ADAPTIVE COMPUTATION OF INVARIANT TORI ACROSS ASTRODYNAMICAL MODELS



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OUTLINE

- INTRODUCTION
  - Astrodynamical models
  - Invariant tori
- METHODOLOGY
  - Computation
  - Continuation
- RESULTS
  - Internal continuation
  - External continuation



Credits: NASA

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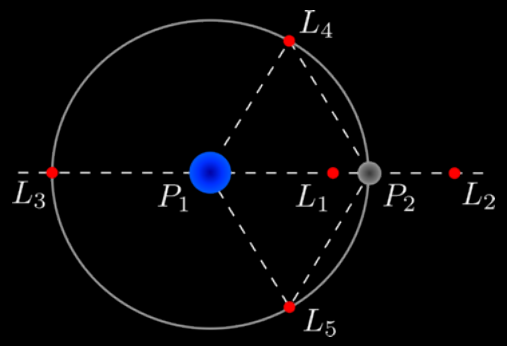


INTRODUCTION

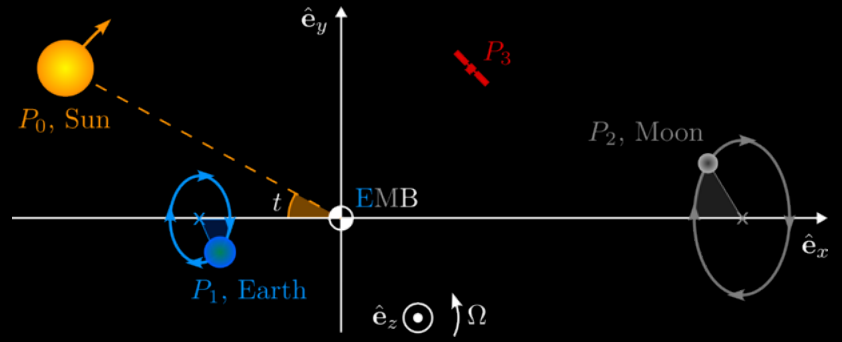
INTRODUCTION

CONTEXT

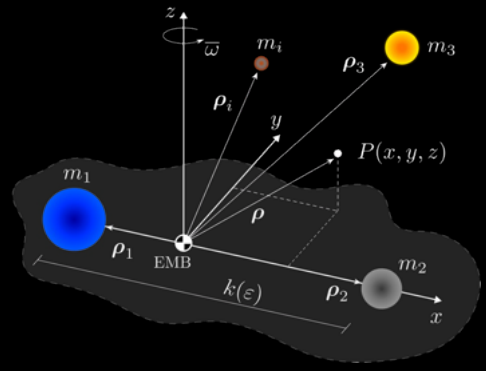
- As space research advances, space missions face ever higher standards in complexity and scientific return
- Such missions demand meticulously designed trajectories, custom-fit to the mission itself
- The astrodynamical model choice is fundamental, especially in multibody regimes (e.g., Sun–Earth–Moon system)



Low-fidelity models



Mid- and high-fidelity models



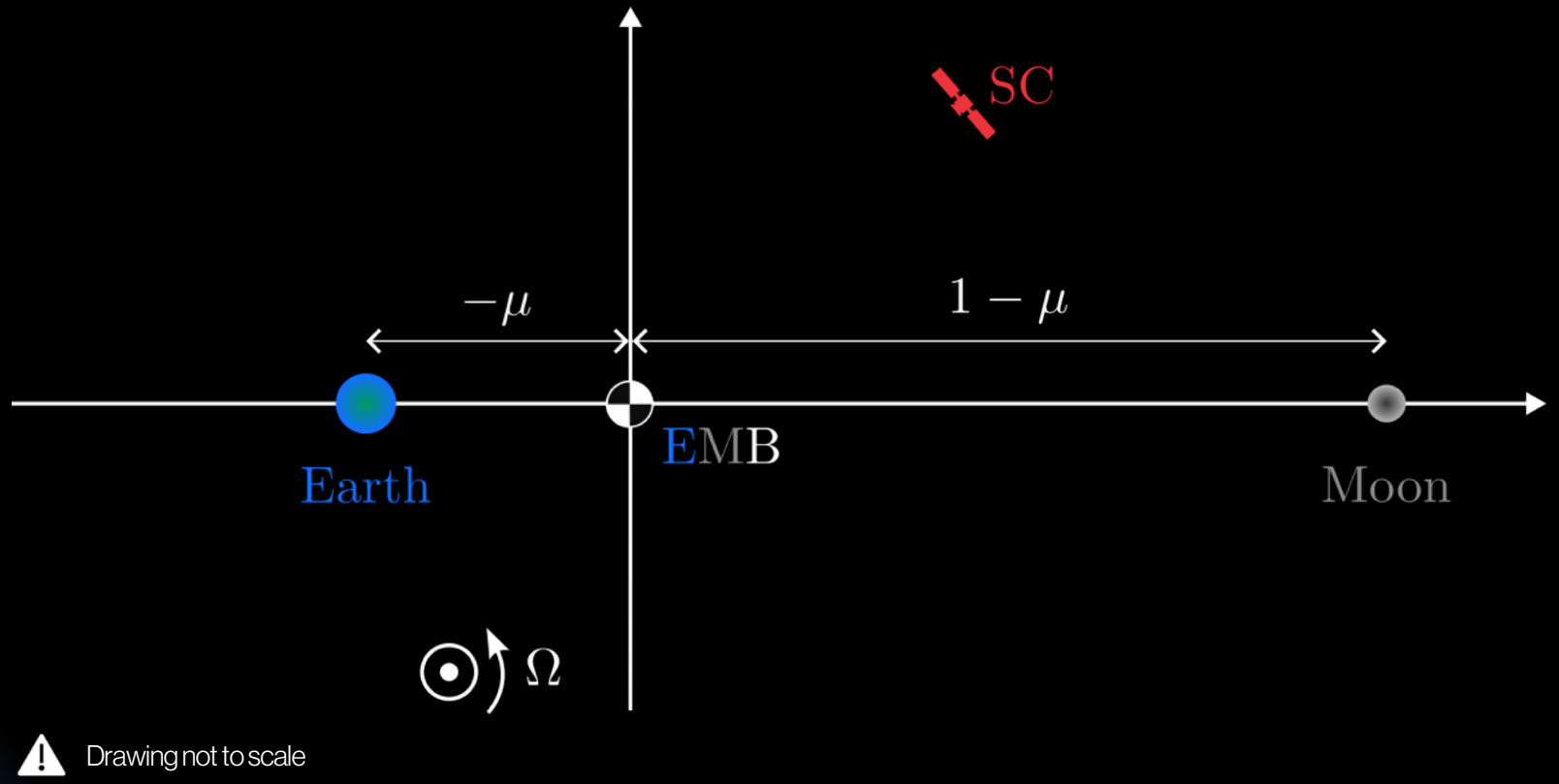
Realistic models

Dynamical content richness +

+ Simplicity of use and understanding

DYNAMICAL MODELS (1/3)

EARTH–MOON CIRCULAR RESTRICTED 3-BODY PROBLEM



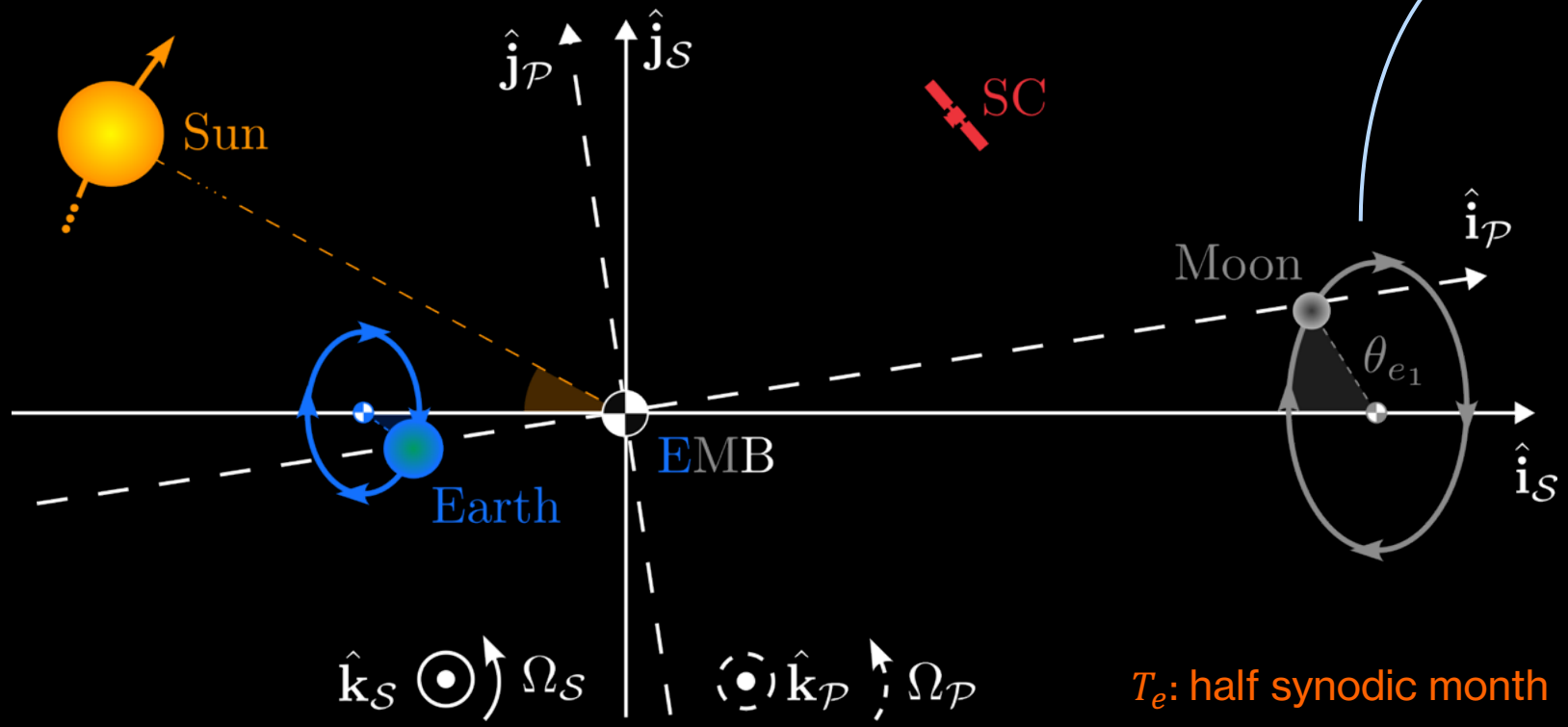
⚠ Drawing not to scale

CR3BP

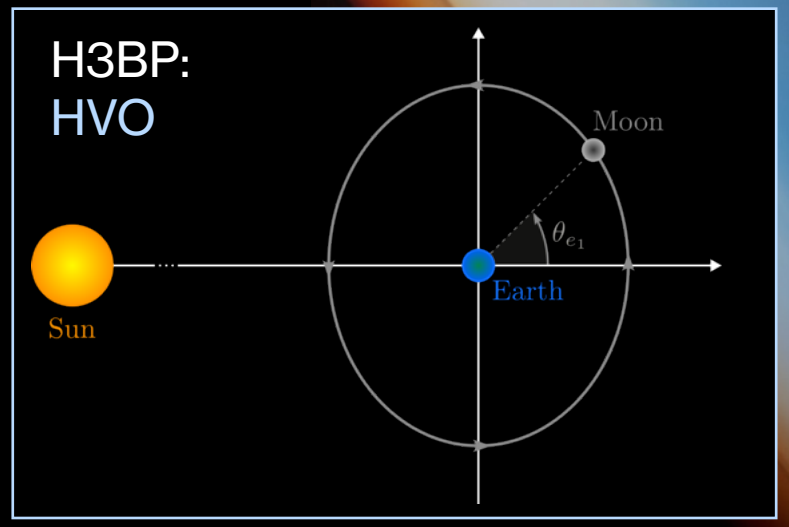
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

DYNAMICAL MODELS (2/3)

SUN–EARTH–MOON HILL RESTRICTED 4-BODY PROBLEM



Drawing not to scale



HR4BP

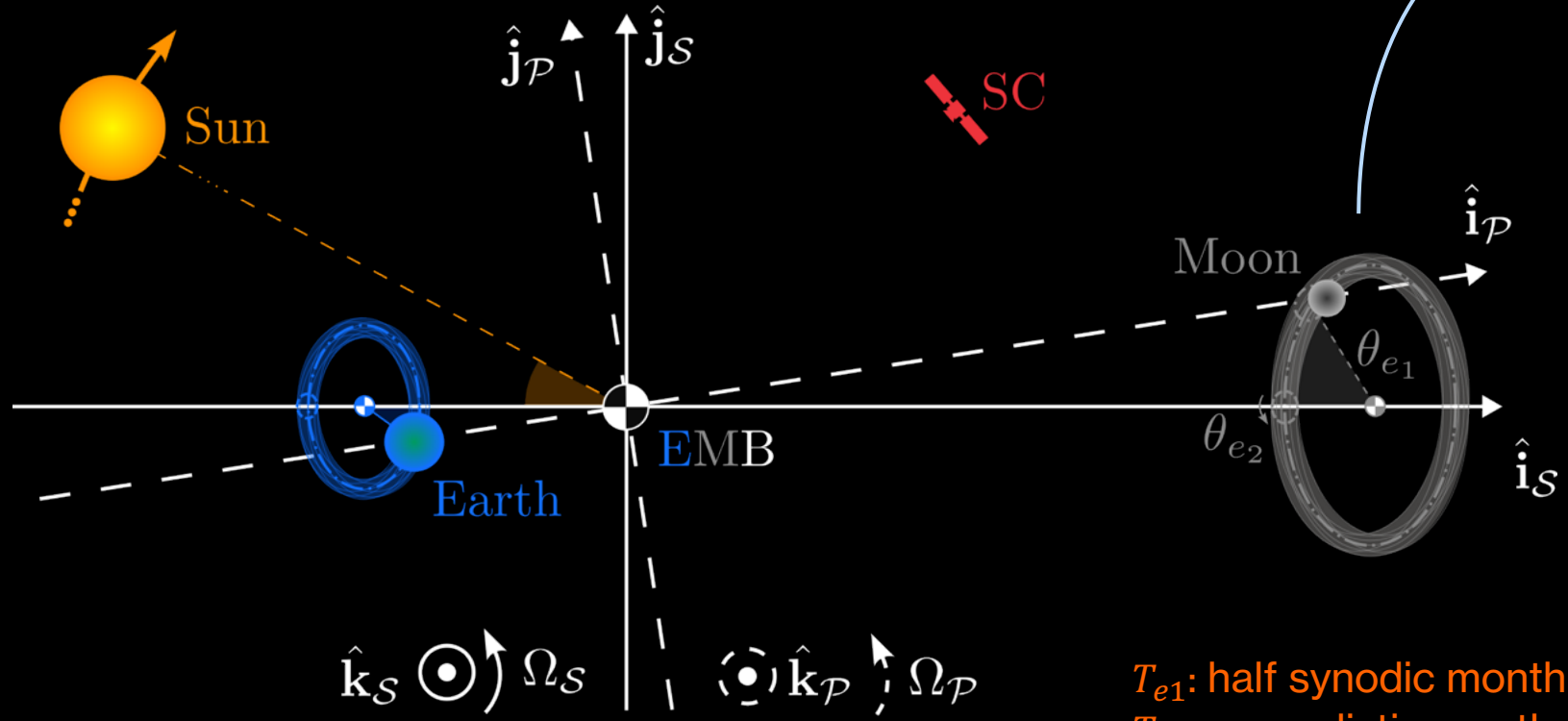
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta_{e_1})$$

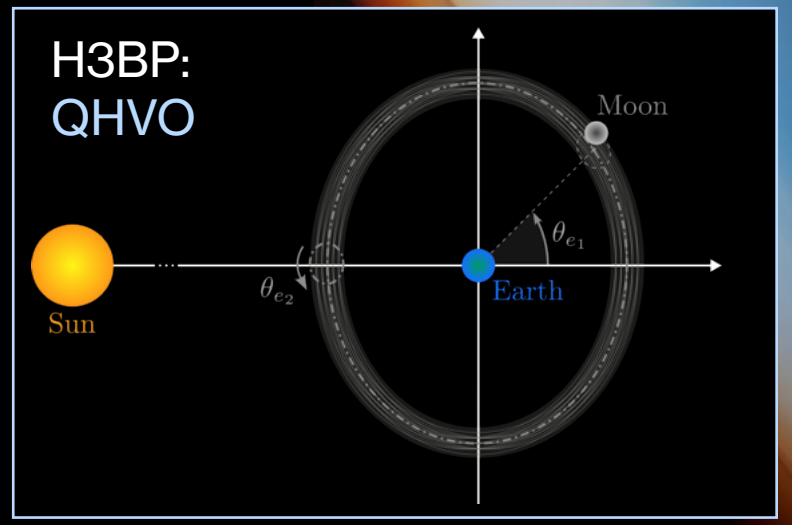
T_e: half synodic month

Scheeres: The Restricted Hill Four-body Problem With Applications To The Earth–Moon–Sun System (1998)

