On Quasi-Bicircular Solutions to the general planar three body problem

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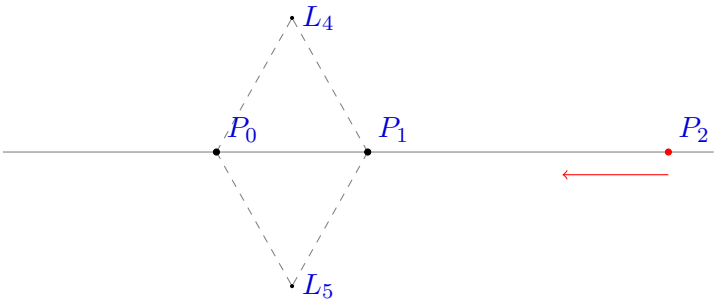
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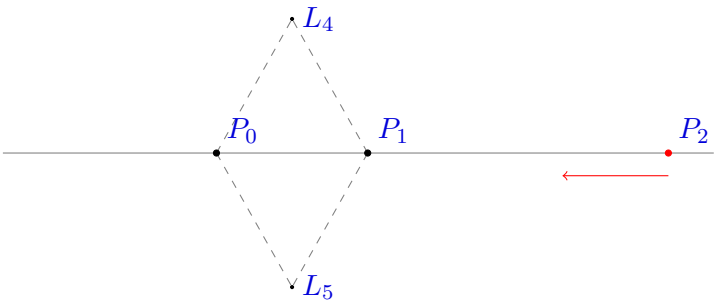
Motivations

How does the presence of a third primary body affect L_4 and L_5 Lagrangian points in the **Restricted Three Body Problem**?



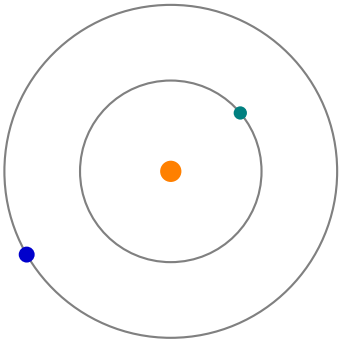
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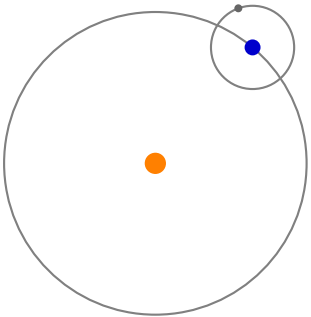


More generally, how to build **coherent** restricted four body problems?

Types of QBCP



Poincaré QBCP



Hill QBCP

Three-Body Problem (3BP)

We consider the general planar 3BP,

$$m_i \ddot{\mathbf{q}}_i = G \sum_{j \neq i} \frac{m_i m_j}{\|\mathbf{q}_{i,j}\|^3} \mathbf{q}_{i,j}, \quad i, j \in \{0, 1, 2\}$$

$\mathbf{q}_i \stackrel{\text{def}}{=} (x_i, y_i)$ and $\dot{\mathbf{q}}_i \stackrel{\text{def}}{=} (\dot{x}_i, \dot{y}_i)$ are the positions and velocities,

$\mathbf{q}_{i,j} \stackrel{\text{def}}{=} \mathbf{q}_j - \mathbf{q}_i$,

G is the gravitational constant.

Hamiltonian with six degrees of freedom.

Conserved Quantities in the TBP

Total energy:

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} - G\sum_{j\neq i}\frac{m_i m_j}{\|\mathbf{q}_{i,j}\|}$$

Center of mass:

$$\sum_{i=0}^{2} m_i \mathbf{q}_i = 0.$$

Total linear Momentum:

$$\sum_{i=0}^{2} \mathbf{p}_i = 0.$$

Total angular Momentum:

$$\mathbf{L} = \sum_{i=0}^{2} \mathbf{q}_i \times \mathbf{p}_i.$$

Periodic motion on the TBP

Absolute Periodic Orbits

Positions and velocities of all the bodies return to their initial values after a certain time, the period.

