Reformulation and Flux-based Statistical Theory of Egalitarian Three-Body System

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17 Sep 2026, CelMec IX, Viterbo

based on
BK, Celest. Mech. Dyn. Astron. **133**, 17 (2021)
Manwadkar, BK, Trani, Leigh, MNRAS **506**, 692 (2021)
BK, CMDA **135**, 29 (2023)
Manwadkar, Trani, BK, CMDA **136**, 4 (2024)
BK, Entropy 28, 610 (2026)
Haham, BK, arXiv:2608.20276
Dandekar, BK et. al. arXiv:2608.30414

flux-based testing
dynam. reduct
measurement story
double-pendulum
planar 3B

See also full talks on YouTube, search “Barak Kol three-body”

Introduce myself

Have worked on Cel. Mec. since 2020

Background: Theoretical high energy physics.
& General Relativity

PhD Stanford 1998,

post-docs Tel-Aviv U, Princeton IAS

Hebrew U, Jerusalem since 2002

Outline

Foundations

Flux-based theory

Reformulation

Developments

Planar 3-body

Flux-based th.

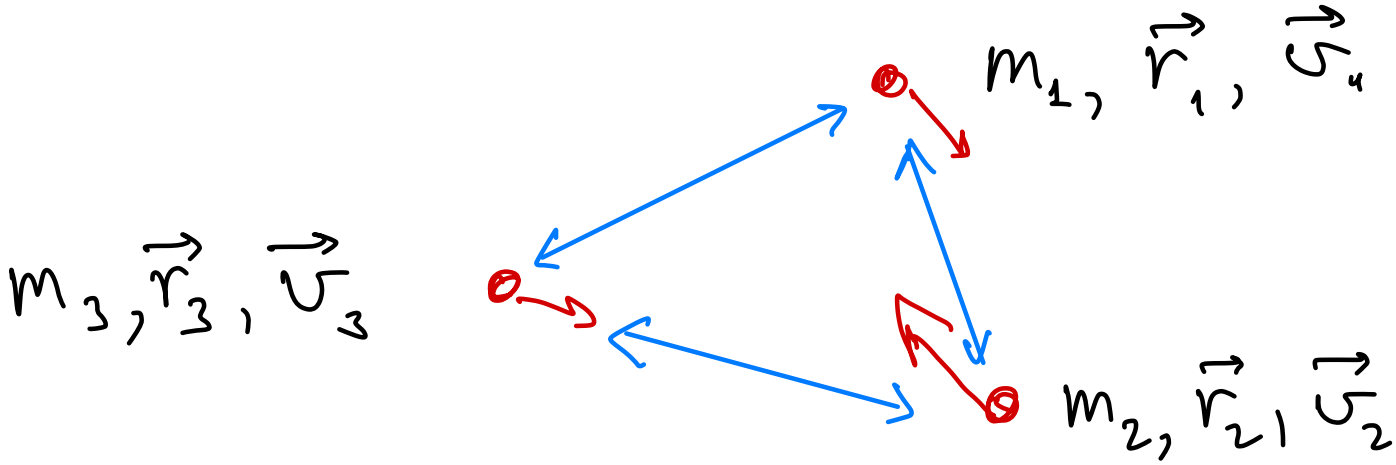
of double pendulum

The problem

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Three-body problem

Given initial conditions



$$F = \frac{G m_a m_b}{r^2}$$

Goal: predict the motion.

Easy to define
not easy to solve.

Non hierarchical — comparable m_a

≡ Egalitarian

" Separations

SKIP

Simulation

Rich

fruitful

& deep

problem.

Started w. Newton, over 330 years ago.

Closely related to

Perturbation theory

Symplectic structure of mechanics

Poisson brackets

chaos

Topology.

Question

Three-body problem believed not to have a deterministic solution. Poincaré 1890

Unsolvable?

Computer simulations

produce statistics

Agekyan & Anosova 1967, Aarseth,
Valtonen, Hut & Bahcall, Boyd & McMillan

Random-like motion

The correct goal:

Given chaotic initial conditions

a statistical solution

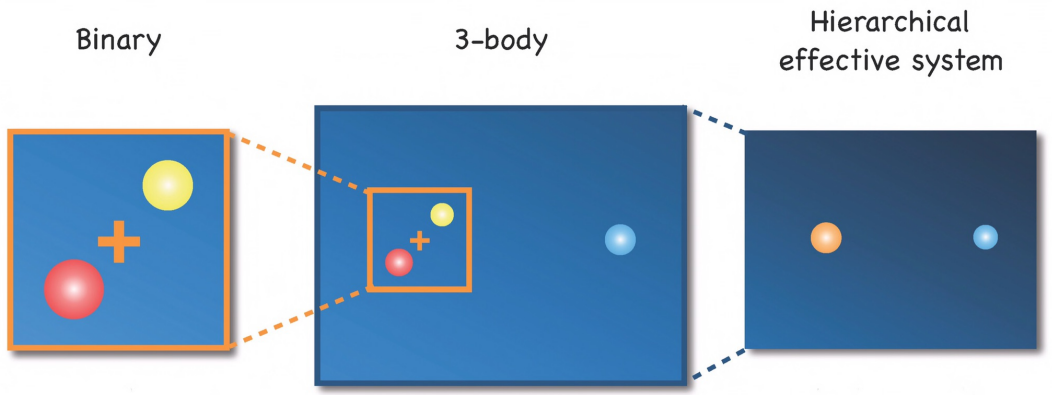
rather than a deterministic one.

Monaghan 1976

Flux-based theory

Main result

Notation for main result



Outcome parameters

$$U = (s, \underbrace{\mathcal{E}_F, \vec{l}_F, \omega_F}_{\text{effective/outer}}, \underbrace{\mathcal{E}, \vec{l}, \omega}_{\text{binary/inner}})$$

angles of pericenter

$s = 1,2,3$ escaper identity

$$\mathcal{E}_F + \mathcal{E} = E, \qquad \vec{l}_F + \vec{l} = \vec{L}$$

Differential decay rate

Flux-emissivity reduction.

(kinematics-dynamics)

$$d\Gamma(u) = \frac{1}{\sigma}\,\mathcal{E}(u)\,dF(u)$$

- $d\Gamma(u)$: probability per unit time
- σ: regularized phase volume.
- $\mathcal{E}(u)$: emissivity ≡ absorptivity
- $dF(u)$: asymp. flux

exact reduction!

$$dF(u) \propto \sqrt{\frac{k_s}{(-2\mathcal{E})^3}}\, d\mathcal{E}\, \frac{d^3l\, d^3l_f}{l \cdot l_f}\, \delta^{(3)}(\vec{l}+\vec{l}_f-\vec{L})$$

$l \cdot l_f$: central force

Insight:

$(d\sigma/d\mathcal{E})_{\text{binary}} = T$ ← binary period

binary constant $k_s = \mu \cdot \alpha^2 \equiv \frac{m_a m_b}{m_a + m_b} \cdot (G m_a m_b)^2$

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Defn. of $\Sigma(u)$

Σ := probability for chaotic absorption (capture) for random binary phase



$0 \leq \Sigma \leq 1$

Kirchhoff's law / detailed balance

Emissivity = Absorptivity

Exact reduction of statistical solution

Theory name:

Flux-based statistical theory.

Evidence

Escape probability P_s

$\mathcal{E}$ blind assumption : $\mathcal{E}(u) = \text{const}$

Marginalize over $di^\top(u)$

For $2|E|L^2 > k_{max}$ *

$$k := \mu\alpha^2 = \frac{m_a m_b}{m_a + m_b}(G m_a m_b)^2$$

$$P_s(E, L^2; \{m_a\}) \propto \frac{l_s^3}{L} \propto k_s^{3/2}$$

$$l_s(E) := \sqrt{\frac{k_s}{-2E}}$$

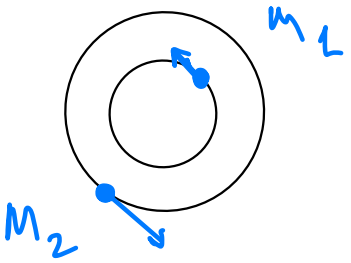
* Impossible for binary to carry all ang. mom.

Validation through measurement of escape probability

Manwadkar, Bk, Trani, Leigh
MNRAS 2021

Set up initial conditions

m_3
at rest



Average over position of m_3 over celestial sphere.
Select for chaotic evolutions.

