Moments and Geometric Equilibrium Equations

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Dedicated to Dieter Schmidt (2025) and Ken Meyer (2026)

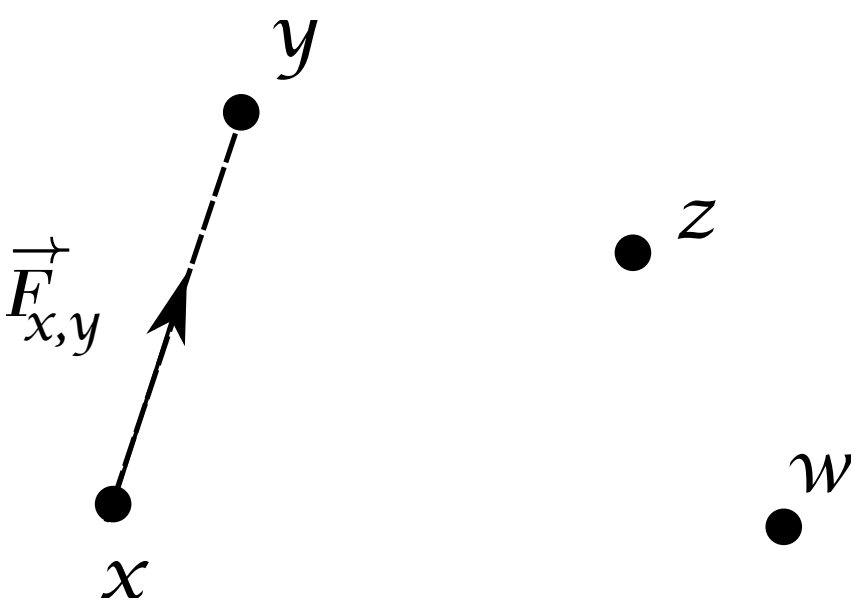


Systems of Interacting Particles

Let X be a finite subset of an affine space $\mathbb{A}$ over a field K.

For every pair $x, y \in X$ there exists an interaction $\overrightarrow{F}_{x,y} = \varphi_{x,y}\overrightarrow{xy}$, $\varphi_{x,y} \in K$.

If X varies, the $\varphi_{x,y}$ are functions of the points in X.



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Typical hypotheses:

(0) $\mathbb{A}$ is an Euclidean affine space.

(1) Newton's 3^{rd} Law: $\overrightarrow{F}_{x,y} = -\overrightarrow{F}_{y,x} \iff \varphi_{x,y} = \varphi_{y,x}$ (symmetry).

(2) Isotropy and homogeneity of space: $\varphi_{x,y}$ depends only on mutual distances between points in X.

Total force on x: $\overrightarrow{F}_x = \sum_{y \neq x} \overrightarrow{F}_{x,y}$.

X is an equilibrium $\iff \overrightarrow{F}_x = \overrightarrow{O}$, for all $x \in X$, i.e.,

$$\sum_{y \in X} \varphi_{x,y} \overrightarrow{xy} = \overrightarrow{O}, \quad \forall x \in X$$

$$\iff \quad \sum_{y \neq x} \varphi_{x,y} y + \left(-\sum_{y \neq x} \varphi_{x,y} \right) x = O, \quad \forall x \in X.$$

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For each $x \in X$, let $w_x : X \longrightarrow K$ be the function

$$w_x(y) = \varphi_{x,y}, \quad \text{if } y \neq x, \quad \text{and} \quad w_x(x) = -\sum_{y \neq x} \varphi_{x,y}.$$

Each pair (X, w_x) is a weighted system with configuration X and weight function w_x.

Example

For central configurations, $\mathbb{A} = \mathbb{E}^N$, and $w_x(y) = m(y)(||\overrightarrow{xy}||^{-3} + \lambda/M)$, if $y \neq x$.

We define the **moments** of a weighted system (X, w) by

$$\mu_0 = \sum_{x \in X} w(x), \quad \mu_1(p) = \sum_{x \in X} w(x)\overrightarrow{px}, \quad \mu_2(p) = \sum_{x \in X} w(x)\overrightarrow{px}^2,$$

$\forall p \in \mathbb{A}$.

Crucial fact

The weighted systems of interacting particles (X, w_x) have $\mu_0 = 0$ and $\mu_1 = \overrightarrow{F}_x$, $\forall x \in X$.

