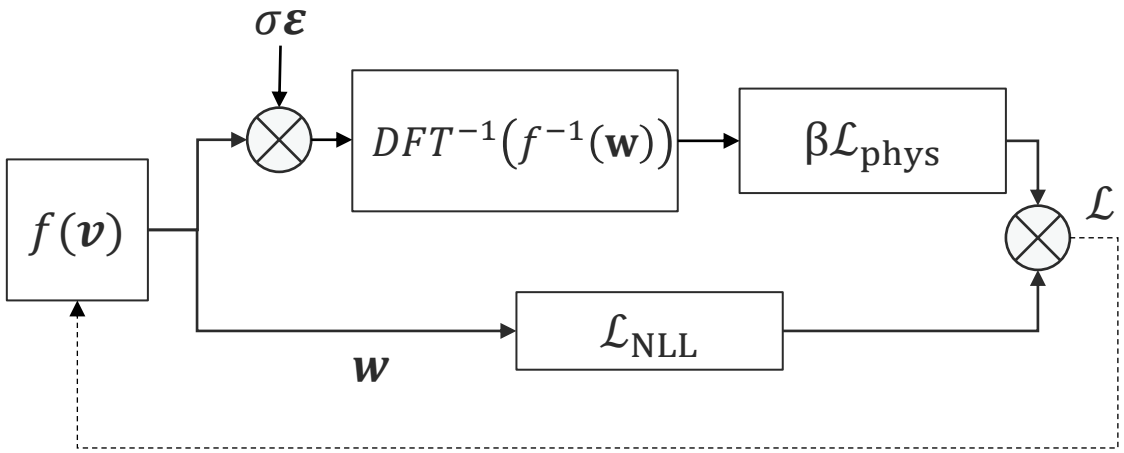
The physics loss is computed on the perturbed decoded samples from a centre point defined on the training batch B_i.

$$\mathbf{w}_{\text{pert}} = \mathbf{w}_{\text{center}} + \sigma\boldsymbol{\epsilon}$$
$$\varepsilon \sim \mathcal{N}(0, \boldsymbol{I}), \sigma = 0.1$$

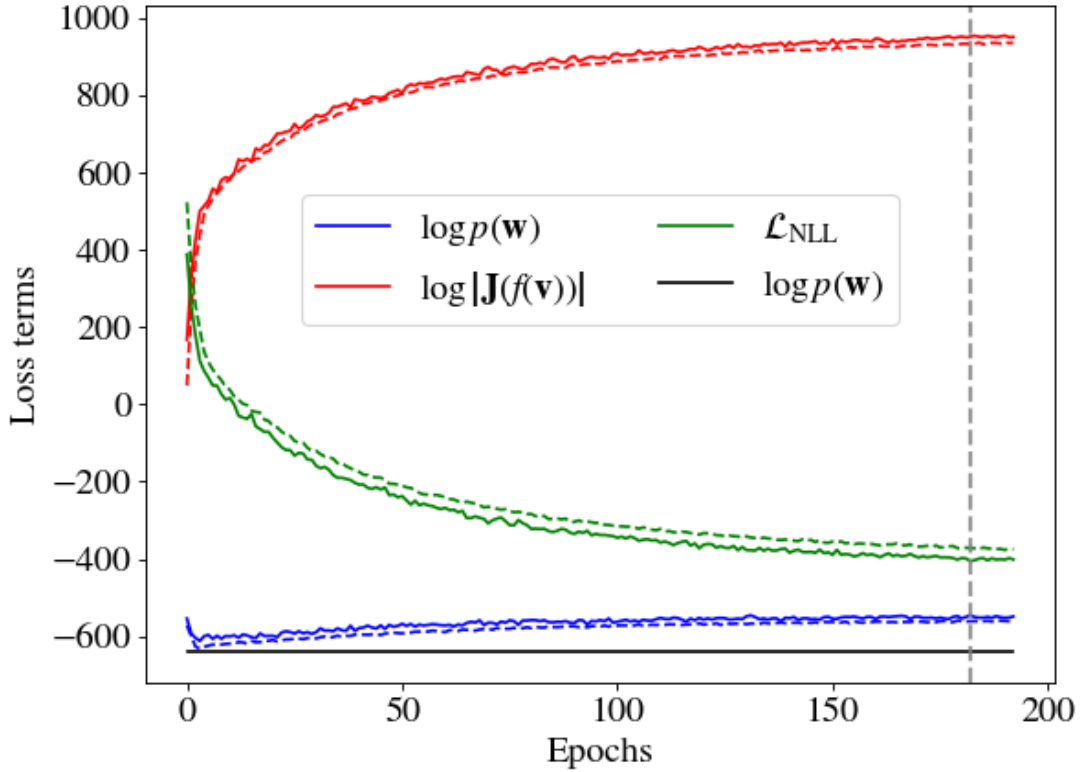


Variational physics loss setting.