Escape probabilities - measured vs predictions

data from Manwadkar, BK, Trani & Leigh 2021

Ring legend: Outer - measured (Manwadkar, BK, Trani and Leigh 2021),
Middle - BK2020 (flux-based), Inner - Valtonen-Karttunen 2006 $P_s \propto m_s^{-2}$

Mass set 15,15,15



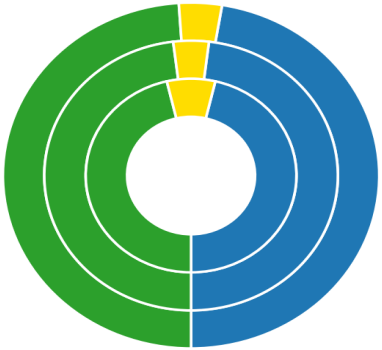
Mass set 12.5,15,17.5



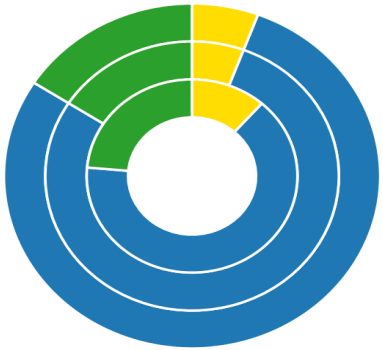
Mass set 12,15,18



Mass set 10,10,20



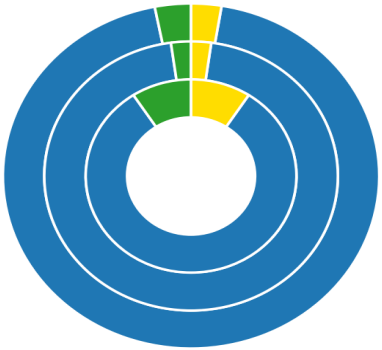
Mass set 10,15,20



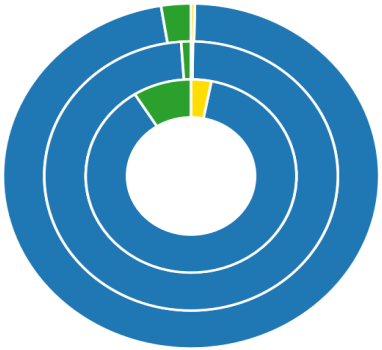
Mass set 10,20,20



Mass set 8,21,21



Mass set 5,15,25



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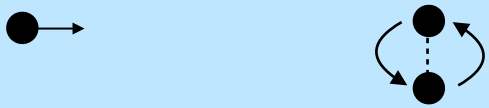
MKTL 2021

Table 2. The escape probabilities P_s and total number of realizations N_s by escaper mass in different three-body systems. The total number of simulations in each set is 10^6. The cut-offs are as follows: [1] The entire set, [2] Removing prompt ejections ($\tau_{\rm D} > 30$yrs), [3] ergodic subset ($\tau_{\rm gap} > 10$yrs and $\tau_{\rm D} > 80$yrs). The ergodic subset (Cut 3) corresponds to the interactions where the final escape happens from the ergodic regime. For discussion on this cut, refer to Section 5. The probabilities of the ergodic subset are being compared to theoretical formalisms: Kol (2020): K20, Ginat & Perets (2020) : GP20, Stone & Leigh (2019): SL19, and Valtonen & Karttunen (2006): VK06. For the equal mass case, the initial binary pair masses are denoted by (1) and (2).

$P_s \propto M_s^{-2}$

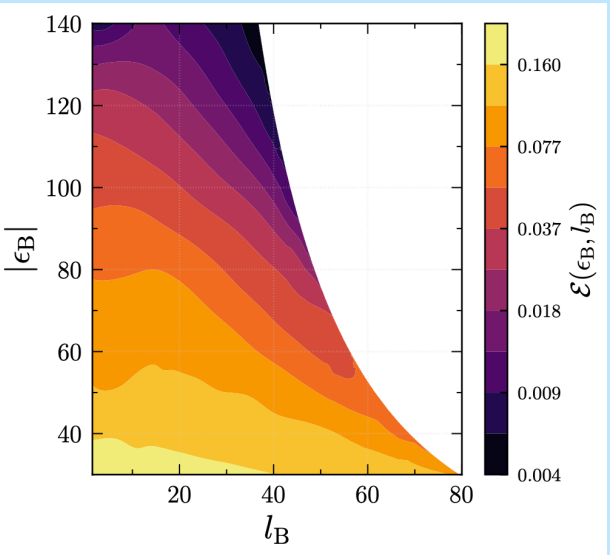
Masses($M_\odot$)	Ejection Mass($M_\odot$)	P_s [1]	N_s [2]	P_s [2]	N_s [3]	P_s [3]	K20	GP20	SL19	VK06
15,15,15	15 (0)	0.199	199045	0.329	45208	0.328	0.333	0.333	0.333	0.333
	15 (1)	0.400	203155	0.336	46655	0.339	0.333	0.333	0.333	0.333
	15 (2)	0.401	203159	0.335	45771	0.333	0.333	0.333	0.333	0.333
12.5,15,17.5	12.5	0.433	433047	0.560	89428	0.572	0.562	0.504	0.639	0.492
	15	0.302	201393	0.261	42525	0.272	0.279	0.297	0.247	0.305
	17.5	0.265	138310	0.179	24429	0.156	0.159	0.199	0.114	0.203
12,15,18	12	0.503	503471	0.608	98807	0.618	0.610	0.542	-	0.525
	15	0.283	198769	0.240	40136	0.251	0.257	0.281	-	0.293
	18	0.214	126009	0.152	20881	0.131	0.133	0.177	-	0.182
10,10,20	10	0.530	524093	0.554	67278	0.4736	0.4805	0.4604	-	0.463
	10	0.418	37013	0.392	69555	0.4896	0.4805	0.4604	-	0.463
	20	0.052	50791	0.054	5233	0.0368	0.0390	0.0792	-	0.074
10,15,20	10	0.727	726841	0.734	113627	0.784	0.784	0.697	0.872	0.649
	15	0.193	182416	0.184	22988	0.159	0.159	0.202	0.103	0.236
	20	0.080	80496	0.081	8306	0.057	0.057	0.101	0.025	0.115
10,20,20	10	0.842	803550	0.838	108978	0.875	0.880	0.796	-	0.726
	20	0.079	78019	0.081	8055	0.065	0.060	0.102	-	0.137
	20	0.079	77933	0.081	7577	0.060	0.060	0.102	-	0.137
8,21,21	8	0.922	862162	0.920	86959	0.943	0.956	0.904	-	0.8158
	21	0.039	37432	0.040	2813	0.031	0.022	0.048	-	0.0921
	21	0.039	37675	0.040	2448	0.026	0.022	0.048	-	0.0921
5,15,25	5	0.9712	826963	0.9664	41960	0.9712	0.9872	0.9667	-	0.8796
	15	0.0248	24724	0.0289	1108	0.0256	0.0108	0.0245	-	0.0895
	25	0.0040	4036	0.0047	137	0.0032	0.0020	0.0088	-	0.0309

1A] Measure Chaotic Absorptivity



Perform *binary-single scatterings* with non-zero relative velocity and different binary energies/angular momentum

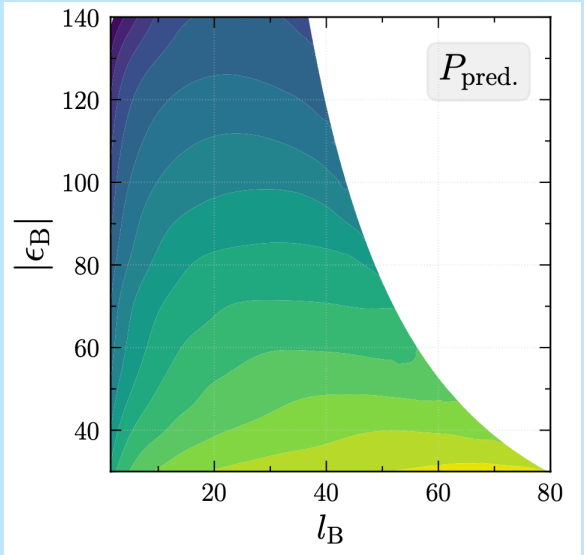
Determine if the scattering resulted in chaotic capture or not to compute absorptivity as a function of binary energy and binary angular momentum (see below plot)



1B] Predict the Outcome Distribution

A finer test

Multiply the absorptivity distribution (measured in left column) by the flux factor (see Equation 5) to get the *predicted* outcome distribution of chaotic three-body interactions in binary energy and binary angular momentum space.



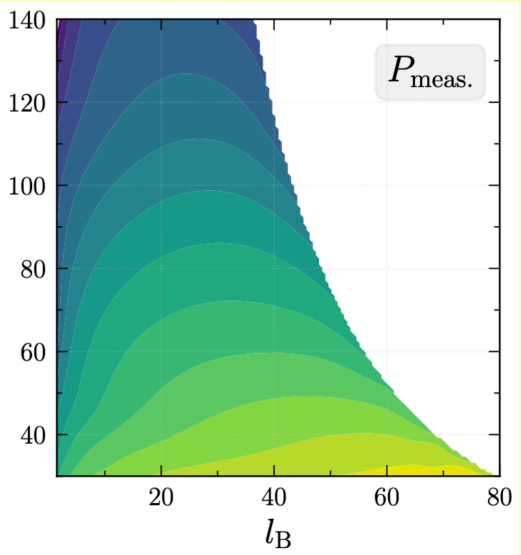
Compare these!

2] Measure the Outcome Distribution

Run large ensemble of three-body simulations and stop simulation when the interaction ends.

