CELMEC IX, San Martino al Cimino, Viterbo, Italy
September 17, 2026

Eccentricities and inclinations are decoupled in super-Earth systems

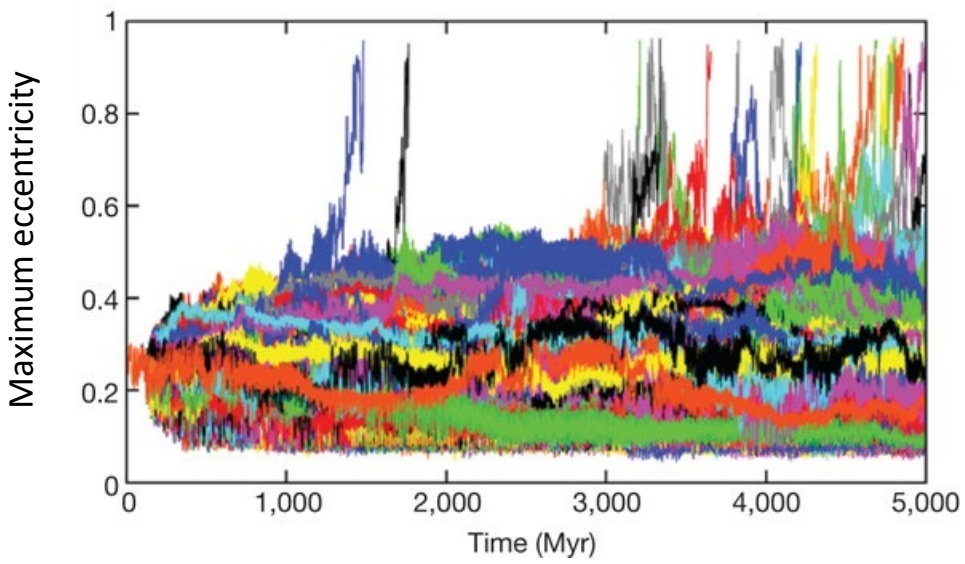
Sacha Kuhn

In collaboration with A. C. Petit, F. Mogavero and A. Crida

Observatoire de la Côte d'Azur
Lagrange Laboratory

A Solar System tale

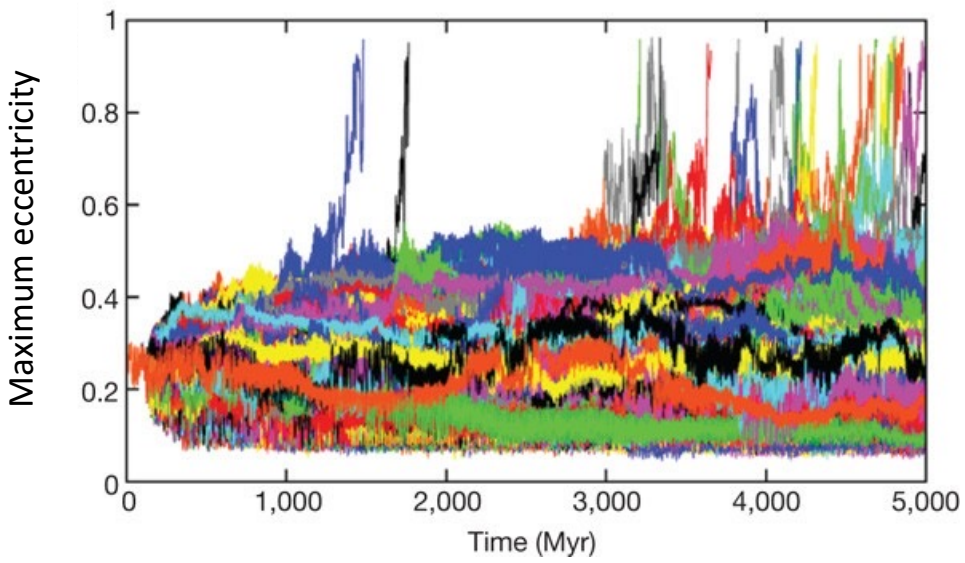
One cannot predict the Solar System's dynamics because of the chaotic nature of the inner Solar System



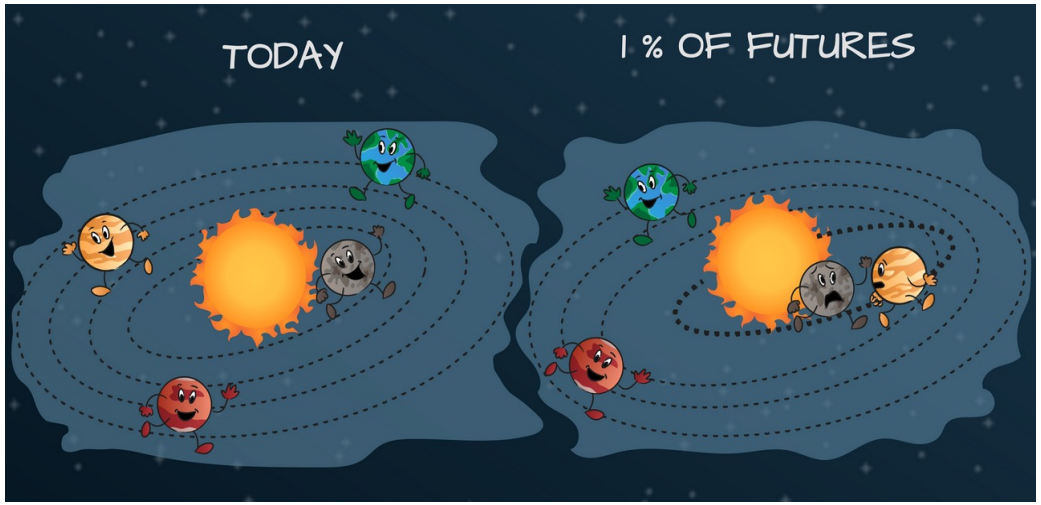
Laskar and Gastineau 2009

A Solar System tale

One cannot predict the Solar System's dynamics because of the chaotic nature of the inner Solar System



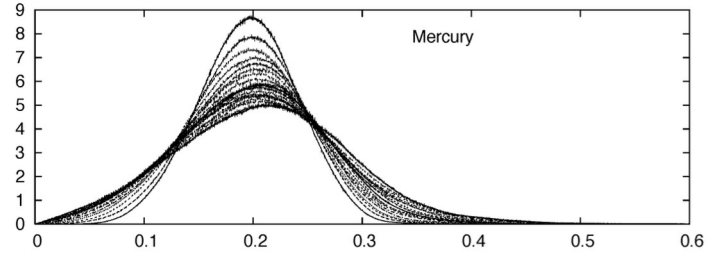
Laskar and Gastineau 2009

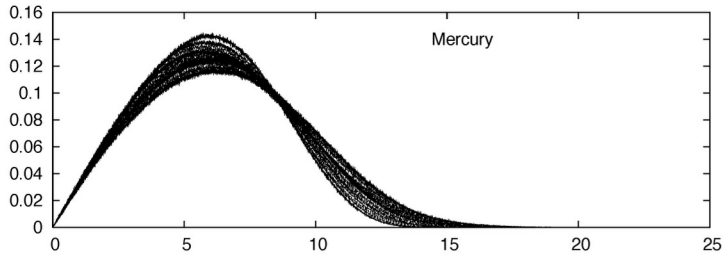


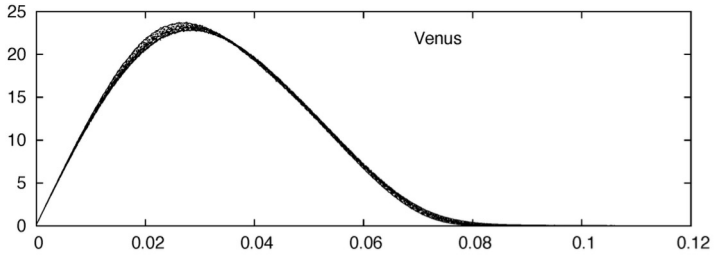
Credits: L. Canil and M. Schirber, “How Our Solar System Avoids Planet Collisions”, physics.aps.org

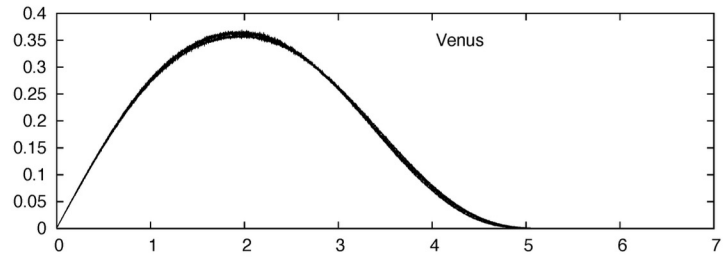
Mercury’s eccentricity can be pumped to values large enough to allow collision with Venus within 5 Gyr

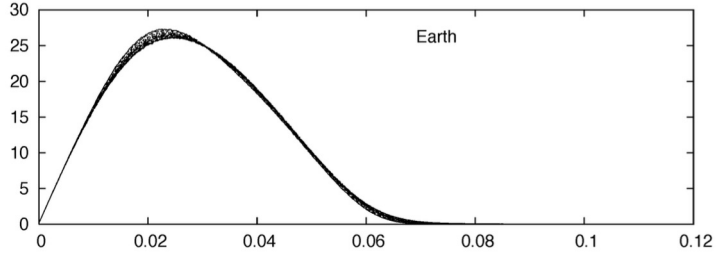
Solar Systems' inner planets e-i density functions