DYNAMICAL MODELS (3/3)

SUN–EARTH–MOON ELLIPTIC HILL RESTRICTED 4-BODY PROBLEM





HR4BP

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta_{e_1})$$

EHR4BP

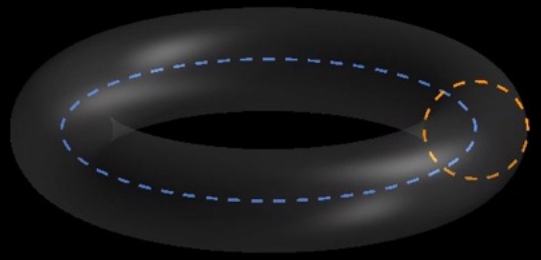
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta_{e_1}, \theta_{e_2})$$

T_{e1}: half synodic month
T_{e2}: anomalistic month

Park et al.: A frequency-based hierarchy of dynamical models in cislunar space: leveraging periodically and quasi-periodically perturbed models (2025)

INVARIANT TORI

INTRODUCTION

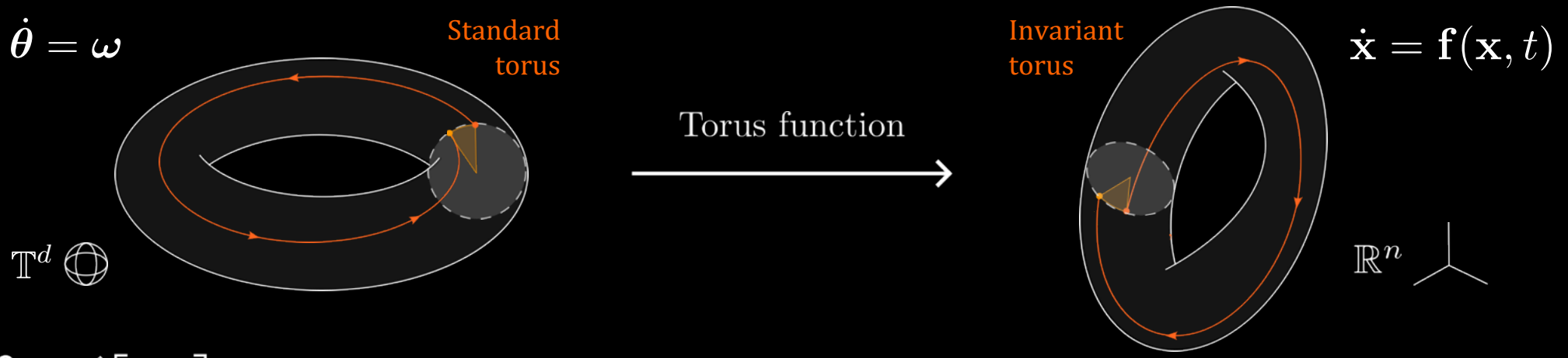


Let $\mathcal{M} \subset \mathbb{R}^n$ be a manifold and let $\varphi_t(\mathbf{x})$ denote the flow of a dynamical system. Then, $\mathcal{M}$ is an **invariant manifold** if

$$\forall\, \mathbf{x} \in \mathcal{M} \quad \Rightarrow \quad \varphi_t(\mathbf{x}) \in \mathcal{M}, \quad \forall\, t \in \mathbb{R}.$$

An invariant manifold $\mathcal{T}$ is called a *D*-**dimensional invariant torus** if the flow on $\mathcal{T}$ is diffeomorphic to the flow induced by $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}$ on the torus. That is, there exists a diffeomorphism $\boldsymbol{\tau} : \mathbb{T}^D \to \mathcal{T} \subset \mathbb{R}^n$:

$$\varphi_t(\boldsymbol{\tau}(\boldsymbol{\theta})) = \boldsymbol{\tau}(\boldsymbol{\theta} + \boldsymbol{\omega} t)$$



INVARIANT TORI

DIMENSIONS

- Hamiltonian system with N DOFs and D_e periodic perturbations
- Torus admissible dimensions: $D_e \leq D \leq D_e + \mathrm{N}$

$$D = D_e + D_i$$

Torus dimension: number of angles necessary to parametrize the manifold

Number of external frequencies (non resonant)

Number of internal frequencies

Invariant tori of internal dimension D_i generally lie in families of dimension D_i



INVARIANT TORI

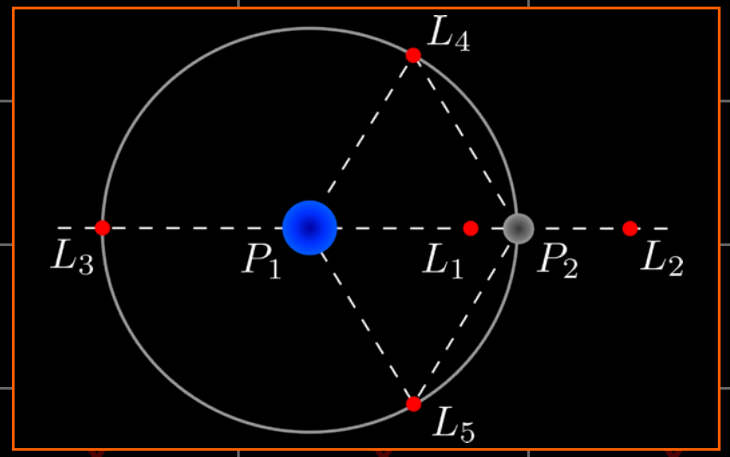
EXISTENCE

D_i internal dim. $D_i = D - D_e$		D torus dimension					
		0 equilibrium point	**1** periodic orbit	**2** quasi-periodic orbit	**3** 3D quasi-periodic orbit	**4** 4D quasi-periodic orbit	**5** ...
D_e external dim.	**0** CR3BP, H3BP	0	1	2	3	X	X
	1 HR4BP	X	0	1	2	3	X
	2 EHR4BP	X	X	0	1	2	3
	...	X	X	X	0	1	2
	N RPRnBP	X	X	X	X	X	X

INVARIANT TORI

EXISTENCE

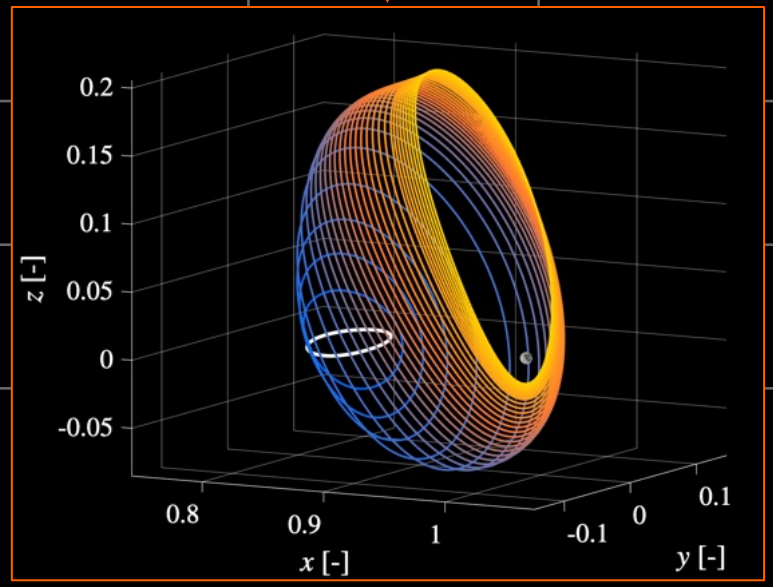
D_i internal dim. $D_i = D - D_e$	D torus dimension: **0** equilibrium point	**1** periodic orbit	**2** quasi-periodic orbit	**3** 3D quasi-periodic orbit	**4** 4D quasi-periodic orbit	**5** ...
D_e external dim. **0** CR3BP, H3BP	0	1	2	3	X	X
1 HR4BP				2	3	X
2 EHR4BP				1	2	3
...				0	1	2
N RPRnBP	X	X	X	X	X	X



INVARIANT TORI

EXISTENCE

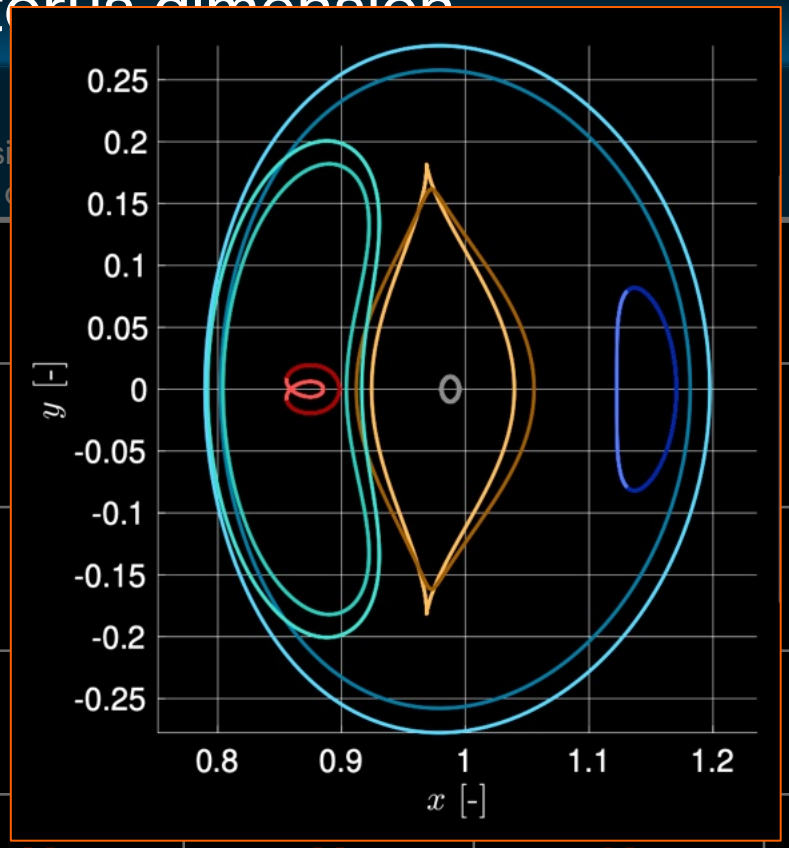
D_i internal dim. $D_i = D - D_e$	D torus dimension					
	0 equilibrium point	**1** periodic orbit	**2** quasi-periodic orbit	**3** 3D quasi-periodic orbit	**4** 4D quasi-periodic orbit	**5** ...
D_e external dim. **0** CR3BP, H3BP	0	1	2	3	X	X
1 HR4BP				2	3	X
2 EHR4BP				1	2	3
...				0	1	2
N RPRnBP				X	X	X



INVARIANT TORI

EXISTENCE

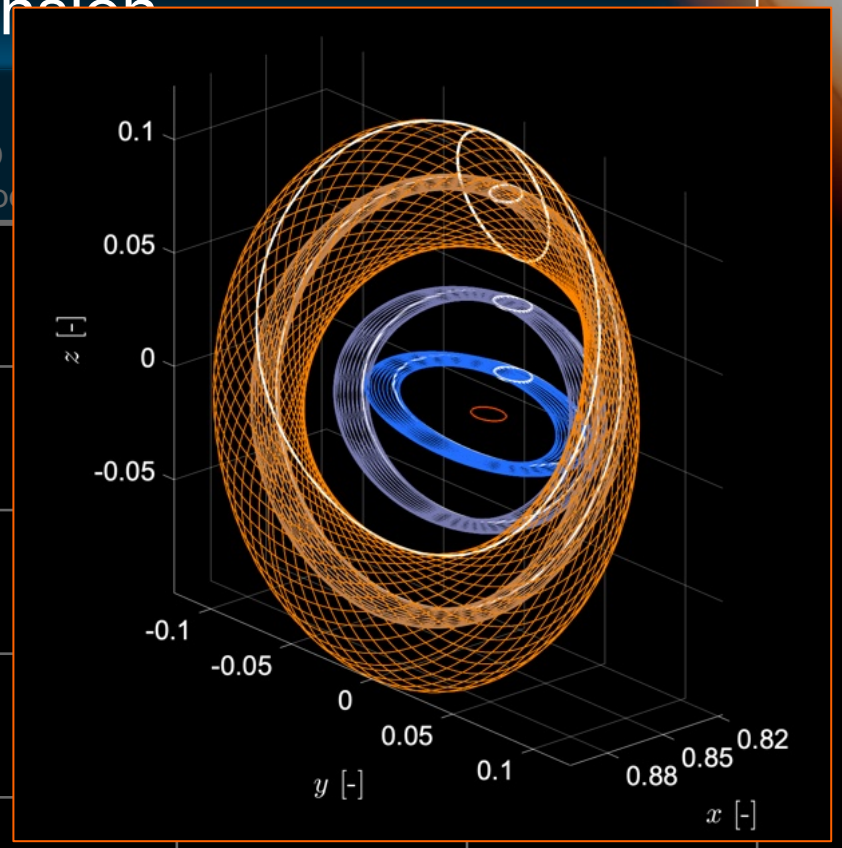
D_i internal dim. $D_i = D - D_e$	D torus dimension: 0 equilibrium point	1 periodic orbit	quasi	5 ...
D_e external dim. 0 CR3BP, H3BP	0	1		X
D_e external dim. 1 HR4BP	X	0 →		X
D_e external dim. 2 EHR4BP	X	X		3
D_e external dim. ...	X	X		2
D_e external dim. N RPRnBP	X	X	X X X	X



INVARIANT TORI

EXISTENCE

D_i internal dim. $D_i = D - D_e$	D torus dimension: **0** equilibrium point	**1** periodic orbit	**2** quasi-periodic orbit	3D perio
D_e external dim. **0** CR3BP, H3BP	0	1	2	
D_e external dim. **1** HR4BP	X	0	1 →	
D_e external dim. **2** EHR4BP	X	X	0	
D_e external dim. ...	X	X	X	
D_e external dim. **N** RPRnBP	X	X	X	X X X



INVARIANT TORI

CONTINUATION

D_i internal dim. $D_i = D - D_e$	D torus dimension: **0** equilibrium point	**1** periodic orbit	**2** quasi-periodic orbit	**3** 3D quasi-periodic orbit	**4** 4D quasi-periodic orbit	**5** ...
D_e external dim. **0** CR3BP, H3BP	0	1	2	EXTERNAL CONTINUATION $D_e + 1$ Add external perturbation		X
1 HR4BP	INTERNAL CONTINUATION $D_i + 1$ Perturb center manifold		1			X
2 EHR4BP			0	1	2	3
...	X	X	X	0	1	2
N RPRnBP	X	X	X	X	X	X

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COMPUTING INVARIANT TORI

INVARIANT TORI

STROBOSCOPIC MAP FOR A 2D TORUS

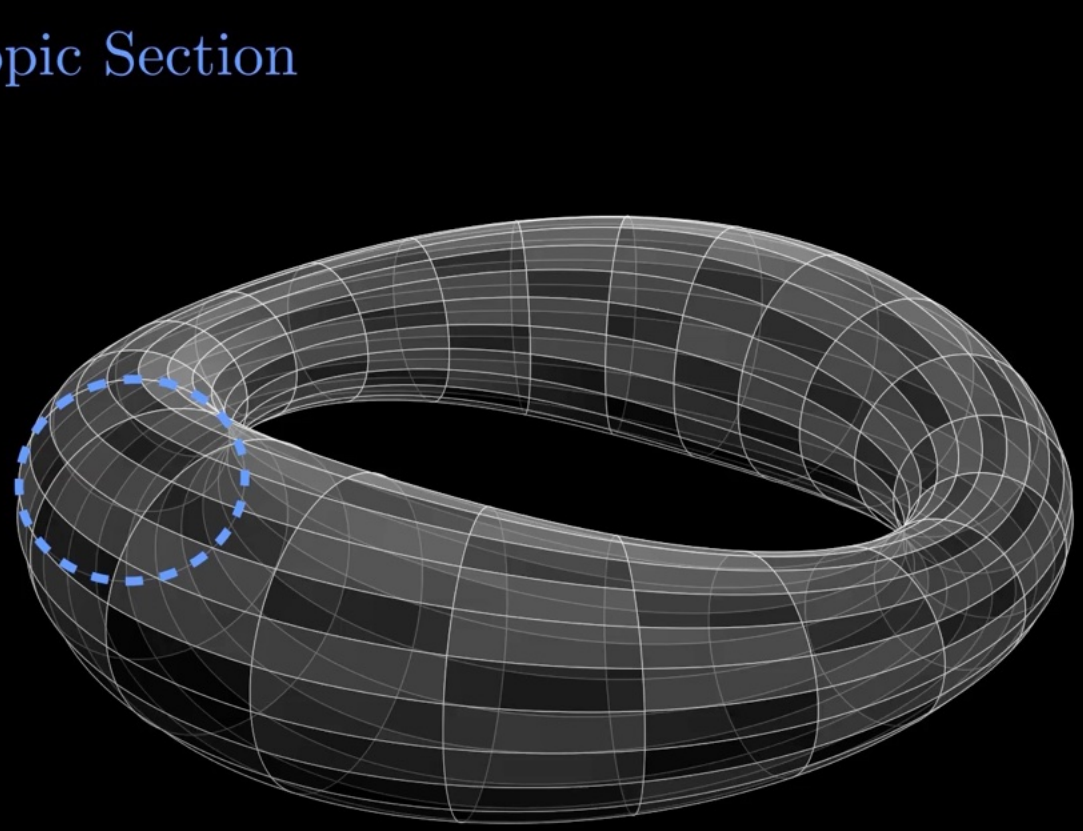
Stroboscopic angle $\theta_s \leftarrow \theta_1$