Orbit Generation Framework
Physics-informed training dynamics



(Backpropagation)

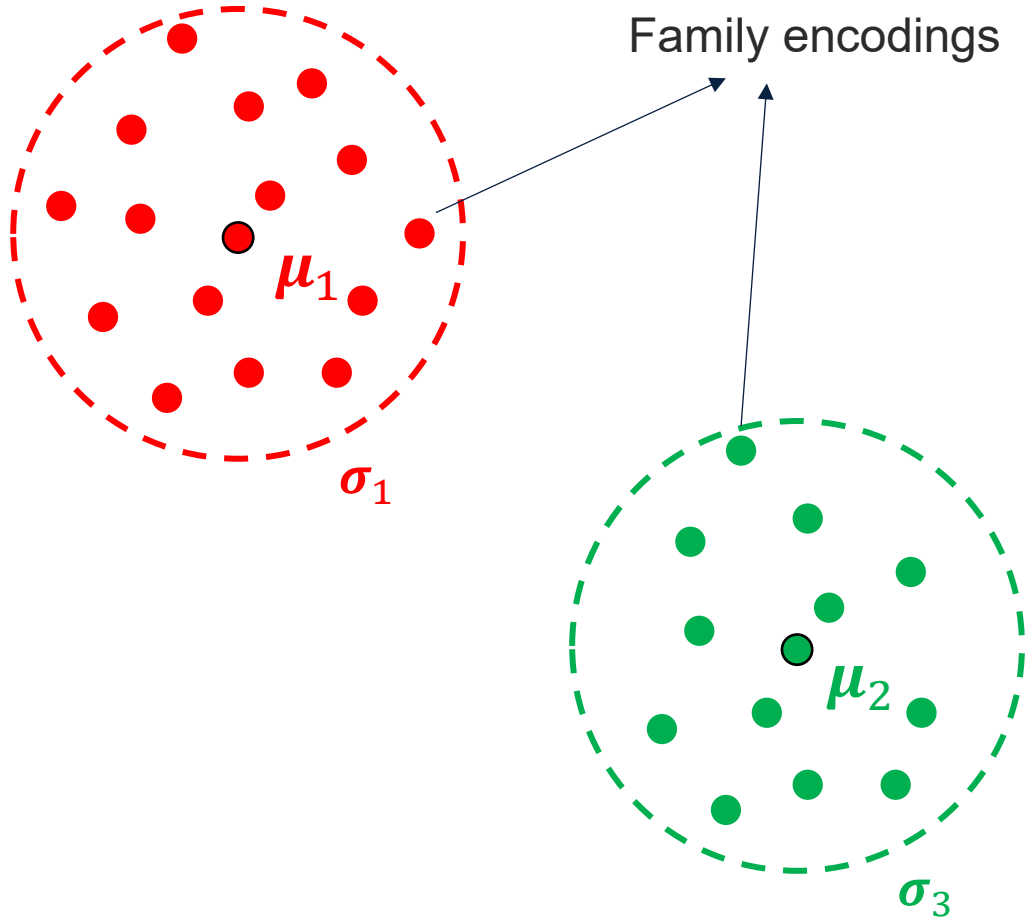


Training dynamics over 200 epochs. The dashed line denotes the training loss; the solid line the validation loss.

Orbit Generation Framework
Generation

- **Gaussian Sampling**
  Sample from a Gaussian distribution in the latent space and reconstruct the samples
  Easy to implement, but not systematic
  $$\mathbf{w}_{\text{inf}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

- **Class-aware** diagonal Gaussian Sampling
  Gaussian Sampling from the class statistics information $\boldsymbol{\mu}_c, \boldsymbol{\sigma}_c$
  $$\mathbf{w}_{\text{inf}} \sim \mathcal{N}(\boldsymbol{\mu}_c, \boldsymbol{\sigma}_c)$$



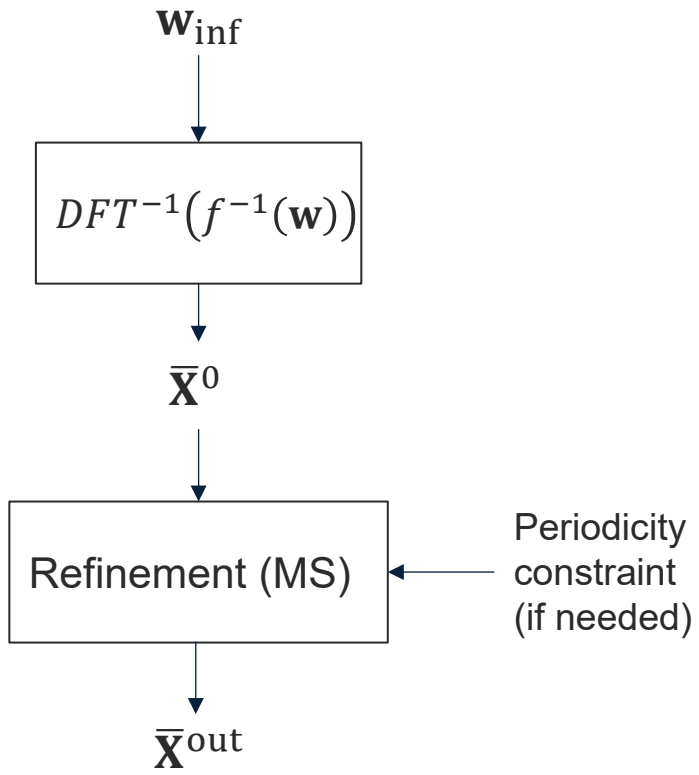
Orbit Generation Framework
Refinement

A **Multiple Shooting** (MS) method enforces continuity of the state of the decoded trajectory

$$\mathbf{F}_k = \mathbf{X}_k^f - \mathbf{X}_{k+1} = 0, \; k = 1, \ldots, N-1$$
$$\mathbf{F}_N = \mathbf{X}_1 - \mathbf{X}_N = 0,$$