Filter for interactions that have an chaotic escape (see Section 4.2.2)

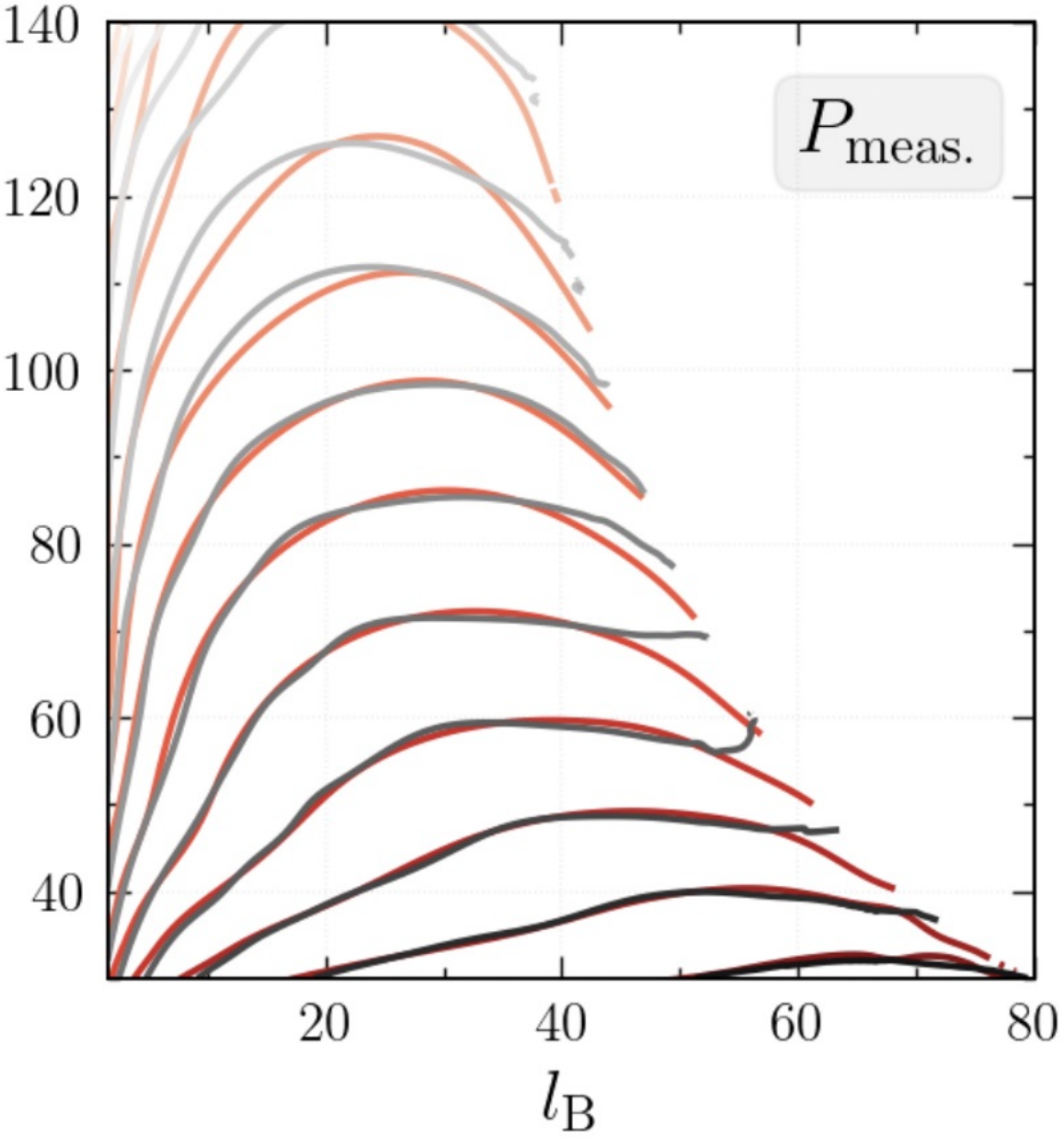
Compute the outcome distribution in binary energy and angular momentum space for these chaotic escapes.



SKIP

Contour maps of predicted & measured outcome distrib.

Binary parameters E_B



Manwadkar, Trani & Bk CMDA 2024

Flux-based statistical theory -

Main idea

Re-examination of theory's foundations

Replace

$P \not\propto \sigma$

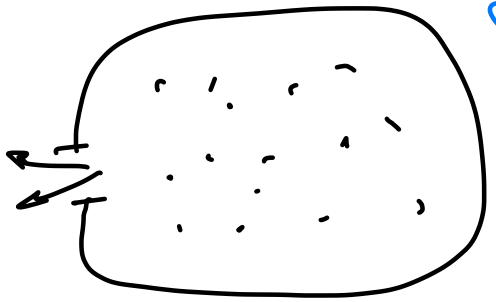
probability — phase-space volume

by

$\Gamma \propto F / \bar{\sigma}$

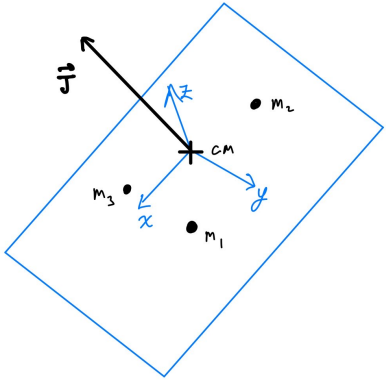
Decay rate — phase-volume flux

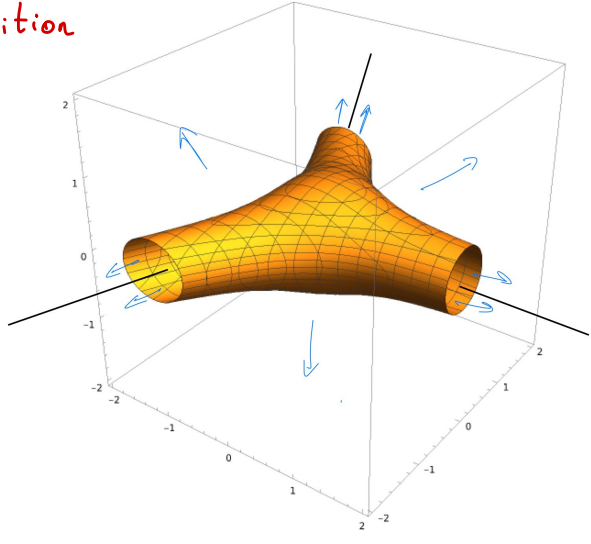
leaky container analogy



Container is pipe joint

Orientation — Geometry decomposition



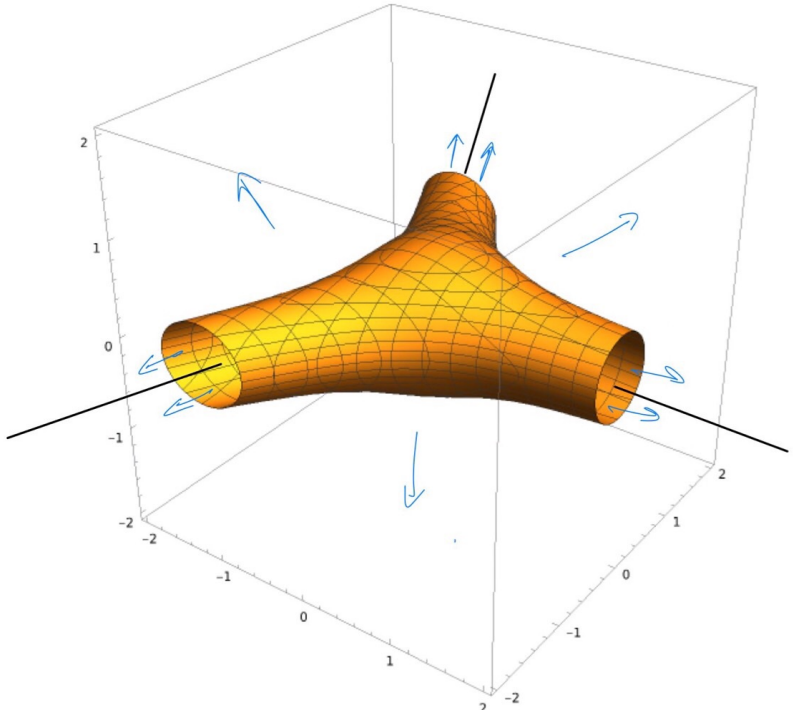


$\vec{J}^2 = \text{const}$

More later...

Reformulation —

Dynamical reduction



BK 2023

Dynamical reduction

* Simplify the dynamics through choice of fewer variables [formulation]

* Conserved quantities enable it

Lagrange 1772
Lemaître 1952
Montgomery & Moeckel 2013

Outline of process

* Translations $\nleftarrow \vec{w}$
* Planar reduction $\nleftarrow \vec{\omega}, \vec{J}$
* In-plane rotation $\nleftarrow (r, \theta, \varphi) \leftrightarrow \vec{q}$

  Triangle geometry space
* Eq. of motion in non-coord. basis

Coordinates for center of mass frame

$\vec{r}_1, \vec{r}_2, \vec{r}_3$ 3 vector d.o.f.

$\vec{0} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3$ 1 vector constraint

In 2-body

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

In 3-body relative positions (transl. invar.)

