

An Arnold diffusion mechanism for the Galileo satellites

Mar Giralt
Universitat Politècnica de Catalunya

Joint work with Elisa Maria Alessi, Inma Baldomá and Marcel Guardia

CELMEC IX

September 17th, 2026

Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

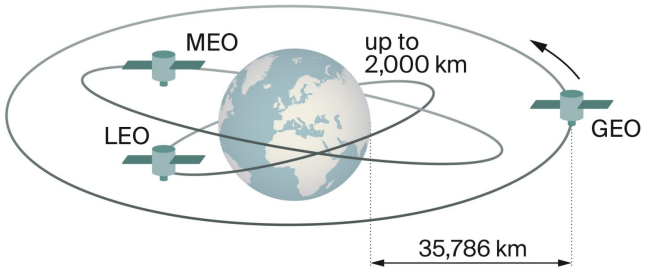
4 Coplanar case

5 Inclined case

6 Conclusions

The space debris problem

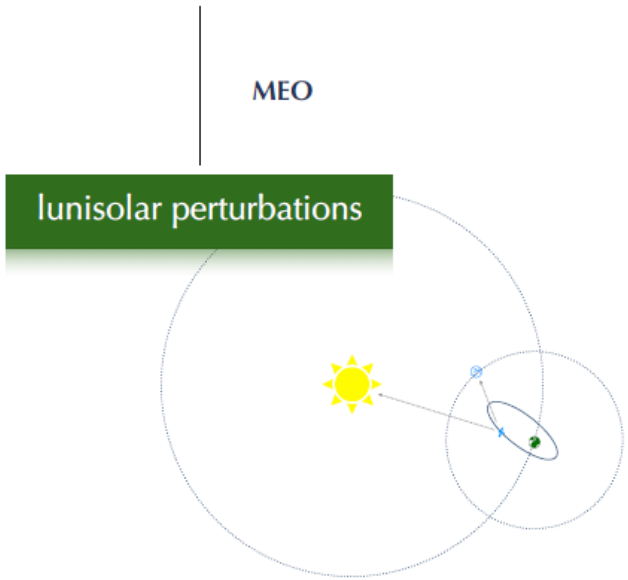
Space debris: all non-functional, artificial objects, including fragments and elements thereof, in Earth orbit or re-entering into Earth's atmosphere.



- There exist well-defined guidelines for end-of-life disposal of LEO and GEO satellites.
- Discussions are still ongoing for the MEO region.
  - Broad region and still not critical in terms of density.
  - Given the high altitude, it is convenient to use circular graveyard orbits.
  - The accumulation of non-operational satellites in a limited region is dangerous (collisions, fragmentation).

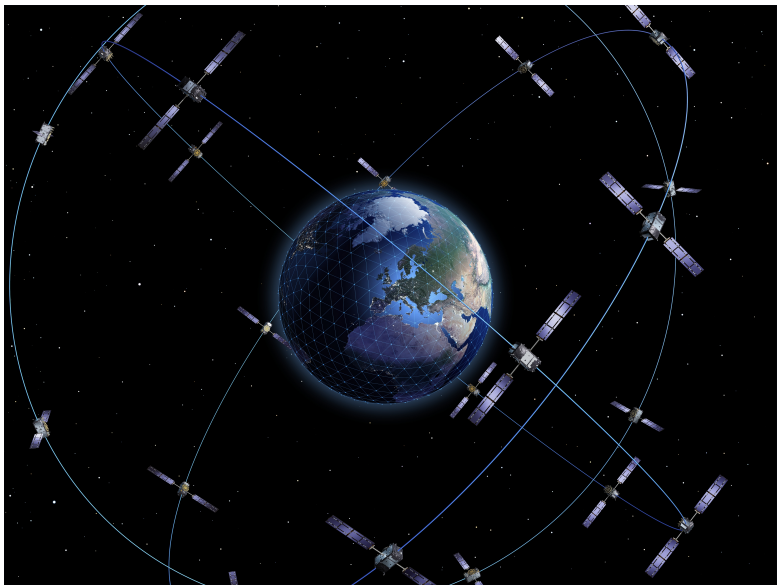
Lunisolar perturbations

In 2005, Jenkin and Gick proposed to dilute the collision probability by increasing the orbital eccentricity of the satellites at the end of life.



Can we exploit natural perturbations to help increase the eccentricity to avoid collision or facilitate reentering?

Objective



Source: P. Carril. ESA.

Galileo is a global navigation satellite system that went live in 2016, created by the European Union through the European Space Agency (ESA).

It consists on a constellation of 30 satellites situated in the MEO region.

Objective: to show how an Arnold diffusion mechanism can trigger the eccentricity growth of the Galileo satellites

Our study focus in Galileo, but can be reproduced for other MEO satellites.

Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

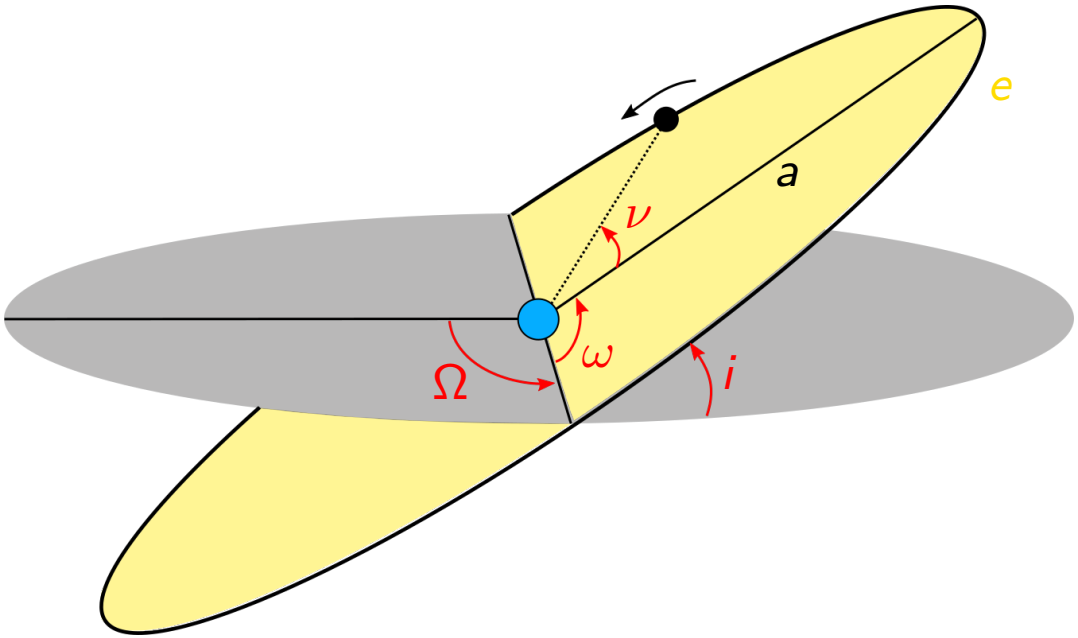
4 Coplanar case

5 Inclined case

6 Conclusions

Orbital elements

We consider the dynamics of the Earth-Satellite in a geocentric equatorial frame:

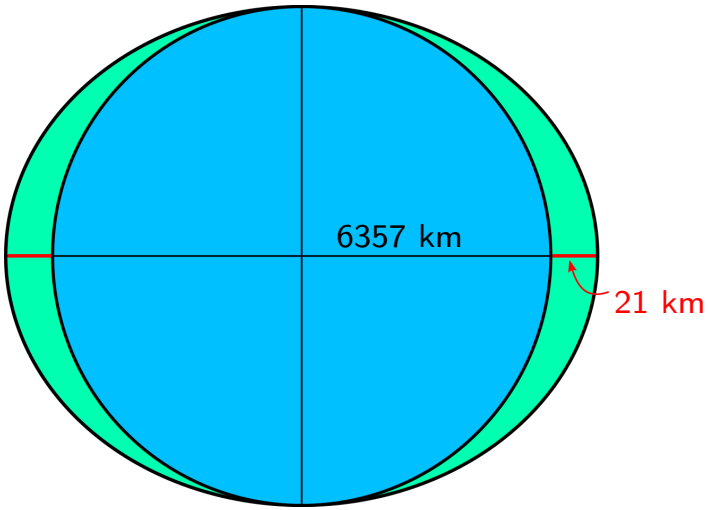


- a: semi-major axis.
- e: eccentricity.
- i: inclination.
- Ω: longitude of the ascending node.
- ω: argument of the periapsis.
- ν: True anomaly $\leftrightarrow$ M: mean anomaly.

Galileo: $a \approx 29.600 km$, $e \approx 0$, $i = 55^\circ \pm 2^\circ$

The J_2 model

We consider a spacecraft that is affected by the **gravitational attraction** of the Earth and the perturbation due to the **Earth's oblateness**.



- We use Delaunay coordinates:

Actions	Angles
$L = \sqrt{\mu a},$	$\ell = M,$
$G = L\sqrt{1-e^2},$	$g = \omega,$
$H = G\cos i,$	$h = \Omega.$

- Secular model: we average over the fast angle ℓ.

$$\mathrm{H}_{\mathrm{secular}}(L,G,H) = \mathrm{H}_{\mathrm{Kepler}}(L) + \mathrm{H}_0(L,G,H)$$

where

$$\mathrm{H}_{\mathrm{Kepler}}(L) = -\frac{\mu^2}{2L^2}, \qquad \mathrm{H}_0(L,G,H) = \frac{\rho_0}{4L^3}\frac{G^2-3H^2}{L^3G^5}.$$

$\rho_0 = \mu^4 J_2 R^2$, μ =mass param. of Earth, J_2 =coeff. of the 2nd zonal harmonic, R =Earth mean equatorial radius.

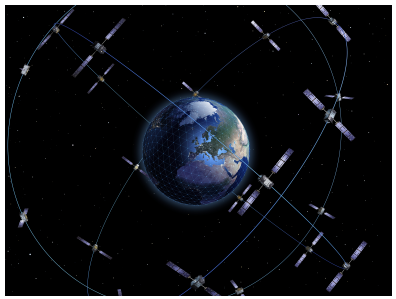
The $2g + h$ resonance

The resonance $2g + h$ is given by the oblateness effect of the Earth.

$$\mathtt{H}_0(L, G, H) = \frac{\rho_0}{4L^3} \frac{G^2 - 3H^2}{L^3 G^5}.$$

It occurs when $5H^2 - G^2 - HG = 0$, which corresponds to inclinations

$$i_\star = 56.06^\circ \quad \text{and} \quad i_\star = 110.99^\circ.$$



Source: P. Carril. ESA.

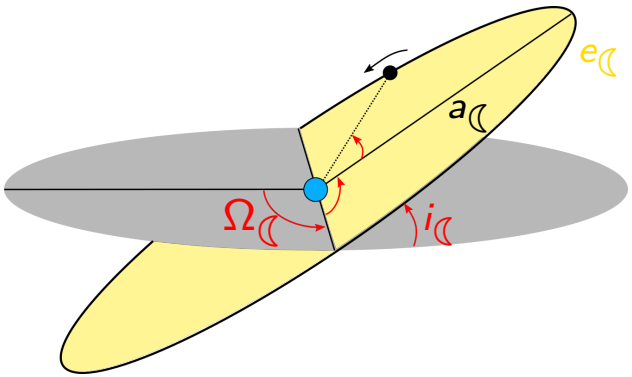
The Galileo satellites are found close to this resonance with $i_\star = 56.06^\circ$.

The goal of our work is to explain how an **Arnold diffusion mechanism** can enhance drift in eccentricity along the $2g + h$ single resonance region.

We drop the dependence of the Hamiltonian on constant L.

Third body effect: the Moon

Consider the secular lunar gravitational pert., in a geocentric ecliptic framework:



- We average over the fast angle, $\ell_{\leftmoon}$.
- $a_{\leftmoon} \approx 384400$ km, $e_{\leftmoon} \approx 0.055$, $\imath_{\leftmoon} = 5.15^\circ$.
- Time dependence of the longitude of the ascending node:

$$\Omega_{\leftmoon}(t) = n_{\Omega_{\leftmoon}}\, t + \Omega_{0,\leftmoon} .$$

The system is described by a 2 and $\frac{1}{2}$ degrees of freedom Hamiltonian:

$$H(G, H, g, h, \Omega_{\leftmoon}; \alpha, i_{\leftmoon}) = H_0(G, H) + \alpha^3 H_1(G, H, g, h, \Omega_{\leftmoon}; i_{\leftmoon})$$

where

$$H_1(G, H, g, h, \Omega_{\leftmoon}; i_{\leftmoon}) = -\frac{\rho_1}{L^2} \sum_{s,m,p=0}^{2} D_{m,p,s}(L, G, H; i_{\leftmoon}) \cos\left(\Psi_{m,p,s}(g, h, \Omega_{\leftmoon})\right] .$$

$\alpha = \frac{a}{a_{\leftmoon}}$, $\rho_1 = \frac{\mu\mu_{\leftmoon}}{(1-e_{\leftmoon}^2)^{3/2}}$ and $\mu_{\leftmoon}$ = mass parameter of the Moon.