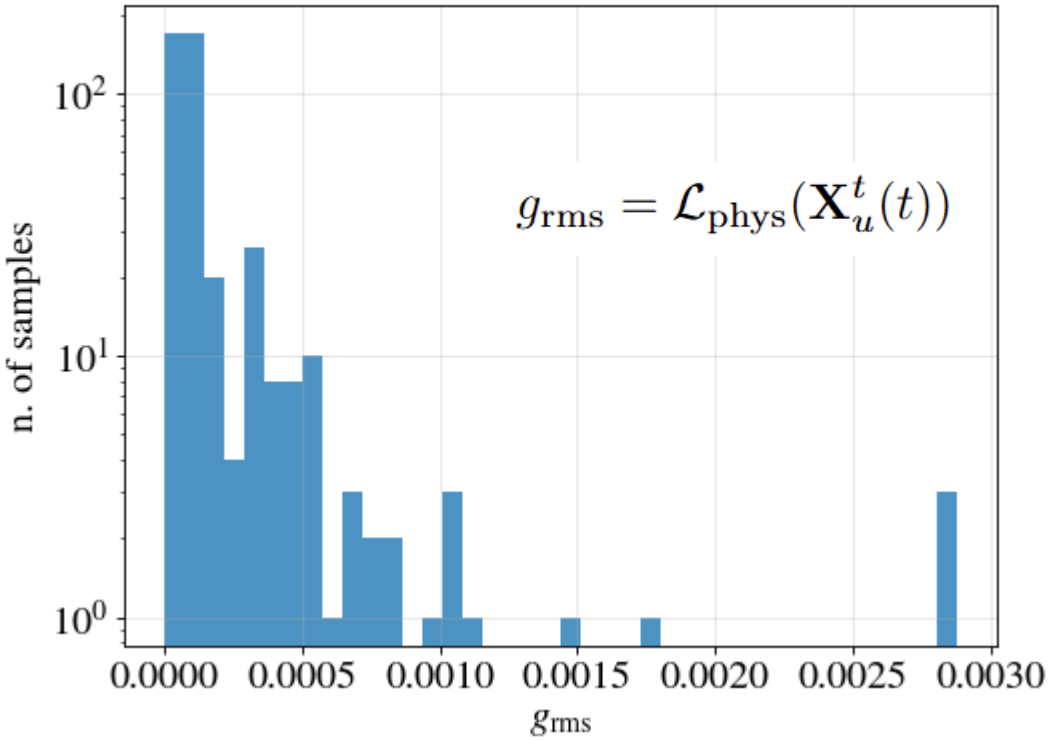
$$\mathbf{F}(\bar{\mathbf{X}}) = \mathbf{0}$$

$$\bar{\mathbf{X}}^r = \bar{\mathbf{X}}^{r-1} - \lambda^r \cdot \mathbf{DF}^{-1}\mathbf{F}(\bar{\mathbf{X}}^{r-1}), \; r = 1, \ldots, N_{max}$$