$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$: $\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{31}$

not independent : $\vec{r}_{12} + \vec{r}_{23} + \vec{r}_{31} = 0$

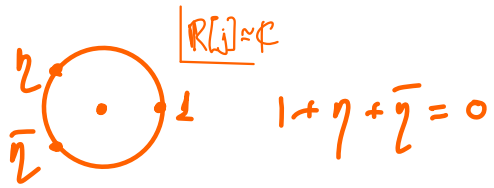
Inspired by Lagrange's solution to the cubic

Define complex relative position

$$\vec{w} = \vec{r}_1 + \eta \vec{r}_2 + \bar{\eta} \vec{r}_3$$

BK2023

$$\eta = \exp(2\pi j/3)$$



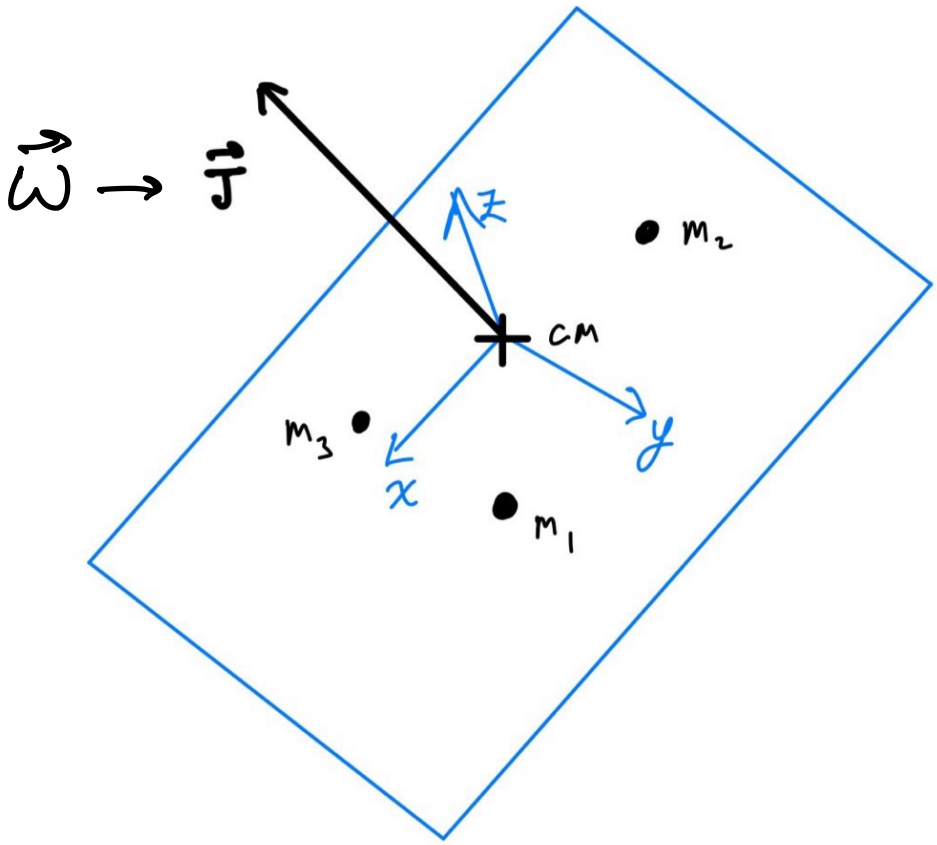
Translation invar.

Transforms nicely under perm.: $1 \to 2 \to 3 \to 1$ $\quad w \to \eta w$

$2 \leftrightarrow 3$ $\quad w \to \bar{w}$

9 d.o.f $\to$ 6 d.o.f !

Motion of plane + within plane



Bi-complex w

Factor rotations ↔ I
Present as phase

$$w := \vec{w} \cdot (\hat{x} + i\hat{y})$$

↑ body axes

So

$$w = w_1 + w_2\, i + w_3\, j + w_4\, ij$$

j - related to permutations

i - plane rotations

$i^2 = j^2 = -1$, $\quad ij = ji$ independent ⇄ commuting.

4 d.o.f + rotating body $(2 \times \frac{1}{2}$ d.o.f $+ 3 \times \frac{1}{2}$ d.o.f$)$
$\Rightarrow 3$
Euler angles

Triangle geometry space

eliminate rotational phase

$4d(w) \rightarrow 3d(Q)$

$$w = \sqrt{\frac{3}{2}}\left(w_1 + j w_2\right)$$

$$Q_s := \begin{bmatrix} w_1^* & w_2^* \end{bmatrix} \tau_s \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad i^* = -i$$

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau_2 = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}, \quad \tau_3 = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

$$\tau_4 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Pauli-like matrices

$$0 = Q_1 + Q_2 + Q_3$$

$$Q_0^2 = \frac{2}{3}\left(Q_1^2 + Q_2^2 + Q_3^2\right) + Q_4^2$$

Inspired by isotropic oscillator, Stokes, Poincare, Pauli, Hopf, Bloch

Spherical coordinates - Triangle geometry Space

size + shape = triangle geometry

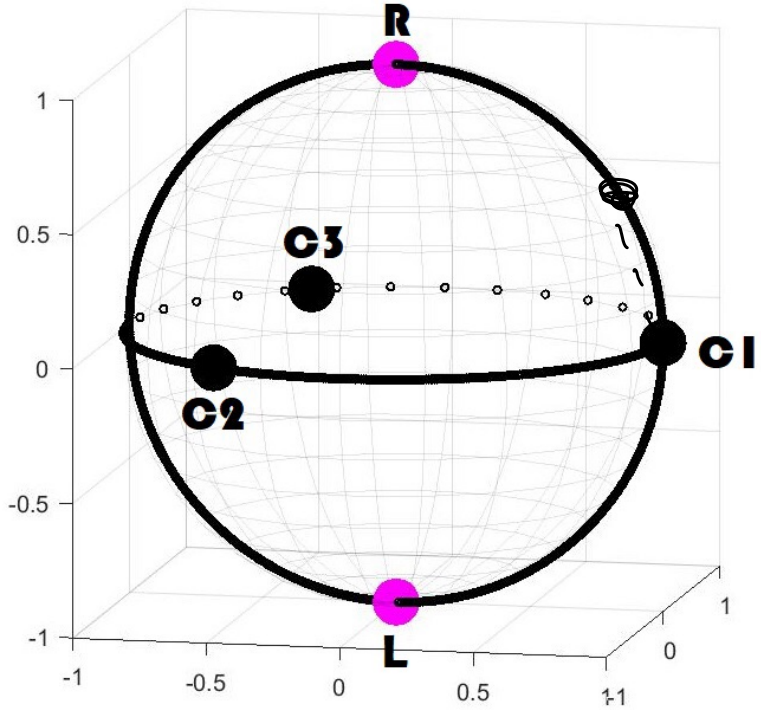
↑ r ↑ θ, φ

SKIP

$$Q_a = r^2 \sin\theta \cos\left(\phi - (a-1)\frac{2\pi}{3}\right) \qquad a = 1, 2, 3$$

equilateral poles

collinear equator



$$r_{23}^2 = r^2 - Q_1$$

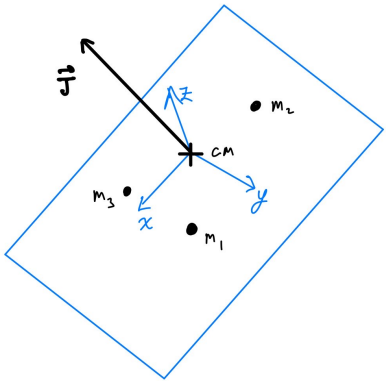
2-3 collision

shape sphere (r, θ, φ)