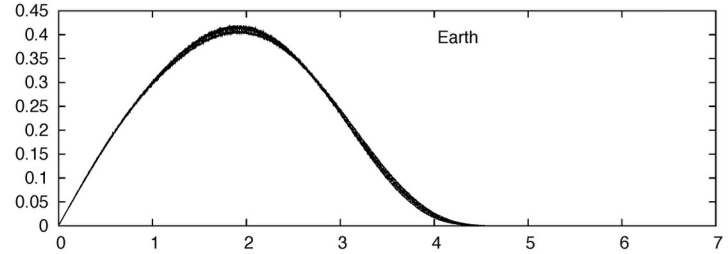


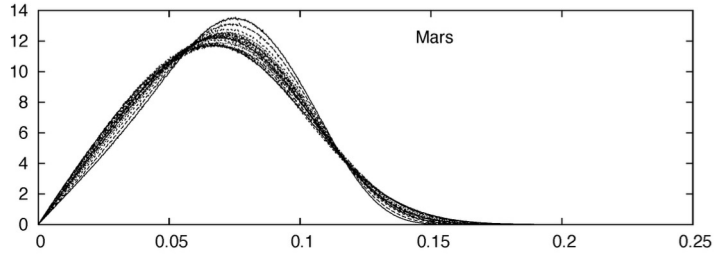


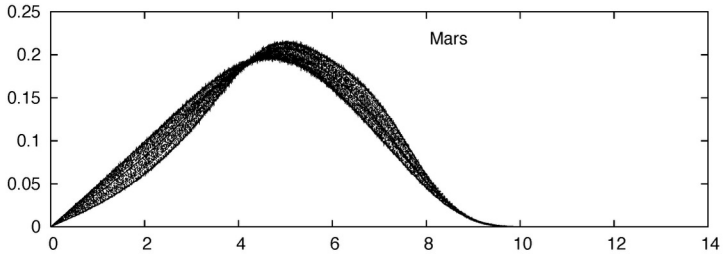










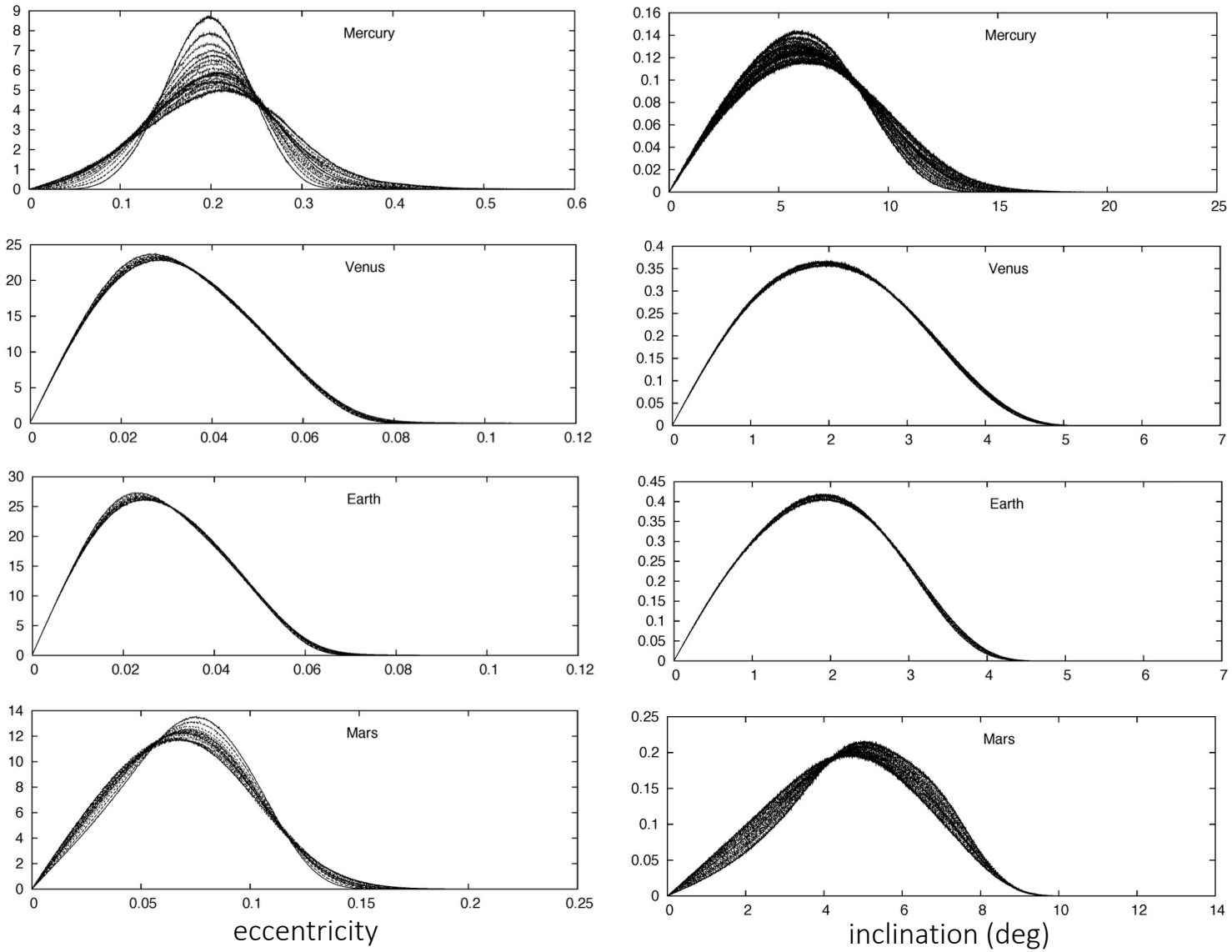


eccentricity

inclination (deg)

Laskar 2008

Solar Systems' inner planets e-i density functions



Laskar 2008

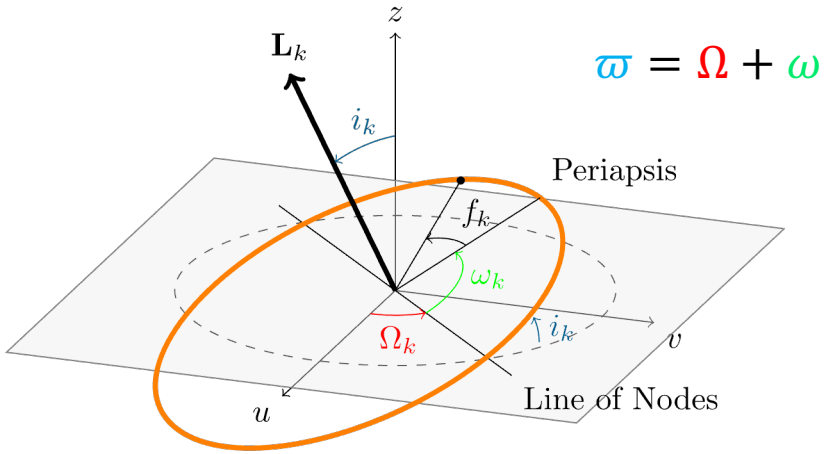
Poincaré variables $(x_k, -\iota\bar{x}_k; y_k, -\iota\bar{y}_k)$

$$x_k \simeq \sqrt{\frac{\Lambda_k}{2}}\, e_k \mathrm{E}^{\iota\varpi_k}$$

$$y_k \simeq \sqrt{\frac{\Lambda_k}{2}}\, i_k\, \mathrm{E}^{\iota\Omega_k}$$

$$\Lambda_k = m_k\sqrt{\mathcal{G}m_0 a_k}$$

Laskar and Robutel 1995



$\varpi = \Omega + \omega$

Laplace-Lagrange (LL) dynamics

Sums of terms of constant amplitudes and frequencies

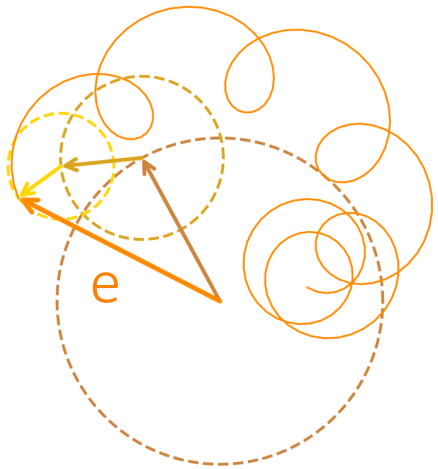
- Secular Hamiltonian truncated at degree 2

$$\mathcal{H}_{\mathrm{LL}} = \bar{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{Q}_{\mathbf{x}} \cdot \mathbf{x} + \bar{\mathbf{y}}^{\mathrm{T}} \cdot \mathbf{Q}_{\mathbf{y}} \cdot \mathbf{y}$$