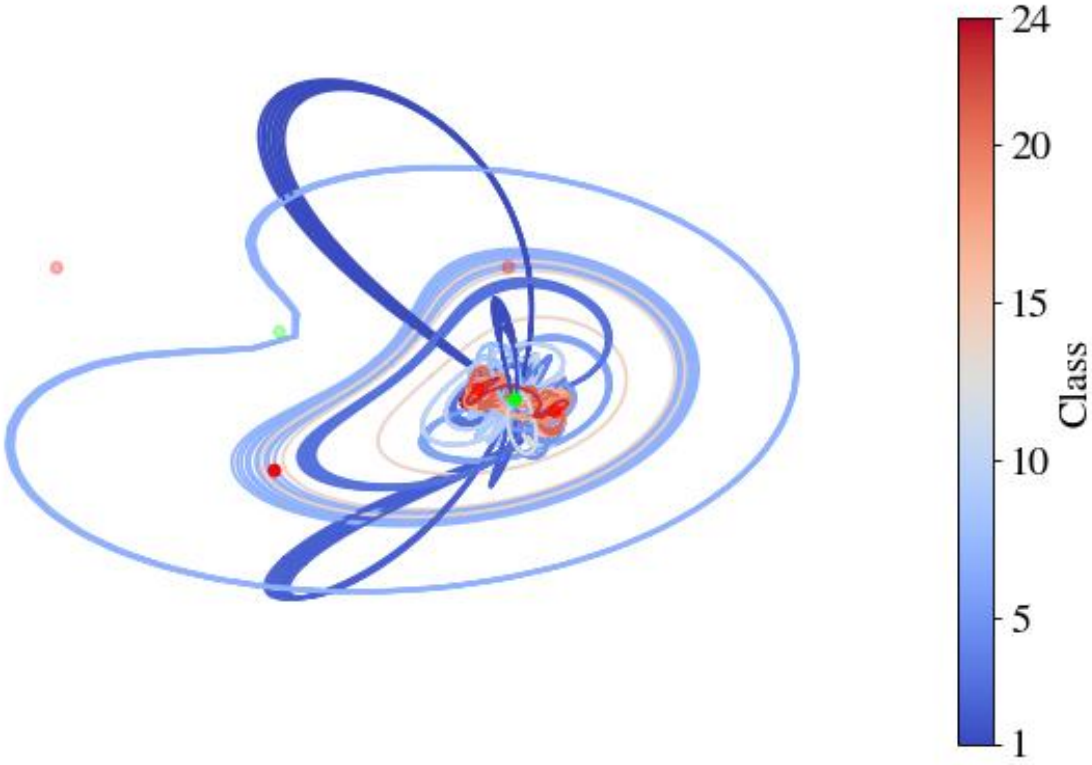


RESULTS

Inversion of the Test Dataset
Physics and similarity metrics

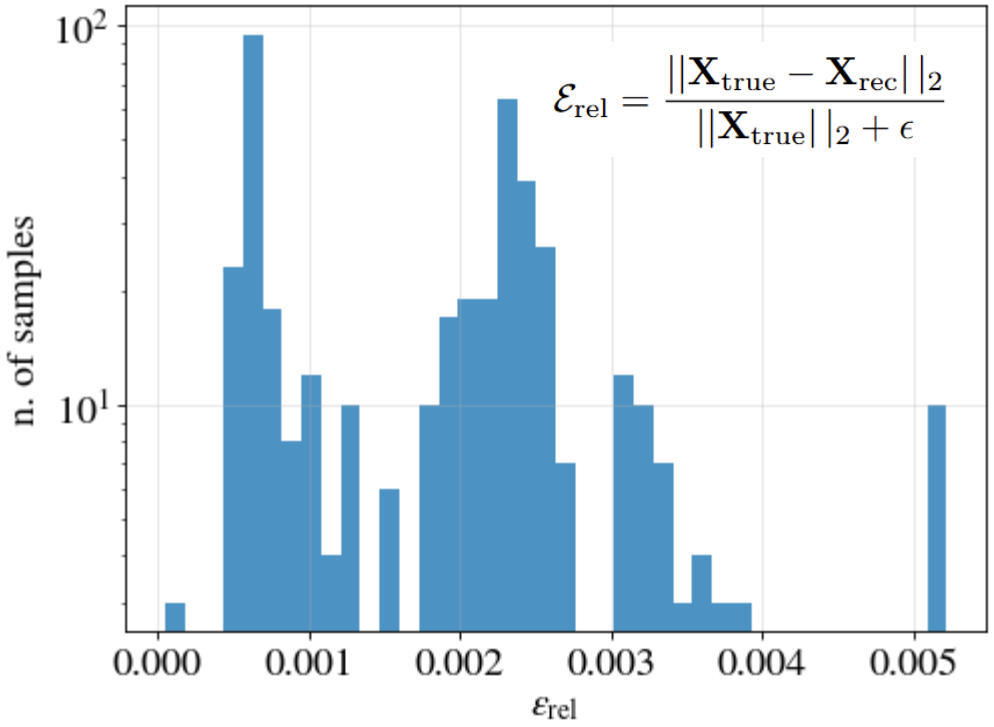


Distribution of the generated test solutions by physics metric.

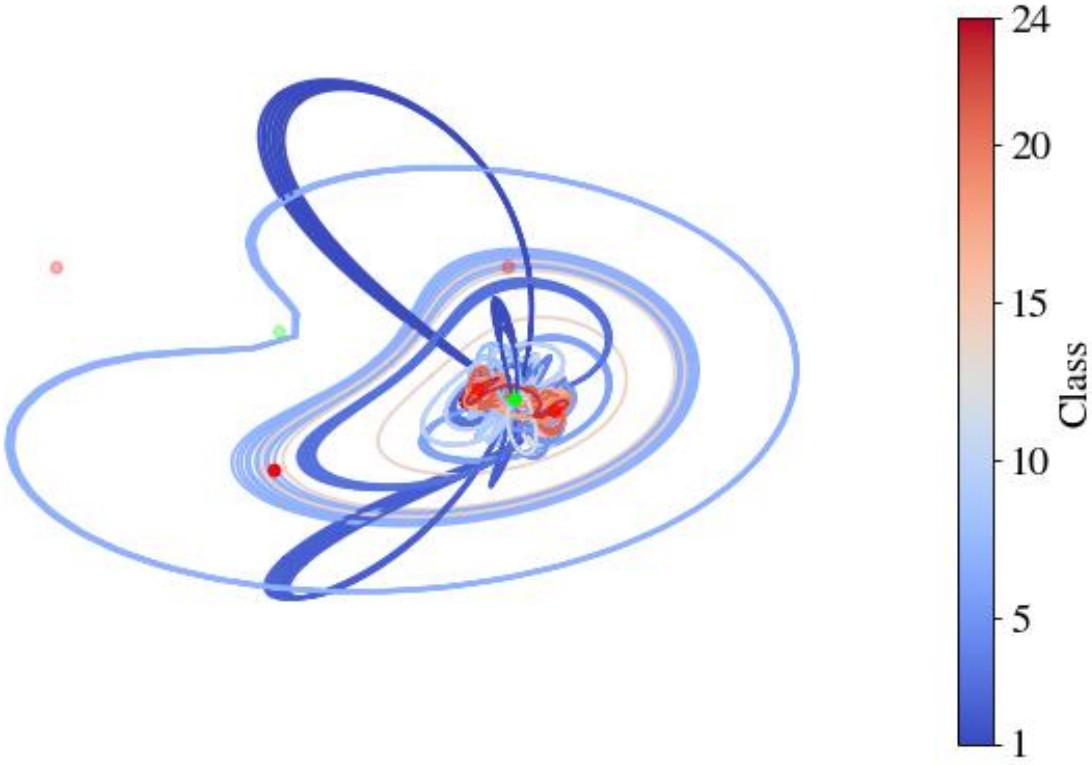


Generation of trajectories from the test set encodings. Colour is assigned by the family (1-24).