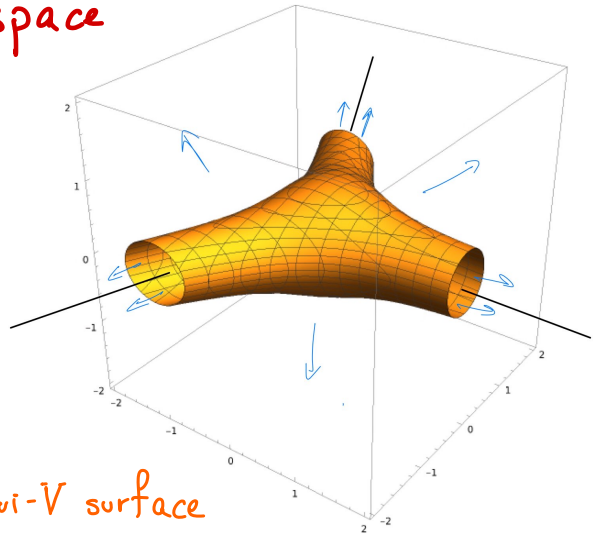
Montgomery - Moeckel 2013

Overall: Orientation — Geometry decomposition

Reduced config. space



$\vec{J}^2 = \text{const}$



Equi-V surface

$\mathcal{L} = \ldots$

Rotating body + 3 d.o.f

Triangle geometry space $\leftrightarrow (r, \theta, \varphi)$

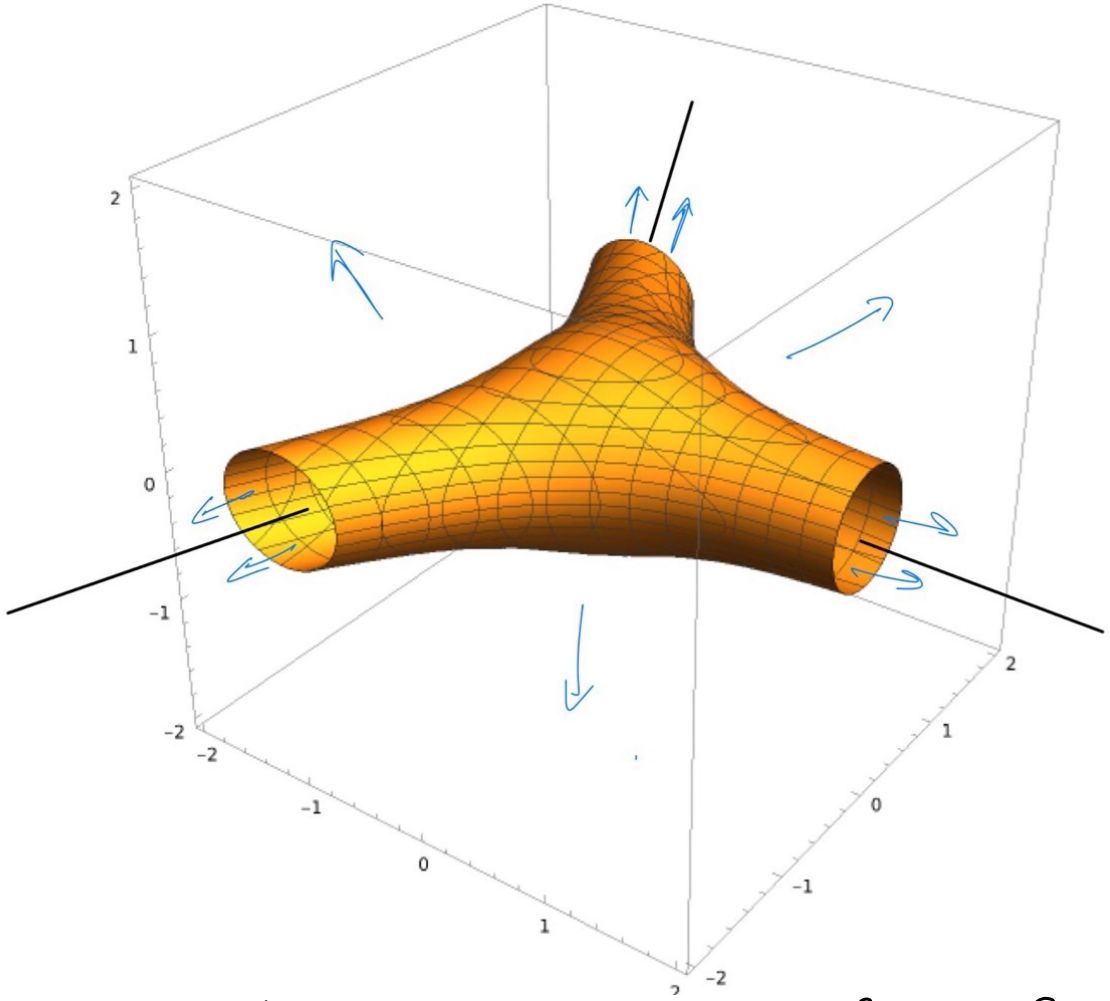
pipe joint

Lemâitre 1952

Montgomery-Moecked 2013

chaotic core

Overall formulation — Orientation — Geometry decomposition



Gravitational attraction : conservative force field toward pipe centers.

Centrifugal potential : outgoing

Coriolis force : magnetic-like field ↗

SKIP

Lagrangian mechanics w. non-coordinate velocities

$$v^a = \beta^a_i(q)\,dq^i$$

Torsion $$dv^c = \frac{1}{2} C^c_{ab}\, v^b v^a$$

Lagrange-Poincaré eq. of motion Poincaré 1901

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right) + C^c_{ab}\, v^b \frac{\partial L}{\partial v^c} = \partial_a L$$

↑ extra term ← $e_a \cdot v^b = \delta^b_a$

e.g. Rigid body $L = \frac{1}{2} I_{ij}\,\omega^i \omega^j$ $\quad H = \frac{1}{2}(I^{-1})^{ij} S_i S_j$

Formulation

SKIP 4

$$\mathcal{L}_J(\vec{J},\vec{g}) := \mathcal{L} - \vec{\omega}\cdot\vec{J} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \qquad \mathcal{O}(J^k)$$

$\vec{g}$: geometry space

motion potential

$$\mathcal{L}_0 = \frac{1}{8I}\dot{I}^2 + \frac{3M_3}{8M}\frac{r^4}{I}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - \left(\frac{G m_2 m_3}{r_{23}} + \text{cyc.}\right)$$

$$I = \sum m_a r_a^2 \overset{\text{cm}}{=} \frac{1}{M}\left(m_2 m_3 r_{23}^2 + \text{cyc.}\right)$$

$$M = m_1 + m_2 + m_3$$

$$M_2 = m_1 m_2 + m_2 m_3 + m_3 m_1$$

$$M_3 = m_1 m_2 m_3$$

$$r_{23}^2 = r^2(1 - \sin\theta\cos\varphi)$$

$$r_{31}^2 = \text{"}(\text{"} - \text{"}\cos(\varphi - \tfrac{2\pi}{3}))$$

shape sphere metric conformal to round S^2.

$$\mathcal{L}_1 = J_3 \frac{L_G}{I} \; ; \quad L_G = -\frac{r^2}{2M}\left[M_2 M_3\left((1-\cos\theta-\sin\theta\cos\varphi)\dot{\phi}-\sin\phi\,\dot{\theta}\right)+\text{cyc.}\right]$$

in "ψ_+-gauge"

Magnetic-like 2-form

$$B = \frac{3}{2} M M_3 J_3 \frac{\sin\theta\, d\varphi\, d\theta}{\left[M_2 m_3 (1-\sin\theta\cos\varphi)+\text{cyc.}\right]^2}$$

monopole charge at $\vec{g}=0$.

$$-L_2 = \frac{1}{2I} J_3^2 + \frac{M}{2M_3} \frac{1}{\Delta^2} \bar{\bar{I}}^{\alpha\beta} J_\alpha J_\beta \qquad \alpha, \beta = 1, 2$$

$$\Delta^2 = \frac{3}{4} r^4 \cos^2\theta$$

Δ is twice triangle area

$$\bar{\bar{I}}^{\alpha\beta} = \frac{1}{M}\left[m_2 m_3 \rho_{23}^{\alpha} \rho_{23}^{\beta} + \text{cyc.}\right] \sim I^{-1}$$

$$\rho_{23} \equiv ir\left[\cos\frac{\theta}{2} - e^{-i\phi}\sin\frac{\theta}{2}\right]$$

in κ_+ gauge

$$\{J_i, J_j\} = -\varepsilon_{ijk} J_k$$

centrifugal potential

Equations of motion

$$\frac{d}{dt} J_i = -\{ J_i, \mathcal{L}_J \}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_J}{\partial \dot{q}^s}\right) = \frac{\partial \mathcal{L}_J}{\partial q^s} \qquad q^s = (r, \theta, \varphi)$$