Section angle $\vartheta \leftarrow \theta_2$

Stroboscopic period $T_s \leftarrow T_1$

$$\varphi_{T_s}(\boldsymbol{\sigma}(\boldsymbol{\vartheta}; \theta_s)) = \boldsymbol{\sigma}(\boldsymbol{\vartheta} + \boldsymbol{\rho}; \theta_s)$$

Rotation number $\rho = \omega_2 T_s$



Olikara and Scheeres: Numerical method for computing quasi-periodic orbits and their stability in the restricted three-body problem (2012)

INVARIANT TORI

MANIFOLD DISCRETIZATION

STROBOSCOPIC SECTION

$$\boldsymbol{\sigma}(\boldsymbol{\vartheta}; \theta_s) = \boldsymbol{\tau}([\theta_s \, \boldsymbol{\theta}^\top]^\top)$$

FOURIER DISCRETIZATION

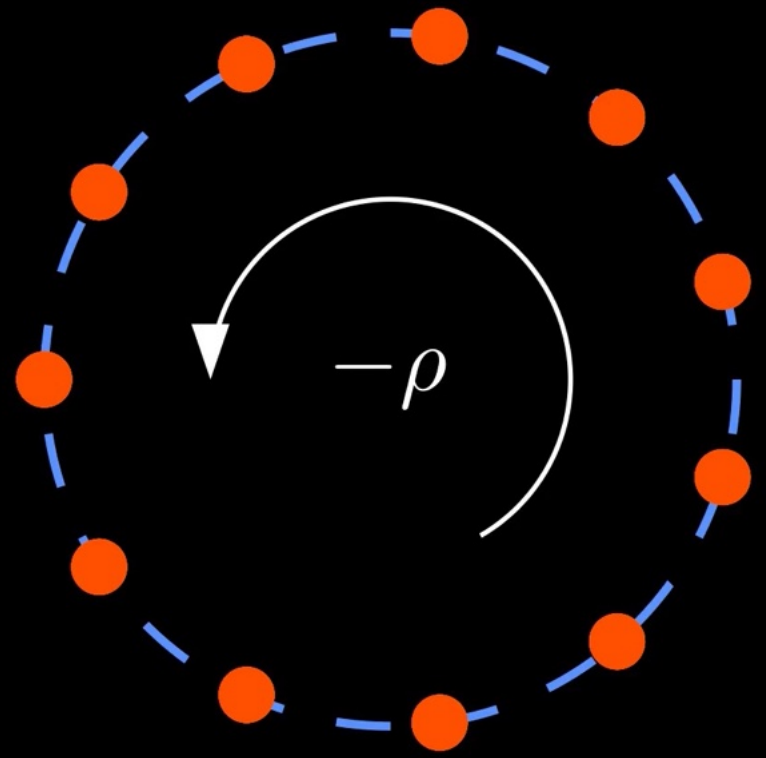
$$\mathbf{c}_j = \frac{1}{\prod_{d=1}^{D} N_{F_d}} \sum_{i \in \mathcal{I}_D} \mathbf{x}_i \, e^{-\mathrm{i} \, j^\top \boldsymbol{\theta}_i}$$

multidimensional DFT

QUASI-PERIODICITY CONDITION

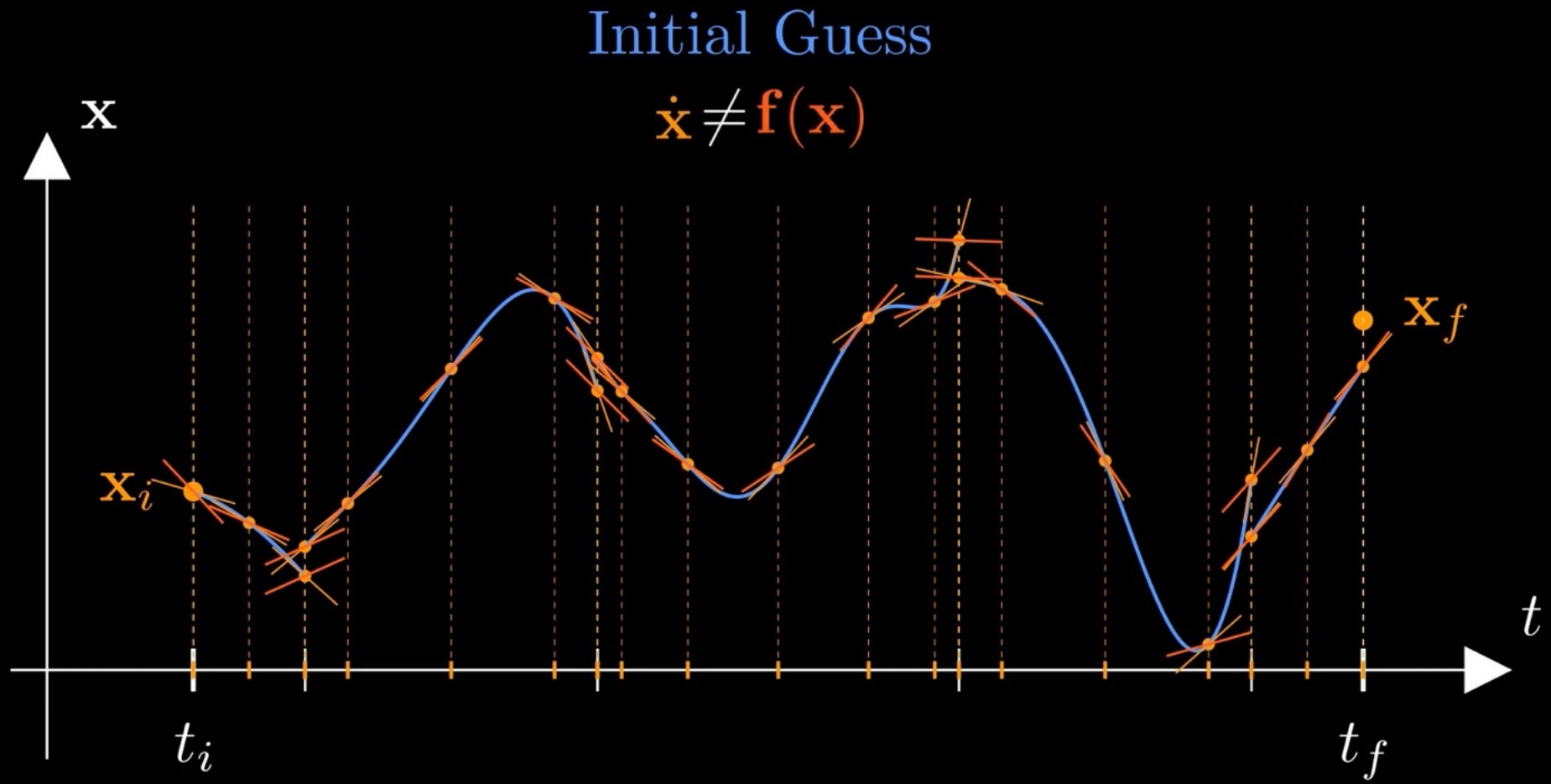
$$R_{-\rho}\left[\boldsymbol{\sigma}\left(\boldsymbol{\vartheta} + \rho; \theta_s\right)\right] = \boldsymbol{\sigma}\left(\boldsymbol{\vartheta}; \theta_s\right)$$

rotation operator



PSEUDOSPECTRAL COLLOCATION

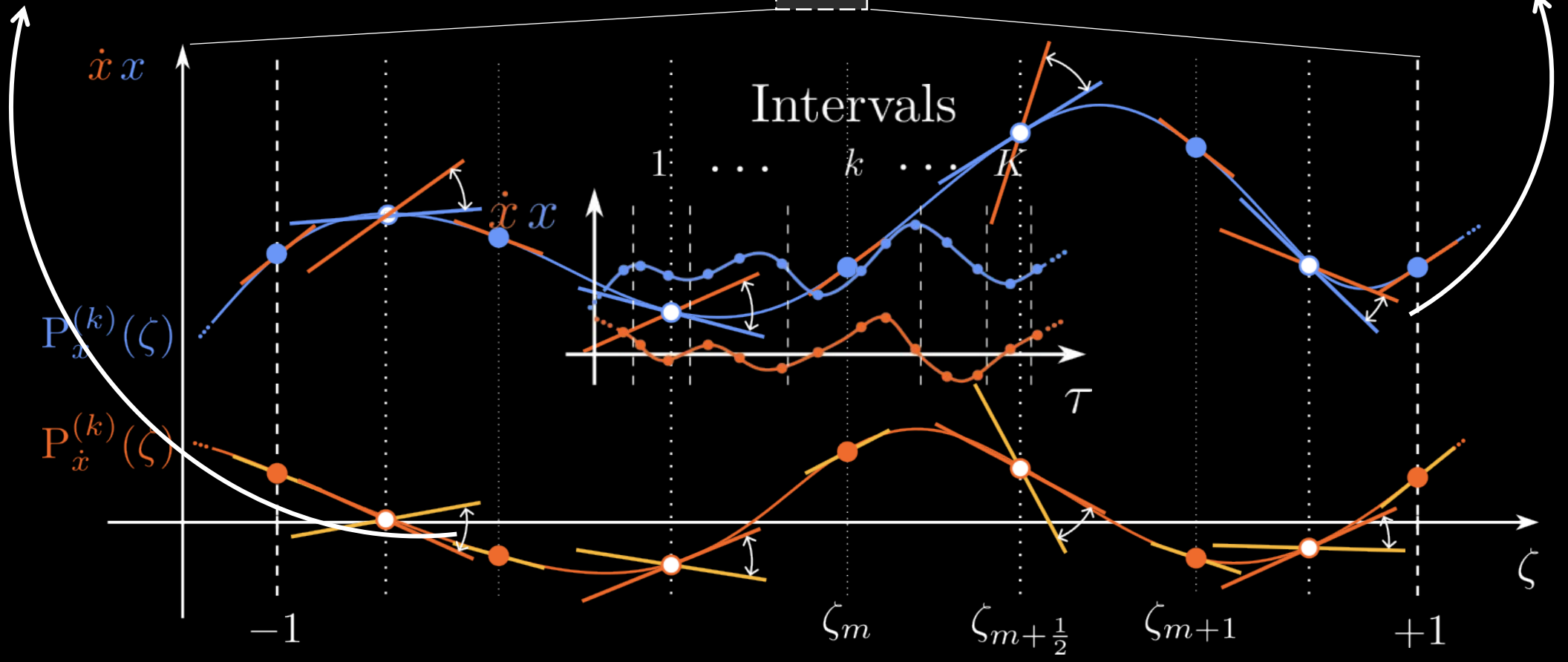
TRAJECTORY PROPAGATION



COLLOCATION ERROR

$$\left|\dot{\mathrm{P}}_{\dot{x}}^{(k)}(\zeta_{m+\frac{1}{2}}) - \mathrm{f}_{\dot{x}}(\mathrm{P}_{\dot{x}}^{(k)}(\zeta_{m+\frac{1}{2}}), \tau(\zeta_{m+\frac{1}{2}}))\right|$$

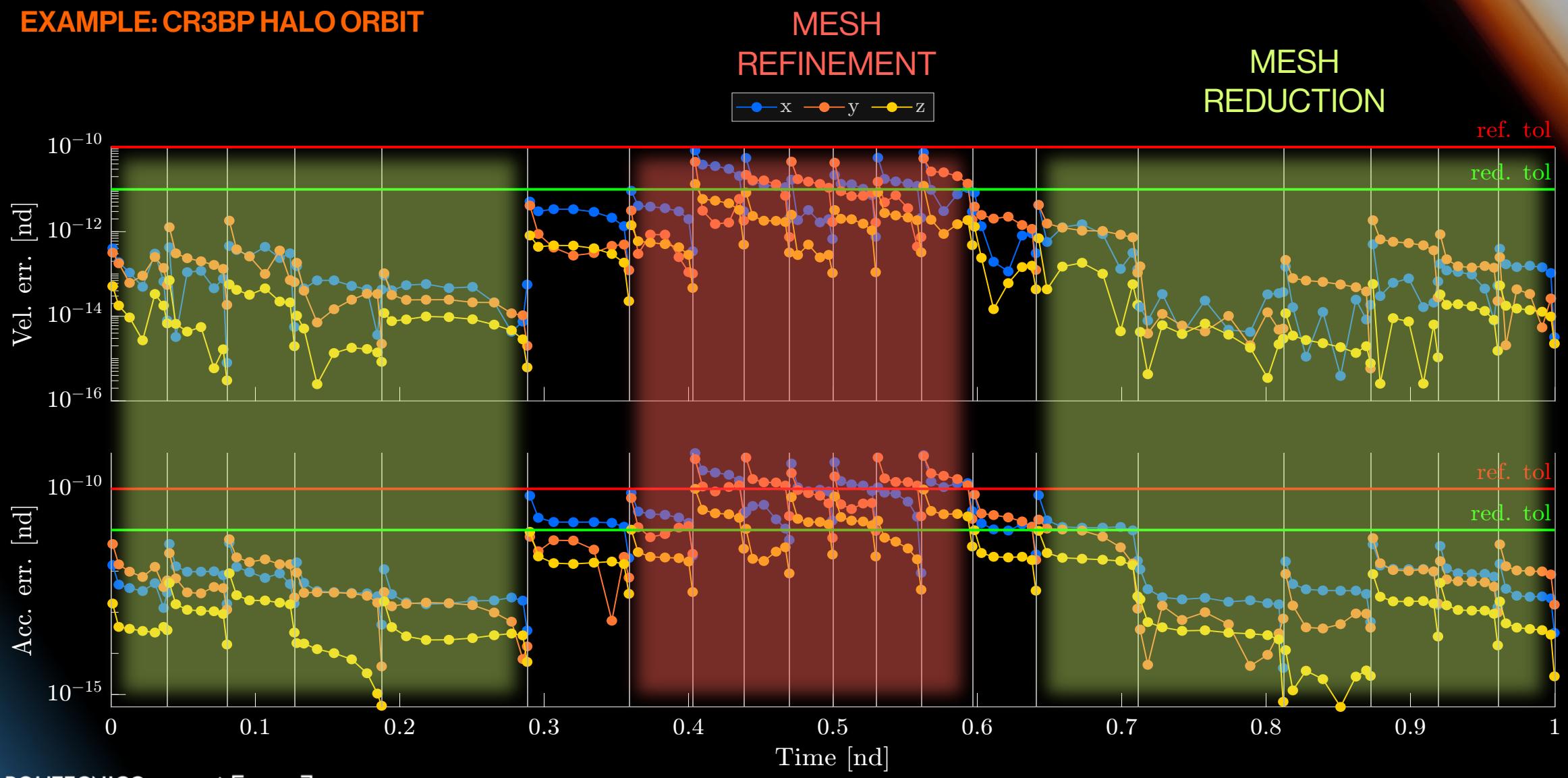
$$\left|\dot{\mathrm{P}}_{x}^{(k)}(\zeta_{m+\frac{1}{2}}) - \mathrm{f}_{x}(\mathrm{P}_{\dot{x}}^{(k)}(\zeta_{m+\frac{1}{2}}), \tau(\zeta_{m+\frac{1}{2}}))\right|$$



Grossi, Baresi & Topputo: Quasi-periodic Invariant Manifolds in the In-plane Quasi-hill Restricted Four-body Problem (2026)