Inversion of the Test Dataset
Physics and similarity metrics

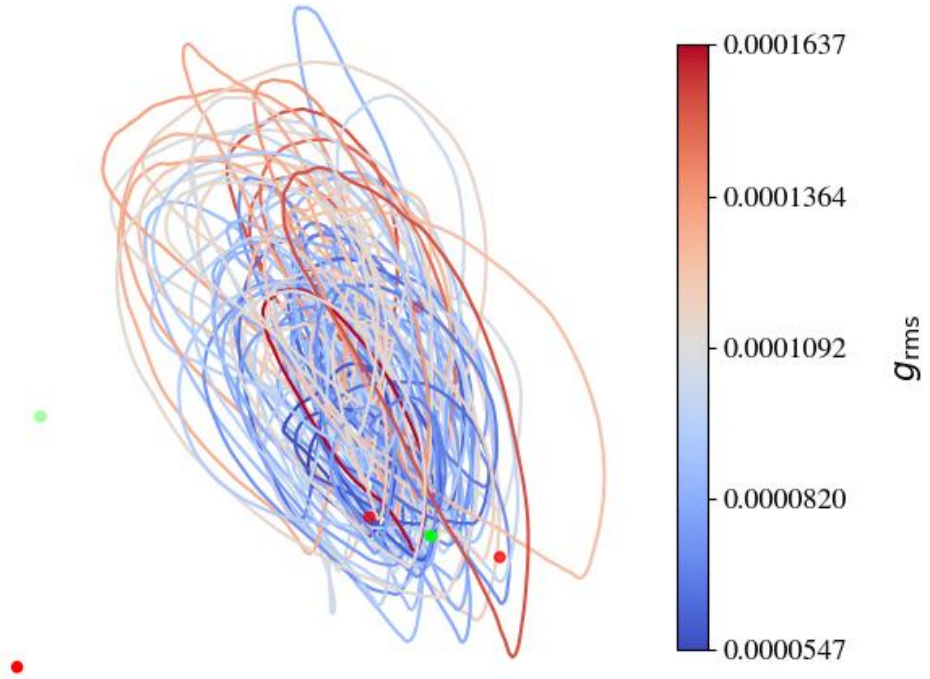


Distribution of the generated test solutions by similarity metric to the reference trajectories.

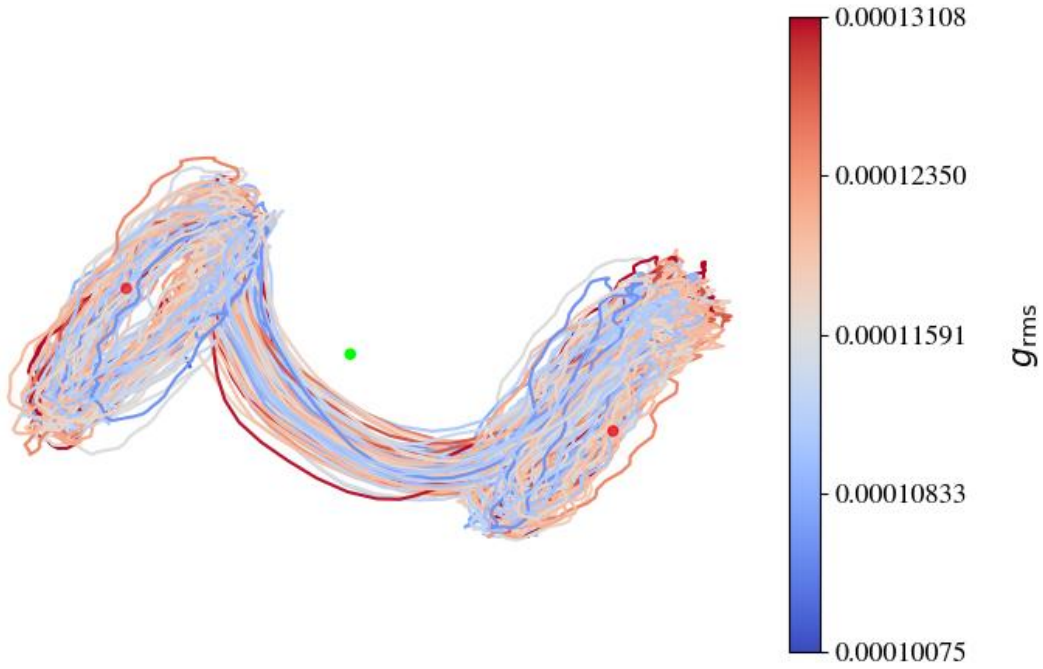


Generation of trajectories from the test set encodings. Colour is assigned by the family (1-24).