Main result

- We take realistic values for α:

$$\alpha = \frac{a}{a_{\leftmoon}} = 0.077.$$

- Realistic value $i_{\leftmoon} \approx 5.15^\circ$. We consider $i_{\leftmoon}$ small enough

$$0 < i_{\leftmoon} \ll 1.$$

Theorem (E.M. Alessi, I. Baldomá, M.G., M. Guardia)

Assume certain numerical ansätze. Then, for $i_{\leftmoon} > 0$ small enough, there exist time $T > 0$ and a trajectory $z(t) = (G(t), H(t), g(t), h(t), \Omega_{\leftmoon}(t))$ such that

$$\mathtt{H}(z(0)) = 0 \qquad \text{and} \qquad \mathtt{H}(z(T)) = 1.4 \cdot 10^{-6}.$$

Moreover, along the energy increase the osculating eccentricity and inclination satisfy

$$e(0) = 0 \qquad \text{and} \qquad e(T) = 0.8.$$

Past Approaches

- Effect of the lunisolar perturbation leading to a natural eccentricity growth up to reentry.

  Rossi 2008. Radtke, Domínguez-González, Flegel, Sánchez-Ortiz, Merz 2015.
  Alessi, Deleflie, Rosengren, Rossi, Valsecchi, Daquin, Merz 2016.
  Armellin, San-Juan 2018. Gondelach, Armellin, Wittig 2019.
  Pellegrino, Scheeres, Streetman 2021 and 2022.

  An exhaustive theoretical explanation of the underlying mechanism is still necessary.

  - Resonance overlap conducting to chaotic regions with eccentricity growth solutions.

    Rosengren, Alessi, Rossi, Valsecchi 2015.
    Daquin, Rosengren, Alessi, Deleflie, Valsecchi, Rossi 2016.

  - Single resonance mechanism by means of analyzing a 1 degree of freedom averaged first order.

    Gkolias, Daquin, Skoulidou, Tsiganis, Efthymiopoulos 2019.
    Daquin, Legnaro, Gkolias, Efthymiopoulos 2022. Legnaro, Efthymiopoulos 2023.

Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

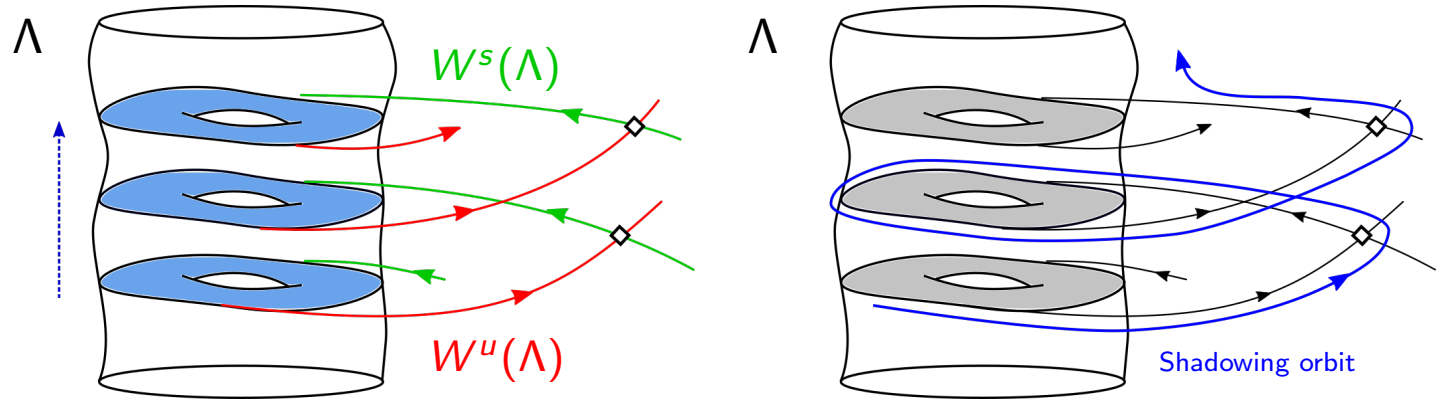
4 Coplanar case

5 Inclined case

6 Conclusions

Arnold diffusion mechanism

Arnold diffusion mechanism: construction of a sequence of heteroclinic orbits that connect invariant tori at different energy levels on a NHIM.



- **Pseudo-orbit**: formed by pieces of orbits in the cylinder (inner dynamics) which are connected by heteroclinic orbits (outer dynamics).
- The final step is to show that there is a true orbit of the Hamiltonian which **"shadows"** (i.e. follows closely) the pseudo-orbit.

Background

- Arnold showed, in a cleverly chosen model, that actions can vary drastically in nearly integrable Hamiltonian systems (of at least 2 and $\frac{1}{2}$ degrees of freedom).

  V. I. Arnold. 1964.
  Instability of dynamical systems with many degrees of freedom.

- The mechanism we propose in this work relies on geometric tools, which have been developed in:

  Bolotin, Treschev 1999. Delshams, de la Llave, Seara 2000 and 2006. Gelfreich, Turaev 2008. Delshams, Huguet 2009. Gelfreich, Turaev 2017. Delshams, Kaloshin, de la Rosa, Seara 2019.

- In this work, we combine analytical and numerical techniques to describe the coplanar model. Following the approach in:

  Féjoz, Guardia, Kaloshin, Roldán, 2016.
  Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem.

Change of coordinates

Two changes of coordinates to focus on the resonant region.

$$(H, G, h, g) \mapsto (y, \Gamma, x, h) \mapsto (\eta, \Gamma, \xi, h)$$

- First change: slow-fast angles

$$x = 2g + h \quad \text{slow (resonant) angle} \qquad h \quad \text{fast angle}$$

$$y = \frac{G}{2} \qquad \Gamma = H - \frac{G}{2}.$$

- Second change: Poincaré coordinates in order to work close to circular orbits

$$\eta = \sqrt{2L - 4y}\cos x, \qquad \xi = \sqrt{2L - 4y}\sin x.$$

The system in new coordinates is described by the 2 and $\frac{1}{2}$ degrees of freedom Hamiltonian:

$$\mathcal{H}(\eta, \Gamma, \xi, h, \Omega_{\mathbb{C}}\,; \alpha, i_{\mathbb{C}}) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_1(\eta, \Gamma, \xi, h, \Omega_{\mathbb{C}}\,; i_{\mathbb{C}})$$

