Effective stability estimates close to resonances with applications to rotational dynamics

Alessandra Celletti, *University of Roma Tor Vergata*
Anargyros Dogkas, *University of Pisa*
Alessia Francesca Guido, *University of Roma Tor Vergata*

Outline

Consider a nearly-integrable Hamiltonian system

$$H(\underline{p},\underline{q}) = h(\underline{p}) + \varepsilon R(\underline{p},\underline{q}) \quad \text{with} \quad (\underline{p},\underline{q}) \in D \subseteq \mathbb{R}^n \times \mathbb{T}^n$$

- Nekhoroshev proved that the actions, under suitable applicability conditions, remain confined over a finite but exponentially long time:

$$||\underline{p}(t) - \underline{p}(0)|| \leq \varepsilon^b \quad \text{for} \quad |t| \leq e^{1/\varepsilon^a}, \quad \text{with } a, b > 0$$

The estimates are obtained by the construction of a suitable canonical transformation.

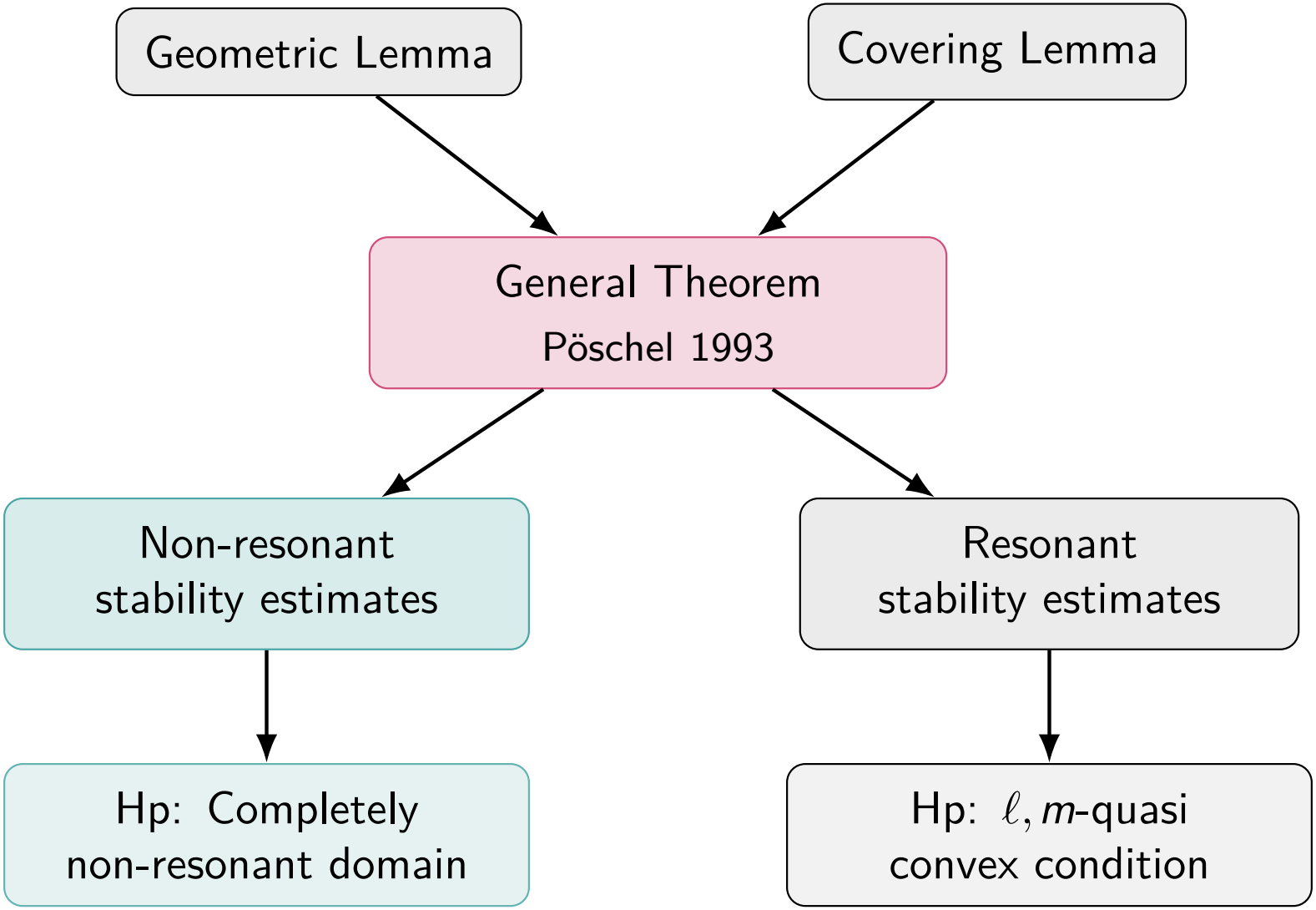
Several works focus on improving the exponential stability exponents, which determine the order of magnitude for both the stability time and the bounds on the actions.