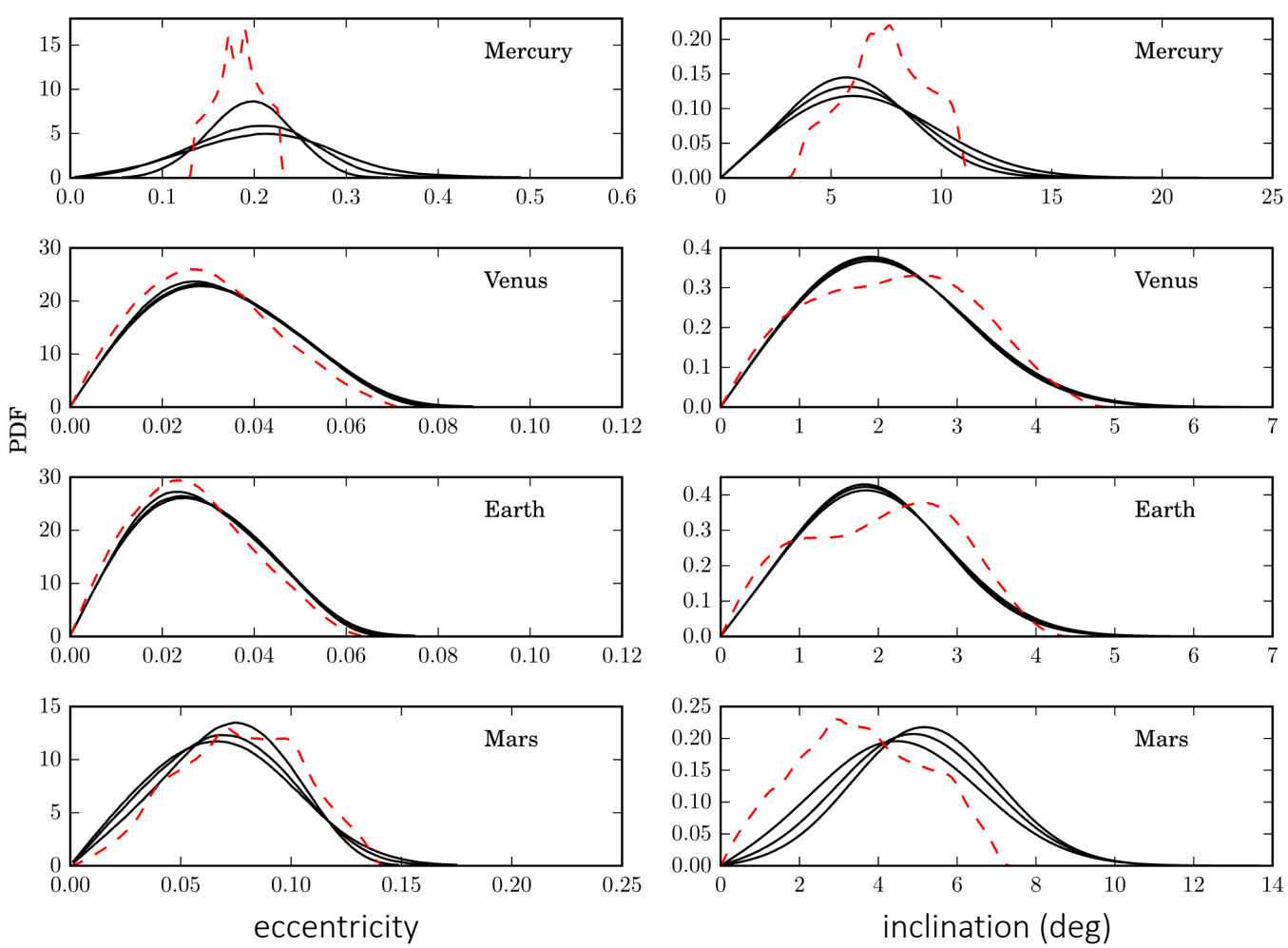
- New variables through diagonalization

$$\mathbf{x}^* = \mathbf{O}_{\mathbf{x}}{}^{\mathrm{T}} \cdot \mathbf{x}, \quad x_j^*(t) = x_j^*(0)\mathrm{E}^{\imath g_j t}$$

$$\mathbf{y}^* = \mathbf{O}_{\mathbf{y}}{}^{\mathrm{T}} \cdot \mathbf{y}, \quad y_j^*(t) = y_j^*(0)\mathrm{E}^{\imath s_j t}$$



$$x_{\mathrm{LL},k}(t) = \sum_j O_{x,k,j}\, x_j^*(0)\mathrm{E}^{\imath g_j t}$$



Mogavero 2017

Ergodic dynamics and Angular Momentum Deficit (AMD)

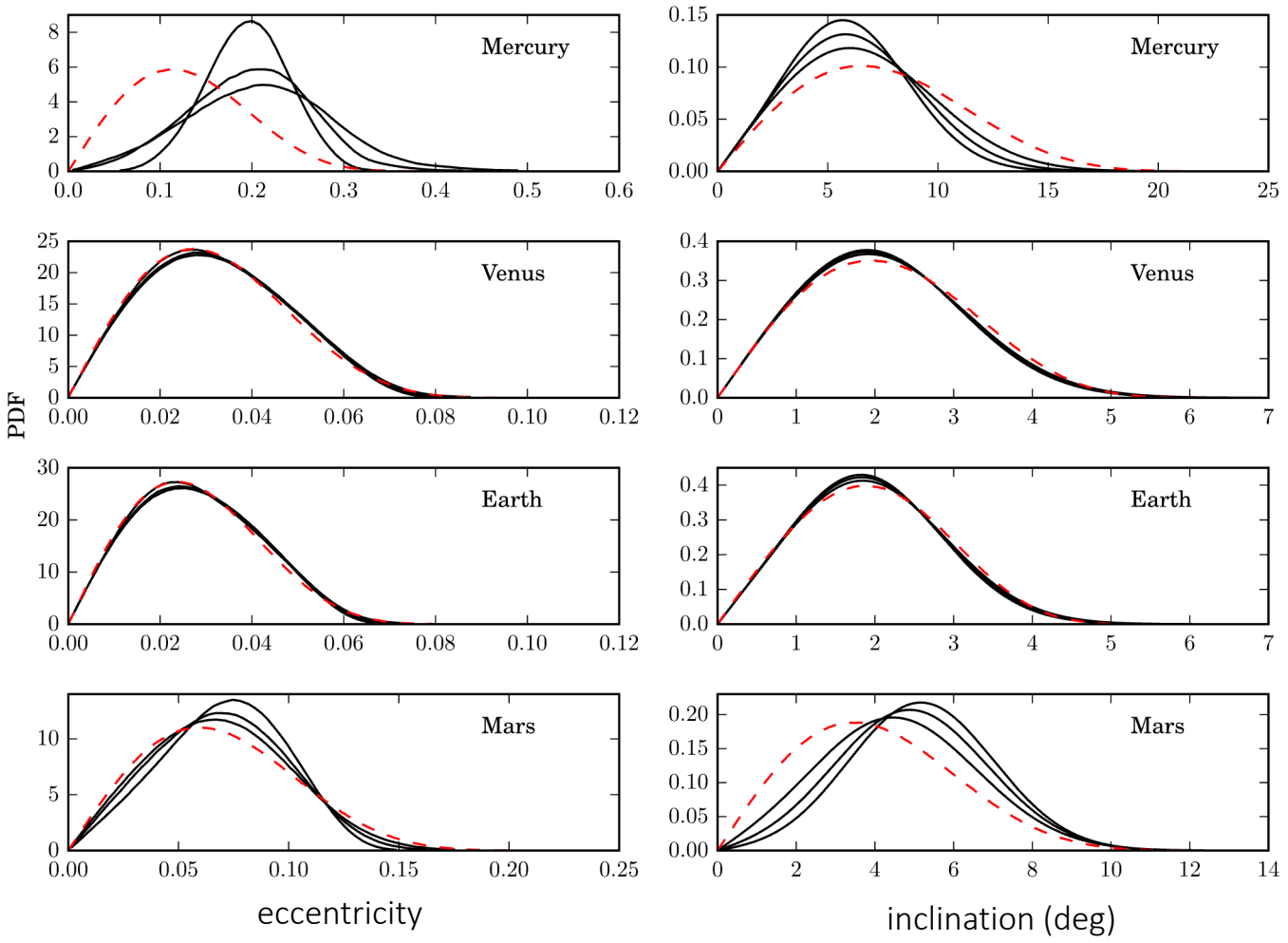
Eccentricities and inclinations are free as long as Angular Momentum Deficit (AMD) is conserved

$$L_{tot} = \sum_{k=1}^{N} \Lambda_k \sqrt{1 - e_k^2} \cos(i_k)$$

$$\mathrm{AMD} = \sum_{k=1}^{N} \underbrace{\Lambda_k}_{\mathrm{cst}} - L_{tot} = \sum_{k=1}^{N} \Lambda_k \left(1 - \sqrt{1 - e_k^2} \cos(i_k)\right)$$

$\sim$ system's dynamical temperature

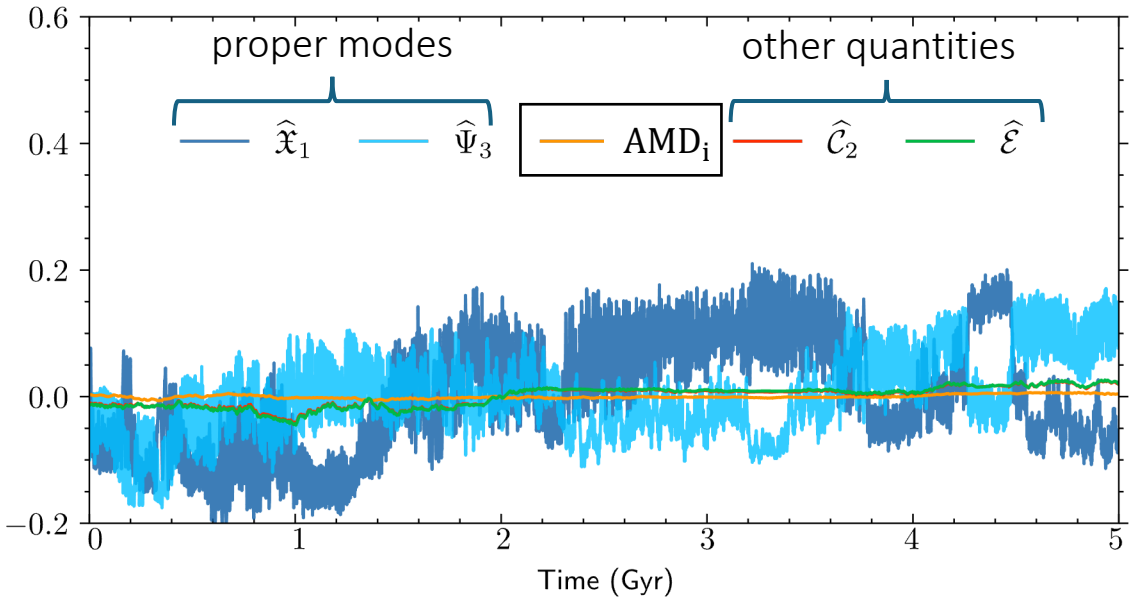
Integral of motion in the secular framework