Developments

Planar (2d) 3-body system

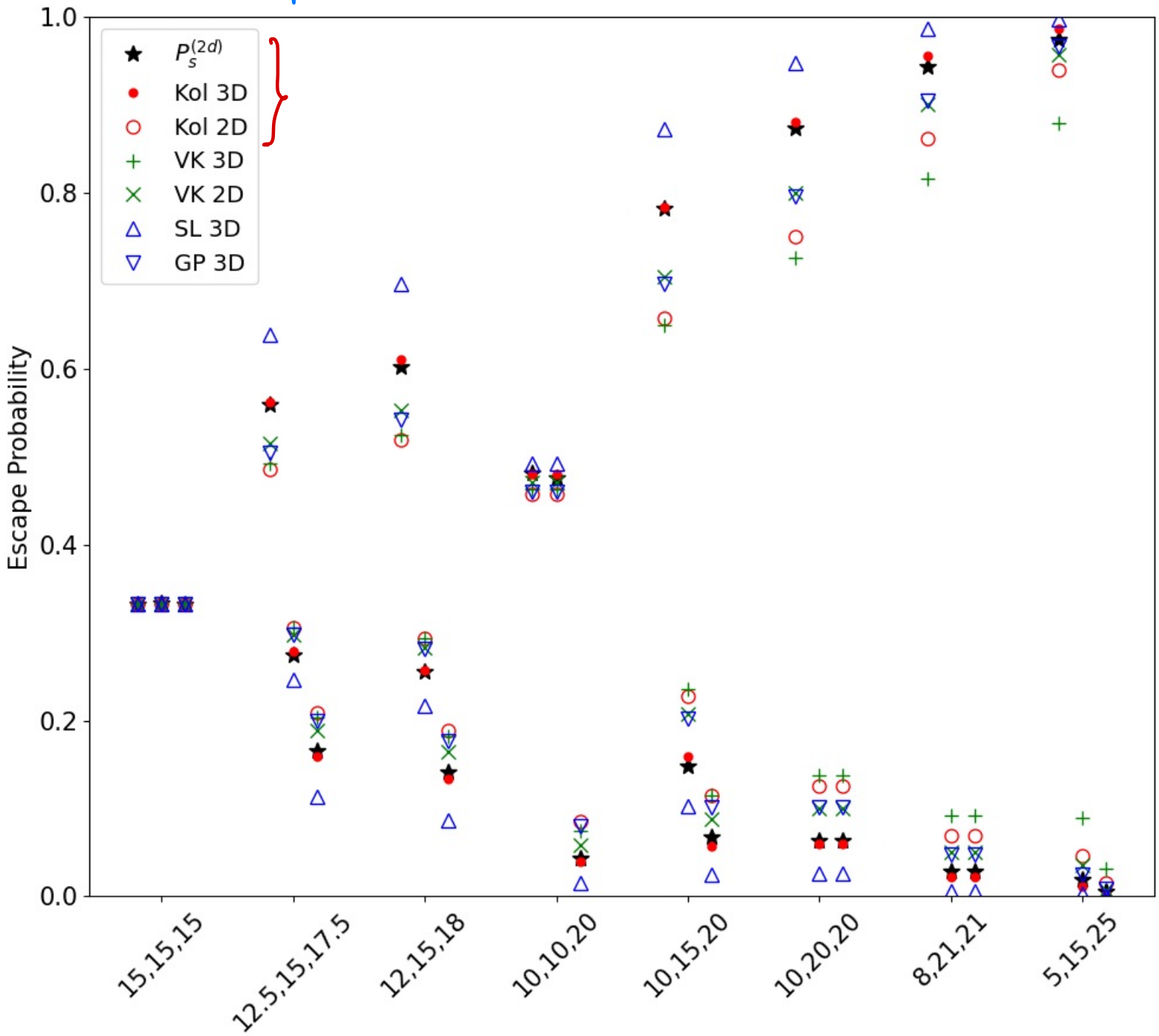
System & natural question

arXiv: 2608

Dandekar, Springer, Zard, Trani, BK, Birnholtz

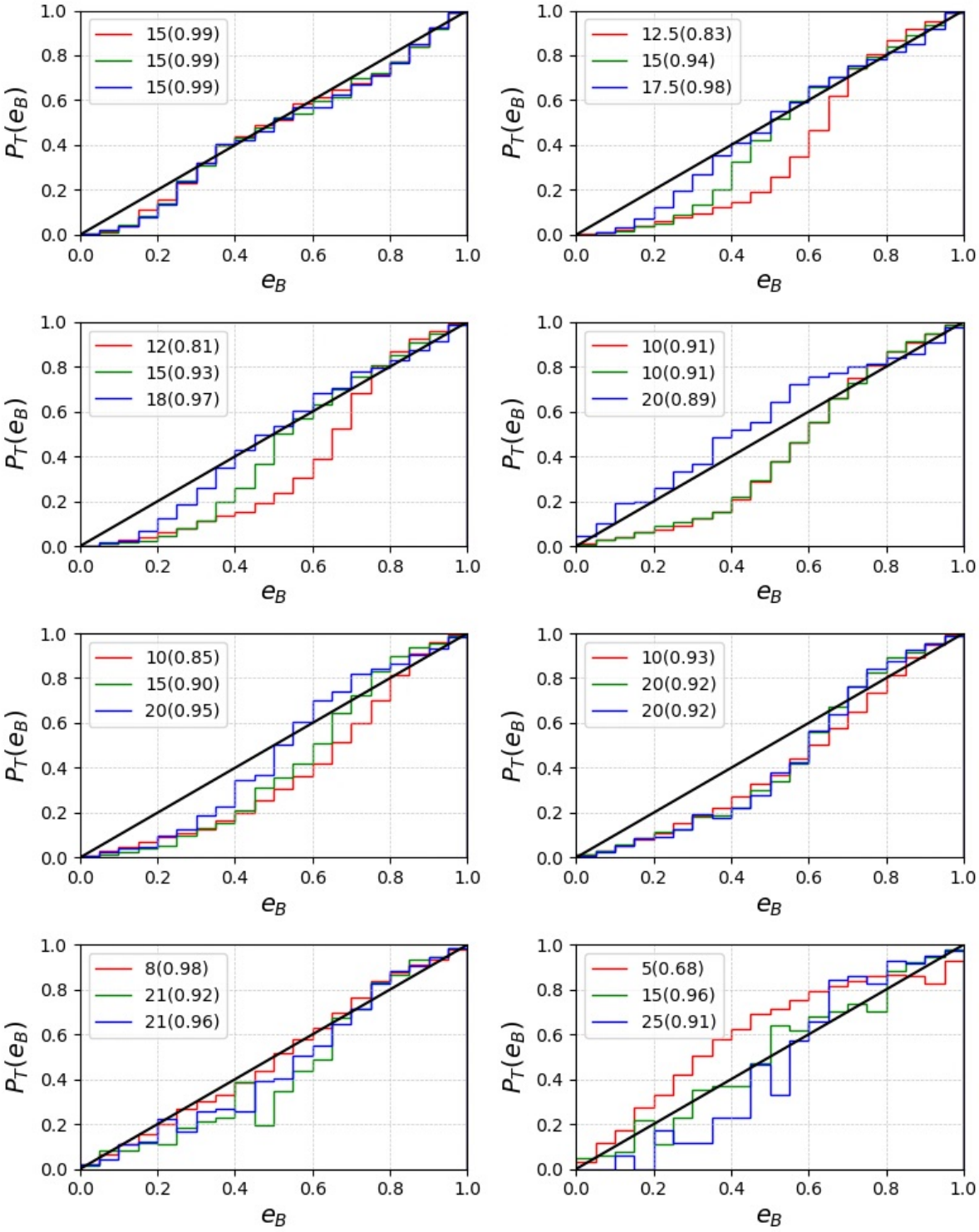
Escape probabilities $\Rightarrow \mathcal{E}$ is not subleading

challenge!