Class-aware sampling
Generation

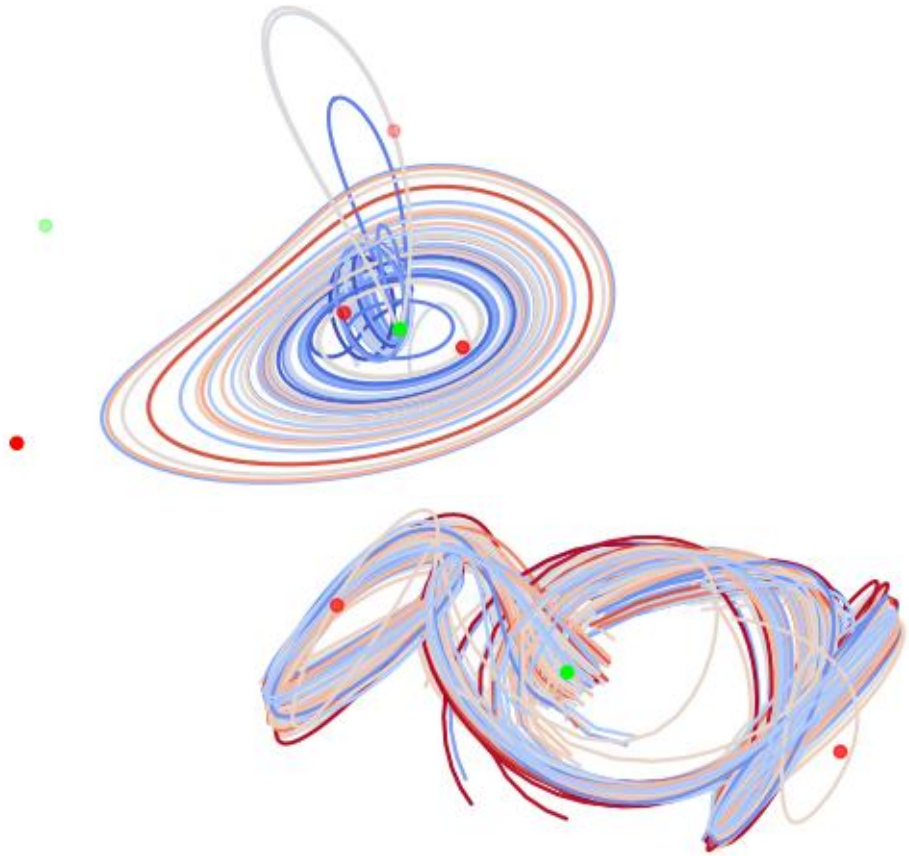


Selection of generations from Class 1 (Halo North L1). Colour is by physic metric

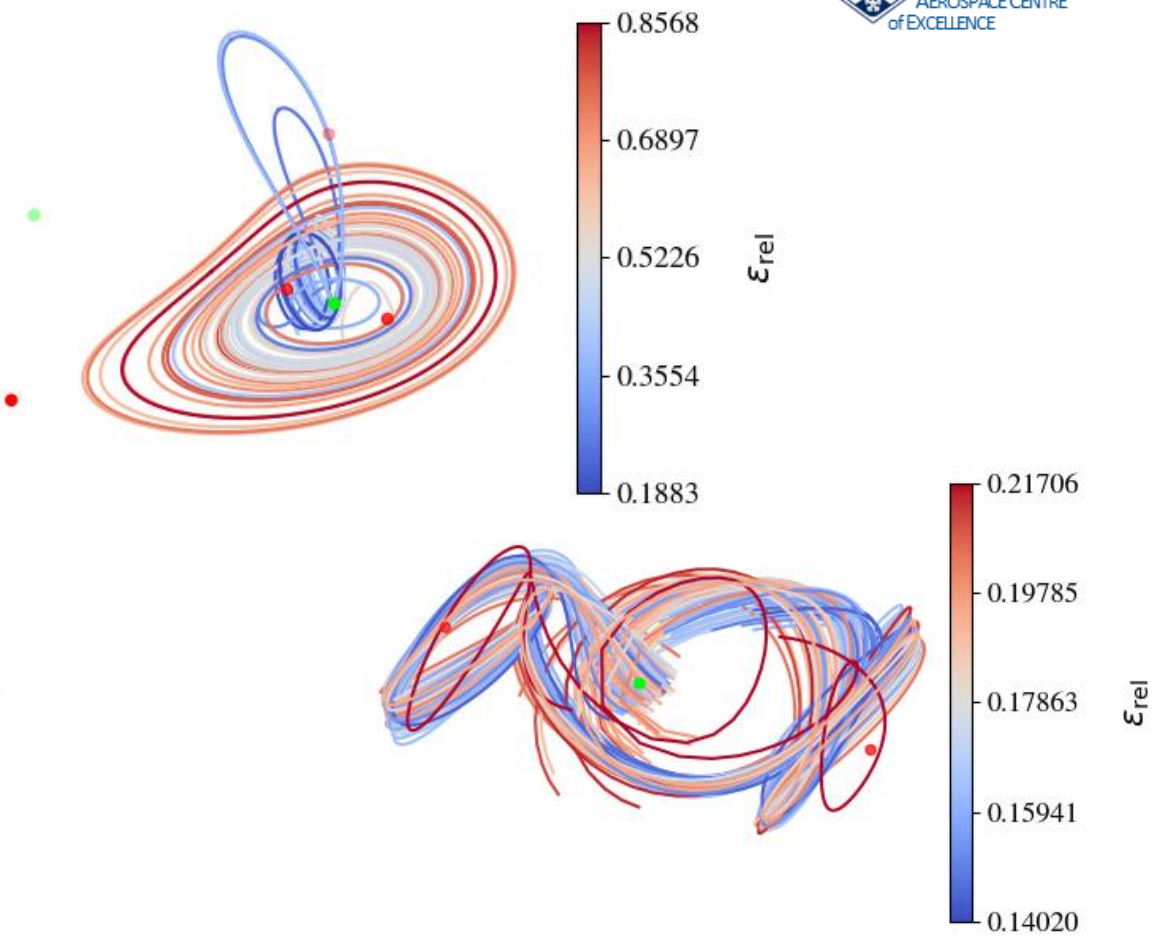


Selection of generations from Class 17 (Halo South L1 to Halo North L2). Colour is by physic metric