Mogavero 2017

Inner Solar System Quasi-Integral

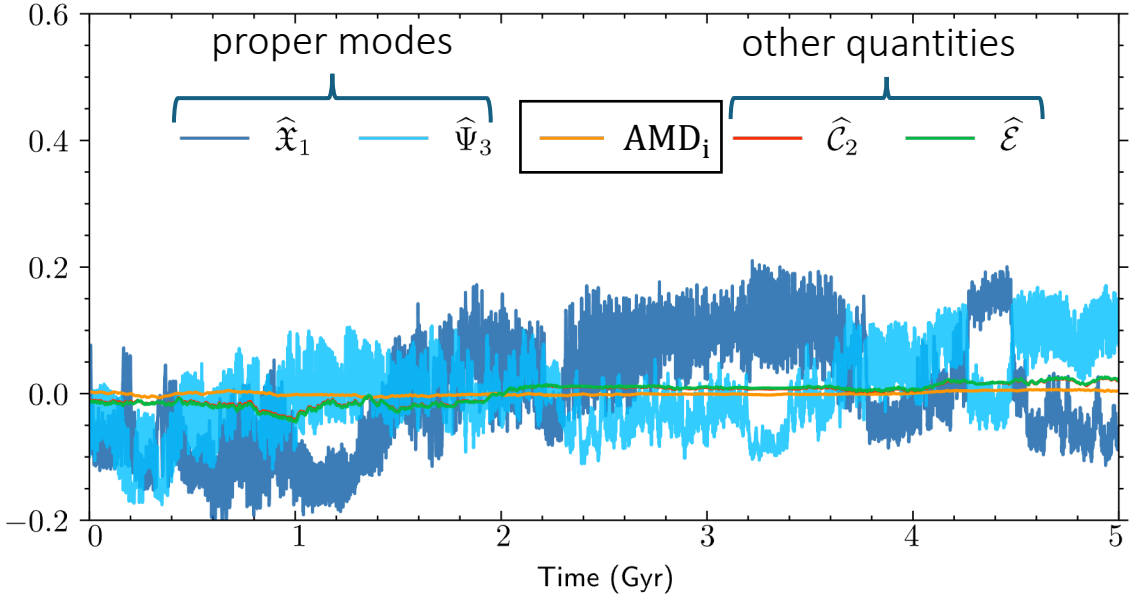
- Mogavero+ 2023:
  - Why is the probability of collisions so low?
  - Longterm evolution of the inner Solar System
  - Existence of a Quasi-Integral (QI) of motion: $\mathbf{AMD_i}$



Mogavero+ 2023

Inner Solar System Quasi-Integral

- Mogavero+ 2023:
  - Why is the probability of collisions so low?
  - Longterm evolution of the inner Solar System
  - Existence of a Quasi-Integral (QI) of motion: $\mathrm{AMD_i}$



Mogavero+ 2023

$$\mathrm{AMD} = \underbrace{\sum_{k=1}^{N} x_k\,\bar{x}_k}_{\mathrm{AMD_e}} + \underbrace{\sum_{k=1}^{N} y_k\,\bar{y}_k}_{\mathrm{AMD_i}} \simeq \sum_{k=1}^{N} \frac{\Lambda_k}{2} e_k^2 + \sum_{k=1}^{N} \frac{\Lambda_k}{2} i_k^2$$

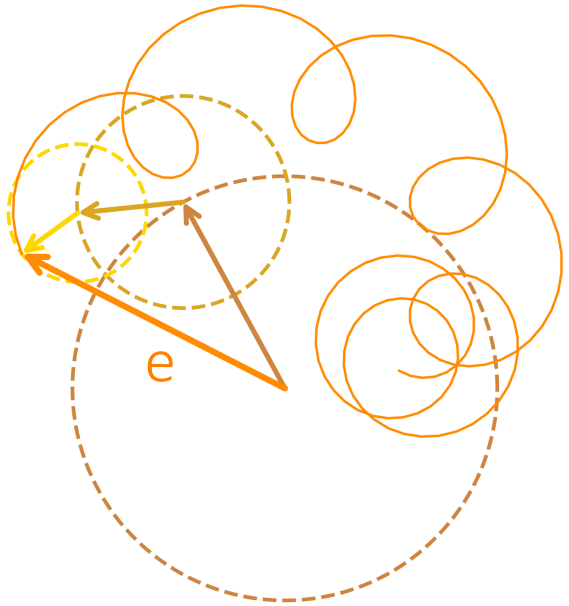
$$\mathrm{AMD_e} = \sum_{j=1}^{N} x_j^*\,\overline{x_j^*} \qquad \mathrm{AMD_i} = \sum_{j=1}^{N} y_j^*\,\overline{y_j^*}$$

Dynamical hypotheses

More restrictive

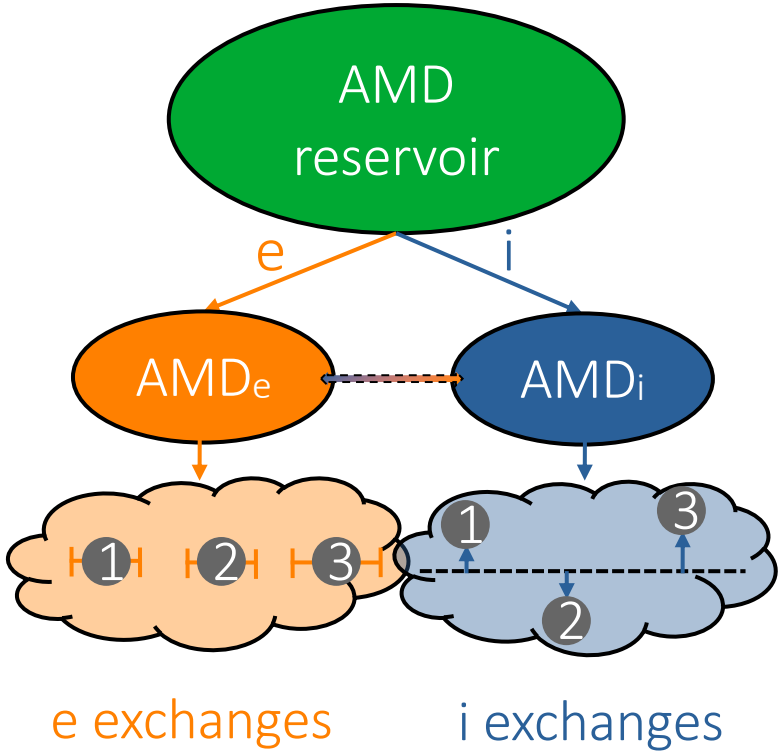
Less restrictive

Regular (quasi-periodic)