Some references

Quasi-convex case Pöschel (1993); Lochak–Neishtadt (1992); Bounemoura–Marco (2011); Zhang (2017)

Steep case Guzzo–Chierchia–Benettin (2016)

Outline


Geometric Lemma
Covering Lemma
General Theorem
Pöschel 1993
Non-resonant
stability estimates
Resonant
stability estimates
Hp: Completely
non-resonant domain
Hp: ℓ, m-quasi
convex condition

Set up

Let $H(\underline{p},\underline{q}) = h(\underline{p}) + \varepsilon R(\underline{p},\underline{q})$ with $(\underline{p},\underline{q}) \in D \subseteq \mathbb{R}^n \times \mathbb{T}^n$.

- We define $\Pi D \subseteq \mathbb{R}^n$ open domain, the projection of D in the action space.
- We consider a complex extension of the domain: $V_{r_0}\Pi D \times W_{s_0}\mathbb{T}^n$ for $r_0, s_0 \in \mathbb{R}_+$.

- We introduce the norm:

$$\|R\|_{r_0,s_0} = \sup_{\underline{p}\in V_{r_0}\Pi D} \sum_{\underline{k}\in\mathbb{Z}^n_K} |R_{\underline{k}}(\underline{p})| \ \exp\left(\|\underline{k}\|_1 s_0\right)$$

  where $\mathbb{Z}^n_K = \{\underline{k} \in \mathbb{Z}^n : \|\underline{k}\|_1 \leq K\}$

- We suppose: $\left\|\varepsilon R(\underline{p},\underline{q})\right\|_{r_0,s_0} \leq E$ and $\sup_{\underline{p}\in V_{r_0}\Pi D} \left\|\partial^2_{\underline{p}} h(\underline{p})\right\|_2 \leq M$

  (E and M are upper bounds on the norms of $\varepsilon R(\underline{p},\underline{q})$ and the Hessian of $h(\underline{p})$)

Definitions

Definition 1: resonant frequency

The frequency vector $\underline{\omega}(\underline{p})$, with $\underline{\omega}(\underline{p}) = \nabla_{\underline{p}} h(\underline{p})$, satisfies a resonance of order $\underline{k} \in \mathbb{Z}^n \setminus \{\underline{0}\}$, if

$$\underline{k} \cdot \underline{\omega}(\underline{p}) = 0 \ .$$

Definition 2: α, K non- resonant condition

A domain $D \subseteq \mathbb{R}^n \times \mathbb{T}^n$ is called α, K non-resonant modulo Λ, for $\Lambda \subseteq \mathbb{Z}^n_K \setminus \{\underline{\mathbf{0}}\}$, if

$$\left|\underline{k} \cdot \underline{\omega}(\underline{p})\right| \geq \alpha \quad \forall \underline{k} \in \mathbb{Z}^n_K \setminus \{\Lambda \cup \underline{0}\} \ , \quad (\underline{p}, \underline{q}) \in D$$

for some $\alpha \in \mathbb{R}_+$. If $\Lambda = \emptyset$, then D is a completely α, K non-resonant domain.

Definition 3: Diophantine condition

A frequency vector $\underline{\omega}(\underline{p})$ is said to satisfy the Diophantine condition if

$$\left|\underline{k} \cdot \underline{\omega}(\underline{p})\right| \geq \frac{C}{\|k\|_1^{\tau}} \quad \forall \ \underline{k} \in \mathbb{Z}^n \setminus \{\underline{0}\}$$

for some constants $C \in \mathbb{R}_+$ and $\tau \geq n-1$.

Non Resonant Stability Estimate

Theorem (Poeschel 1993)

Suppose the domain D completely α, K-non-resonant.
For some positive parameters j and ℓ, such that $1/j + 1/\ell = 1$, and r such that

$$r \leq \min\left(\frac{\alpha}{M K^j}, r_0\right) ,$$

let E be bounded by

$$\|R\|_{r_0,s_0} \leq E \leq \frac{1}{2^7 \ell} \frac{\alpha r}{K} .$$

Then, for any initial condition $(\underline{p}_0, \underline{q}_0) \in D$, the evolution of the system in the action space is bounded as

$$\|\underline{p}(t) - \underline{p}_0\| \leq r \qquad \text{for all} \quad |t| \leq T \equiv \frac{s_0 r}{5E} \exp\left(\frac{K s_0}{6}\right) .$$

This theorem provides explicit estimates on the parameters ensuring the stability.

These parameters $(r_0, s_0, M, K, \alpha, E, \ell)$ must satisfy certain applicability conditions. Since these conditions allow some freedom in parameter choice, we propose an optimization algorithm to maximize the stability time.