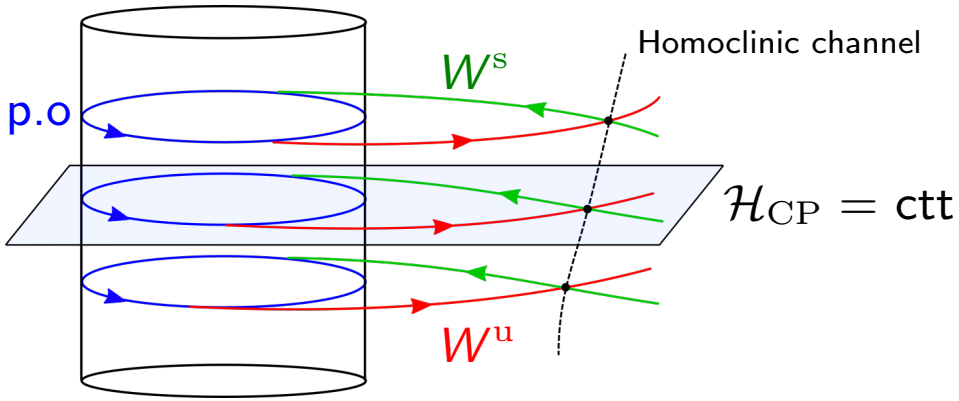
The coplanar model

Since we are assuming $0 < i_{\leftmoon} \ll 1$, first we study the coplanar model, $i_{\leftmoon} = 0$:

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}(\eta, \Gamma, \xi, h, \Omega_{\leftmoon}\,; \alpha, 0).$$

The coplanar model is a 2 degrees of freedom **autonomous** Hamiltonian system. It is independent on the time given by the coordinate $\Omega_{\leftmoon}$.

We **check numerically** that it has a normally hyperbolic cylinder with stable and unstable manifolds that intersect transversally.



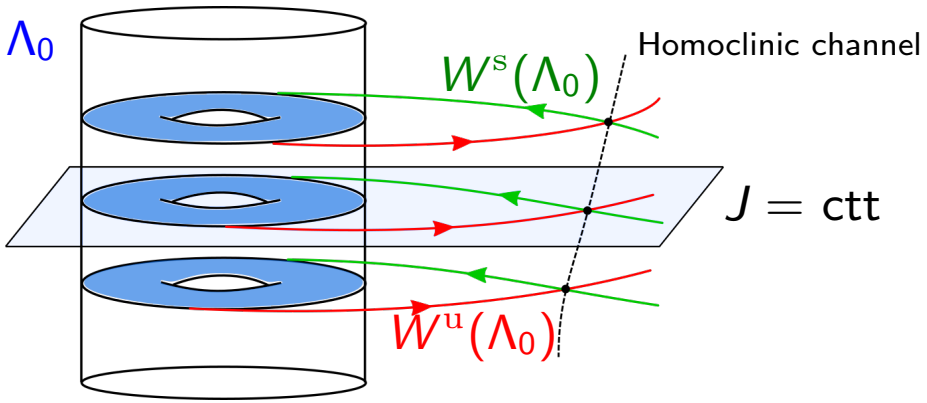
The coplanar model

Since we are assuming $0 < i_{\text{☾}} \ll 1$, first we study the coplanar model, $i_{\text{☾}} = 0$:

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}(\eta, \Gamma, \xi, h, \Omega_{\text{☾}}\,; \alpha, 0).$$

The coplanar model is a 2 degrees of freedom **autonomous** Hamiltonian system. It is independent on the time given by the coordinate $\Omega_{\text{☾}}$.

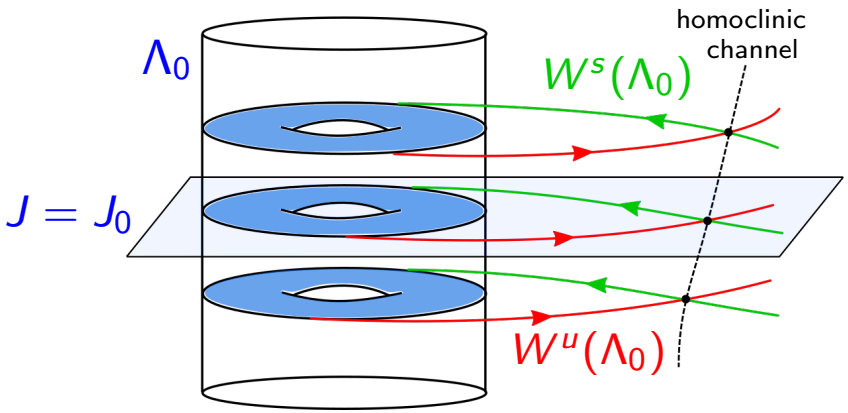
We **check numerically** that it has a normally hyperbolic cylinder with stable and unstable manifolds that intersect transversally.

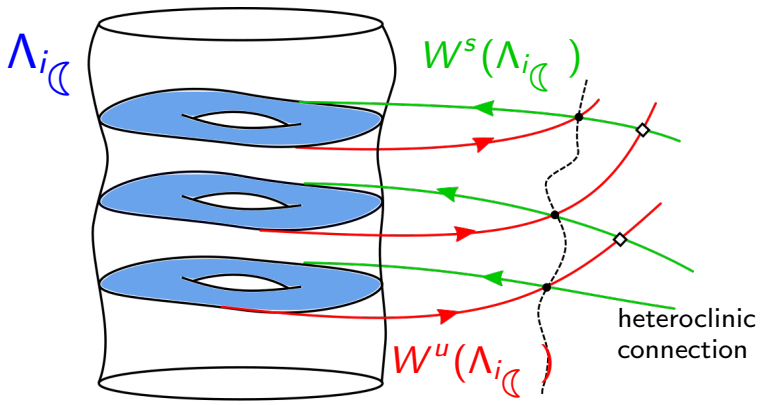


We consider the **extended phase space**, adding the pair $(J, \Omega_{\text{☾}})$, where J is related to the energy Hamiltonian level $\mathcal{H}_{\mathrm{CP}}$.

Inclined model

Classical Fenichel Theory implies that the invariant cylinder and its stable and unstable manifold persists when adding a small perturbation.