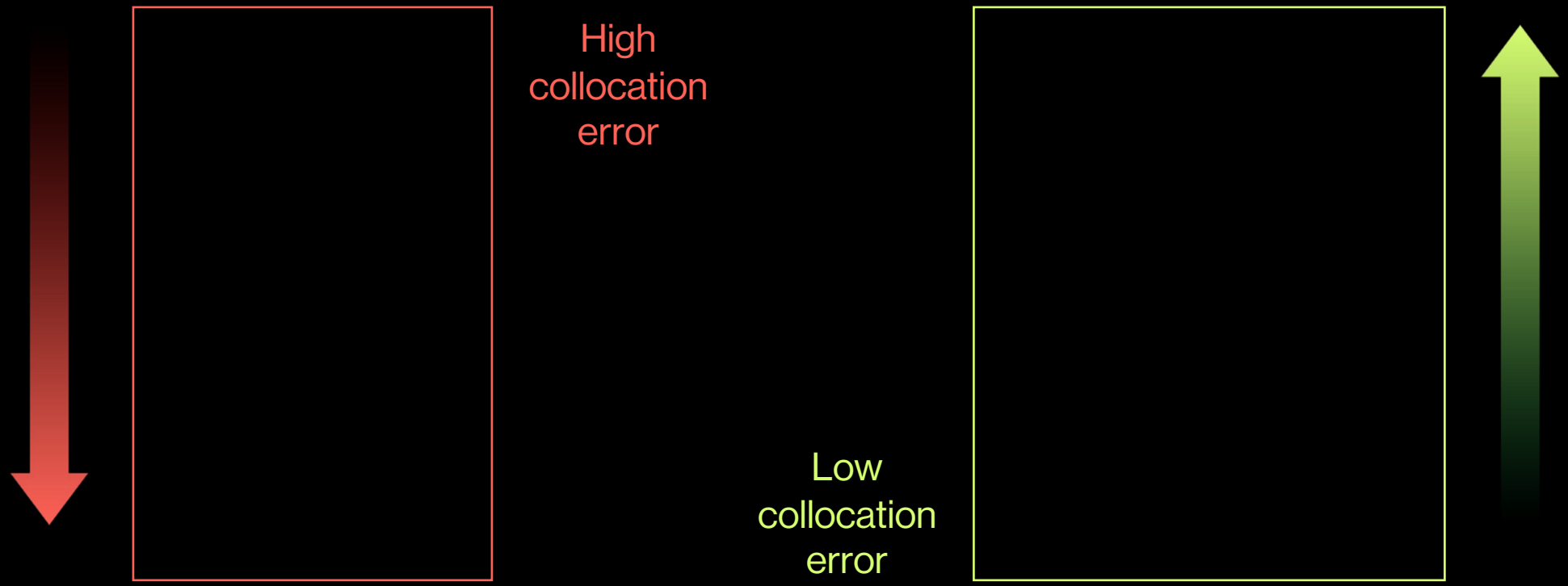
ADAPTIVE COLLOCATION

EXAMPLE: CR3BP HALO ORBIT



ADAPTIVE COLLOCATION

STRATEGIES



Haman & Rao: Adaptive Mesh Refinement and Error Estimation Method for Optimal Control Using Direct Collocation (2025)

PSEUDO-ARCLENGTH CONTINUATION

OVERVIEW

PEDICTOR (secant)

$$\mathbf{y} = \widetilde{\mathbf{y}} + \delta s \, \frac{d\widetilde{\mathbf{y}}}{ds}$$

CORRECTOR (Newton)

$$\begin{aligned}
&\textbf{while } \|\mathbf{H}(\mathbf{y})\| > \mathrm{tol} \textbf{ do} \\
&\quad \mathbf{y} \leftarrow \mathbf{y} - [D_{\mathbf{y}}\mathbf{H}(\mathbf{y})]^{-1} \, \mathbf{H}(\mathbf{y}) \\
&\textbf{end while}
\end{aligned}$$

Seydel: Practical Bifurcation and Stability Analysis (2010)

INVARIANT TORI

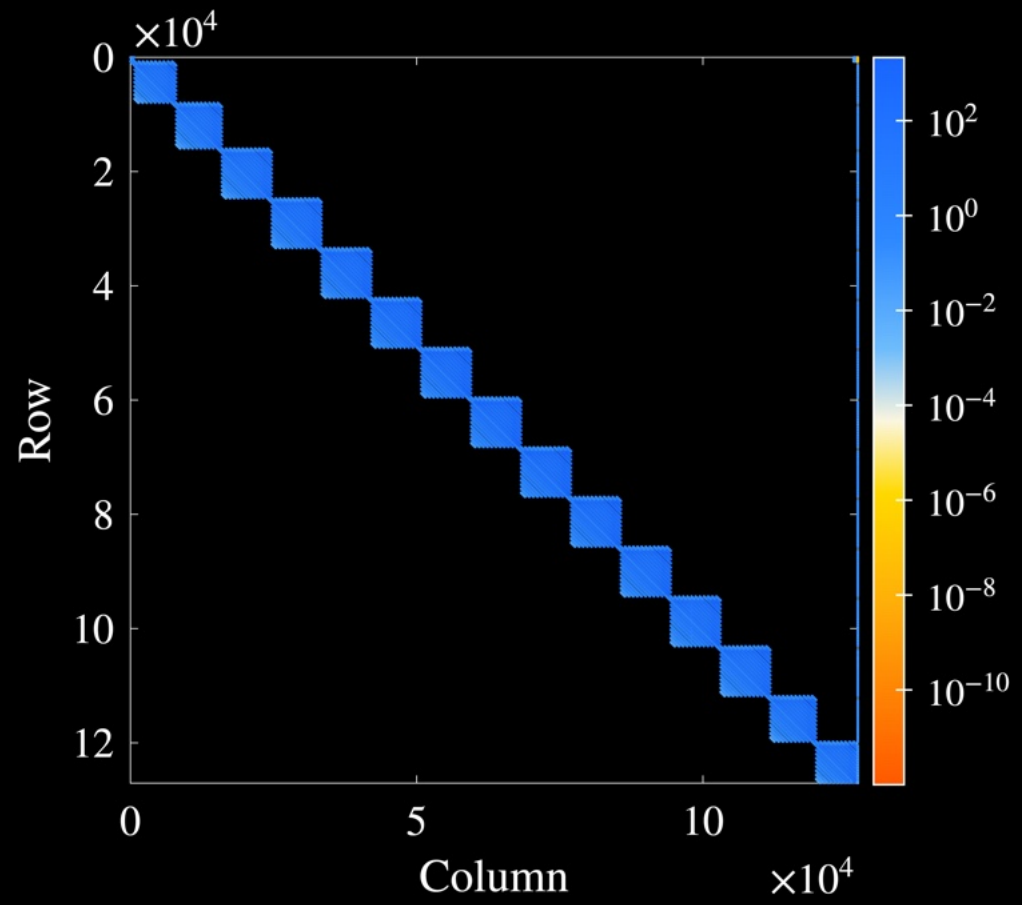
SET OF EQUATIONS

Unknowns:

1. States at collocation points
2. Unfolding parameters
3. Continuation parameter (e.g., ρ, m, e)

Constraints:

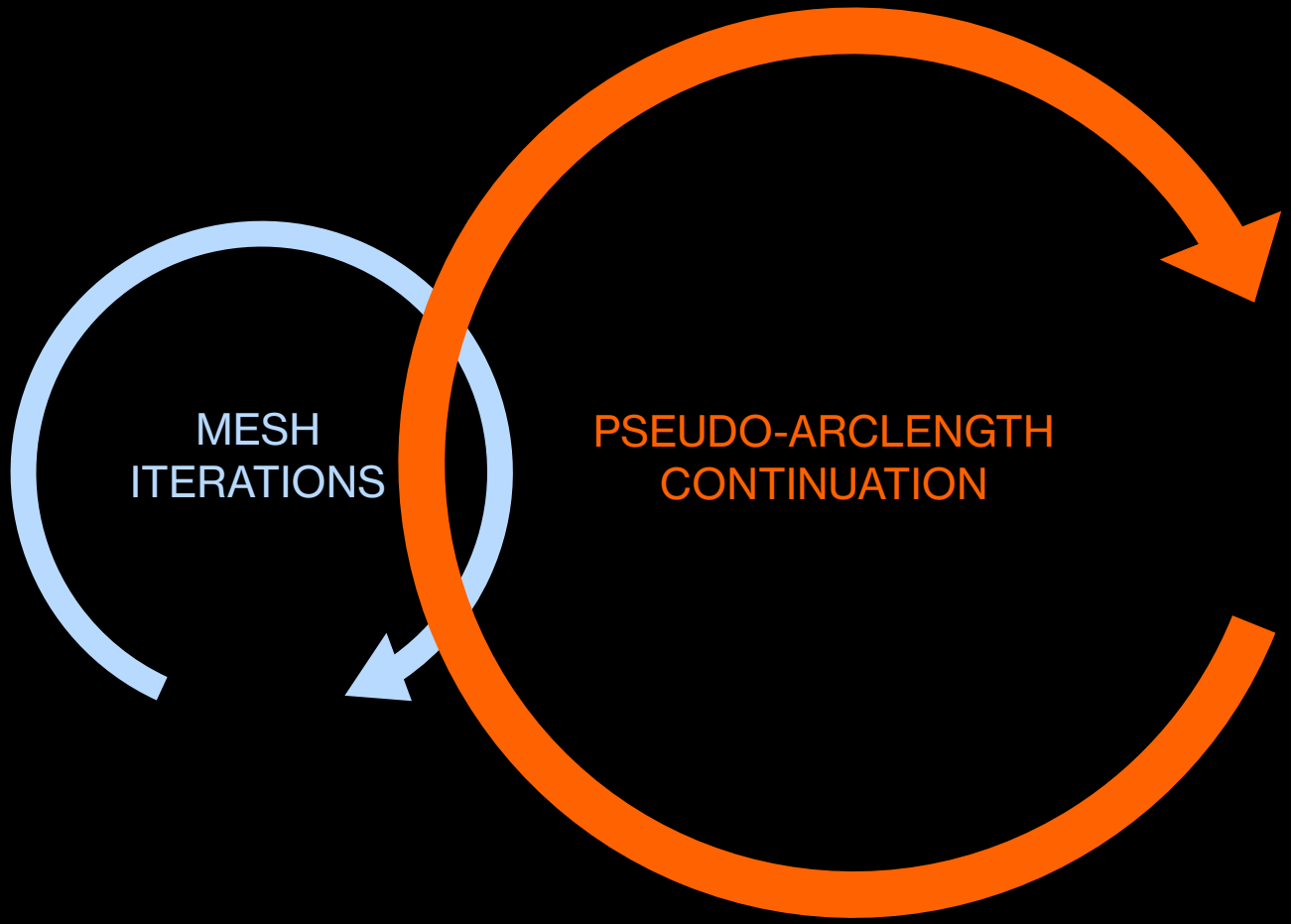
1. Collocation
2. Quasi-periodicity
3. Phase-locking condition
4. Pseudo-arclength



Easily reach 100,000 unknowns!

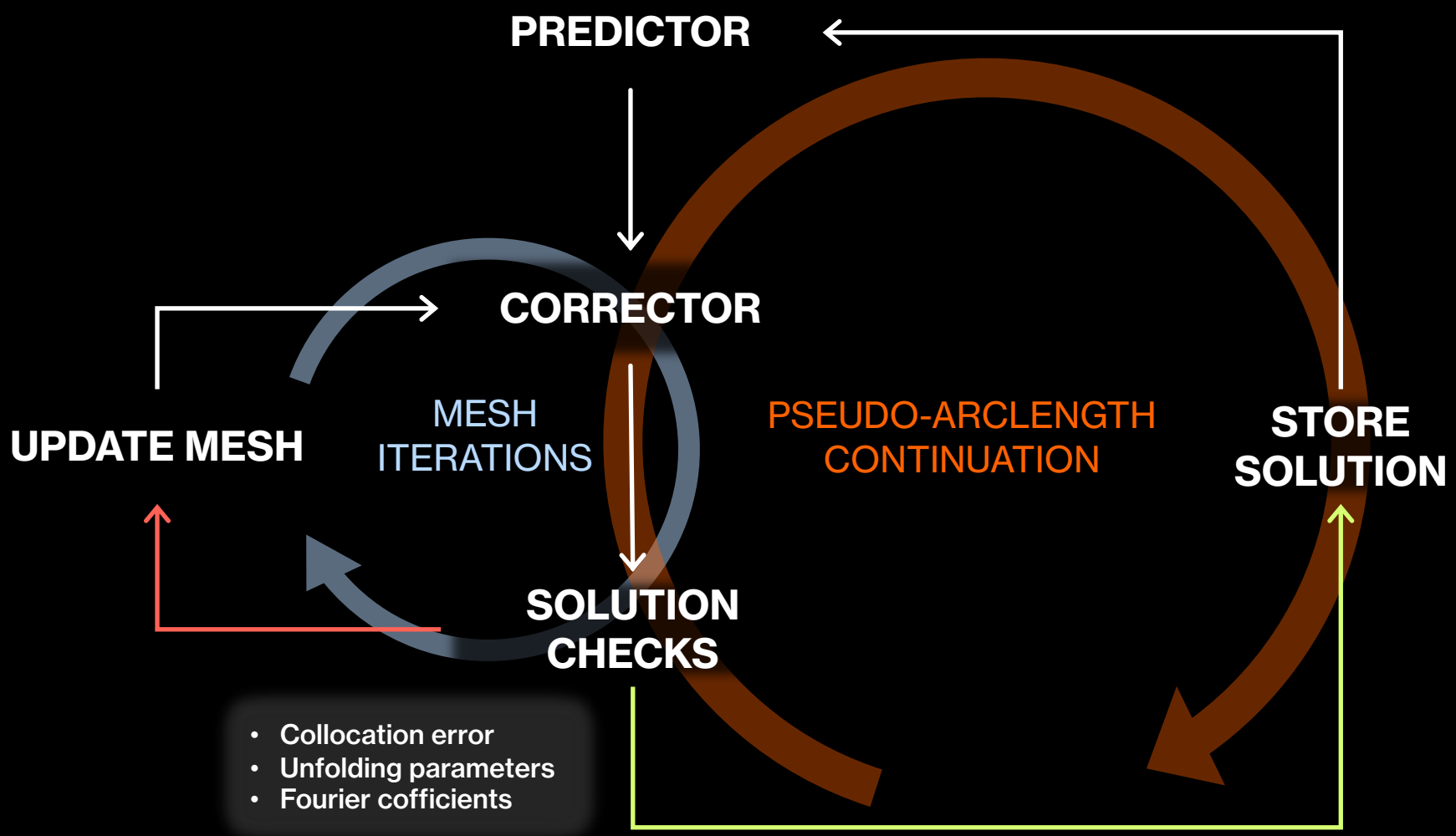
CONTINUATION SCHEME

TWO NESTED CYCLES



CONTINUATION SCHEME

DETAILED



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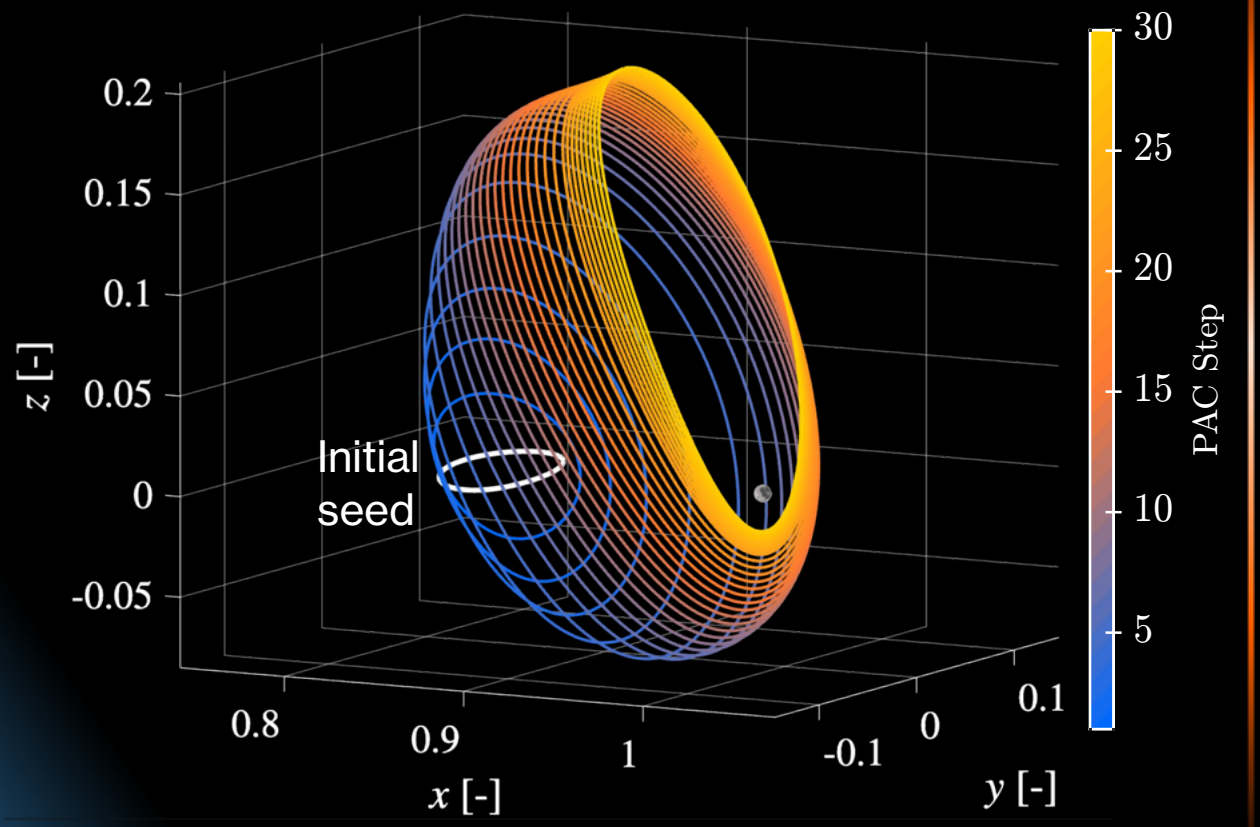