Eccentricities (and inclinations) are sums of Laplace-Lagrange modes

Stable secular dynamics

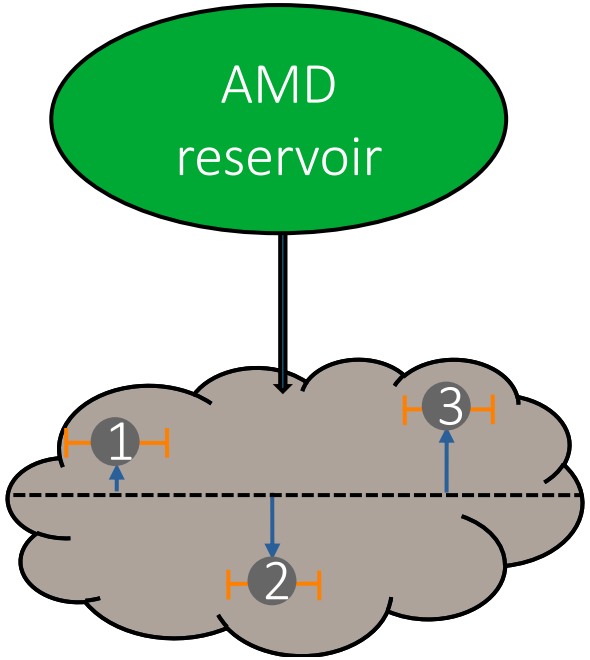


e exchanges

i exchanges

AMD_e and AMD_i are Quasi-Integrals of motion

Ergodic

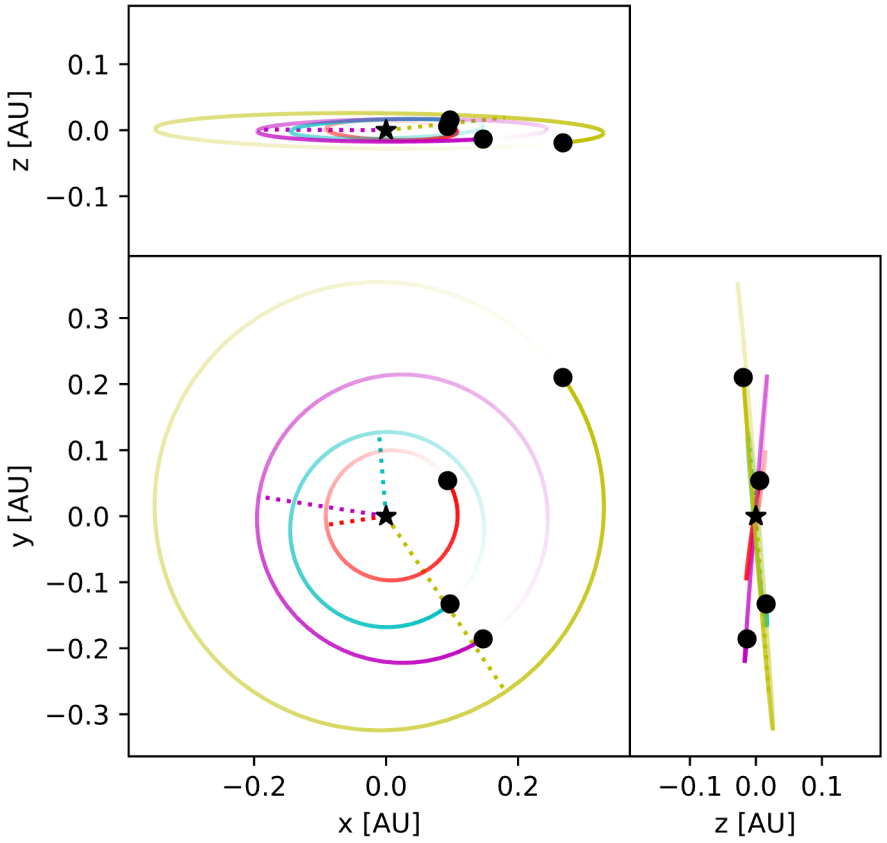


Sub-reservoirs merging

Exploration of all accessible configurations

Is $AMD_{e,i}$ conservation valid beyond the Solar System?

Dynamical setup



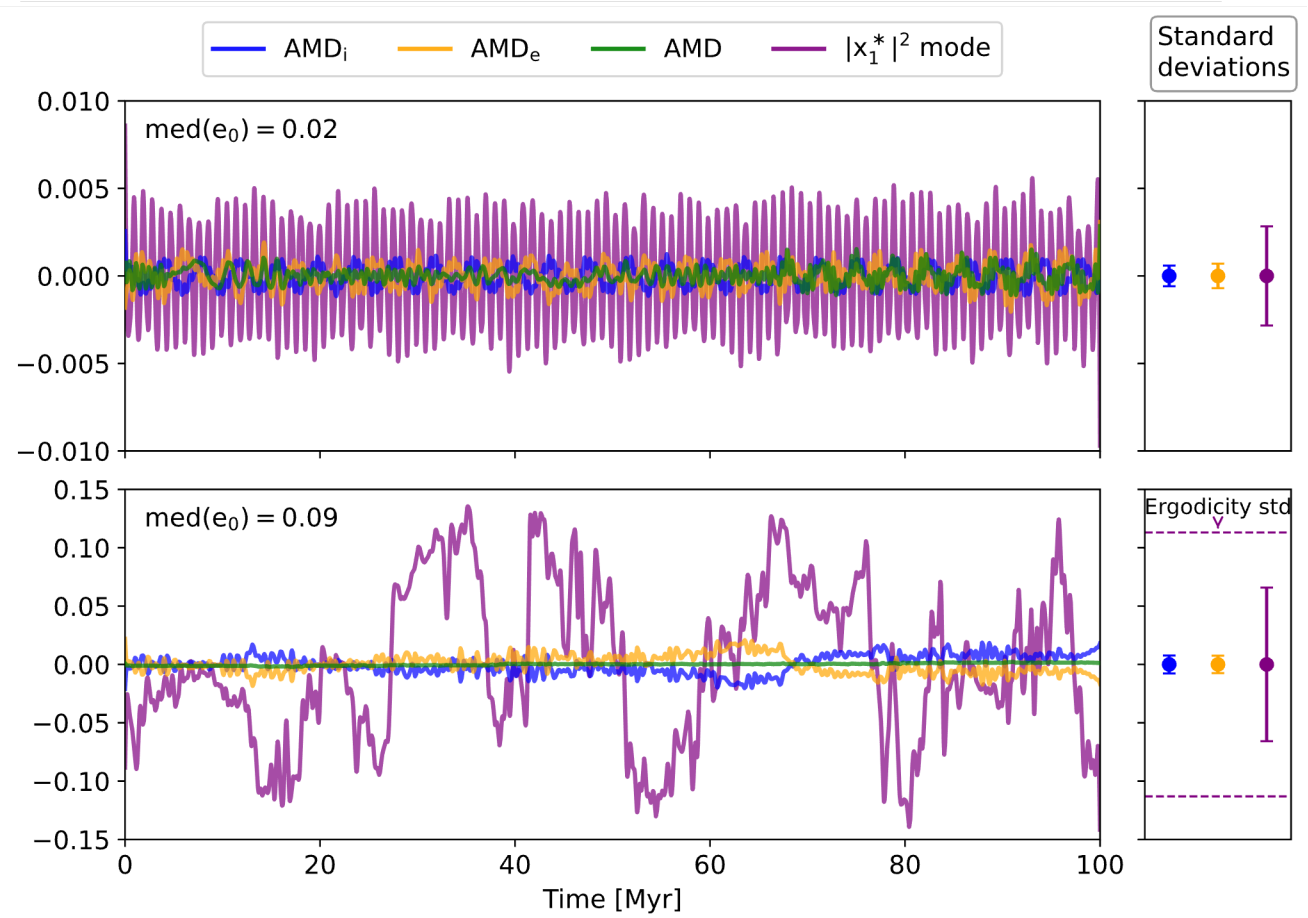
- 4-planet systems
  - 3 $M_\oplus$
  - a_{inner} =0.1 AU
  - Period ratio ~U(1.77, 1.93)

- Integrations
  - N-body (REBOUND) and secular integrations (Gauss method), GR
  - 7 sets of 250 systems
  - Each set has a different AMD/AMD_H ratio
  - Increase AMD/AMD_H ratio up to instability
  - Uniform distribution of AMD between planets

- Critical Hill AMD: AMD_H
  - Proxy for stability
  - Limiting AMD ensuring stability in the Hill framework

Rein+ 2012, Petit+ 2018

Time series example



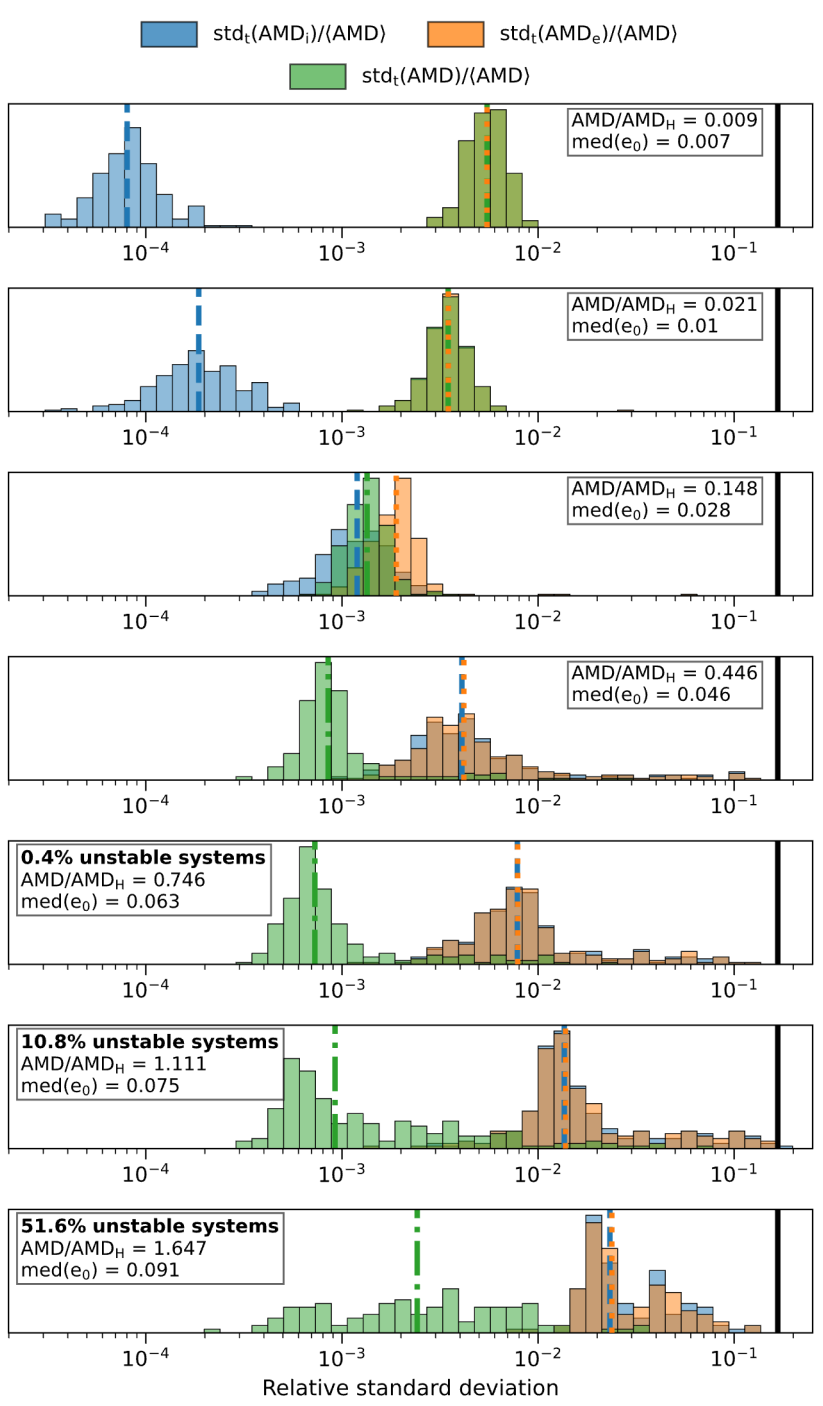
Each quantity has the mean over 100 Myr subtracted and is normalized by the AMD