For $i_{\mathbb{C}} = 0$:

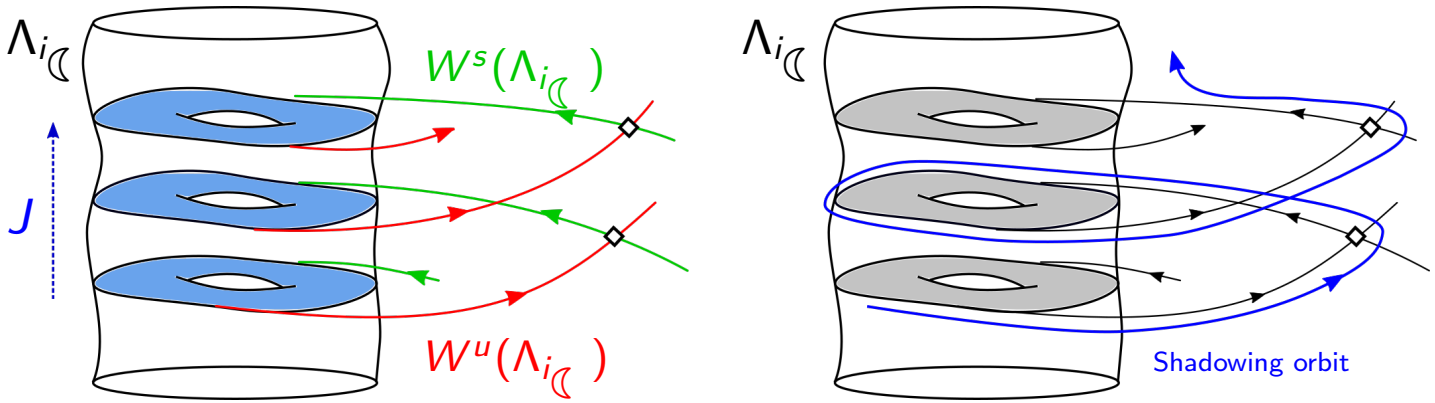
- There are not enough dimensions to achieve Arnold diffusion.

For $0 < i_{\mathbb{C}} \ll 1$:

- J is not a first integral anymore.
- One can construct heteroclinic connections between different tori.

Arnold diffusion mechanism

Arnold diffusion mechanism: construction of a sequence of heteroclinic orbits that connect invariant tori at different energy levels on a NHIM.



- **Pseudo-orbit**: formed by pieces of orbits in the cylinder (inner dynamics) which are connected by heteroclinic orbits (outer dynamics).
- The final step is to show that there is a true orbit of the Hamiltonian which **"shadows"** (i.e. follows closely) the pseudo-orbit.

Increasing the energy level J implies larger homoclinic excursions which leads to **larger oscillations in eccentricity**.

Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

4 Coplanar case

5 Inclined case

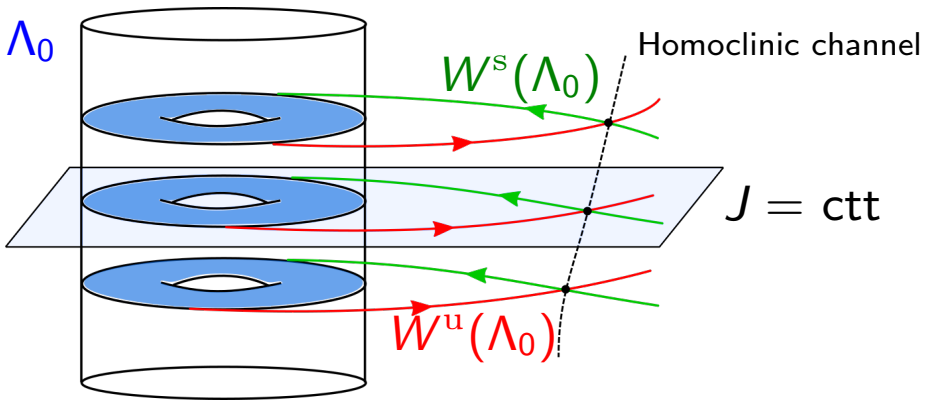
6 Conclusions

The coplanar model

The coplanar model is a 2 degrees of freedom **autonomous** Hamiltonian system. It is independent on the time given by the coordinate $\Omega_{\text{(}}$, since $i_{\text{(}} = 0$.

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}(\eta, \Gamma, \xi, h, \Omega_{\text{(}}\,; \alpha, 0).$$

We **check numerically** that it has a normally hyperbolic cylinder with stable and unstable manifolds that intersect transversally.



We consider the **extended phase space**, adding the pair $(J, \Omega_{\text{(}})$, where J is related to the energy Hamiltonian level $\mathcal{H}_{\mathrm{CP}}$.

The coplanar Hamiltonian

The coplanar Hamiltonian:

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

- Since we consider $\alpha = 0.07$ as a fixed parameter, the application of perturbative techniques to analyze $\mathcal{H}_{\mathrm{CP}}$ are not possible.

- We assume certain ansätze on the Hamiltonian $\mathcal{H}_{\mathrm{CP}}$ and verify them numerically.

Periodic orbits

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

Ansatz 1: In every energy level $E \in [E_-, E_+] = [-2 \times 10^{-7}, 1.35 \times 10^{-6}]$, there exists a hyperbolic periodic orbit $(\eta, \Gamma, \xi, h) = (0, \Gamma_E(t), 0, h_E(t))$.

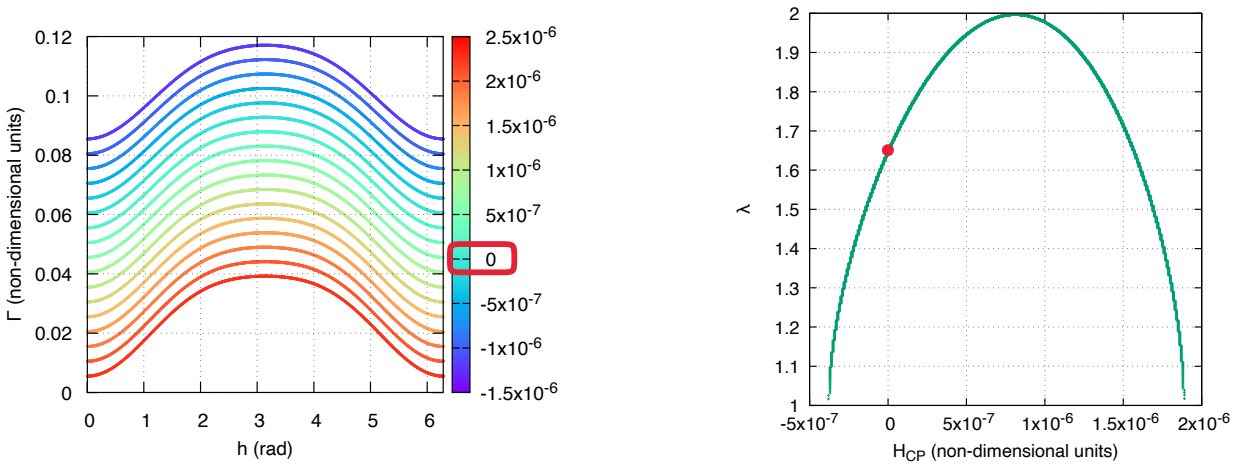


Figure: (Left) Examples of periodic orbits, the colorbar reports the value of $\mathcal{H}_{\mathrm{CP}}$. (Right) The value of the eigenvalue greater than 1 as a function of the $\mathcal{H}_{\mathrm{CP}}$.