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A Key Result on μ_0, μ_1 and μ_2

Theorem 1 (Huygens-Leibniz-König)

For every $p, q \in \mathbb{A}$, we have that

$$\mu_2(p) - \mu_2(q) = \overrightarrow{pq} \cdot (\mu_1(p) + \mu_1(q)). \tag{1}$$

Therefore

1. *if $\mu_0 \neq 0$ and $G = bar(X, w)$, then $\mu_2(p) - \mu_2(G) = \mu_0 \overrightarrow{pG}^2$, for all $p \in \mathbb{A}$,*
2. *if $\mu_0 = 0$ and $\mu_1 = \overrightarrow{v}$, then $\mu_2(p) - \mu_2(q) = 2\overrightarrow{v} \cdot \overrightarrow{qp}$, for all $p, q \in \mathbb{A}$, and*
3. *for every $q \in \mathbb{A}$, the Leibniz's identity*

$$\mu_0\mu_2(q) - \mu_1(q)^2 = \sum_{\{x,y\} \subset X} w(x)w(y)\overrightarrow{xy}^2$$

holds.

Moments and Mutual Distances

Corollary 2

$\mu_1 = \overrightarrow{O} \iff \mu_0 = 0$ *and* $\sum_{\{x,y\}\subset X} w(x)w(y)\overrightarrow{xy}^2 = 0.$

Corollary 3

Suppose $\mu_0 = 0$. *The following are equivalent:*

(i) $\mu_1 = \overrightarrow{O}$;

(ii) μ_2 *is constant;*

(iii) *the restriction of* μ_2 *to* X *is constant.*

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The equivalence (i) $\Longleftrightarrow$ (iii) in Corollary 3 corresponds to

Corollary 4

The weighted system (X, w) *has* $\mu_1 = \overrightarrow{O}$ *if and only if* $\mu_0 = 0$ *and*

$$\sum_{z \in X} w(z)(\overrightarrow{xz}^2 - \overrightarrow{yz}^2) = 0 \text{ or } \sum_{z \in X \setminus \{x\}} w(z)(\overrightarrow{xy}^2 - \overrightarrow{yz}^2 + \overrightarrow{zx}^2) = 0, \tag{2}$$

holds for every $x, y \in X$.

Equations (2) generalize to systems (X, w) with $\mu_1 = \overrightarrow{O}$ a famous set of identities known in Celestial Mechanics as Albouy-Chenciner equations for central configurations.

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View weight functions w as vectors in the weight space $K^n =: \mathbb{W}$.

Let δ_x be the weight function: $\delta_x(y) = 1$ if $x = y$, and $\delta_x(y) = 0$ if $x \neq y$.

Finally let $\mathcal{B}(X) = \left(\overrightarrow{xy}^2\right)_{x,y\in X}$ be the so-called relative configuration matrix.

The first set of formulæ in (2) in Corollary 4, can be written as

$$(\delta_x - \delta_y)\mathcal{B}(X)w = 0, \qquad x, y \in X.$$

Differences $\delta_x - \delta_y$, $x \neq y$, form a basis of the space $\mathbb{D} = \mathrm{Ker}(\mu_0)$. So the kernel of $\mathcal{B}(X)$ as a bilinear form restricted to $\mathbb{D}$ is the subspace

$$\mathbb{W}_0(X) = \{w \in \mathbb{W} \text{ s.t. } \mu_1 = \overrightarrow{O}.\}$$

In particular, $\mathcal{B}(X)$ is the **null form** on $\mathbb{W}_0(X)$!

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Theorem 5

A configuration X of interacting particles is an **equilibrium** $\iff$ *the corresponding weighted systems (X, w_x) are such that* $\boldsymbol{w_x \in \mathbb{W}_0(X), \forall x \in X}$.

Combined with the theory of moments, we obtain the following equilibrium equations.

Corollary 6

A configuration X is an equilibrium of a system of interacting particles $\iff$ *there holds one of the following sets of identities*

$$\sum_{z \in X \setminus \{x\}} \varphi_{x,z}(\overrightarrow{xy}^2 - \overrightarrow{yz}^2 + \overrightarrow{zx}^2) = 0, \tag{3}$$

or

$$\sum_{\{u,z\} \subset X} \varphi_{x,u}\varphi_{y,z}\overrightarrow{uz}^2 = 0. \tag{4}$$

for every pair of particles $x, y \in X$.