RESULTS

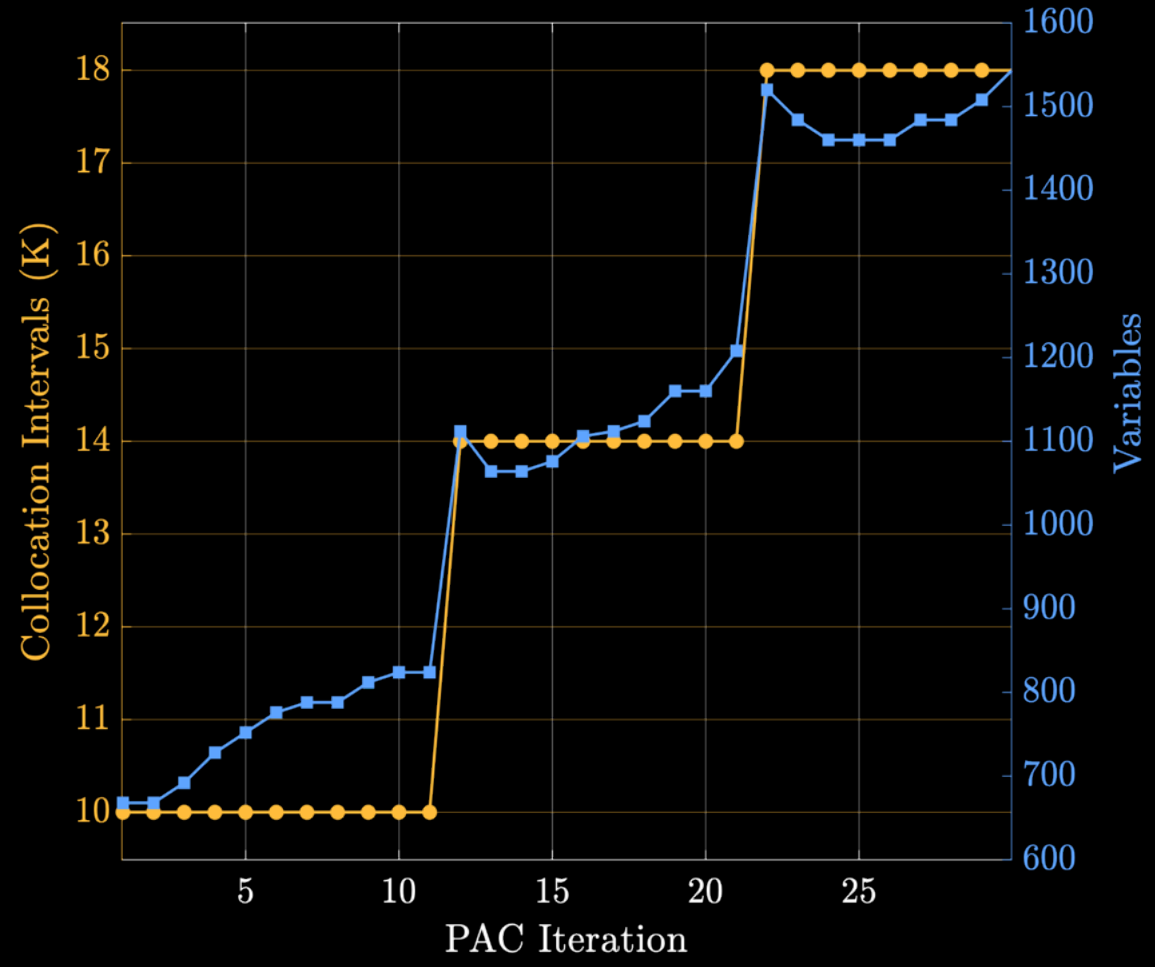
INTERNAL CONTINUATION

CR3BP PO FAMILY

HALO ORBIT FAMILY

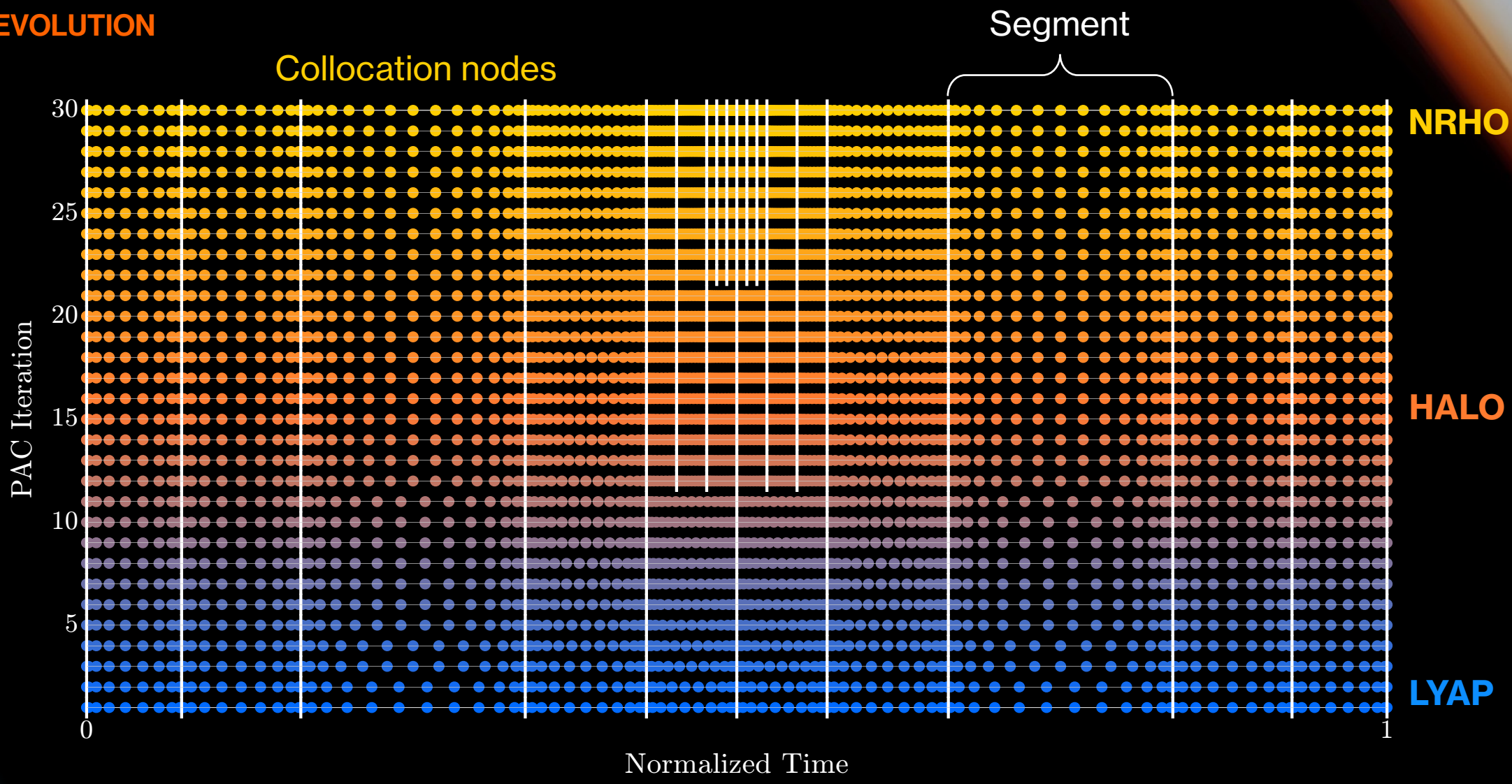


Mesh during iterations



CR3BP PO FAMILY

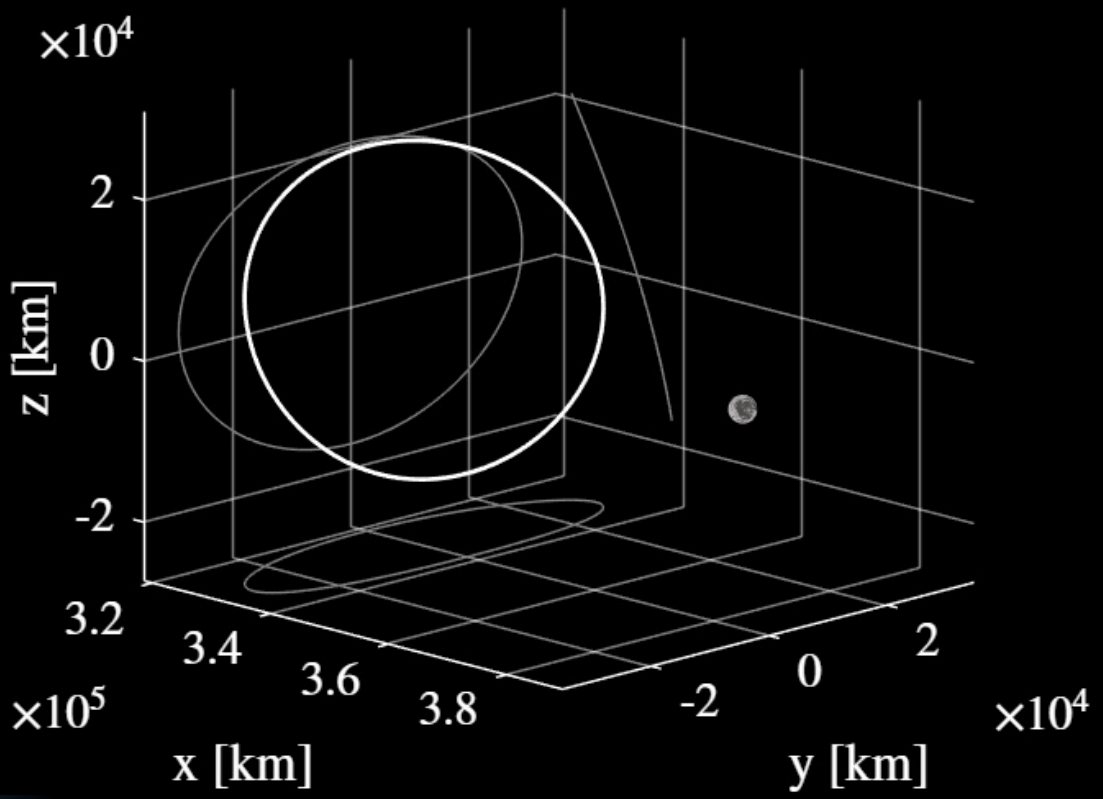
MESH EVOLUTION



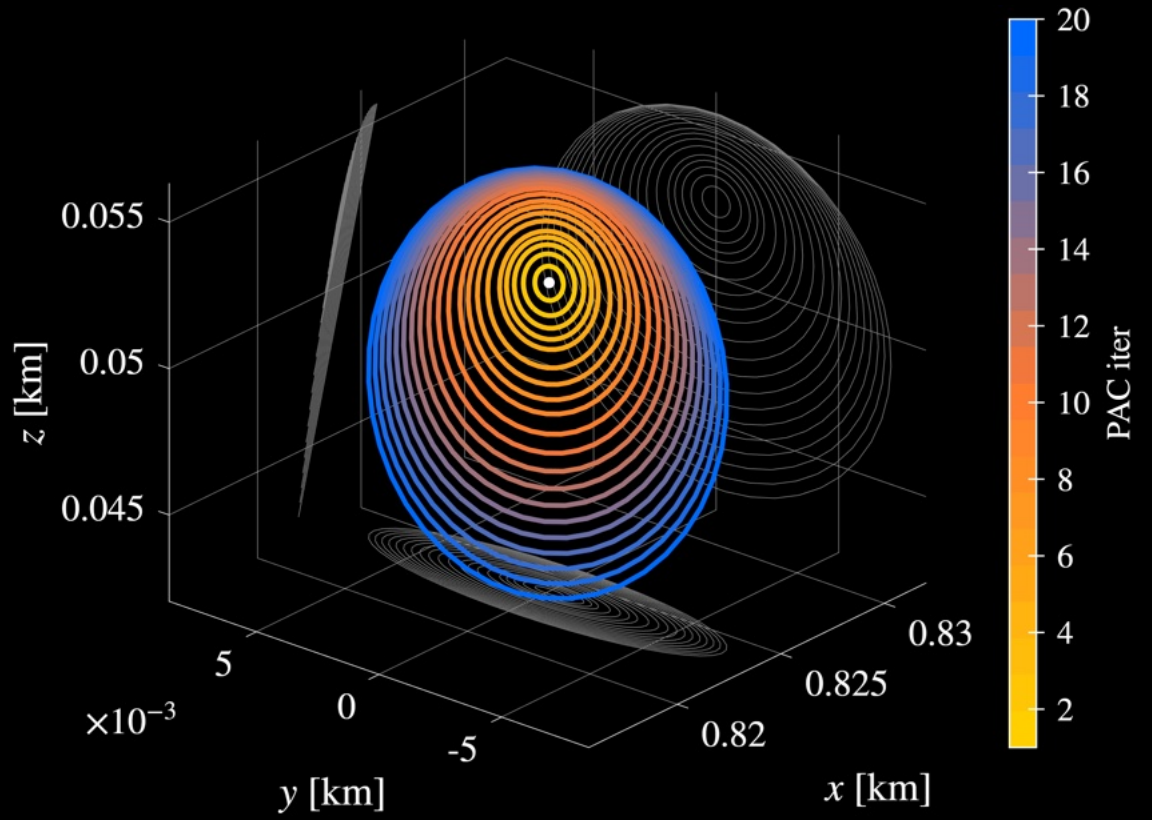
CR3BP 2D QPO

HALO L_1, $T_1 = 12.2$ days

Quasi-periodic orbits



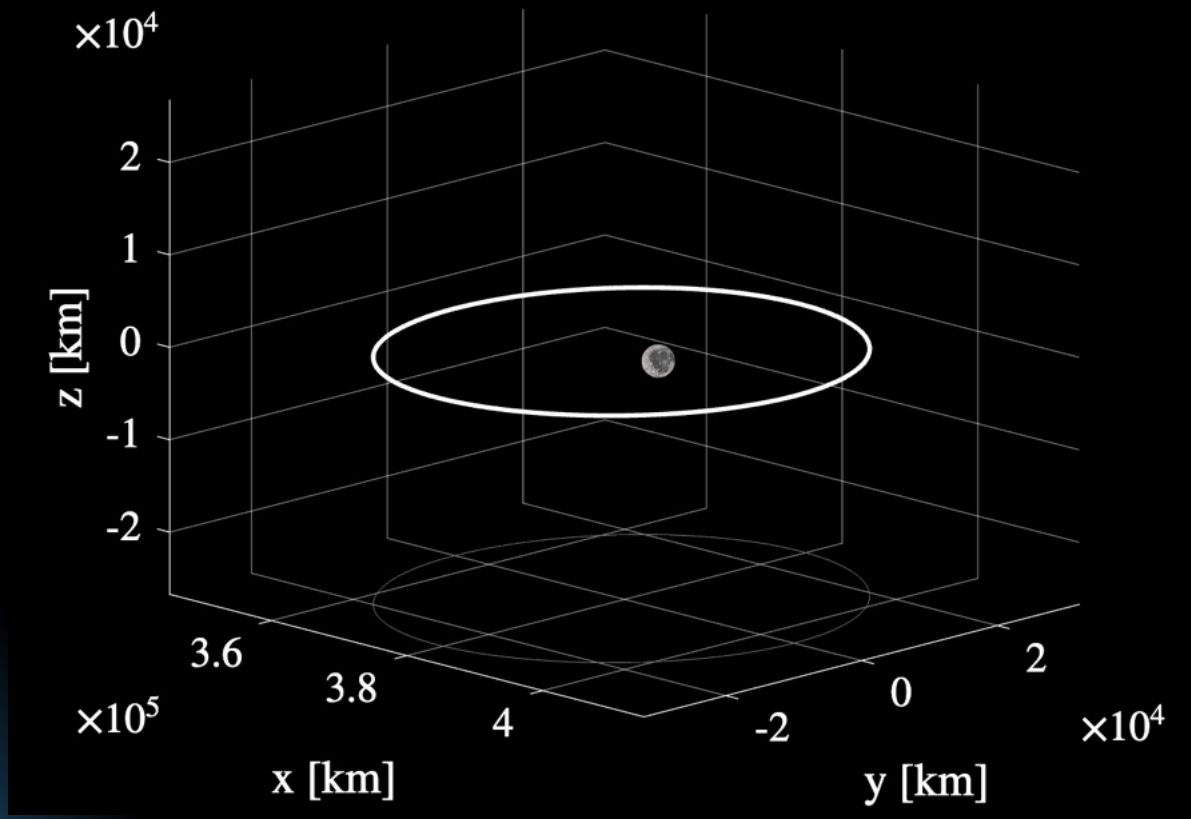
Invariant stroboscopic surfaces



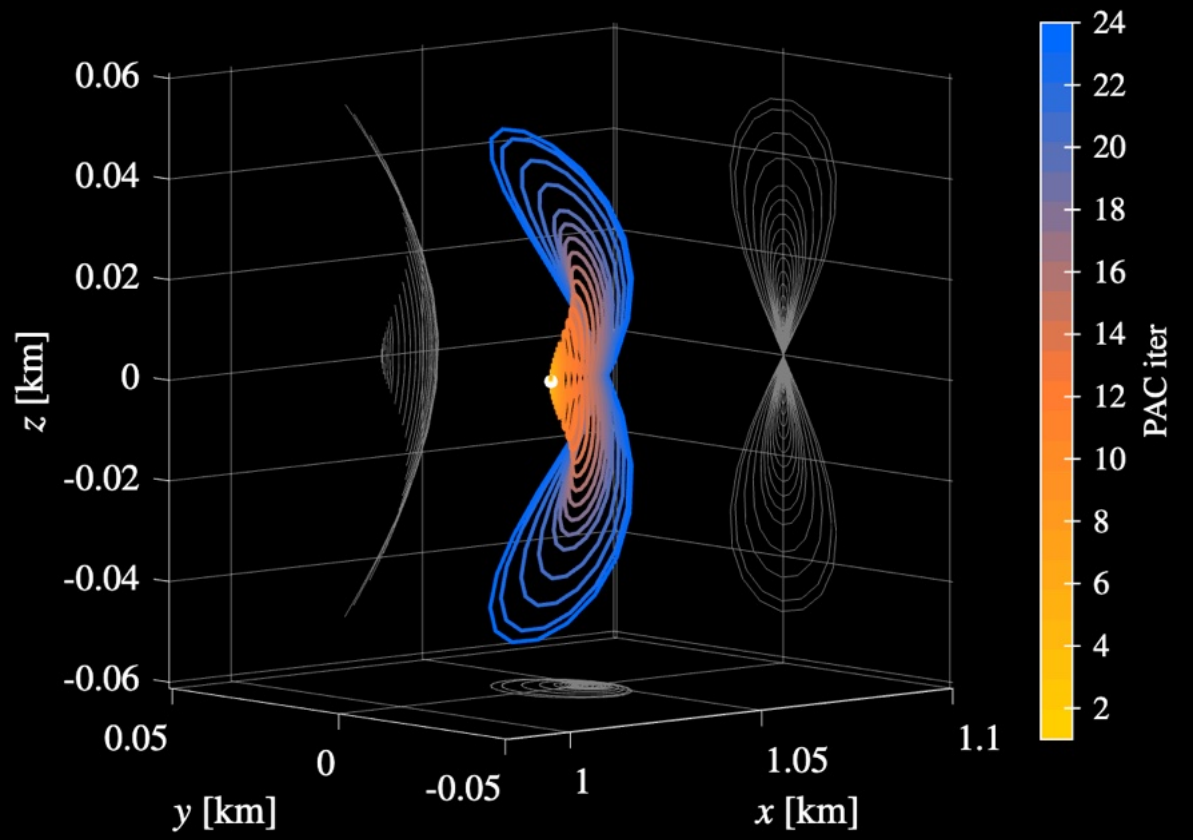
CR3BP 2D QPO

DPO $T_1 = 5.3$ days

Quasi-periodic orbits



Invariant stroboscopic surfaces





RESULTS

EXTERNAL CONTINUATION

EXTERNAL CONTINUATION

CR3BP

Autonomous Hamiltonian system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

Isolated equilibrium points
0D tori

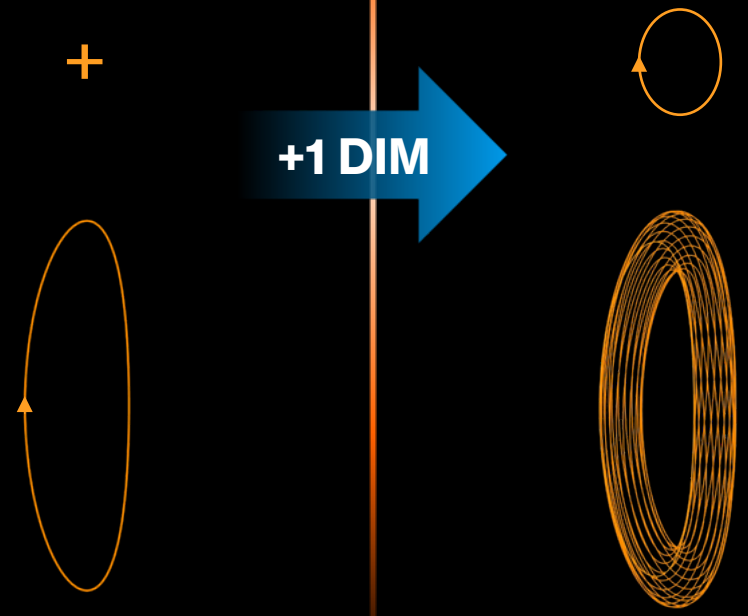