Outgoing binary eccentricity

- preserves agreement

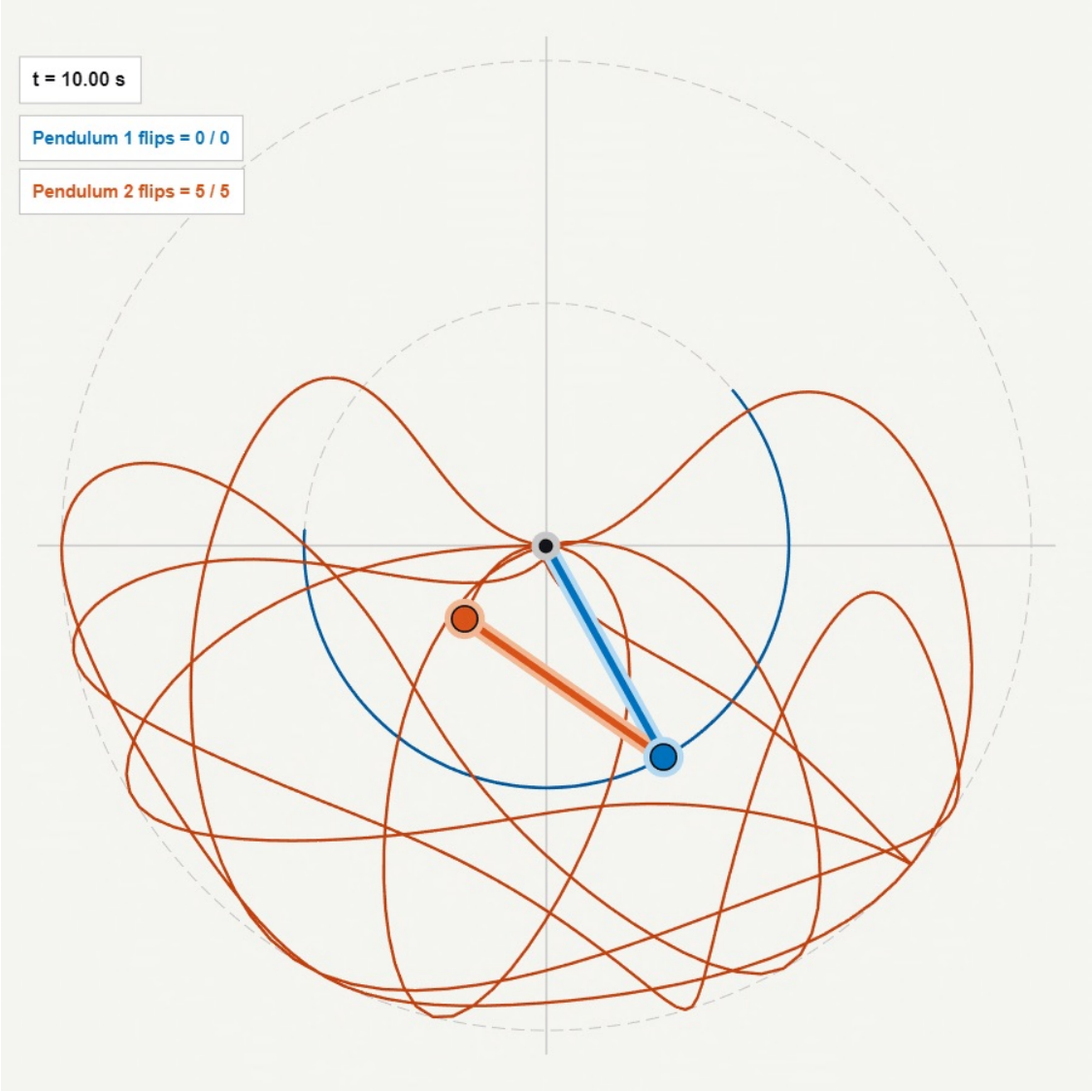


Flux-based theory predicts chaos in the double pendulum

arXiv:2608

Peleg Haham & BK

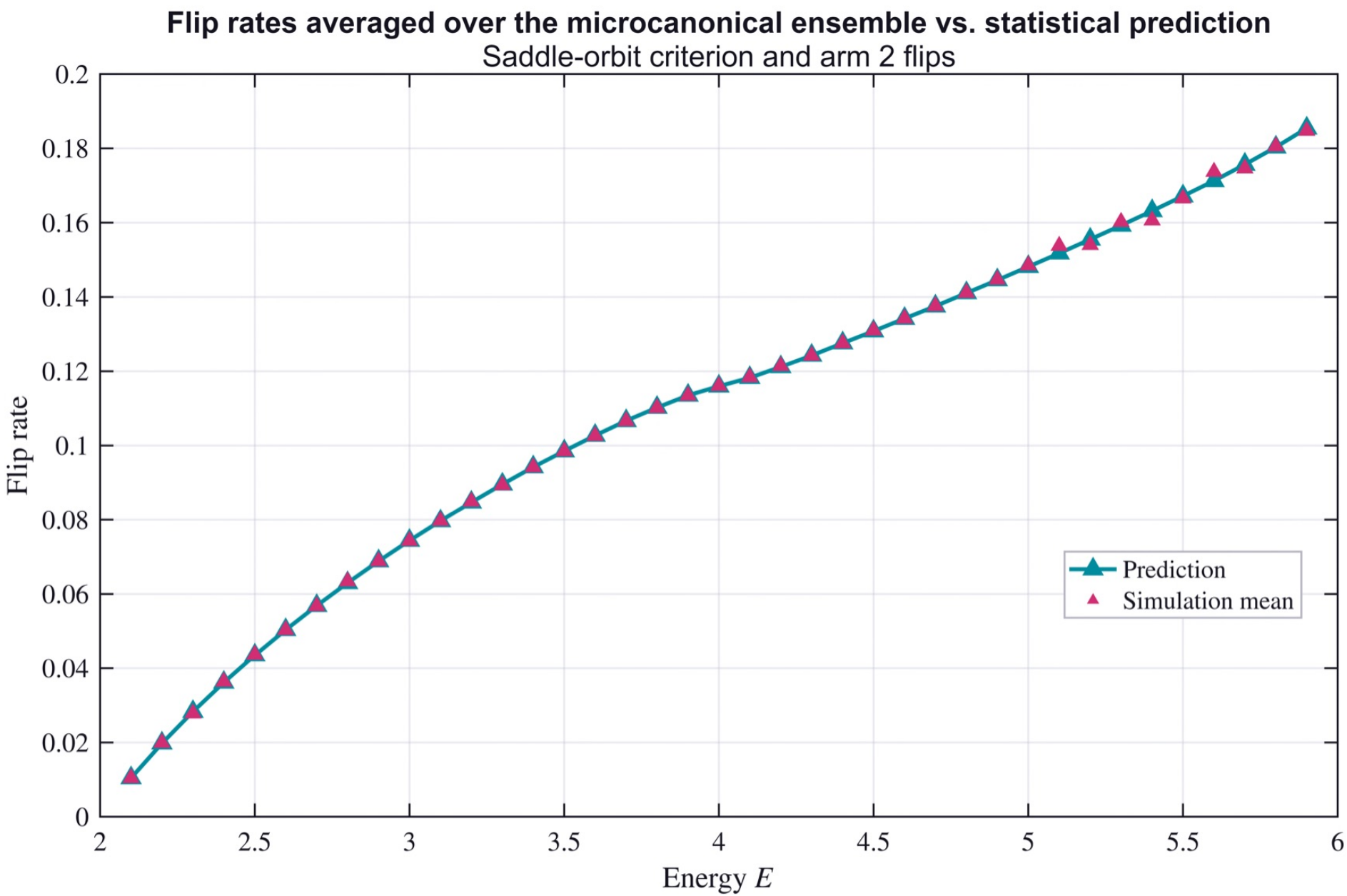
The double pendulum



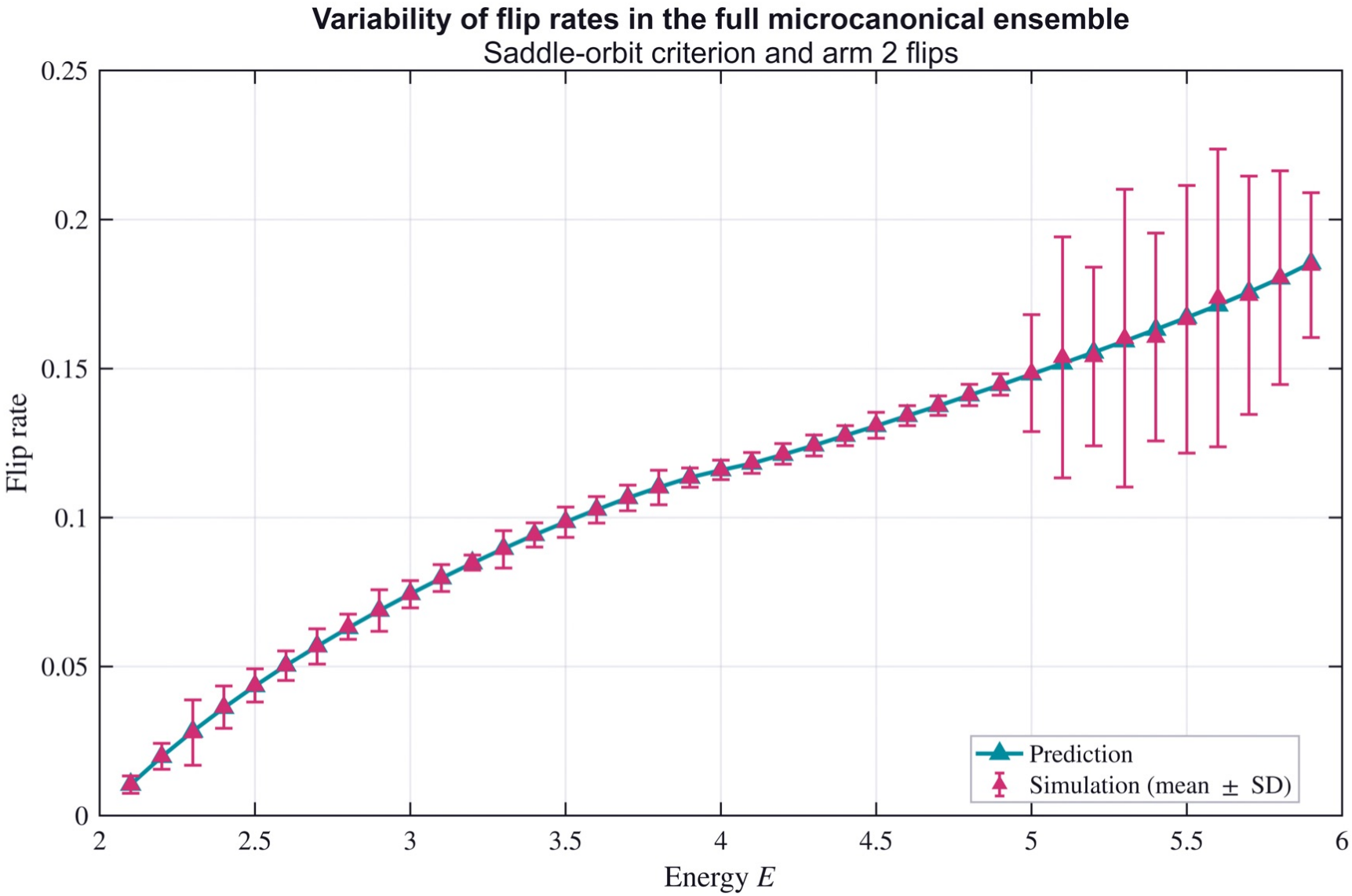
- Prototypical chaotic system
- Flip events : an arm turns over
- Flips persist
- Mean flip rate = ?