Optimization of Non Resonant Stability Estimates

For a given initial condition $(\underline{p}_0, \underline{q}_0)$ we aim to choose the parameters that maximize the stability time. The conditions to be satisfied are:

$$\begin{cases} E_{min} \equiv \|R(\underline{p}_0, \underline{q}_0)\|_{r_0,s_0} \leq E \leq \dfrac{1}{2^7 \ell} \dfrac{\alpha\, r}{K} \\ \alpha \leq \alpha_{max} \equiv \inf\limits_{\underline{k} \in \mathbb{Z}^n_K} |\underline{k} \cdot \underline{\omega}(\underline{p}_0)| \\ M \geq M_{min} \equiv \sup\limits_{\|\underline{p}-\underline{p}_0\| \leq r_0} \left(\|\partial^2_{\underline{p}} h(\underline{p})\|_2 \right) \\ r \leq r_{max} \equiv \min\left(\dfrac{\alpha}{MK} \left(1 - \dfrac{1}{\ell}\right), r_0 \right) \\ K \geq \sup_{\underline{k} \in \Lambda_R} (\|\underline{k}\|_1) = K_{min} \end{cases}$$

Choice of arbitrary parameters

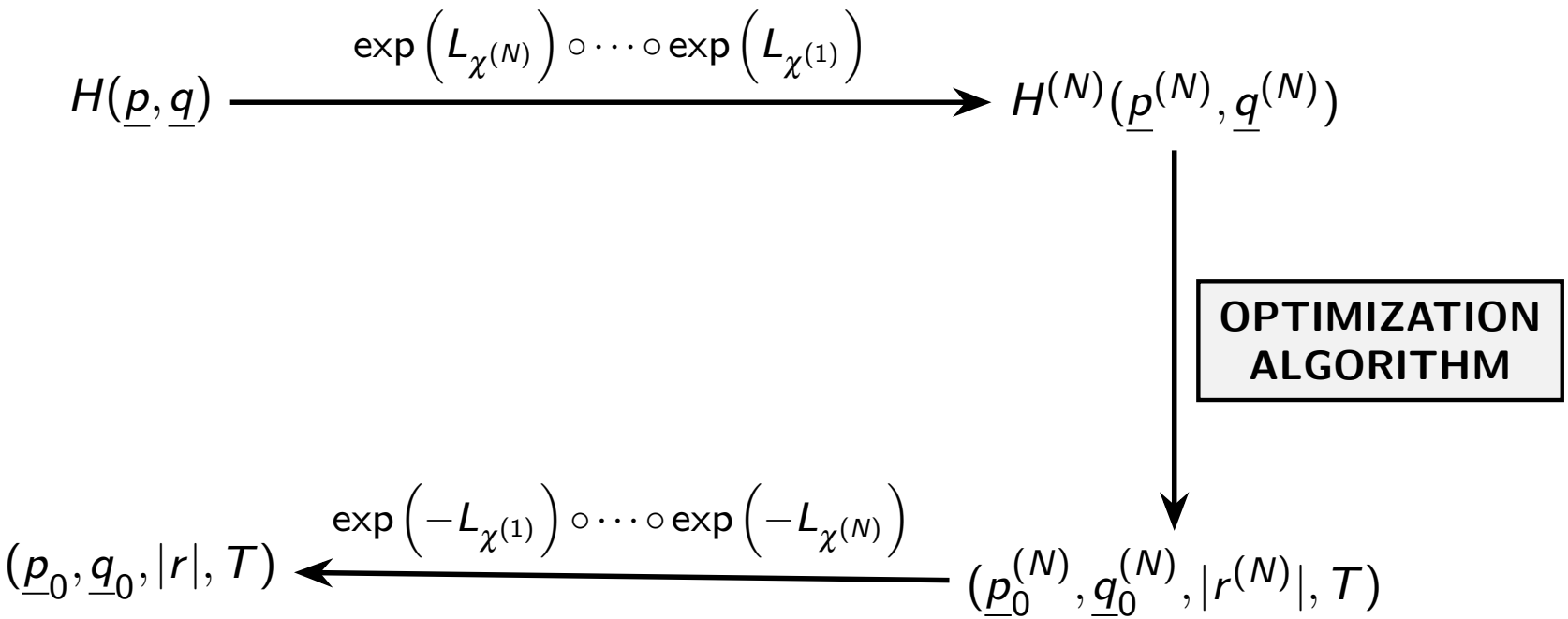
$$(r_0, s_0, M, K, \alpha, E, \ell)$$
$$\downarrow$$
$$E = E_{min} \text{ and } r = r_{max},$$
$$\downarrow$$
$$M = M_{min} \text{ and } \alpha = \alpha_{max}$$
$$\downarrow$$
$$(r_0, s_0, K, \ell)$$

such that $T \equiv \dfrac{s_0 r}{5E} \exp\left(\dfrac{K s_0}{6}\right)$ is maximized.

Techniques: Implementation of Perturbation Theory

Perturbation Theory

It is convenient to reduce the norm of the perturbing function by implementing **perturbation theory**, based on the construction of near-identity canonical transformations defined by the generating functions $\{\chi^{(i)}\}_{i=1:N}$.