Relative Periodic Orbits

Relative positions and velocities of all the bodies w.r.t. the centre of masses are periodic functions.

Relative Periodic Orbits are said to be periodic on a **rotating frame**.

Let $\gamma \in \mathbb{R}^{12}$ denote a solution of TBP, then it is a relative periodic orbit if there exist a block-diagonal rotation $\mathbf{R}_\theta \in \mathbb{M}_{12}(\mathbb{R})$ of angle θ and a value $T > 0$ such that

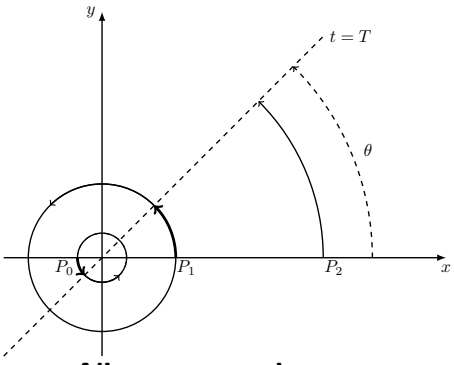
$$\gamma(t + T) = \mathbf{R}_\theta \gamma(t).$$

Relative Periodic Orbits of the TBP based on alignments

- We use an **inertial reference frame** and **coplanar** setup.
- The **origin is fixed** at the center of mass.
- Our RPO start with the 3 bodies aligned (*syzygy configuration*).
- The second alignment does not necessarily occur along the same initial line.
- At time T, the bodies will align again, at a line at an angle θ from the initial line.
- If a **Relative Periodic Orbits (RPO)** occurs, a $-\theta$ **planar rotation** restores the system to its initial configuration, maintaining the same alignment pattern.

Construction of Relative Periodic Orbits of the TBP - I

- Variables $P_j = (x_j, y_j, \dot{x}_j, \dot{y}_j)$ for $j = 0, 1, 2$.
- Full phase space is 12-dimensional.
- Center of mass fixed at the origin.
- One body (P_0) is deducible from the other.
- Reduced phase space is 8-dimensional.



Alignment section:

$$\Sigma = \{x_1 y_2 = x_2 y_1\}$$

(*Syzygy condition*)

Besides:

- Normalization $G = 1$ and $m_0 + m_1 + m_2 = 1$.
- Initial position at $\{y = 0\}$ and $x_1 = 1$.

Construction of Relative Periodic Orbits of the TBP - II

1. Given initial $(x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = (1, 0, 0, \dot{y}_1, x_2, 0, 0, \dot{y}_2)$ in Σ.
2. $(X_j, Y_j, \dot{X}_j, \dot{Y}_j)$ for $j = 1, 2$, TBP until second cut on Σ (output: flying time T).
3. Compute $\theta = \texttt{atan2}(Y_2, X_2)$.
4. Impose **equal relative configuration**:
   - 4.1 Same distance of P_1 to the CM: $\|(X_1, Y_1)\| = \|(x_1, y_1)\|$ $\quad \|(X_1, Y_1)\| = 1$
   - 4.2 Same P_1 velocity: $\|(\dot{X}_1, \dot{Y}_1)\| = \|(\dot{x}_1, \dot{y}_1)\|$ $\quad \|(\dot{X}_1, \dot{Y}_1)\| = |\dot{y}_1|$
   - 4.3 Same distance of P_2 to the CM: $\|(X_2, Y_2)\| = \|(x_2, y_2)\|$ $\quad \|(X_2, Y_2)\| = |x_2|$
   - 4.4 Same P_2 velocity: $\|(\dot{X}_2, \dot{Y}_2)\| = \|(\dot{x}_2, \dot{y}_2)\|$ $\quad \|(\dot{X}_2, \dot{Y}_2)\| = |\dot{y}_2|$
   - 4.5 Same angle of P_1 w.r.t. $\{y = 0\}$: $X_1\dot{X}_1 + Y_1\dot{Y}_1 = x_1\dot{x}_1 + y_1\dot{y}_1$ $\quad X_1\dot{X}_1 + Y_1\dot{Y}_1 = 0$
   - 4.6 Same angle of P_2 w.r.t. $\{y = 0\}$: $X_2\dot{X}_2 + Y_2\dot{Y}_2 = x_2\dot{x}_2 + y_2\dot{y}_2$ $\quad X_2\dot{X}_2 + Y_2\dot{Y}_2 = 0$