The period

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

Ansatz 2: The period of the periodic orbit satisfy that $n_{\Omega_{\leftmoon}} \mathcal{T}_E \in [3.9\pi, 4.16\pi]$. In addition, $\mathcal{T}_E$ is an strictly increasing function with respect to E.

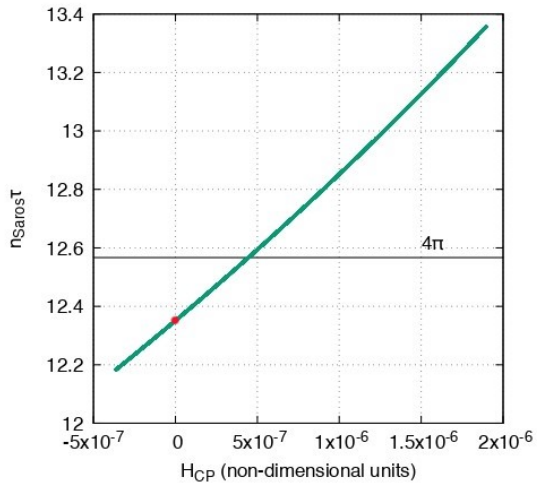


Figure: The period $n_{\Omega_{\leftmoon}} \mathcal{T}_E$ of the hyperbolic periodic orbits as a function of $\mathcal{H}_{\mathrm{CP}}$.

When the period is 4π, corresponds to the (weak) **double resonance** $2h - \Omega_{\leftmoon}$.

Stable and unstable manifolds

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

Ansatz 3: Every p.o. has two branches of stable and unstable manifolds.

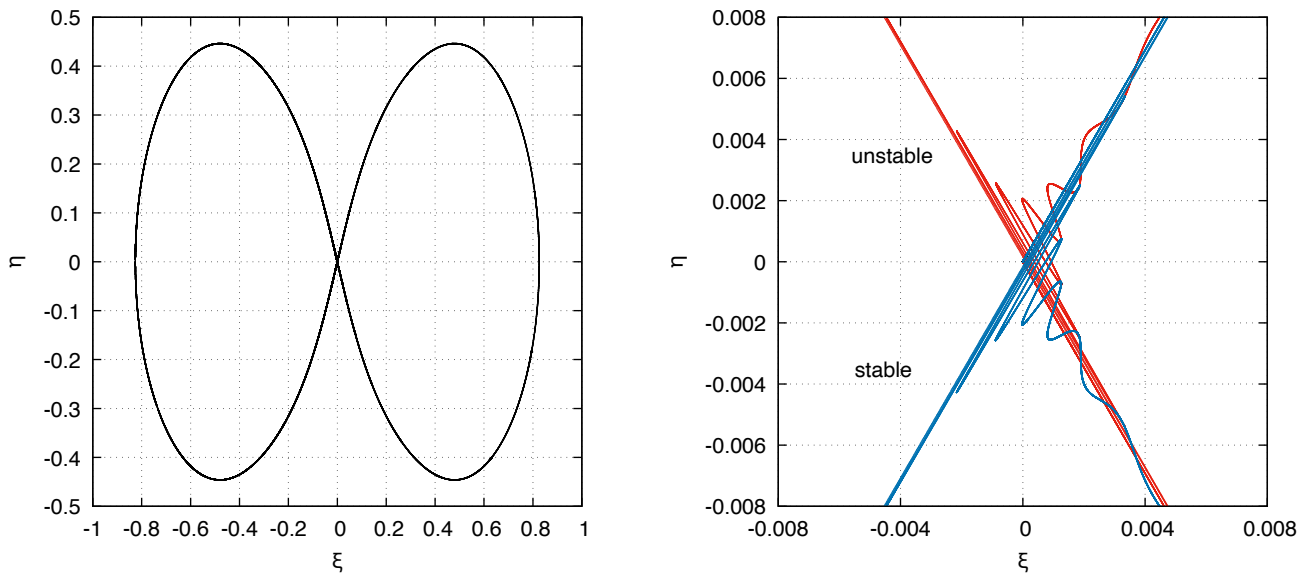


Figure: The behavior of the hyperbolic manifolds on the Poincaré section $\{h = 0\}$.

Transversal intersections

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

Ansatz 4: At least one of the branches of stable and unstable manifolds has a transverse intersection.

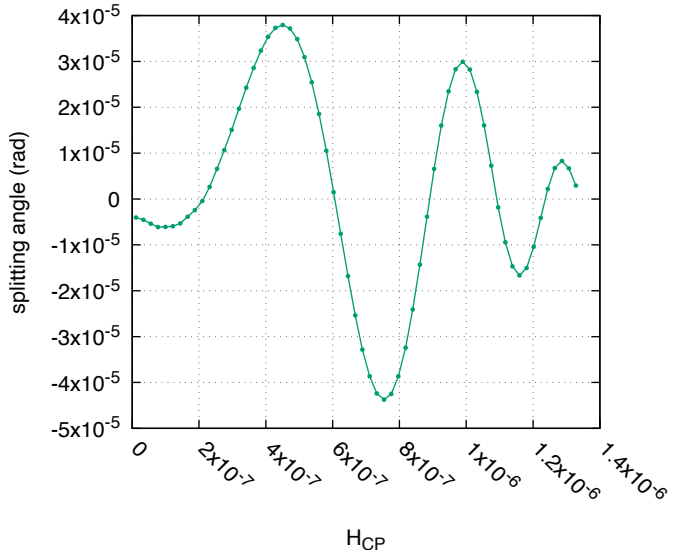
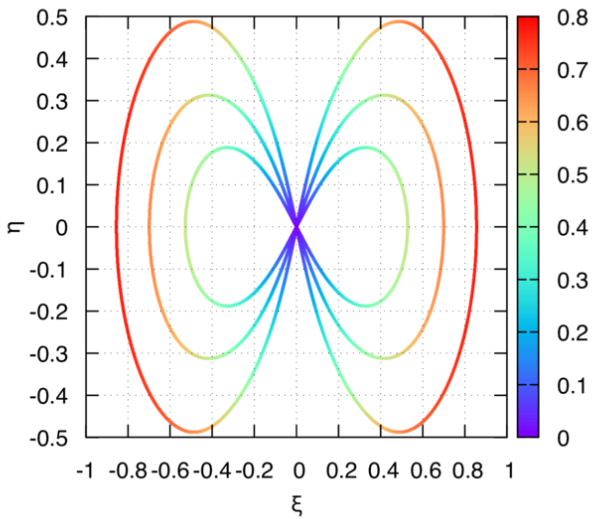


Figure: Splitting angle between one branch of the stable and the unstable manifolds.