Techniques: Sequences of Diophantine frequencies

Non-autonomous 1D system: $\underline{\omega}(\underline{p}) = (\omega_1(p_1), 1)$

Close to a resonance of order (k_1, k_2), we construct sequences of irrational frequencies that converge to the resonant frequency $\omega_1 = -k_2/k_1$:

$$\Gamma_{z,s}^{(k_1,k_2)} = -\frac{k_2}{k_1} - \frac{s}{z+\gamma} \xrightarrow{z\to\infty} (-\frac{k_2}{k_1})^- \ , \qquad \Delta_{z,s}^{(k_1,k_2)} = -\frac{k_2}{k_1} + \frac{s}{z+\gamma} \xrightarrow{z\to\infty} (-\frac{k_2}{k_1})^+$$

for $z \in \mathbb{Z}\backslash\{0\}$, $s \in \mathbb{R}_+$, γ being the golden ratio, $\gamma = (\sqrt{5}-1)/2$

Proposition

The numbers $\Gamma_{z,s}^{(k_1,k_2)}$, $\Delta_{z,s}^{(k_1,k_2)}$ satisfy the inequality

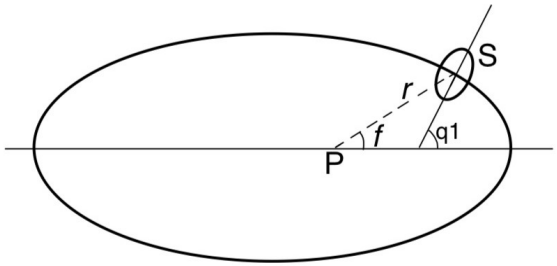
$$|k_1\omega_1 + k_2| \geq \frac{C}{|k_1|}, \quad \forall (k_1, k_2) \in \mathbb{Z}^2\backslash\{\underline{0}\} \ .$$

From Liouville Theorem: an irrational algebraic number ω_1 of degree 2 satisfies the Diophantine condition for some positive constant C and for $\tau = 1$.

Conservative Spin orbit model

Assumptions

- Triaxal satellite S
- Spin axis $\perp$ to the orbital plane
- S moves on a Keplerian orbit around a central planet P



1D time dependent Hamiltonian

$$H(q_1, p_1, t) = \frac{p_1^2}{2} - \frac{\varepsilon}{2}\frac{a^3}{r(t)^3}\cos(2q_1 - 2f(t))$$

with $\varepsilon = \frac{3}{2}(I_b - I_a)/I_c$, $\underline{\omega} = (\omega_1, 1) = (p_1, 1)$

Spin orbit resonance of type $k_2 : k_1$

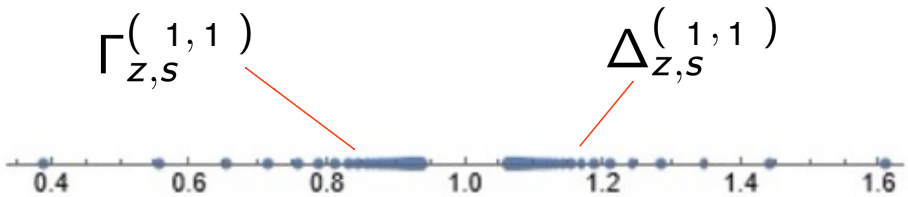
A spin-orbit resonance of type $k_2 : k_1$ occurs if (Definition 1):

$$\underline{\omega} \cdot \underline{k} = k_1 \omega_1 - k_2 = 0$$

which implies that the body S_1 makes k_1 rotations within k_2 orbital revolutions.

• We want to approach a specific resonance without intersecting any other resonances:

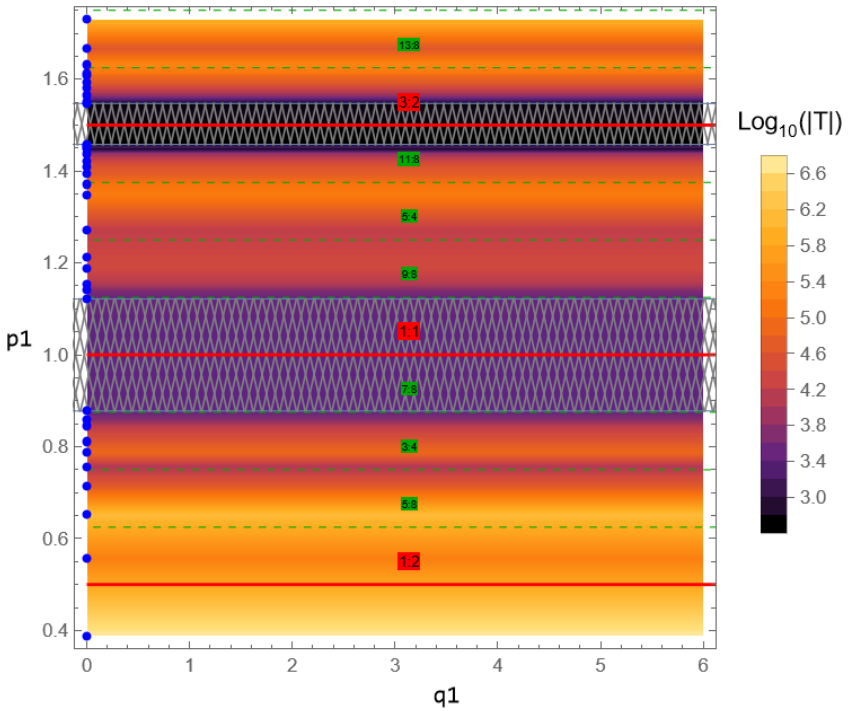
Sequences of Diophantine frequencies $\rightarrow$ Resonant frequency k_2/k_1