Class-aware sampling
Refinement

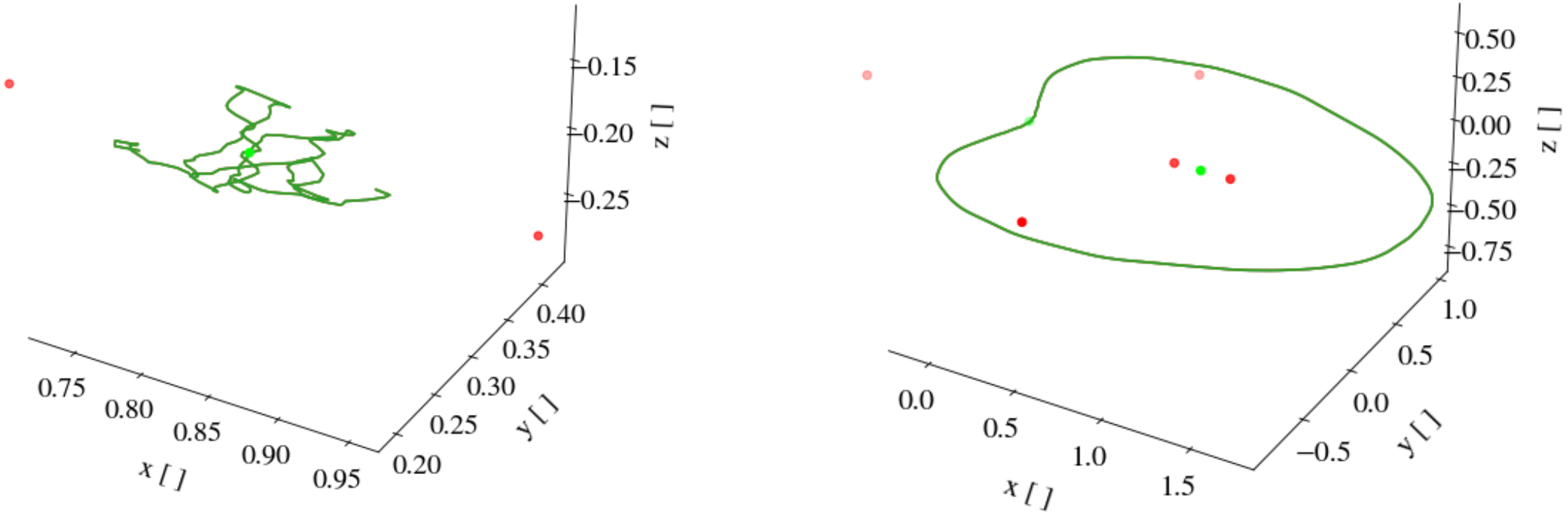


Selection of refined generations, coloured by physic metric of the generated orbits.



Selection of refined generations, coloured by similarity metric to the generated trajectories.

On the role of the Physics loss
It enforces physical consistency of the trajectories



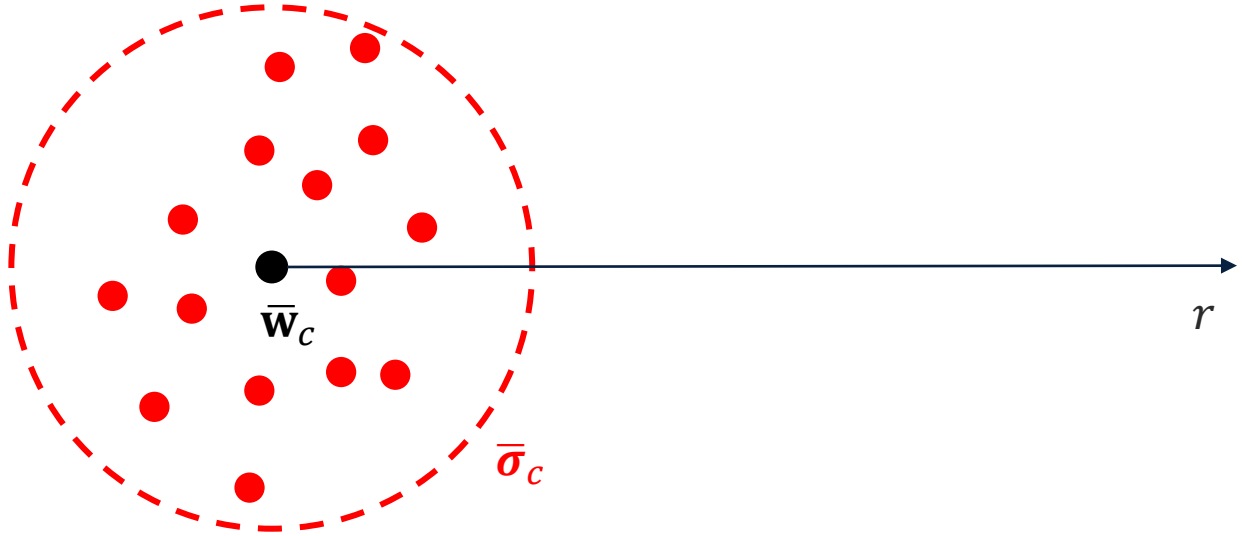
Worst generated trajectories under class-aware conditional sampling.
Left: $\lambda_{\text{phys}} = 0, g_{\text{rms}} = 3.8 \times 10^{-1}$(class 24).
Right: $\lambda_{\text{phys}} = 1000, g_{\text{rms}} = 6.5 \times 10^{-2}$(class 7).