Growth in eccentricity

$$\mathcal{H}_{\mathrm{CP}}(\eta, \Gamma, \xi, h; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h).$$

Remark: Increasing the energy level implies larger homoclinic excursions which leads to larger oscillations in eccentricity



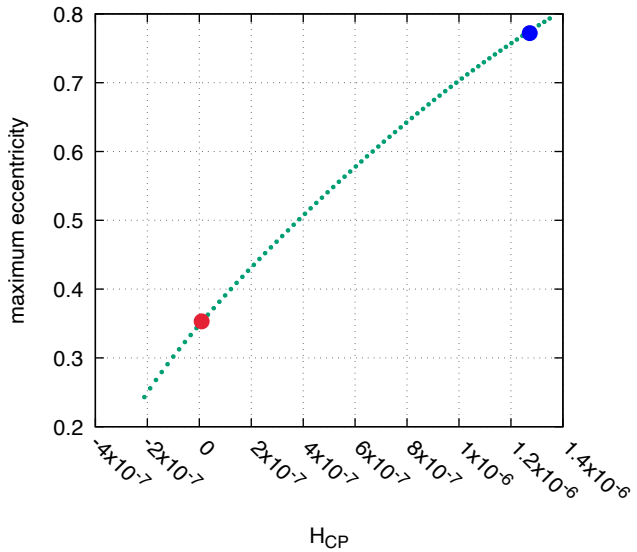
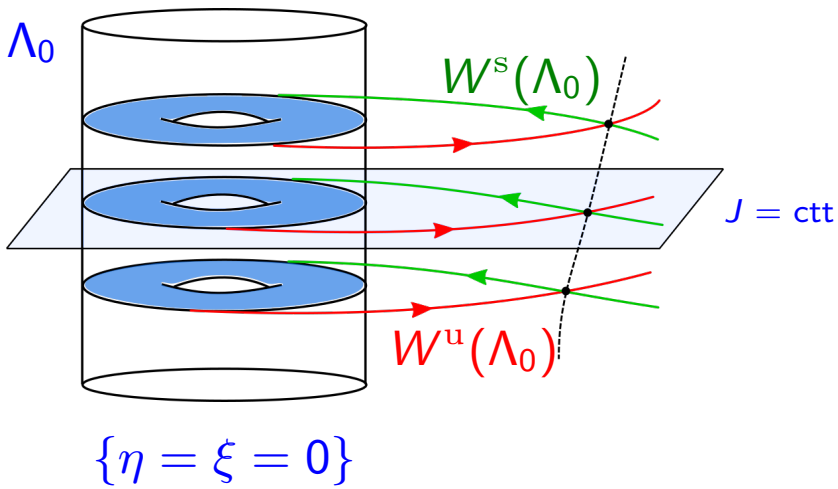


Figure: (Left) example of the eccentricity growth (colorbar) along an unstable invariant manifold on the section $\{h = 0\}$.
(Right) the maximum eccentricity achieved in each energy level.

The inner and the outer dynamics

Extended coplanar Hamiltonian:

$$\mathcal{K}_{\mathrm{CP}}(\eta, \Gamma, J, \xi, h, \Omega_{\mathbb{C}}\,; \alpha) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\mathrm{CP},1}(\eta, \Gamma, \xi, h) + n_{\Omega_{\mathbb{C}}}\, J.$$



We reduce the dimension of the model:

- A Poincaré map at section $\{h = 0\}$.
- Eliminating the variable Γ through energy conservation.

The cylinder Λ_0 becomes two dimensional. A good system of coordinates for it is $(J, \Omega_{\mathbb{C}})$.

The inner and outer dynamics

- The **inner map**:

$$\mathcal{F}_0^{\text{in}} : \begin{pmatrix} J \\ \Omega_{\leftmoon} \end{pmatrix} \mapsto \begin{pmatrix} J \\ \Omega_{\leftmoon} + n_{\Omega_{\leftmoon}} \mathcal{T}_0(\mathcal{J}) \end{pmatrix},$$

  where $\mathcal{T}_0(J)$ correspond with the period of the periodic orbits.

- The **scattering map** (for the two branches $j = 1, 2$):

$$\mathcal{F}_0^{\text{out},j} : \begin{pmatrix} J \\ \Omega_{\leftmoon} \end{pmatrix} \mapsto \begin{pmatrix} J \\ \Omega_{\leftmoon} + n_{\Omega_{\leftmoon}} \zeta^j(J) \end{pmatrix},$$

  where $\zeta^j(J)$ is the solution of a Melnikov-like integral computed with the help of the numerical ansätze.

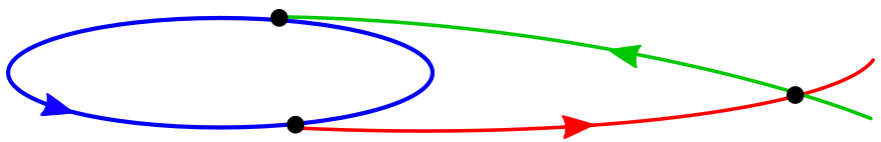


Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

4 Coplanar case

5 Inclined case

6 Conclusions

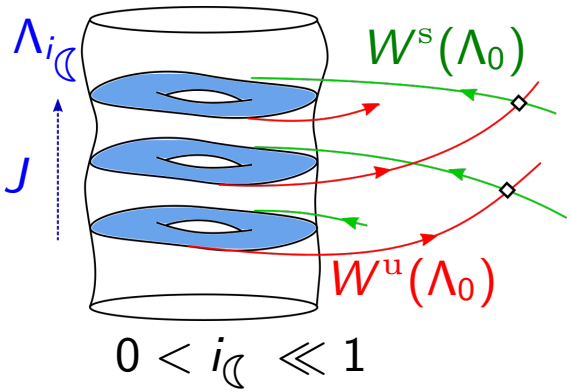
The inclined Hamiltonian

The inclined Hamiltonian:

$$\begin{aligned}\mathcal{H}(\eta,\Gamma,\xi,h,\Omega_{\text{(}};\alpha,i_{\text{(}}) =& \mathcal{H}_0(\eta,\Gamma,\xi)+\alpha^3\mathcal{H}_{\mathrm{CP},1}(\eta,\Gamma,\xi,h)\\ &+ i_{\text{(}}\,\alpha^3\mathcal{R}_1(\eta,\Gamma,\xi,h,\Omega_{\text{(}})\\ &+ i_{\text{(}}^2\,\alpha^3\mathcal{R}_2(\eta,\Gamma,\xi,h,\Omega_{\text{(}})\\ &+ \mathcal{O}(i_{\text{(}}^3)\end{aligned}$$