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Constraints on Mutual Distances

Traditionally the constraints for n bodies forming a configuration $X = \{x_1, \ldots, x_n\}$ to have dimension $d < n - 1$ are given by the annihilation of Cayley-Menger determinants

$$D(X) = \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & s_{12} & \ldots & s_{1n-1} & s_{1n} \\ 1 & s_{21} & 0 & \ldots & s_{2n-1} & s_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & s_{n1} & s_{n2} & \ldots & s_{nn-1} & 0 \end{vmatrix}, \quad \text{where } s_{ij} = \overrightarrow{x_i x_j}^2.$$

The volume V of the $(n-1)$-dimensional simplex X with vertices $x_1, \ldots, x_n$ satisfies

$$V^2 = \frac{(-1)^n}{2^{n-1}(n-1)!^2} D(X).$$

X has dimension less than $n - 1 \iff D(X) = 0$.

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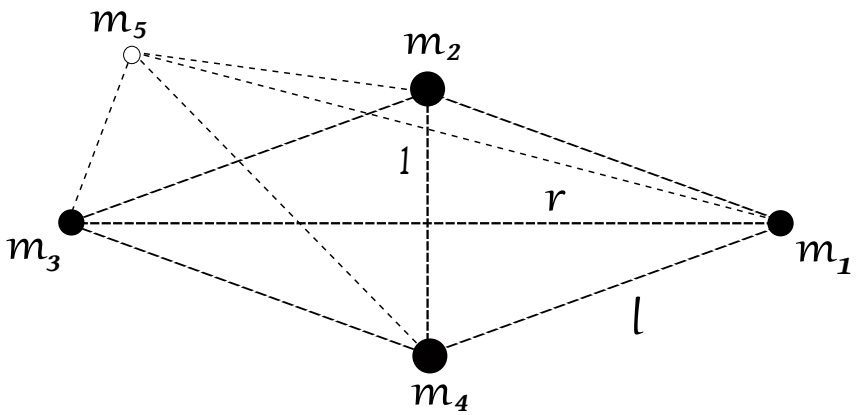
Direct Constraints

Issue: Cayley-Menger determinants are large! For some problems, there is a simpler alternative.

Corollary 7

Let (X, w) and (X', w') be weighted systems with the same nonzero total weight and the same barycenter. The difference between the respective μ_2 has the same value at all points.

Example: the (Rhombus+1)-Body Problem



Parameters: $m_1 = m_3 = m$, $m_2 = m_4 = 1$, $l^2 = 1 + r^2$,
$\lambda = -\frac{1}{4} - \frac{2m}{l^3} = -\frac{2}{l^3} - \frac{m}{4r^3}$.

Relation: $m = \frac{r^3(8 - l^3)}{8r^3 - l^3}$.

Let s_i be the squared distance from m_i to m_5. Corollary 7 implies

$$s_1 - s_2 + s_3 - s_4 = 2(r^2 - 1).$$

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Number of Independent Constraints

Definition

The codimension of a configuration X of n points and dimension d is $(n-1)-d =:$ codim(X).

Theorem 9

Let X be a configuration of dimension $d < n-1$. The mutual distances of X are such that precisely $\binom{\text{codim}(X)+1}{2}$ functionally independent Cayley-Menger determinants of subconfigurations of X vanish.

Moments and Barycentric Coordinates

"We expose the theory of Dziobek configurations more simply (no Cayley-Menger determinants!) and more completely than in any published paper." (A. Albouy, 2002)

Proposition 10

Let $p \in \mathbb{A}, x \in X$.

1. *If* $\mu_0 \neq 0$ *and* $G = bar(X, w)$, *then* $\mu_1(p) = \mu_0 \overrightarrow{pG}$.
2. *If* $\mu_0 = 0$ *and* $w(x) \neq 0$, *then* $\mu_1(p) = w(x)\overrightarrow{G_x x}$, *where* G_x *is the barycenter of the weighted system* $\left(X \setminus \{x\}, w\big|_{X\setminus\{x\}}\right)$.

Corollary 11

If $\mu_1 = \overrightarrow{O}$, *then for every* $x \in X$ *with* $w(x) \neq 0$, *we have that* $x = bar\left(X \setminus \{x\}, w\big|_{X\setminus\{x\}}\right)$.

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$\mathbb{W}_0(X)$ and Volume Barycentric Coordinates

A basic fact and the "codimension exterior-power formula"

$$\dim \mathbb{W}_0(X) = \mathsf{codim}(X) \quad \therefore \quad \bigwedge^{\mathsf{codim}(X)+1} \mathbb{W}_0(X) = \{O\}.$$

If $X \setminus \{x\}$ is a simplex with the *same dimension*, d, as X, then Corollary 11 serves as a definition of barycentric coordinates for x with respect to $X \setminus \{x\}$. Möbius introduced nice, geometrically meaningful homogeneous barycentric coordinates.