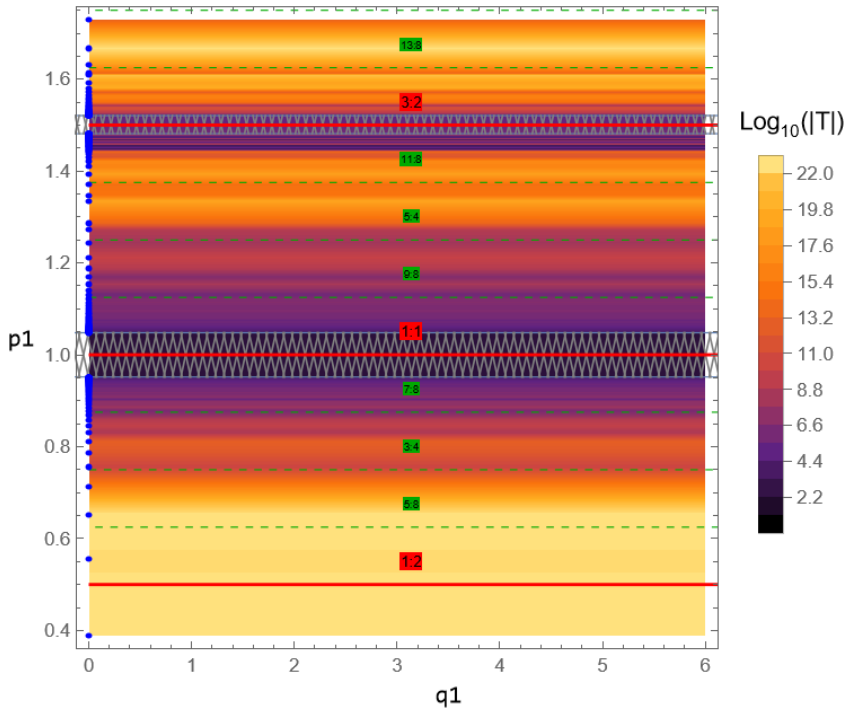
Results

We consider as initial conditions $(\underline{p}(0), \underline{q}(0)) = (p_1(0), 0, 0, 0)$ such that

$\underline{\omega}(p_1(0))_z = (\Gamma_{z,s}^{(k_1,k_2)}, 1)$ or $\underline{\omega}(p_1(0))_z = (\Delta_{z,s}^{(k_1,k_2)}, 1)$

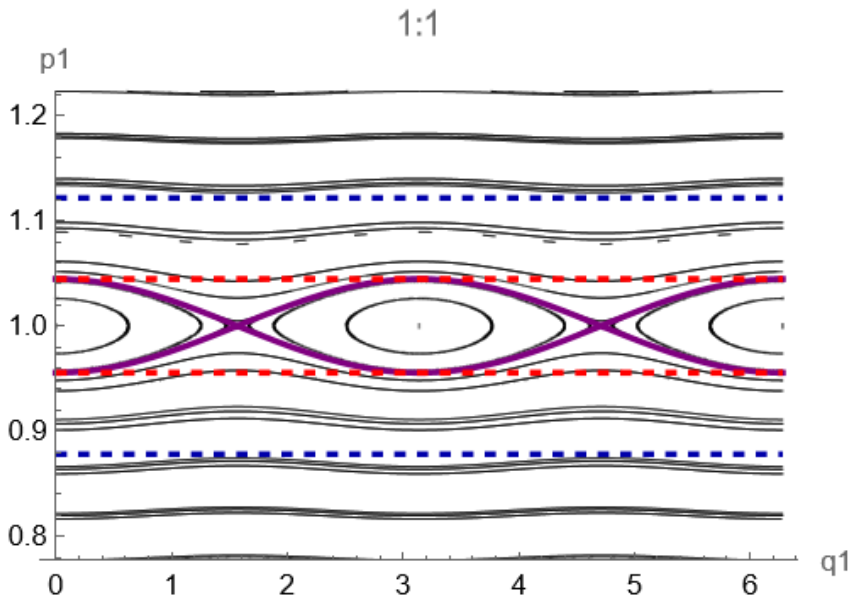


a)