Exploratory **generation**
Sampling out of the known family manifold

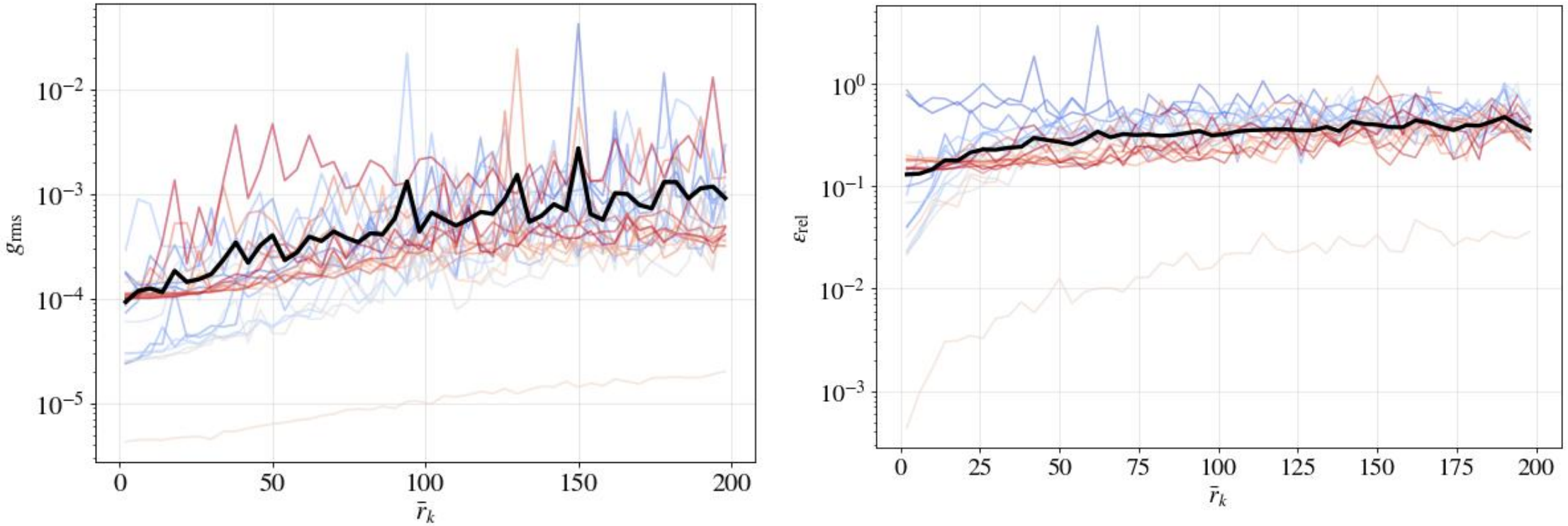
1. Identify the closest family sample $\bar{\mathbf{w}}_c$ to the family centroid $\boldsymbol{\mu}_c$
2. Sample at increasing distance r from the family centroid

$$\mathbf{w}_c^r = \bar{\mathbf{w}}^c + r\bar{\boldsymbol{\sigma}}_c \odot \boldsymbol{\rho}$$

$$\boldsymbol{\rho} = \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}, \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$



***Exploratory* generation**
Sampling out of the known family manifold



Generation of trajectories at increasing normalised radial distance from the family prototype.
Left: Per-class physics loss (coloured from blue to red) and aggregated mean (thick black line).
Right: Per-class normalised Euclidean distance between the generated and refined orbits (coloured from blue to red), and aggregated mean (thick black line).

CONCLUSION

Conclusion and

An unconditioned Flow Model trained on a dataset of varieties of periodic solutions to the CR3BP can generate samples conditioned to the specific class, by a class-aware sampling strategy.

The gradient of the action functional provides an integration-free physic metric which improved the quality of the generations.

Systematic exploration at high distance from the dataset reveals orbits with increasingly high associated physics loss.

Future Work

Implement more expressive representations to deal with POs, QPOs, HCs in a unified framework Ideas?

Search for *novelty* in the dataset

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A Pale Blue Dot
Voyager 1, 1990

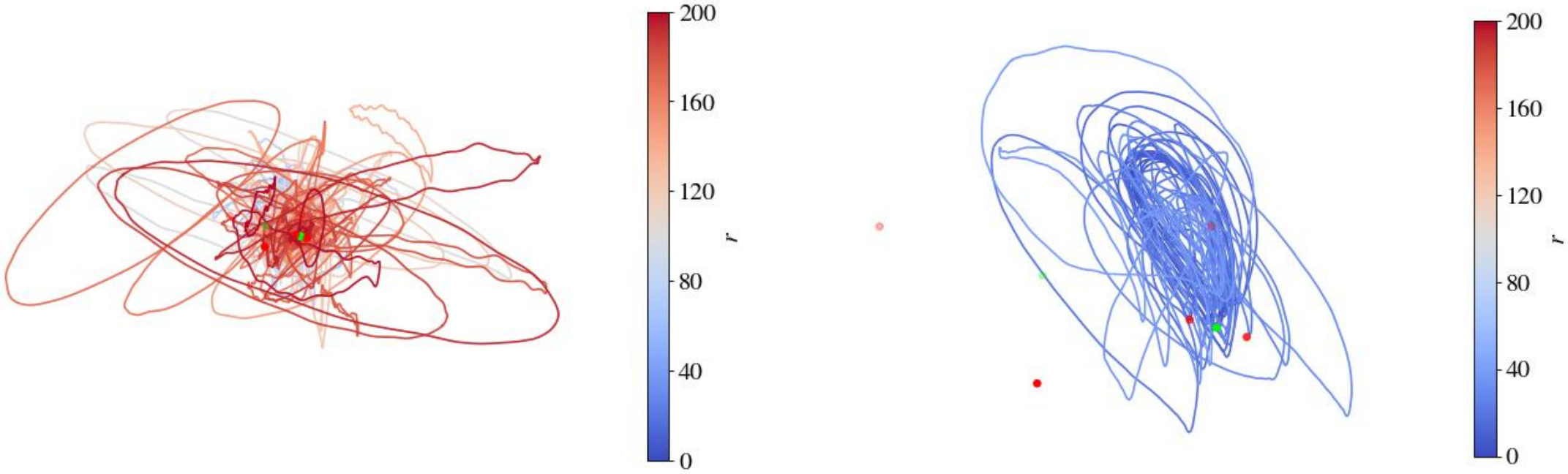
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Exploratory **generation**
Sampling out of the known family manifold



Detail of the generated trajectories from exploratory generation. Right: detail up to $r = 100$.