See video

Prediction vs. simulated full ensemble average

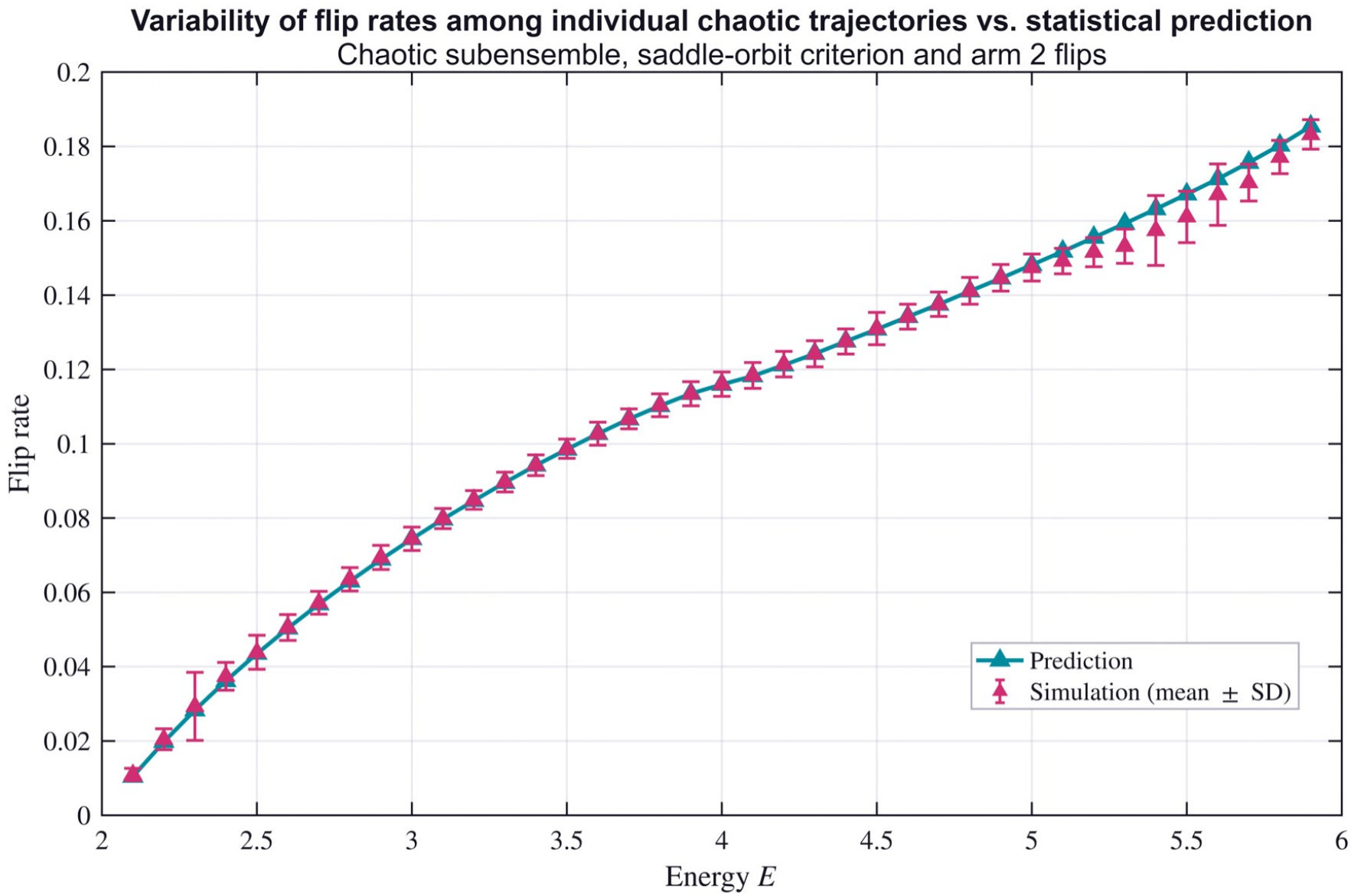


w. ensemble scatter shown

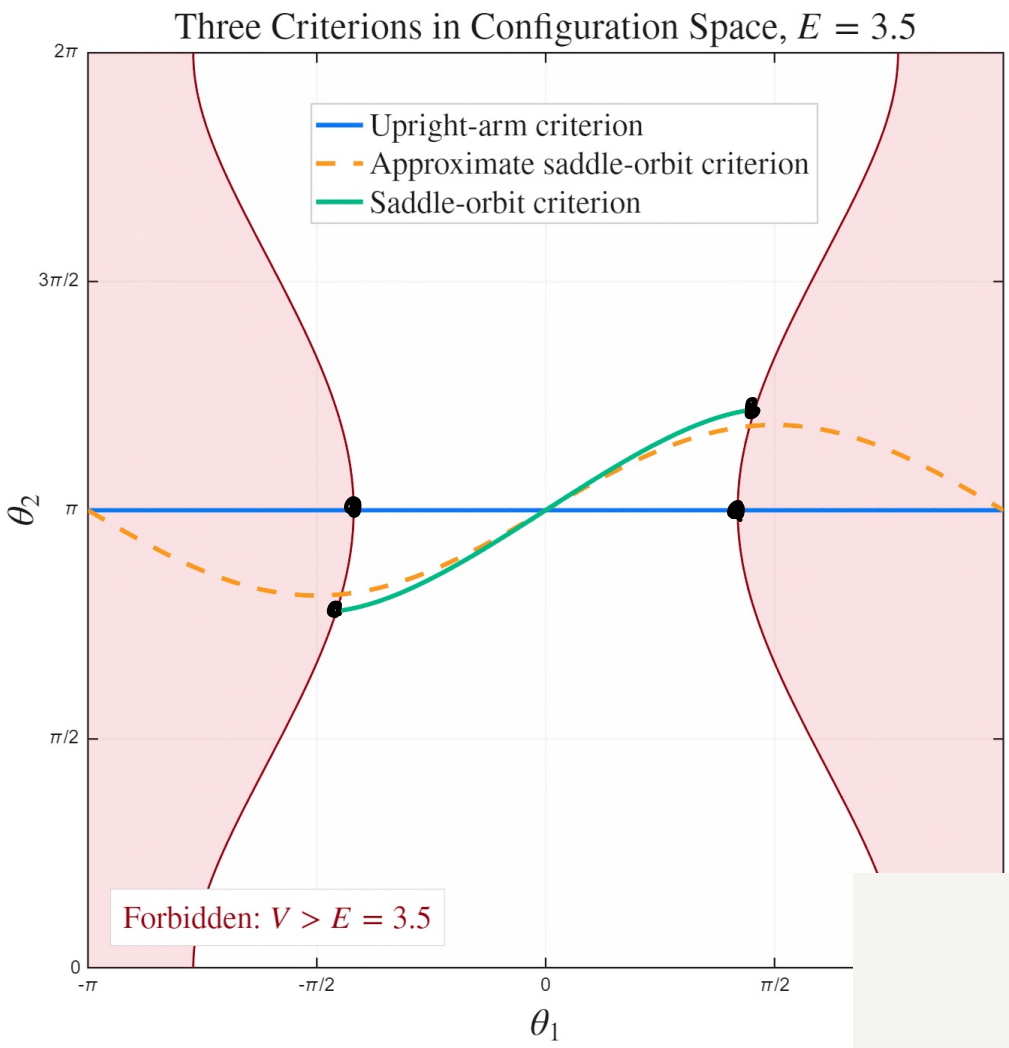


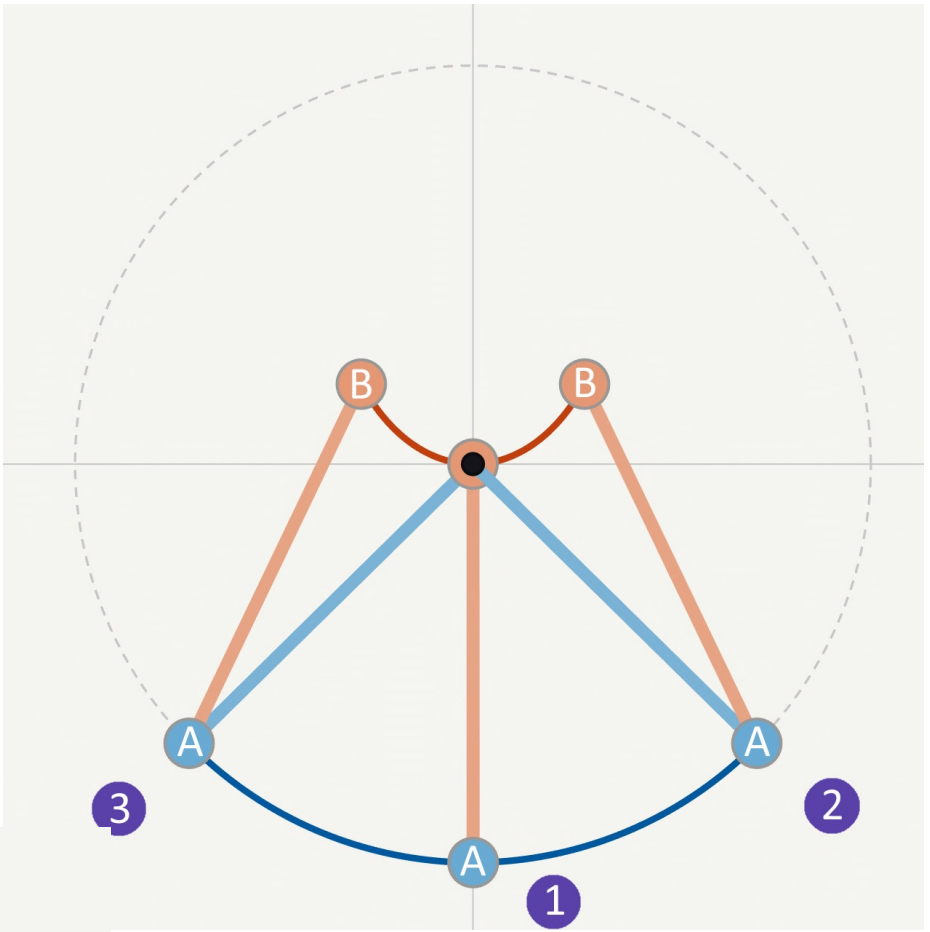
due to the regular component in phase space

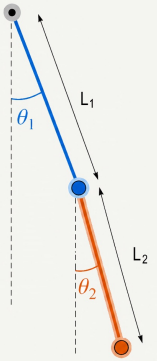
Prediction vs. individual trajectories MFR from the chaotic ensemble



A natural flip criterion and the saddle orbit







$E = 2.5$

- See two videos and website

The story of the three-body problem

arXiv:2510

Thank you for your attention!

Grazie per l'attenzione!