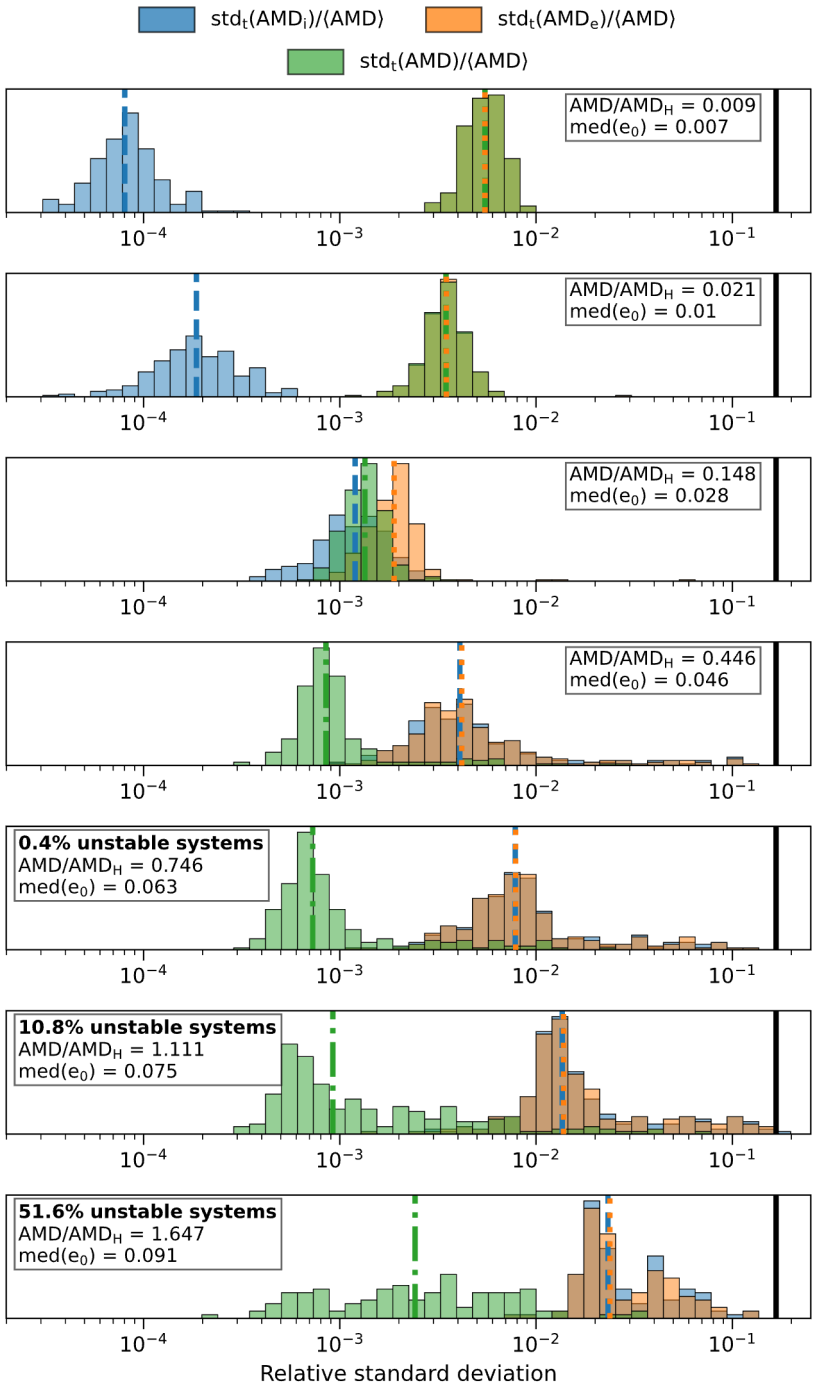
Systems from 2 ≠ sets (i.e. ≠ AMD)

- Low AMD
  - Very good conservation of the mode
  - Excellent conservation of AMD_e and AMD_i
  - LL-like dynamics
- High AMD
  - Secular chaotic dynamics
  - Large mode variations
  - Small changes in AMD_e and AMD_i

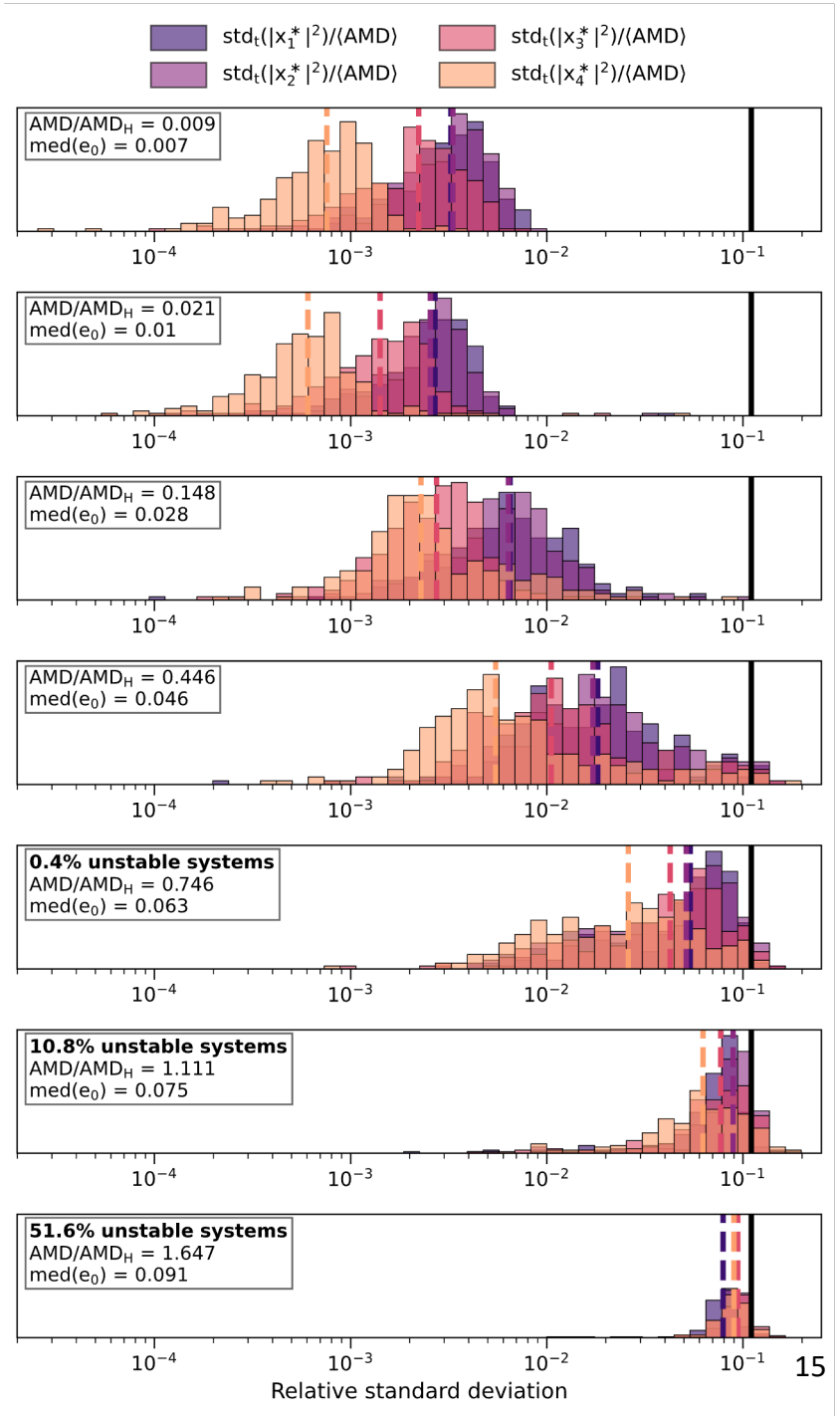
AMD, AMD_e, AMD_i conservation from stability to instability

- Bulk of distributions is lognormal
- Conservation to ~ percent level for stable systems → QIs