Proposition 12

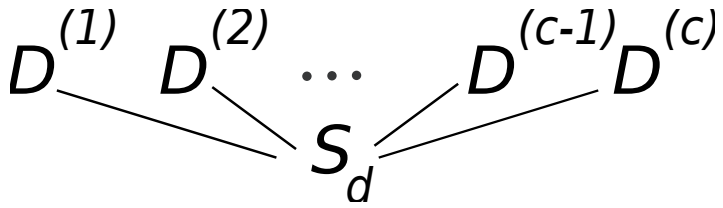
Let $X = \{x_1, \ldots, x_n\}$, and let X_j be the subconfiguration $X \setminus \{x_j\}$. Let $\mathcal{X}_j$ be the augmented configuration matrix of X_j, and let $\Delta_j = \det(\mathcal{X}_j)$. We have that

$$\mathit{codim}(X) = 1 \;\Leftrightarrow\; (-\Delta_1, \Delta_2, -\Delta_3, \ldots, (-1)^n \Delta_n) \text{ spans } \mathbb{W}_0(X).$$

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Bases of $\mathbb{W}_0(X)$

We call X a Dziobek configuration when $\operatorname{codim}(X) = 1$.
A Dziobek fan for a configuration X of codimension c is a diagram:



S_d is a d-dimensional simplex S_d called a core of X.
$D^{(l)}$, $l = 1, \ldots, c$ are all Dziobek subconfigurations containing S_d.
Each Dziobek fan produces a basis of $\mathbb{W}_0(X)$!
For instance, for $n = 7$, $d = 2$, if $S_2 = \{x_1, x_2, x_3\}$, then

$$W_4 = (-\Delta_{1567}, \Delta_{2567}, -\Delta_{3567}, \Delta_{4567}, 0, 0, 0),$$
$$W_5 = (-\Delta_{1467}, \Delta_{2467}, -\Delta_{3467}, 0, \Delta_{4567}, 0, 0),$$
$$W_6 = (-\Delta_{1457}, \Delta_{2457}, -\Delta_{3457}, 0, 0, \Delta_{4567}, 0),$$
$$W_7 = (-\Delta_{1456}, \Delta_{2456}, -\Delta_{3456}, 0, 0, 0, \Delta_{4567})$$

form a basis (since $\Delta_{4567} \neq 0$).

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Membership in $\mathbb{W}_0(X)$

Theorem 13

Let X be a configuration of dimension d such that $\Delta_{d+2\ldots n} \neq 0$. We have that $(w_1, \ldots, w_n) \in \mathbb{W}_0(X)$ if and only if the identities

$$(-1)^{i-d} w_i \Delta_{d+2\ldots n} = \sum_{l=d+2}^{n} w_l \Delta_{i\, d+2\ldots \widehat{l}\ldots n}, \quad i = 1, \ldots, d+1, \quad (5)$$

hold

There is a nice connection with Algebraic Geometry: changes of bases of $\mathbb{W}_0(X)$ produce the classical Plücker relations for Grassmannians by means of the above formulæ!

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Generalized Dziobek Equations

A family $\mathfrak{F} = \{(X, f_j)\}_{j=1}^n$ of weighted systems is symmetric if the corresponding weight matrix $\mathcal{F} = (f_j(x_i))_{1 \leq i,j \leq n}$ is symmetric.

Theorem 14

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let F_j, $j = 1, \ldots, n$ be the weight vectors associated with the symmetric family $\mathfrak{F} = \{(X, f_j)\}_{j=1}^n$ of weighted systems. The following assertions are equivalent:

(a) $F_j \in \mathbb{W}_0(X), \forall j = 1, \ldots, n.$

(b) *The equation $\mathcal{F} = \mathcal{W}\Sigma\mathcal{W}^T$ is satisfied by the weight matrix $\mathcal{F}$ of $\mathfrak{F}$, for some $n \times c$ basis matrix $\mathcal{W}$ of $\mathbb{W}_0(X)$ and some $c \times c$ symmetric matrix Σ of scalars, where $c = codim(X)$.*

Corollary 15

Dziobek central configurations ($c = 1$) satisfy

$$m_i m_j (s_{ij}^{-3/2} + \lambda/M) = (-1)^{i+j} \sigma \Delta_i \Delta_j, \quad 1 \leq i < j \leq n.$$

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Thank You!