Stability of Relative Periodic Orbits of the TBP

Given a RPO γ with angle θ, we define

- **Rotated monodromy matrix** in the full phase space:

$$\mathcal{M} = \mathbf{R}_{-\theta}\, \mathrm{D}\, \phi_T(\gamma(0))$$

where $\mathbf{R}_\theta = \mathrm{diag}(\mathcal{R}_\theta, \ldots, \mathcal{R}_\theta) \in \mathbb{M}_{12}(\mathbb{R})$ and $\mathcal{R}_\theta \in \mathbb{M}_2(\mathbb{R})$.

- **Rotated monodromy matrix** in the reduced phase space:

$$\mathcal{M}_{\mathsf{red}} = \mathbf{R}_{-\theta}\, \mathrm{D}\, \phi_T(\mathbf{z})$$

where $\mathbf{z} = (\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2) \in \mathbb{R}^8$, $\mathbf{R}_\theta = \mathrm{diag}(\mathcal{R}_\theta, \ldots, \mathcal{R}_\theta) \in \mathbb{M}_8(\mathbb{R})$ and $\mathcal{R}_\theta \in \mathbb{M}_2(\mathbb{R})$.

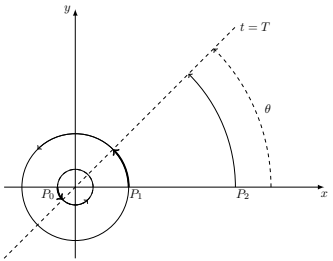
The **Reduced Monodromy matrix** has 8 eigenvalues, being 4 of them 1's.

Poincaré solutions

Poincaré $m_0 >> m_1, m_2$

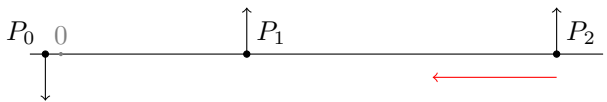
By considering:

- $m_0 >> m_1, m_2$,
- $m_0 + m_1 + m_2 = 1$, $G = 1$ and $x_1 = 1$.



Period of body 1 is 2π meanwhile body 1 describes a circular orbit.

Sense of the continuation, P_2 approaches P_1:



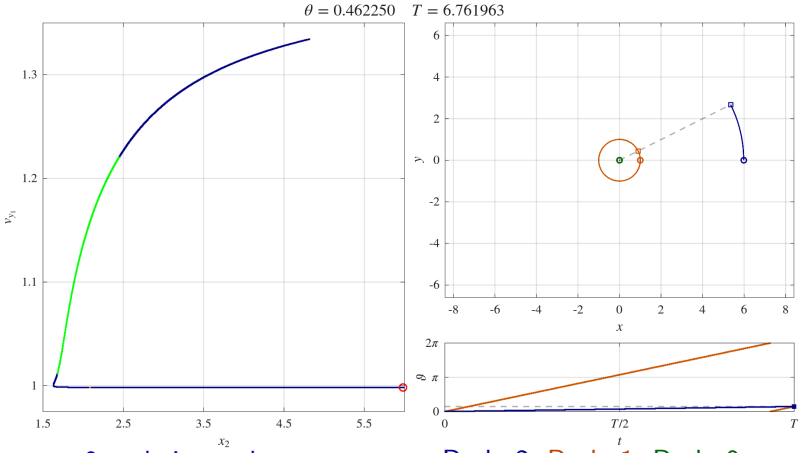
Poincaré example: $m_0 = 1 - m_1 - m_2$, $m_1 = 10^{-3}$, $m_2 = 10^{-4}$

Each point (x_2, v_{y_1})

is a

Relative Periodic Orbit

of the 3BP.