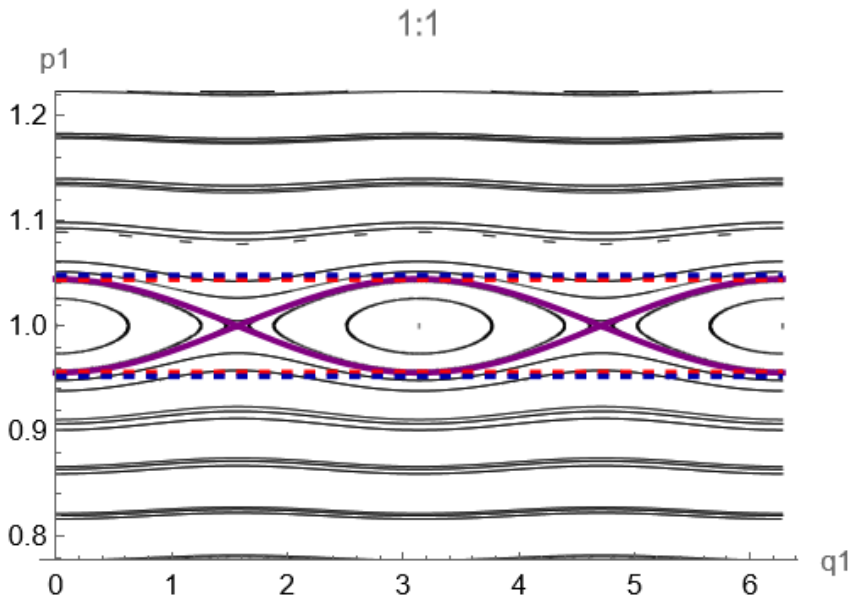


b)

Stability time for $z = 2, \ldots, 100$ with $s = 1.6$ $(1:1$ resonance) and $s = 0.6$ $(3:2$ resonance) for (a) $\varepsilon = 10^{-3}$ after $N = 2$ perturbative steps, (b) $\varepsilon = 10^{-3}$ after $N = 3$ perturbative steps.



a)



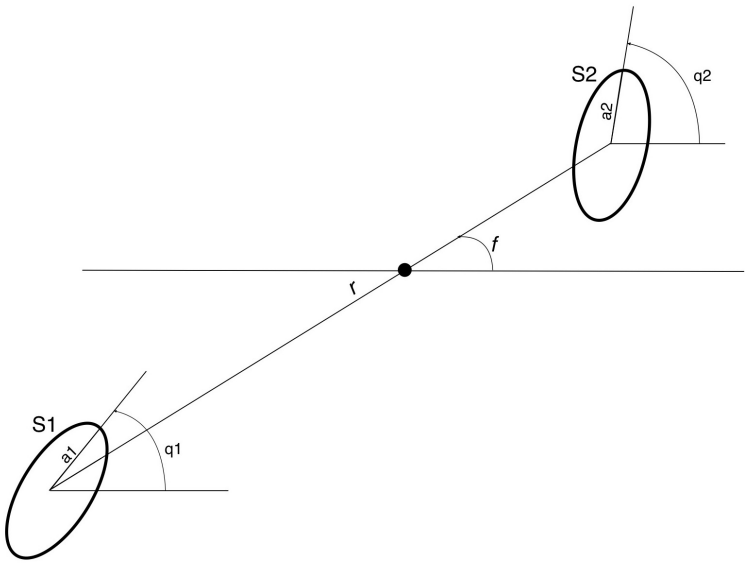
b)

The Comparison between the last value of p_1 for which the algorithm provides results (blue line) and a resonant normal form computation of the width of the 1:1 resonance (red line) for (a) $\varepsilon = 10^{-3}$ after $N = 2$ perturbative steps, (b) $\varepsilon = 10^{-3}$ after $N = 3$ perturbative steps.

Conservative Spin Spin orbit model

Assumptions

- Triaxal satellites S_1, S_2
- Respective spin axes $\perp$ to the orbital plane
- S_1, S_2 move around each other on Keplerian orbits around their barycenter



2D time-dependent Hamiltonian

$$H(q_1,q_2,p_1,p_2,t)=\frac{p_1^2}{2I_c^{(1)}}+\frac{p_2^2}{2I_c^{(2)}}+\underbrace{V_1(r,f,q_1,q_2)}_{\text{Spin-orbit terms}}+\underbrace{V_2(r,f,q_1,q_2)}_{\text{Spin-spin-orbit terms}}$$

$$V_1=-\frac{m}{r^3(t)}\sum_{k=1,2}\frac{\varepsilon_k I_c^{(k)}}{2M_{S_k}}\cos\big(2q_k-2f(t)\big)$$

$$-\frac{m}{r^5(t)}\sum_{k=1,2}\left(\frac{\varepsilon_k I_c^{(k)}}{M_{S_k}}\right)^2\left(\frac{5}{112}+\frac{25}{48}\cos(4q_k-4f(t))\right)$$

$$V_2=-\frac{m}{r^5(t)}\left(\frac{\varepsilon_1 I_c^{(1)}}{M_{S_1}}\right)\left(\frac{\varepsilon_2 I_c^{(2)}}{M_{S_2}}\right)\left(\frac{35}{24}\cos\big(4f(t)-2q_1-2q_2\big)+\frac{1}{8}\cos(2q_1-2q_2)\right)$$

with $\varepsilon_k=\dfrac{3}{2}\dfrac{I_b^{(k)}-I_a^{(k)}}{I_c^{(k)}}$, $m=GM_{S_1}M_{S_2}$, and $\{I_c^{(k)}\}_{k=1,2}$ the principal moments of inertia.