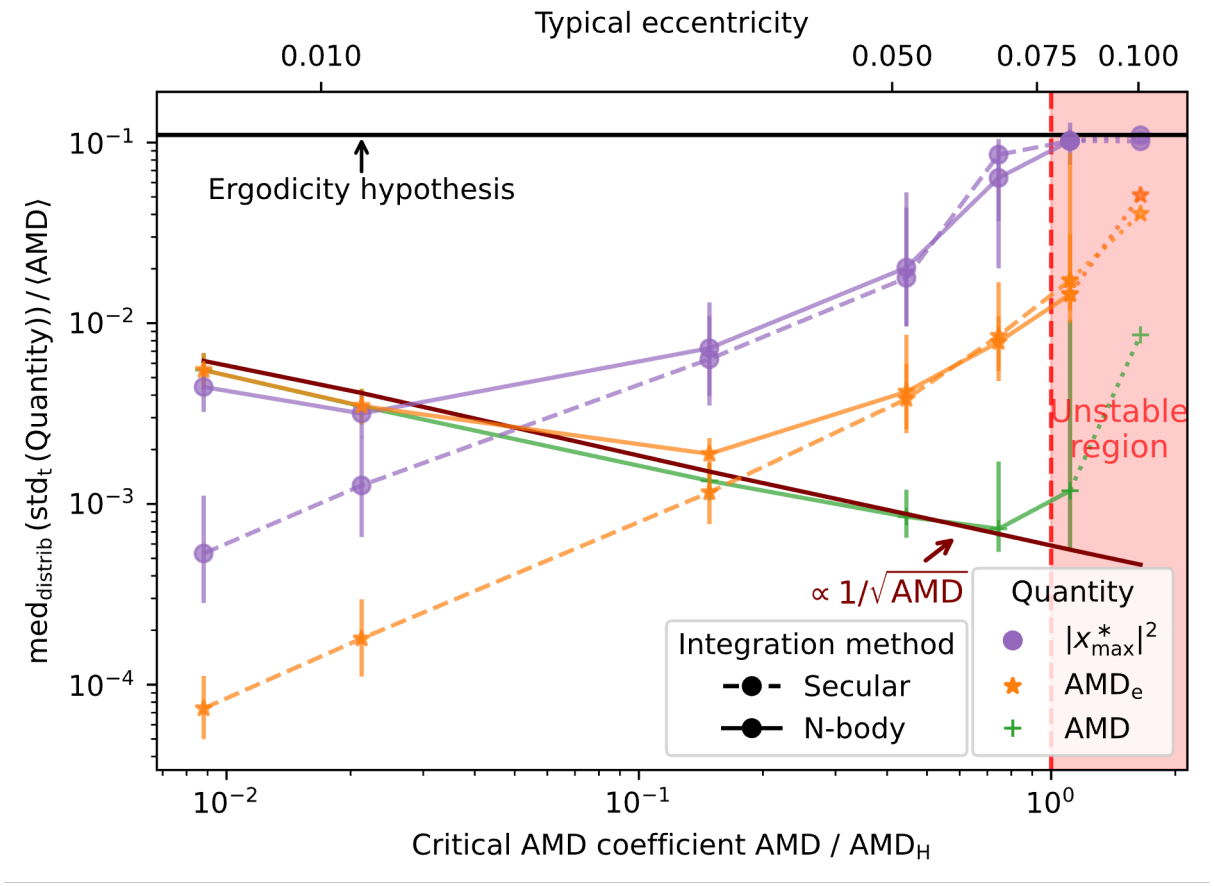


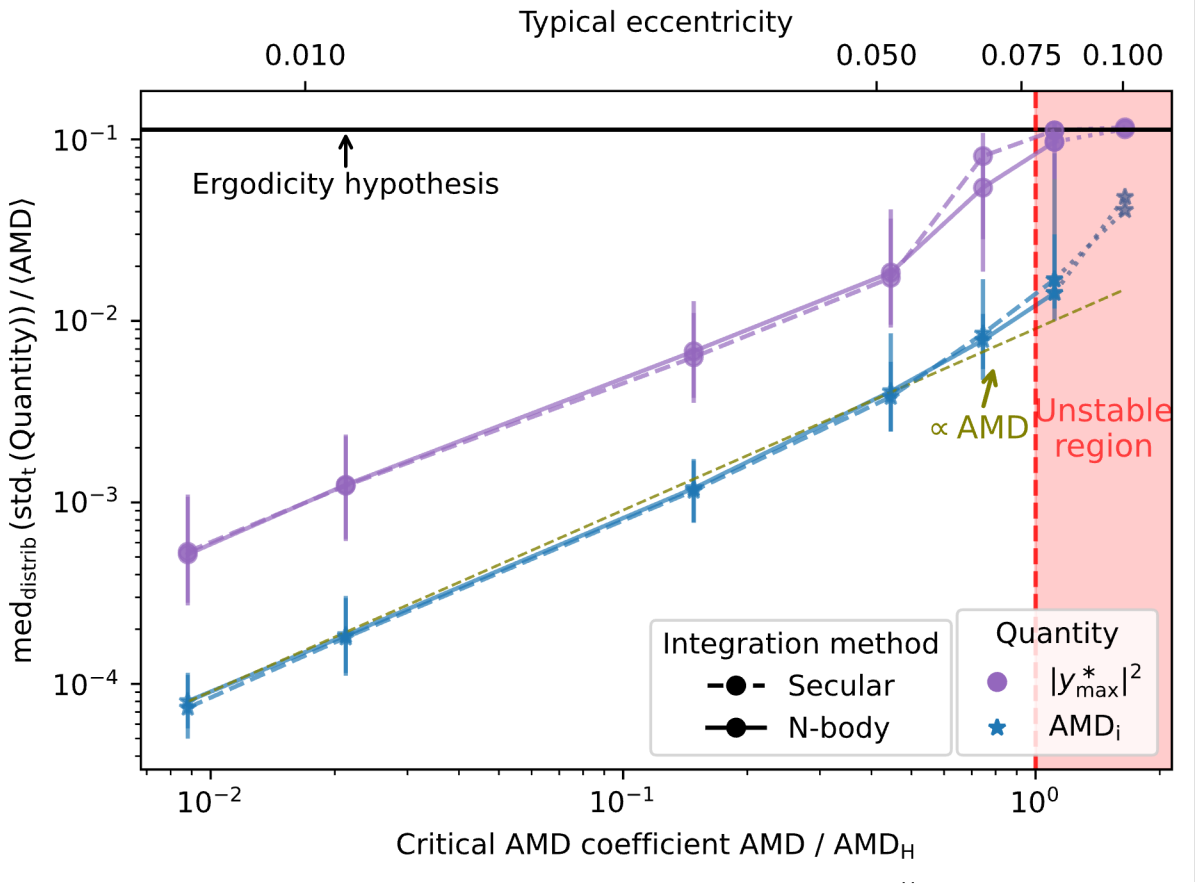


- Modes less conserved than AMD components
- As AMD increases distributions shrink, modes saturate
- Modes: Ergodicity limit reached
- QIs: Always lower than ergodicity limit



AMD, AMD_e, AMD_i and modes conservation trends





Difference of behavior modes and $AMD_{e,i}$ → Quasi-integrals of motion

Power series form of secular Hamiltonian

$\mathcal{H}_{\mathrm{sec}}$ in power series of $(x_k, -\iota\bar{x}_k; y_k, -\iota\bar{y}_k)$

$$\mathcal{H}_{\mathrm{sec}} = \sum_{\mathbf{l},\bar{\mathbf{l}},\mathbf{l}',\bar{\mathbf{l}}'} h^{\mathbf{l},\bar{\mathbf{l}},\mathbf{l}',\bar{\mathbf{l}}'}(\boldsymbol{\Lambda}) \prod_{j=1}^{N} x_j{}^{l_j}\, \bar{x}_j^{\bar{l}_j} y_j{}^{l'_j} \bar{y}_j^{\bar{l}'_j}, \quad \mathbf{l}, \bar{\mathbf{l}}, \mathbf{l}', \bar{\mathbf{l}}' \in \mathbb{N}^N$$

dark magic

Proper mode variables:

$$\mathbf{x}^* = \sqrt{\mathbf{I}^*}\, \mathrm{E}^{-\iota\boldsymbol{\theta}^*}, \quad \mathbf{y}^* = \sqrt{\mathbf{I}'^*}\, \mathrm{E}^{-\iota\boldsymbol{\theta}'^*}$$

Power series form of secular Hamiltonian

$\mathcal{H}_{\mathrm{sec}}$ in power series of $(x_k, -\iota\bar{x}_k; y_k, -\iota\bar{y}_k)$

$$\mathcal{H}_{\mathrm{sec}} = \sum_{\mathbf{l},\bar{\mathbf{l}},\mathbf{l}\prime,\bar{\mathbf{l}}\prime} h^{\mathbf{l},\bar{\mathbf{l}},\mathbf{l}\prime,\bar{\mathbf{l}}\prime}(\boldsymbol{\Lambda}) \prod_{j=1}^{N} x_j{}^{l_j}\, \bar{x}_j^{\bar{l}_j} y_j{}^{l\prime_j} \bar{y}_j^{\bar{l}\prime_j}, \quad \mathbf{l}, \bar{\mathbf{l}}, \mathbf{l}', \bar{\mathbf{l}}' \in \mathbb{N}^N$$

dark magic

Proper mode variables:

$$\mathbf{x}^* = \sqrt{\mathbf{I}^*}\, \mathrm{E}^{-\iota\boldsymbol{\theta}^*}, \quad \mathbf{y}^* = \sqrt{\mathbf{I}'^*}\, \mathrm{E}^{-\iota\boldsymbol{\theta}\prime^*}$$

$$\mathcal{H}_{\mathrm{sec}} = \sum_{\mathbf{k},\mathbf{k}\prime} \mathrm{E}^{\iota(\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k}\prime\cdot\boldsymbol{\theta}\prime^*)} \prod_{j=1}^{N} I_j^{*\frac{|k_j|}{2}} I'^{*\frac{|k\prime_j|}{2}}_j \left(\tilde{h}^{*\mathbf{k},\mathbf{k}\prime}(\boldsymbol{\Lambda}, \mathbf{I}^*, \mathbf{I}'^*)\right) \qquad \mathbf{k}, \mathbf{k}' \in \mathbb{Z}^N$$

$$C_{\boldsymbol{\gamma},\boldsymbol{\gamma}\prime} = (\boldsymbol{\gamma}, \boldsymbol{\gamma}') \cdot (\mathbf{I}^*, \mathbf{I}'^*), \quad \boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathbb{R}^N$$