+

Families of periodic orbits*
1D tori

*non-resonant

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \theta_S)$$
$$\dot{\theta}_S = \omega_S$$

+1 DIM



HR4BP

Periodic Hamiltonian system

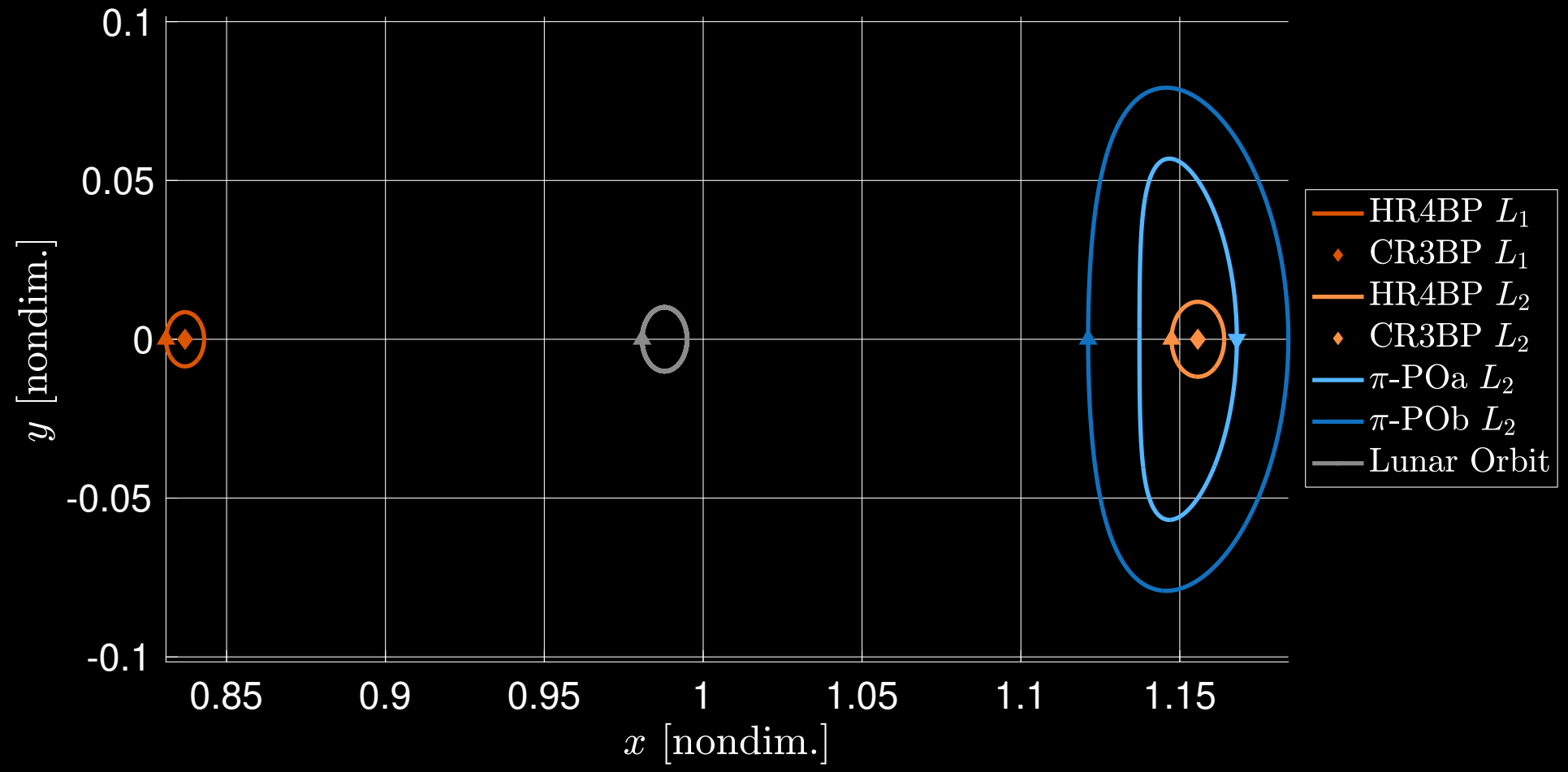
Isolated periodic orbits
1D tori

Families of quasi-periodic orbits
2D tori

Jorba, Villanueva.: On the Persistence of Lower Dimensional Invariant Tori under Quasi-Periodic Perturbations (1997)

HR4BP PO

L_1, L_2 EP DYNAMICAL SUBSTITUTES

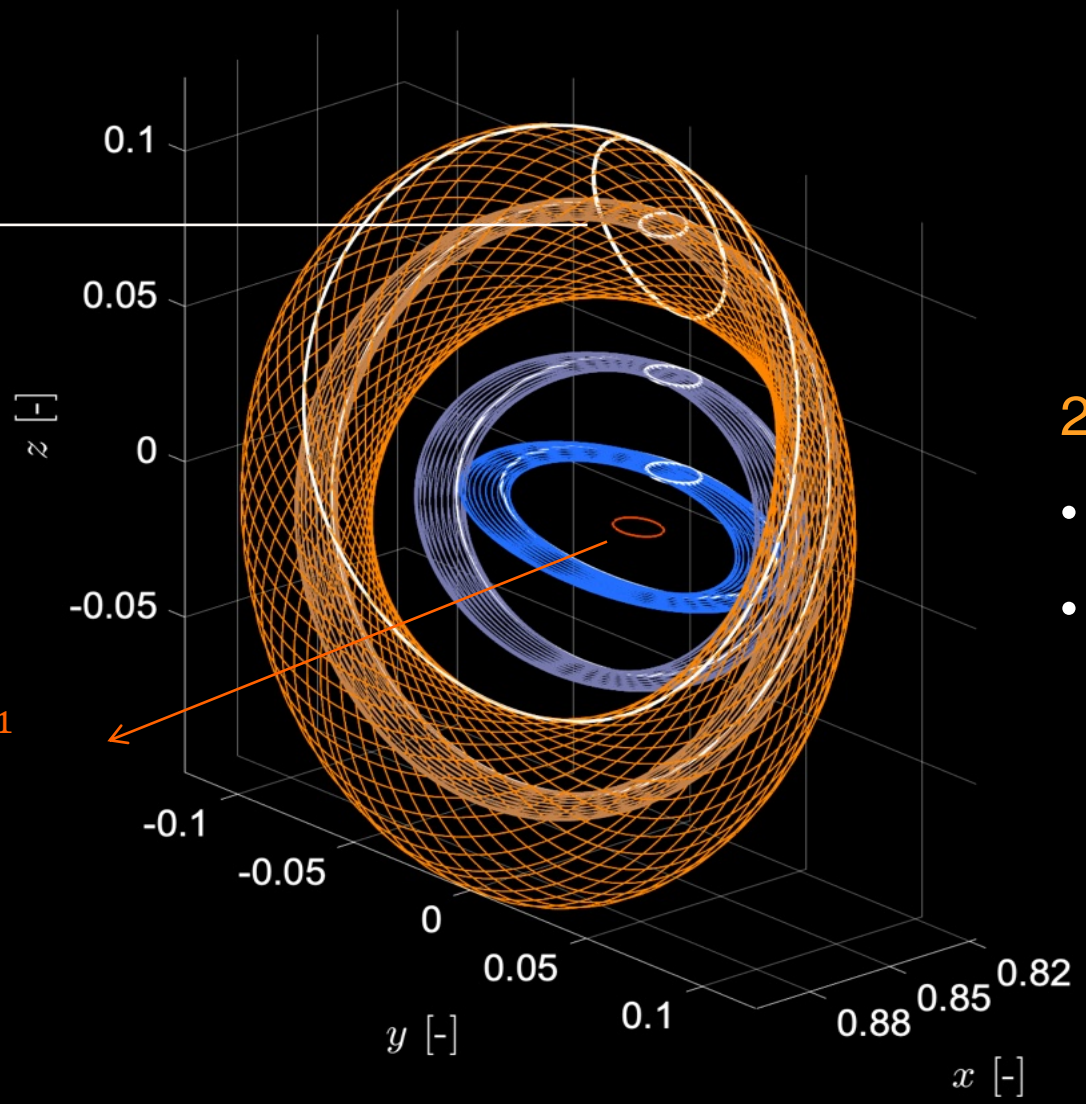


Peterson et al.: The vicinity of Earth–Moon L1 and L2 in the Hill restricted 4-body problem (2023)

HR4BP 2D QPO

HALO L_1

Invariant curves



Dynamical substitute of L_1

is a PO with $T_1 = \pi$

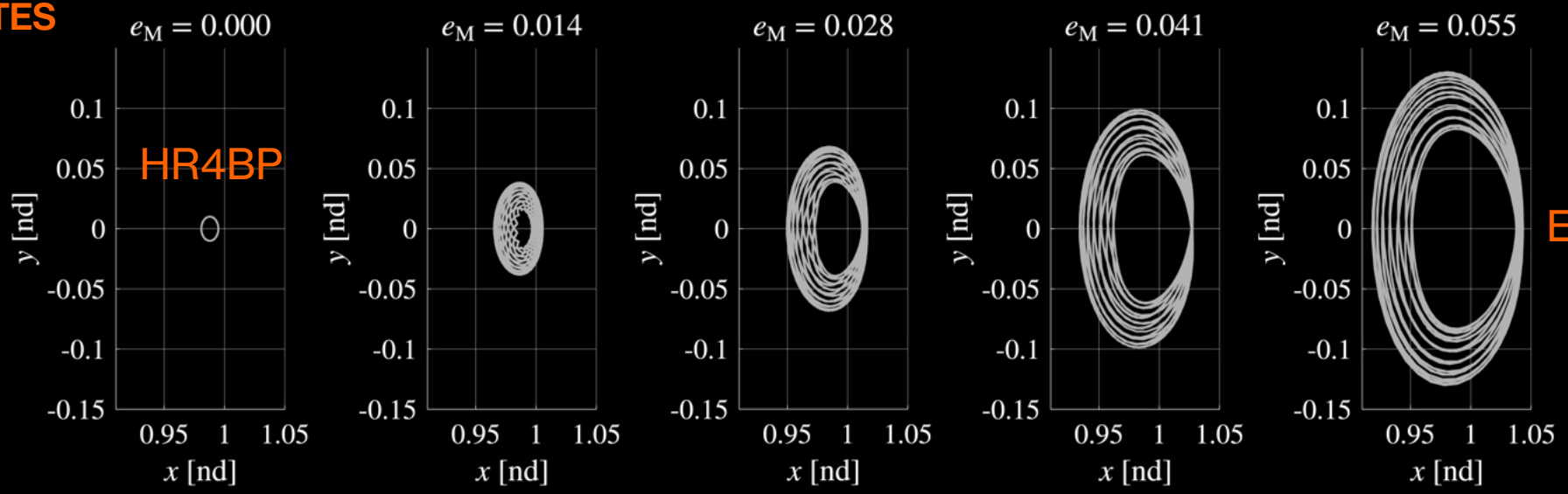
2D TORI:

- $T_1 = \pi$
- T_2 varying along the family

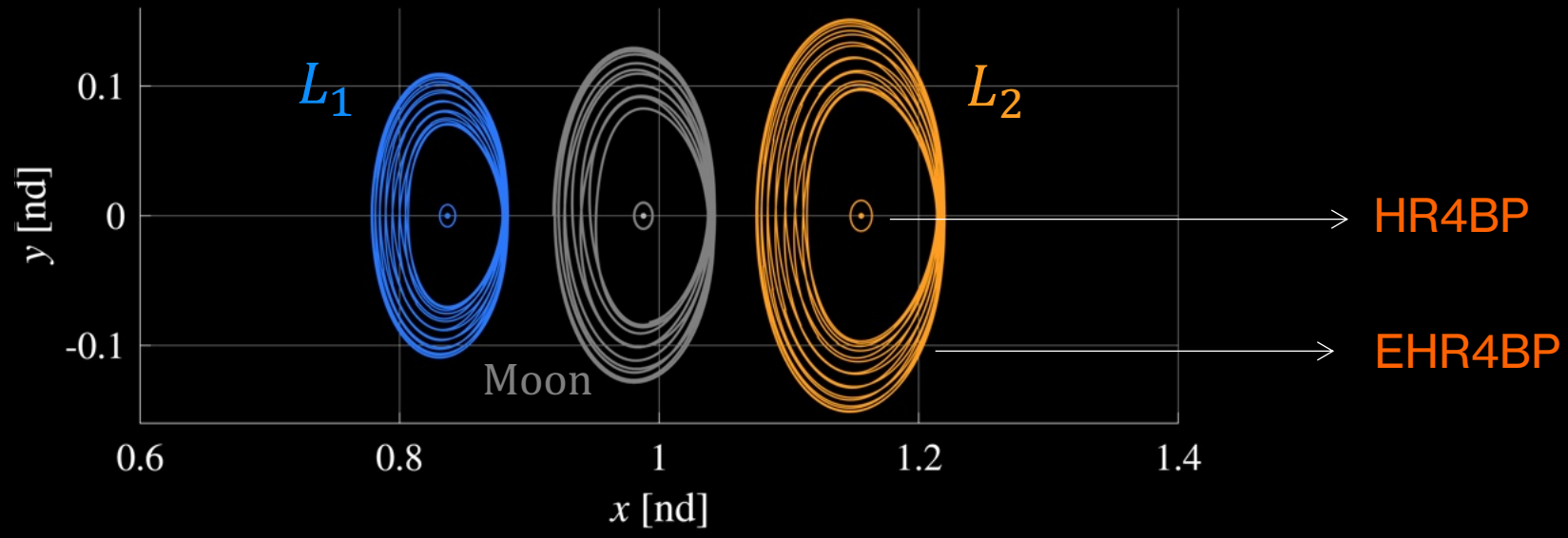
EHR4BP 2D QPO

DYNAMICAL SUBSTITUTES

LUNAR ORBIT



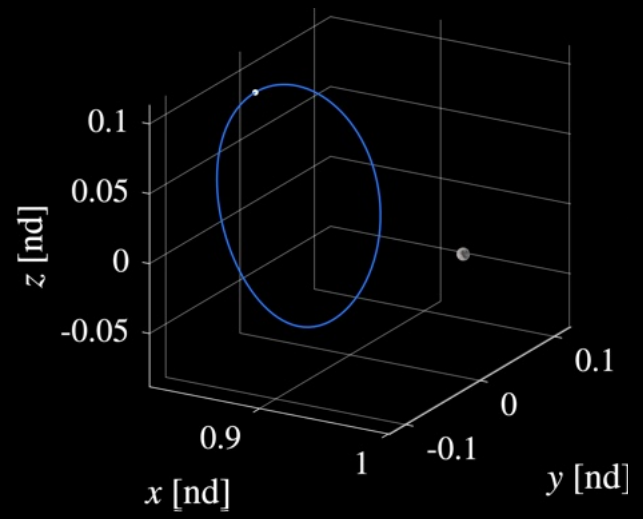
EP DYNAMICAL SUSTITUTS



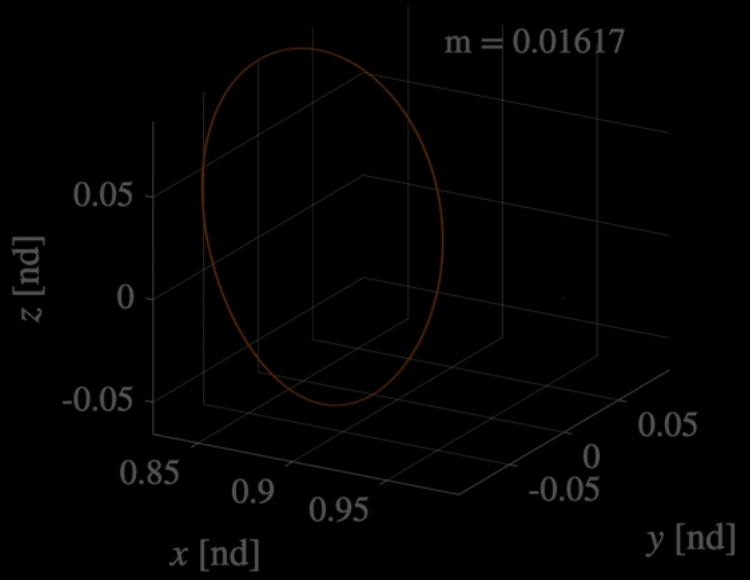
EHR4BP 3D QPO

2 EXTERNAL CONTNUATIONS

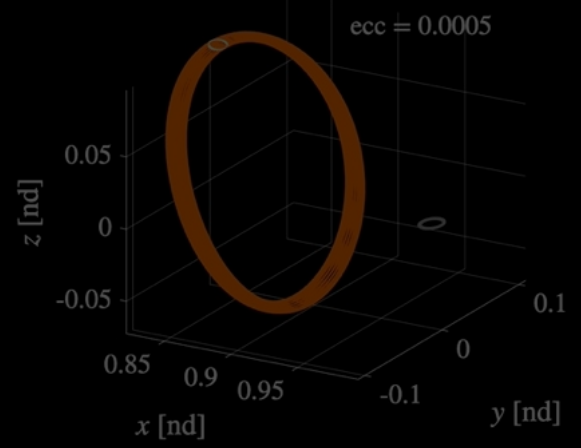
CR3BP



HR4BP



EHR4BP

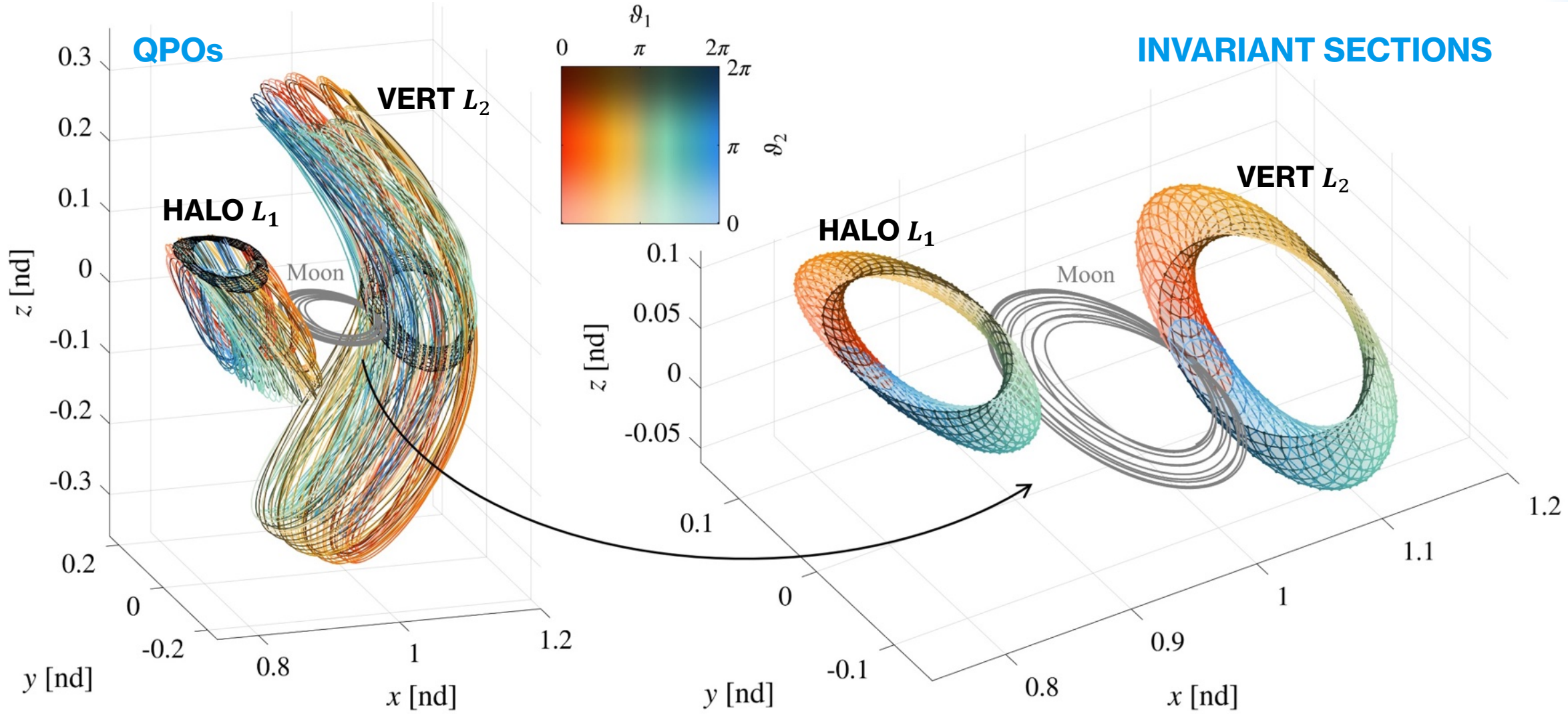


few seconds
5-10 PAC iterations
10k variables

few minutes
10-15 PAC iterations
100k variables

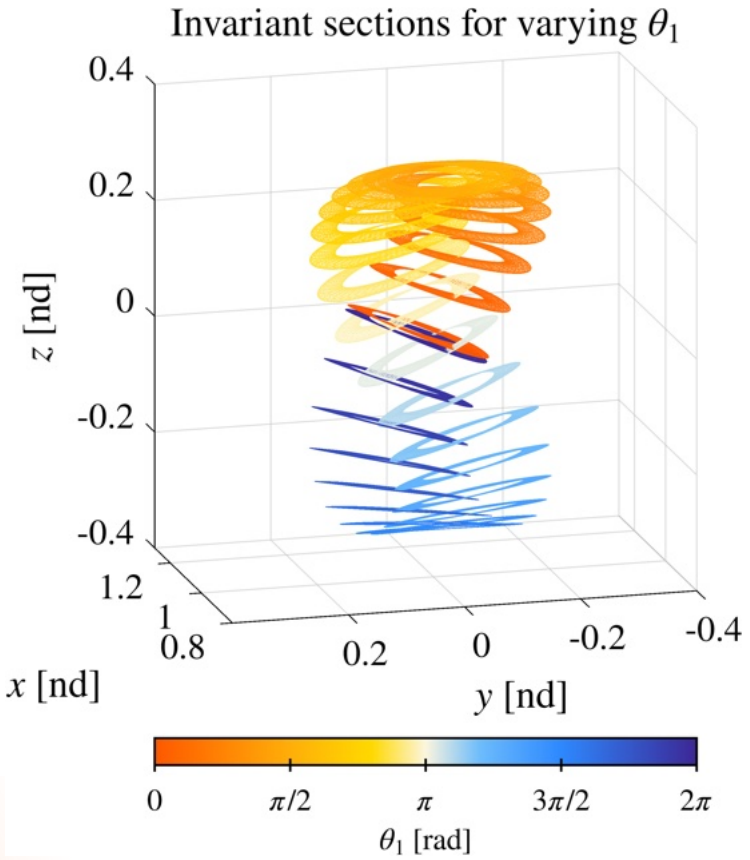
3D INVARIANT TORI

QPO AND INVARIANT CURVES IN THE ROTATING FRAME

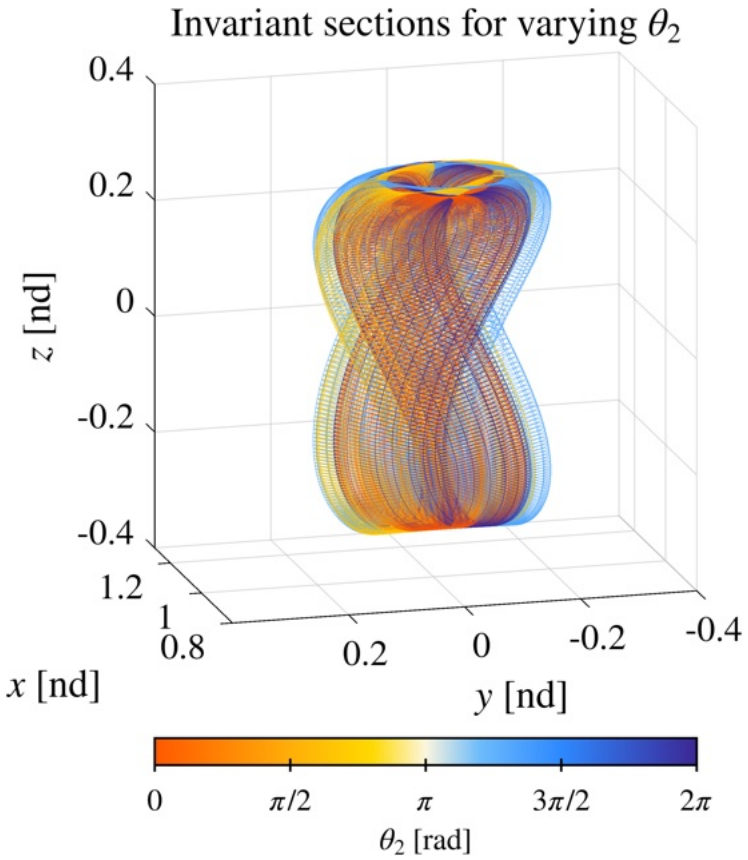


INVARIANT SECTIONS

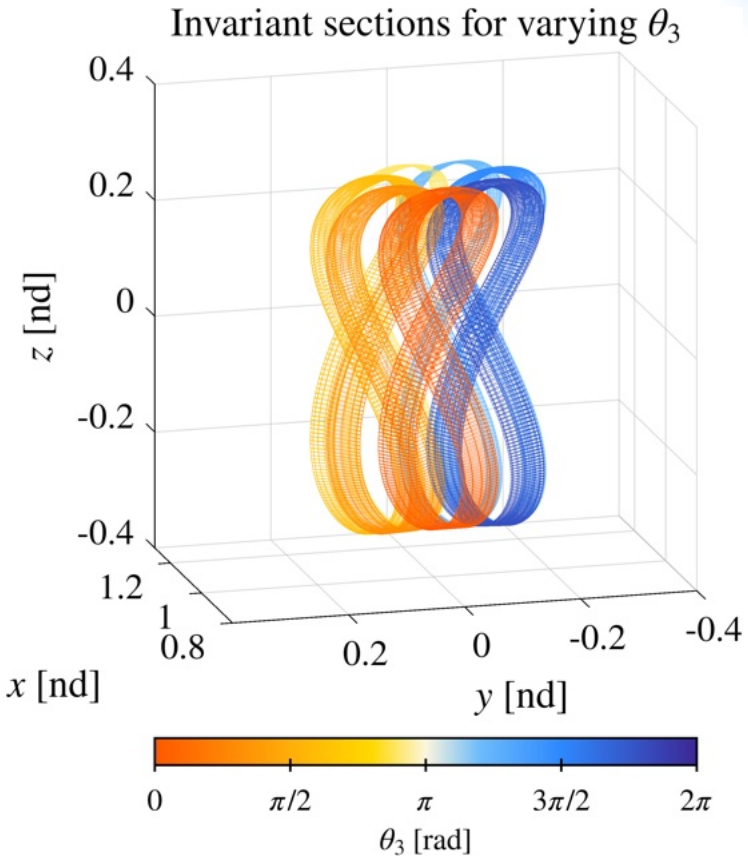
L_2 VERTICAL LYAPUNOV ORBIT



Effect of the uniform-rotating frame



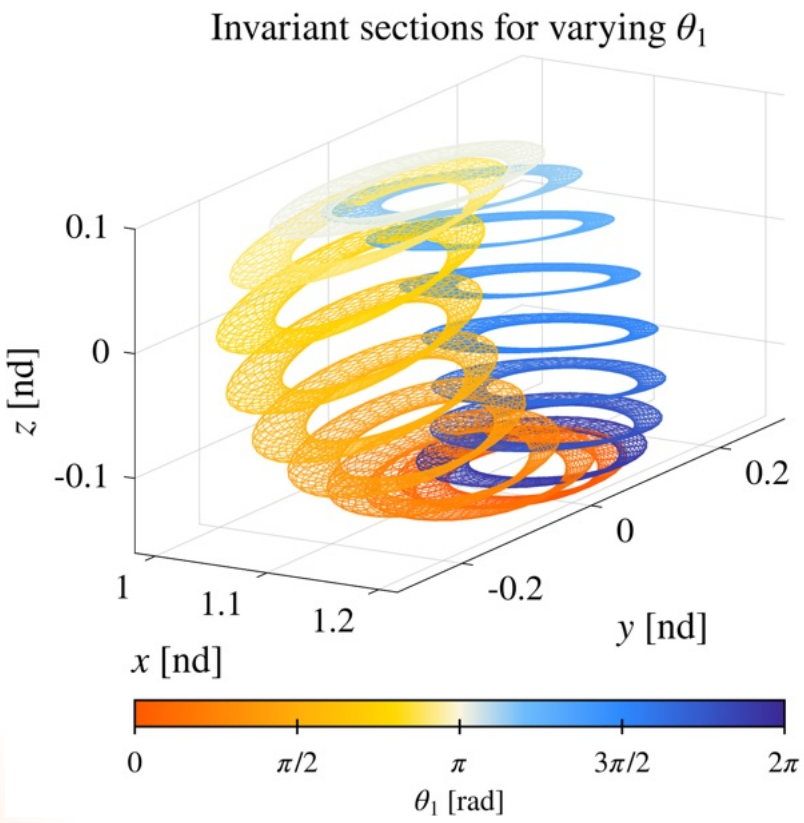
Effect of the Sun's perturbation



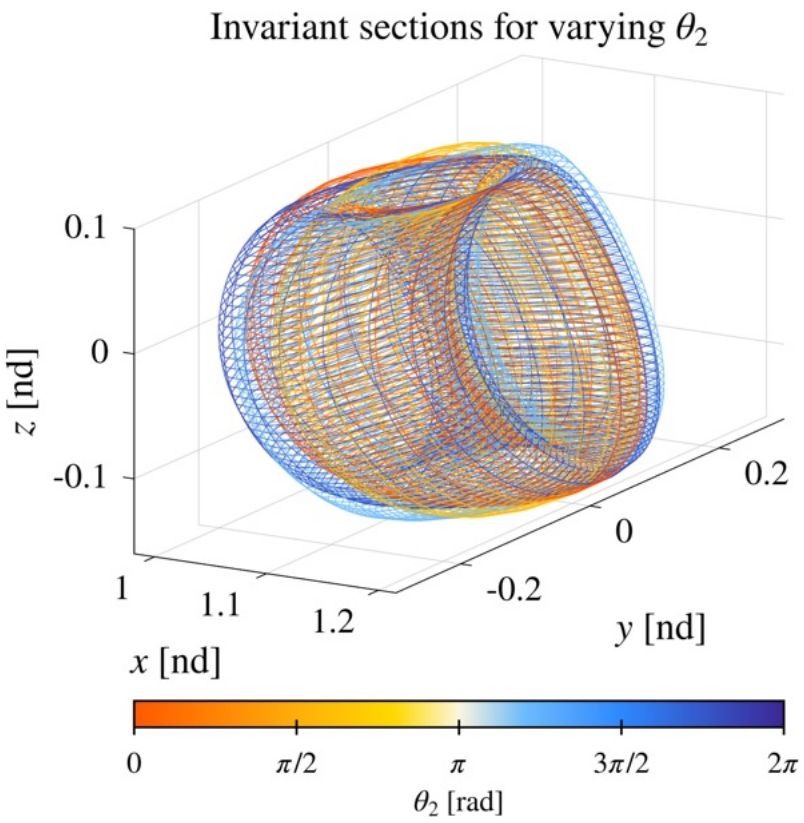