Units

$$M_{S_1}+M_{S_2}=1,\quad I_c^{(1)}+I_c^{(2)}=1,\quad G=a^3.$$

Techniques: Sequences of Diophantine frequencies

Non-autonomous 2D system: $\underline{\omega}(\underline{p}) = (\omega_1(\underline{p}), \omega_2(\underline{p}), 1)$

Definition

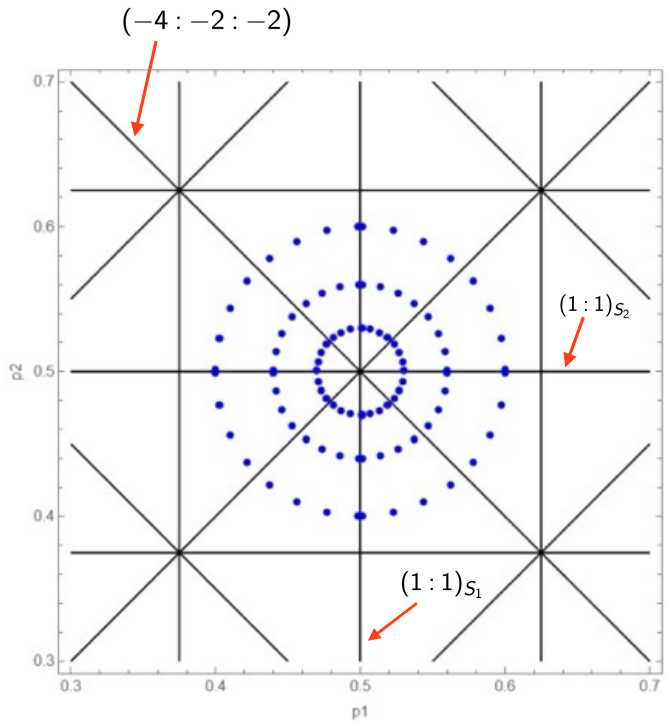
A spin-spin-orbit resonance of type $k_3 : k_1 : k_2$ occurs when (Definition 1):

$$\underline{\omega} \cdot \underline{k} = 0 \qquad \text{with} \quad \underline{\omega} = (\omega_1, \omega_2, 1) \quad \text{and} \quad \underline{k} = (k_1, k_2, -k_3)$$

Definition

A spin-orbit/spin-orbit resonance of type $(k_{3,1} : k_1)_{S_1}$, $(k_{3,2} : k_2)_{S_2}$ is an intersection between the spin-spin-orbit resonances of types $k_{3,1} : k_1 : 0$ and $k_{3,2} : 0 : k_2$:

$$k_1 \omega_1 - k_{3,1} = 0 \ , \qquad k_2 \omega_2 - k_{3,2} = 0$$



Sequence of Diophantine frequencies

In the general n-dimensional case, we define the vector $\underline{\omega} = (\omega_1, \ldots, \omega_{n-1}, 1) \in \mathbb{R}^n$ such that:

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 & & & \\ \vdots & & \mathscr{A} & \\ b_{n-1} & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{n-1} \end{pmatrix}, \tag{1}$$

where $(b_1, \ldots, b_{n-1}) \in \mathbb{R}^n$, the $(n-1)\times(n-1)$ dimensional matrix $\mathscr{A}$ has rational coefficients a_j with $\det \mathscr{A} \neq 0$, and α is a real algebraic number of degree n. The vectors defined by (1) satisfy the Diophantine condition (Definition 3).

$n = 3$ encompasses the case of a 2D non-autonomous system.

Diophantine frequencies for 2D time-dependent Hamiltonian

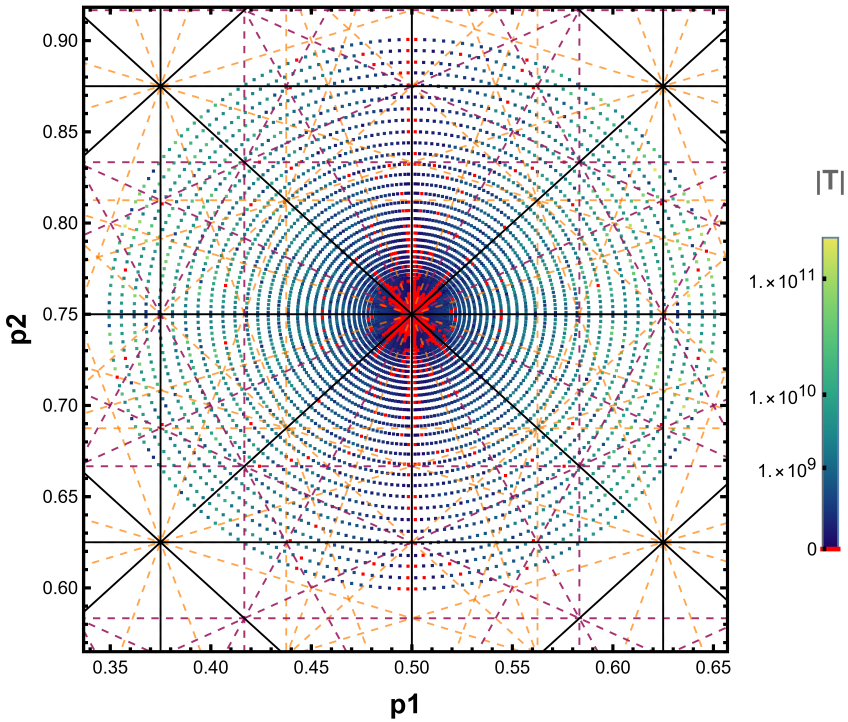
Given a spin-orbit/spin-orbit resonance of type $(k_{3,1} : k_1)_{S_1}, (k_{3,2} : k_2)_{S_2}$:

$$\omega_{1,z}^{(k_1,k_{3,1})} = \frac{k_{3,1}}{k_1} \pm \left(\frac{\tilde{a}_1}{z}\alpha + \frac{\tilde{a}_2}{z}\alpha^2 \right) \xrightarrow{z\to\infty} \left(\frac{k_{3,1}}{k_1} \right)^{\pm},$$

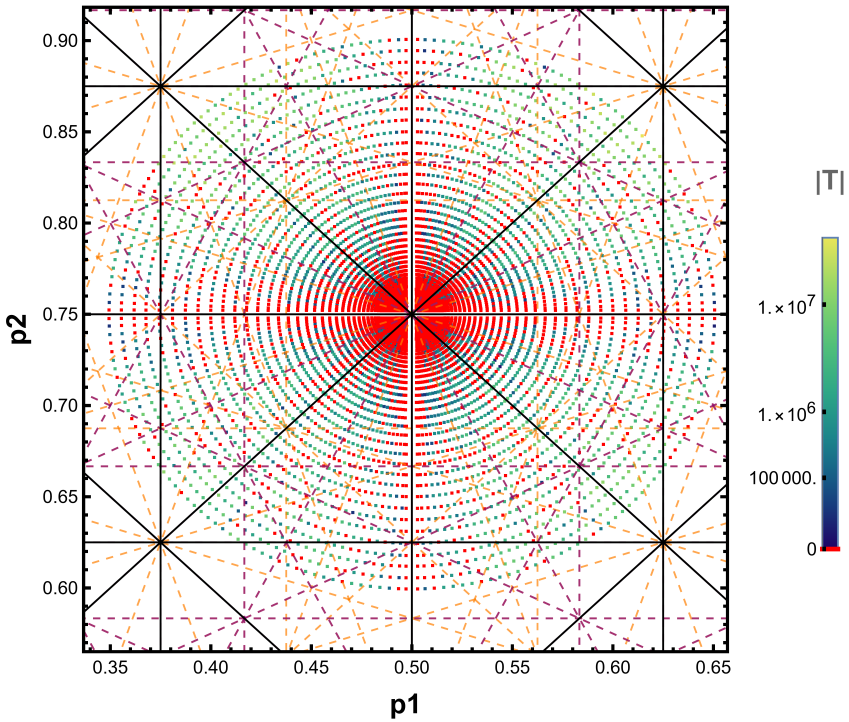
$$\omega_{2,z}^{(k_2,k_{3,2})} = \frac{k_{3,2}}{k_2} \pm \left(\frac{\tilde{a}_3}{z}\alpha + \frac{\tilde{a}_4}{z}\alpha^2 \right) \xrightarrow{z\to\infty} \left(\frac{k_{3,2}}{k_2} \right)^{\pm},$$

Results

We consider as initial conditions $(\underline{p}(0), \underline{q}(0)) = (p_1(0), p_2(0), 0, 0)$ such that $\underline{\omega}(p_1(0), p_2(0))_z = (\omega_{1,z}^{(k_1, k_{3,1})}, \omega_{2,z}^{(k_2, k_{3,2})}, 1)$



a)



b)

Stability time around the spin-spin-orbit resonance $(1:1)_{S_1}, (3:2)_{S_2}$, (a) for $\varepsilon_1 = \varepsilon_2 = 10^{-5}$ after two perturbative steps and (b) for $\varepsilon_1 = 3 \cdot 10^{-5}$ and $\varepsilon_2 = 10^{-4}$ after two perturbative steps.

Conclusions and future works

Method and Applications

- Effective stability estimates near resonances, based on the non-resonant formulation of Nekhoroshev's theorem given in Poeschel.
- Optimization algorithm to maximize the stability time, combined with a reduction of the norm of the perturbing function via perturbation theory.
- Two model problems in rotational dynamics, the spin-orbit and the spin-spin-orbit problems, with long stability times computed on a grid of points along the Diophantine sequences.

Future works

- Implement a similar method for the resonant case.
- Extend and optimize the non-resonant method for observed binary systems.

Reference

Celletti, A., Dogkas, A., Guido, A.F. Effective stability estimates close to resonances with applications to rotational dynamics. Nonlinearity, 2026.