Time derivative of a dynamical quantity

$$C_{\boldsymbol{\gamma},\boldsymbol{\gamma}'} = (\boldsymbol{\gamma},\boldsymbol{\gamma}') \cdot (\mathbf{I}^*, \mathbf{I}'^*), \quad \boldsymbol{\gamma},\boldsymbol{\gamma}' \in \mathbb{R}^N$$

For a given harmonic $(\mathbf{k}, \mathbf{k}')$ $\dfrac{\mathrm{d}C_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}}{\mathrm{d}t} = 2(\boldsymbol{\gamma},\boldsymbol{\gamma}') \cdot (\mathbf{k},\mathbf{k}') \prod_{j=1}^{N} {I_j}^{*\frac{|k_j|}{2}} {I'_j}^{*\frac{|k'_j|}{2}} \tilde{h}^{*\mathbf{k},\mathbf{k}'} \sin(\underbrace{\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k}'\cdot\boldsymbol{\theta}'^*}_{\phi = \text{Cst?}})$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \approx \mathbf{k}\cdot\boldsymbol{g} + \mathbf{k}'\cdot\boldsymbol{s} \approx 0$$

Time derivative of a dynamical quantity

$$C_{\boldsymbol{\gamma},\boldsymbol{\gamma}'} = (\boldsymbol{\gamma},\boldsymbol{\gamma}')\cdot(\mathbf{I}^*,\mathbf{I}'^*), \quad \boldsymbol{\gamma},\boldsymbol{\gamma}' \in \mathbb{R}^N$$

For a given harmonic $(\mathbf{k},\mathbf{k}')$ $\dfrac{\mathrm{d}C_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}}{\mathrm{d}t} = 2(\boldsymbol{\gamma},\boldsymbol{\gamma}')\cdot(\mathbf{k},\mathbf{k}')\prod_{j=1}^{N} I_j^{*\frac{|k_j|}{2}} I'^{*\frac{|k'_j|}{2}}_j \tilde{h}^{*\mathbf{k},\mathbf{k}'} \sin(\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k}'\cdot\boldsymbol{\theta}'^*)$

$$\mathrm{AMD_e} = \sum_{j=1}^{N} x_j^*\,\bar{x}_j^* = \sum_{j=1}^{N} I_j^* = C_{\mathbf{1},\mathbf{0}}, \quad \mathrm{AMD_i} = C_{\mathbf{0},\mathbf{1}}, \quad \mathrm{AMD} = C_{\mathbf{1},\mathbf{1}}$$

$$\frac{\mathrm{dAMD_e}}{\mathrm{d}t} \propto \sum_j k_j\,, \quad \frac{\mathrm{dAMD_i}}{\mathrm{d}t} \propto \sum_j k'_j, \quad \frac{\mathrm{dAMD}}{\mathrm{d}t} \propto \sum_j k_j + k'_j = 0 \text{ (D'Alembert)}$$

Resonances and $AMD_{e,i}$ exchanges

$$\sin(\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k}'\cdot\boldsymbol{\theta}'^*)$$

$$k = (1,-1,0,0), \quad k' = (1,-1,0,0)$$

$$\sin(\phi) = \sin(+1\theta_1^* - 1\theta_2^* + 1\theta'^*_1 - 1\theta'^*_2)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = g_1 - g_2 + s_1 - s_2$$ possibly resonant

$$\frac{\mathrm{dAMD_e}}{\mathrm{d}t} \propto \sum_j k_j = 0, \quad \frac{\mathrm{dAMD_i}}{\mathrm{d}t} \propto \sum_j k'_j = 0$$

conserving

Resonances and $AMD_{e,i}$ exchanges

$$\sin(\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k'}\cdot\boldsymbol{\theta'}^*)$$

$$k = (1,1,0,0), \quad k' = (-1,-1,0,0)$$

$$\sin(\phi) = \sin(+1\theta_1^* + 1\theta_2^* - 1\theta'^*_1 - 1\theta'^*_2)$$

$$\frac{\mathrm{dAMD_e}}{\mathrm{d}t} \propto \sum_j k_j \neq 0, \quad \frac{\mathrm{dAMD_i}}{\mathrm{d}t} \propto \sum_j k'_j \neq 0$$

non conserving

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \underbrace{g_1 + g_2}_{>0} \underbrace{-s_1 - s_2}_{>0}$$

not resonant

Resonances and $AMD_{e,i}$ exchanges

$$\sin(\mathbf{k}\cdot\boldsymbol{\theta}^* + \mathbf{k'}\cdot\boldsymbol{\theta'}^*)$$

$$k = (1,1,0,0), \quad k' = (-1,-1,0,0)$$

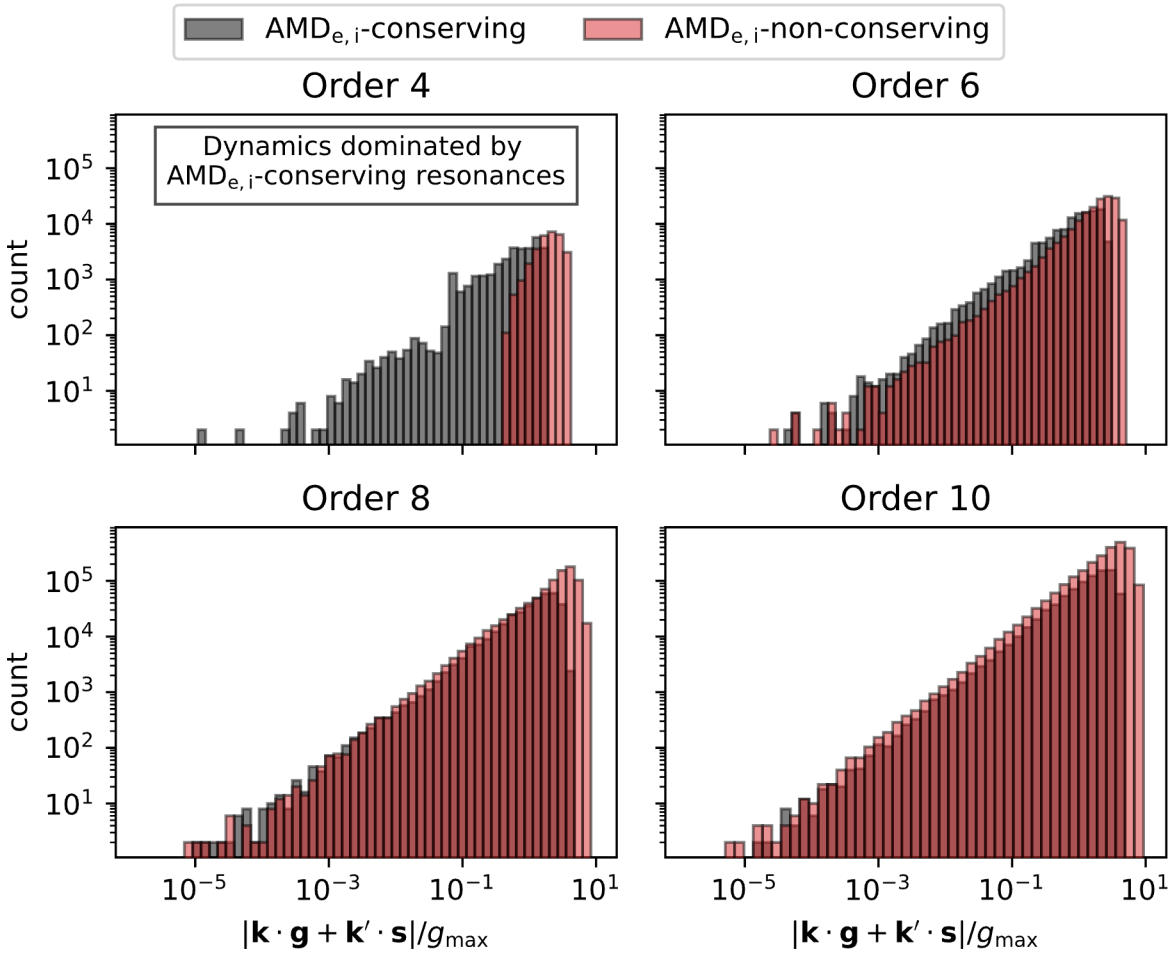
$$\sin(\phi) = \sin(+1\theta_1^* + 1\theta_2^* - 1\theta'^*_1 - 1\theta'^*_2)$$

$$\frac{\mathrm{dAMD_e}}{\mathrm{d}t} \propto \sum_j k_j \neq 0, \quad \frac{\mathrm{dAMD_i}}{\mathrm{d}t} \propto \sum_j k'_j \neq 0$$

non conserving

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \underbrace{g_1 + g_2}_{>0} \underbrace{-s_1 - s_2}_{>0}$$

not resonant



$$\mathrm{Order} \equiv \|(\mathbf{k}, \mathbf{k}')\|_1$$

Conclusions

- Conservation of $AMD_{e,i}$ beyond the Solar System ⟶ Quasi integrals of motion
- Conserved since last perturbation, $AMD_{e,i}$ are tracers of the dynamical history
- $AMD_{e,i}$ are easy to compute
- Low AMD systems have LL-like dynamics despite confined chaos