- Since we consider $0 < i_{\text{(}} \ll 1$, we take a series in $i_{\text{(}}$ of the Hamiltonian.
- Our analysis relies in the understanding of the harmonic structures:
  - The term $\mathcal{R}_1$ only has harmonics $\{e^{\pm i\Omega_{\text{(}}}\}$.
  - The term $\mathcal{R}_2$ has harmonics $\{0, e^{\pm 2i\Omega_{\text{(}}}\}$.

Construction of the pseudo-orbit



Pseudo-orbit: formed by pieces of orbits in the cylinder (inner dynamics) which are connected by heteroclinic orbits (outer dynamics)

- The **inner map**:

$$\mathcal{F}^{\mathrm{in}}_{i_{\mathfrak{C}}} : \begin{pmatrix} J \\ \Omega_{\mathfrak{C}} \end{pmatrix} \mapsto \begin{pmatrix} J + i_{\mathfrak{C}} \left(e^{i\Omega_{\mathfrak{C}}} A_1^+(J) + e^{-i\Omega_{\mathfrak{C}}} A_1^-(J) \right) + \mathcal{O}(i_{\mathfrak{C}}^2) \\ \Omega_{\mathfrak{C}} + n_{\Omega_{\mathfrak{C}}} \mathcal{T}_0(\mathcal{J}) + \mathcal{O}(i_{\mathfrak{C}}) \end{pmatrix}.$$

- The **scattering map** (for the two branches $j = 1, 2$):

$$\mathcal{F}^{\mathrm{out},j}_0 : \begin{pmatrix} J \\ \Omega_{\mathfrak{C}} \end{pmatrix} \mapsto \begin{pmatrix} J + i_{\mathfrak{C}} (e^{i\Omega_{\mathfrak{C}}} B_1^{+,j}(J) + e^{-i\Omega_{\mathfrak{C}}} B_1^{-,j}(J)) + \mathcal{O}(i_{\mathfrak{C}}^2) \\ \Omega_{\mathfrak{C}} + n_{\Omega_{\mathfrak{C}}} \zeta^j(J) + \mathcal{O}(i_{\mathfrak{C}}) \end{pmatrix}.$$

The values $A_1^{\pm}(J)$ and $B_1^{\pm}(J)$ are the solutions of Melnikov-like integrals computed with the help of the numerical ansätze for the coplanar model.

Pseudo-orbit result

Ansatz 5: The following functions do not vanish (at least in one branch) for the energy levels $J \in [J_-, J_+]$:

$$f_\pm^j(J) = \left(e^{\pm i n_{\Omega_{\mathbb{C}}} \mathcal{T}_0(J)} - 1\right) B_1^{\pm,j}(J) - \left(e^{\pm i n_{\Omega_{\mathbb{C}}} \zeta^j(J)} - 1\right) A_1^{\pm}(J).$$

Theorem (E.M. Alessi, I. Baldomá, M.G., M. Guardia)

Assume the previous numerical ansätze. For $i_{\mathbb{C}} > 0$ small enough, there exist a sequence $z_0, \ldots, z_N \in \Lambda_{i_{\mathbb{C}}}$ such that

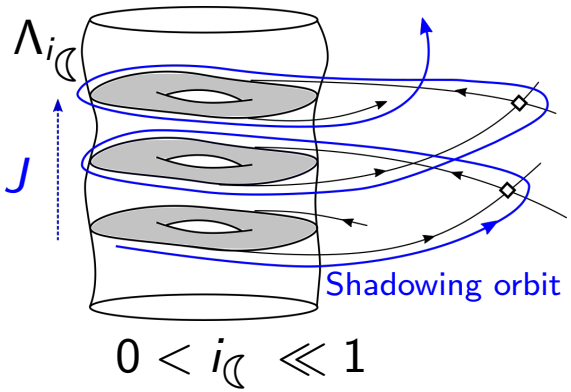
$$z_m = \left(\mathcal{F}^{\text{in}}_{\Omega_{\mathbb{C}}}\right)^{i_m} \circ \mathcal{F}^{\text{out},j_m}_{\Omega_{\mathbb{C}}} \circ \left(\mathcal{F}^{\text{in}}_{\Omega_{\mathbb{C}}}\right)^{k_m} (z_{m-1})$$

and the initial and final points satisfy

$$|\pi_J z_0 - J_-| \leq \delta \quad \text{and} \quad |\pi_J z_N - J_+| \leq \delta,$$

for some small $\delta > 0$.

Shadowing orbit



Our shadowing result relies on the shadowing Theorem in



Gidea, de la Llave, Seara 2020.

A general mechanism of diffusion in Hamiltonian systems: qualitative results.

Corollary (E.M. Alessi, I. Baldomá, M.G., M. Guardia)

Assume the previous numerical ansätze. Then, for $i_{(\!(} > 0$ small enough, there exists an orbit of the corresponding Poincaré map shadowing the pseudo-orbit found in the previous Theorem.

Table of contents

1 Background

2 Statement of the problem

3 Arnold diffusion mechanism

4 Coplanar case

5 Inclined case

6 Conclusions

Conclusions

- **Resonance overlapping at** $2h - \Omega_{\leftmoon}$: Since $i_{\leftmoon} > 0$ small, the second resonance is weak and it does not break up the invariant cylinder that exists along the $2g + h$ resonance.

- For $i_{\leftmoon} > 0$ small, the mechanism is rather slow. The drifting time is

$$T \approx \mathcal{O}\left(\frac{1}{i_{\leftmoon}}\right).$$

  However, this work lays the foundations to study the problem with the actual value of $i_{\leftmoon}$.

  We expect that the mechanism will persist and that the time of diffusion will reduce.

- The procedure developed can be applied to other cases, for instance to show where to locate initially the satellites to facilitate their end of life phase.

Thanks for your attention!