Effect of the lunar eccentricity

INVARIANT SECTIONS

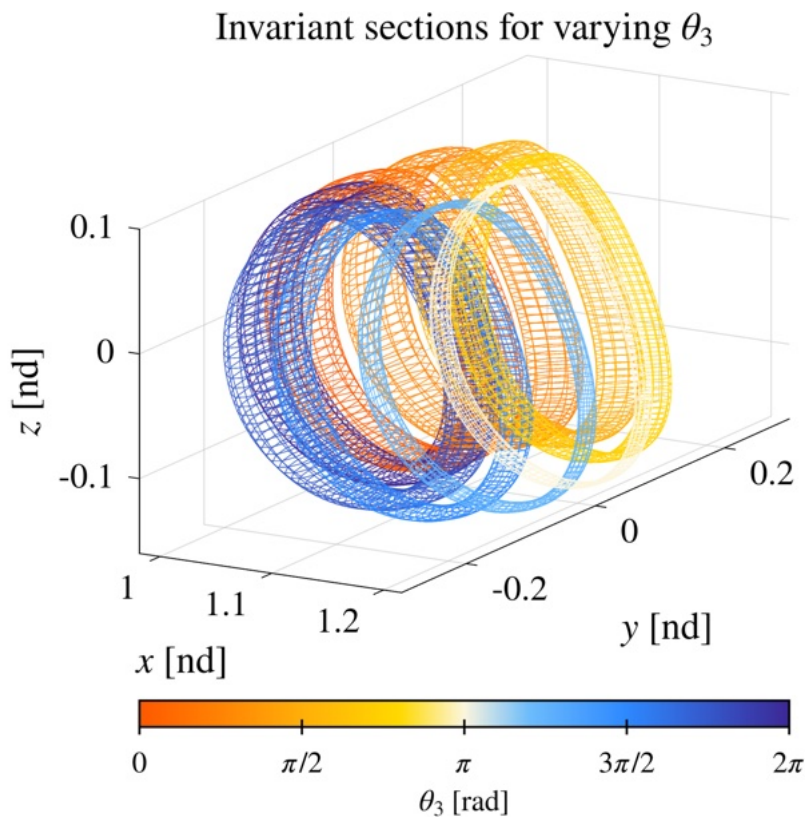
L_2 HALO ORBIT



Effect of the uniform-rotating frame



Effect of the Sun's perturbation



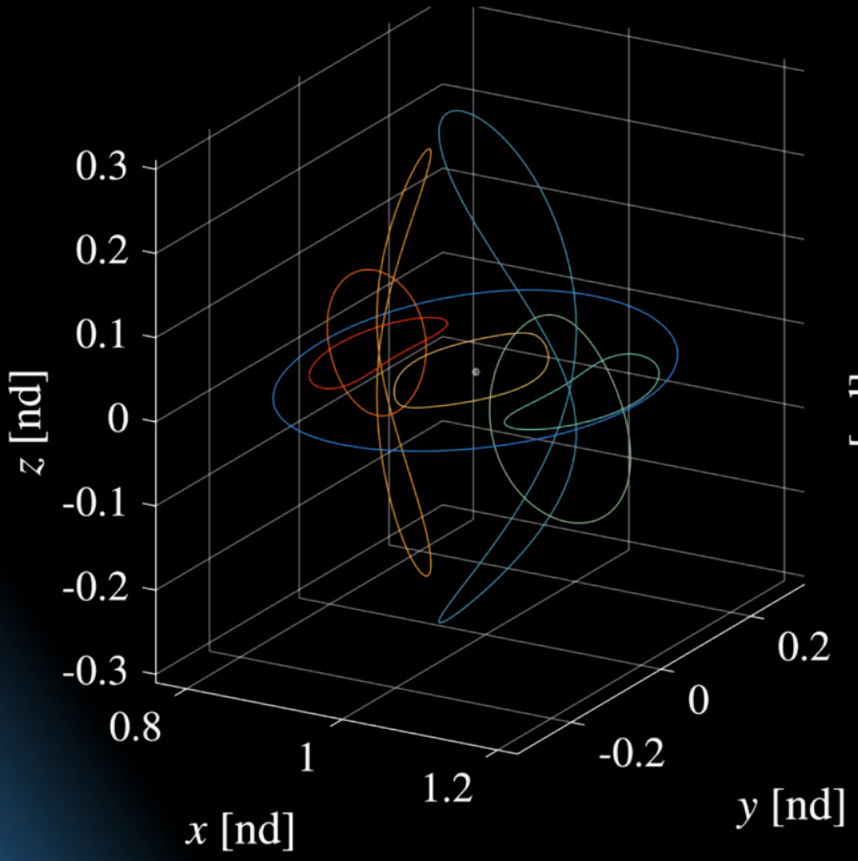
Effect of the lunar eccentricity

EHR4BP 3D QPO

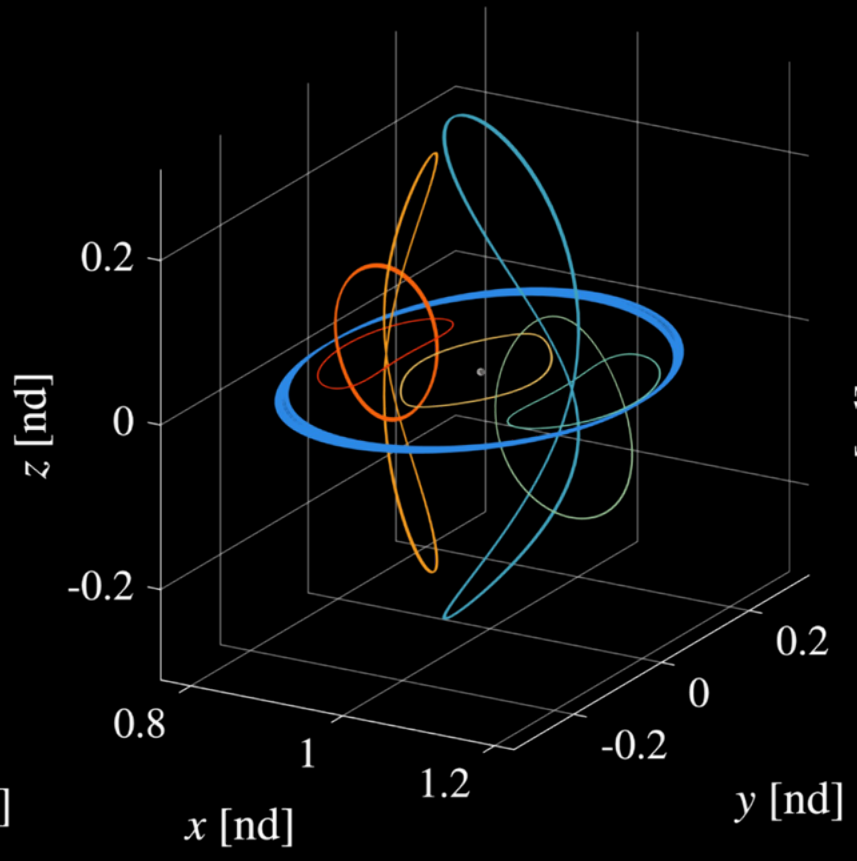
ROTOPULSATING FRAME

CR3BP PERIODIC ORBTS (period in days)		
	LYAP L_1 (13.0)	HALO L_2 (14.0)
	HALO L_1 (12.1)	LYAP L_2 (15.2)
	VERT L_1 (18.3)	VERT L_2 (21.1)
	DPO (12.2)	DRO (15.3)

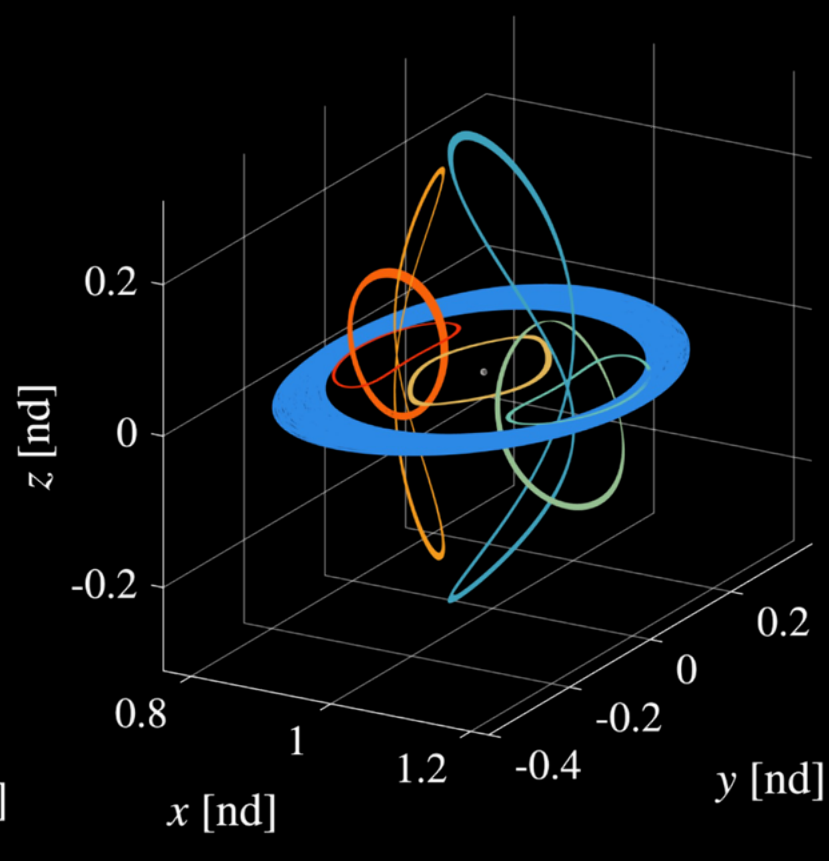
CR3BP



HR4BP

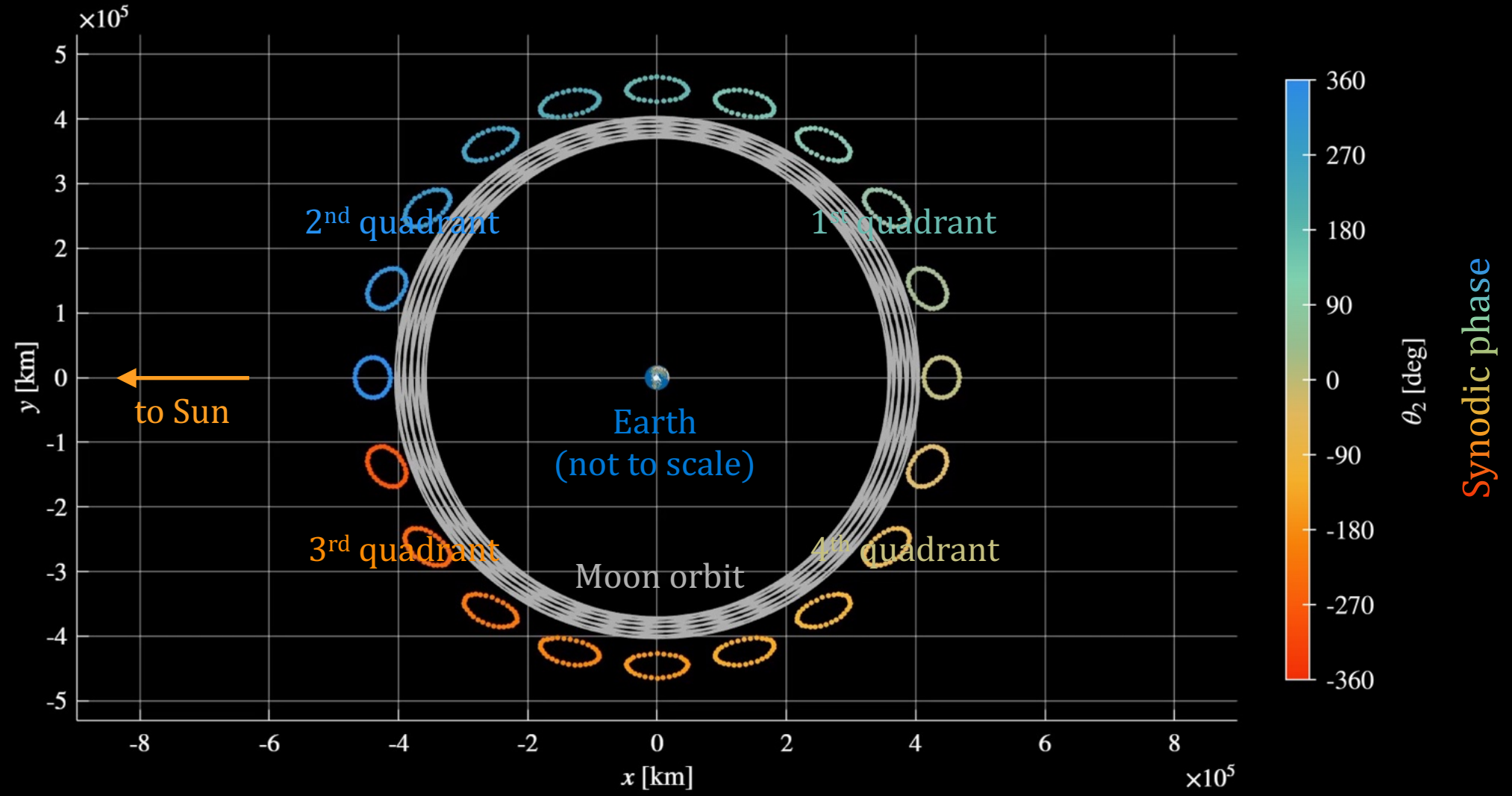


EHR4BP



STABLE WHISKERS

EHR4BP, L_2 HALO ORBIT, SUN–EARTH FRAME



POLITECNICO MILANO 1863

DART Group

THANK FOR THE ATTENTION



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Ninth International Meeting on Celestial Mechanics

September 2026,
San Martino al Cimino, Italy