0 real eigenvalues

2 real eigenvalues

4 real eigenvalues

Body 2, Body 1, Body 0

$\vartheta(t)$: angular position at t.

$\vartheta(t = T) = \theta$

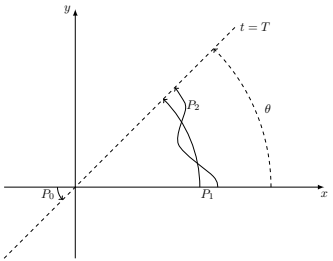
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Hill solutions

Hill Solutions $m_0 >> m_1 >> m_2$

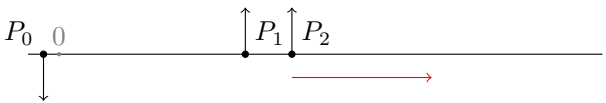
By considering:

- $m_0 >> m_1 >> m_2$,
- $m_0 + m_1 + m_2 = 1$, $G = 1$ and $x_1 = 1$.



Period of body 1 is 2π meanwhile body 1 describes a circular orbit.

Sense of the continuation, P_2 gets apart from P_1:



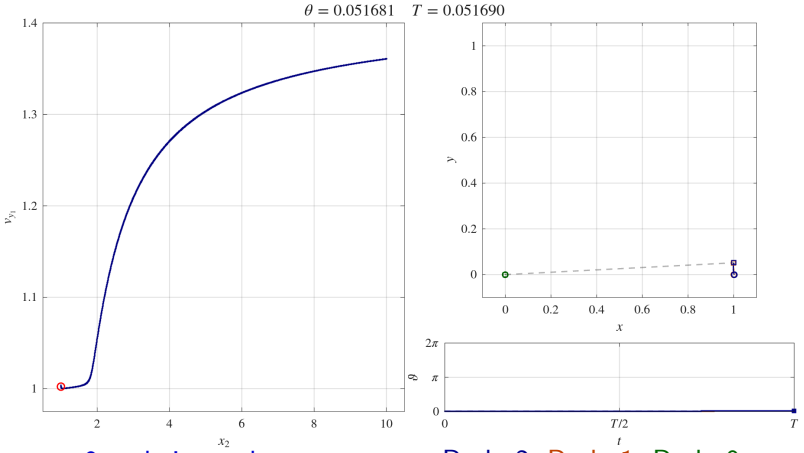
Hill example: $m_0 = 1 - m_1 - m_2$, $m_1 = 10^{-4}$, $m_2 = 10^{-6}$

Each point (x_2, v_{y_1})

is a

Relative Periodic Orbit

of the 3BP.



0 real eigenvalues

2 real eigenvalues

4 real eigenvalues

Body 2, Body 1, Body 0

$\vartheta(t)$: angular position at t.

$\vartheta(t = T) = \theta$

Click for video

Conclusions & Future work

- **Systematic construction** of coplanar RPO to explore planetary systems; inside or outside our solar system.
- Different families of RPO continued to reasonable limits.
- Solutions found features natural phenomena of many real systems; they can be used as **basis model** to their study.
- Find **bounded chaotic configuration**.
- Given a RPO, the **dynamical equivalents** of L_4 and L_5 is now being explored.



Àngel Jorba (1963-2025)



"No, you weren't downloaded.
You were born."

Jerry King

https://www.maia.ub.es/~angel/

Thank You for Your Attention!

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