# Normal Form of the Regularized Primary in the Spatial CR3BP

## IX International Meeting on Celestial Mechanics

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Ongoing work with **J.D. Mireles** (FAU) and **L. Peterson** (UT@Austin)

# The Circular Restricted Three-Body Problem

$$m_E = 1-\mu, \qquad m_M = \mu,$$

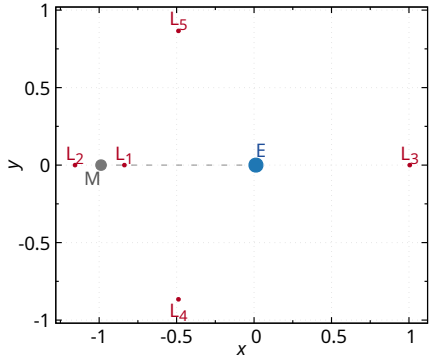
$$\mathbf{r}_E = (\mu, 0, 0), \qquad \mathbf{r}_M = (-(1-\mu), 0, 0).$$

$$r_E^2 = (x-\mu)^2+y^2+z^2, \quad r_M^2 = (x+1-\mu)^2+y^2+z^2,$$

$$\Omega(x,y,z) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_E} + \frac{\mu}{r_M}.$$

$$\ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y, \qquad \ddot{z} = \Omega_z.$$

$$\boxed{C = 2\Omega - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}$$




Earth–Moon: $\mu = 0.0121505856$.

# Overview




**KS lift**
charts, fiber,
regularized time

**physical subset**
first integrals
and constraints

**origin theorem**
critical value
and resonance

**normal form**
Hamiltonian
and time-h map

**next steps**

# Two spaces, one physical problem

R3BP lives in synodic variables;
KS lives in a larger regularized phase space with a **physical submanifold**.

**Synodic RTBP**

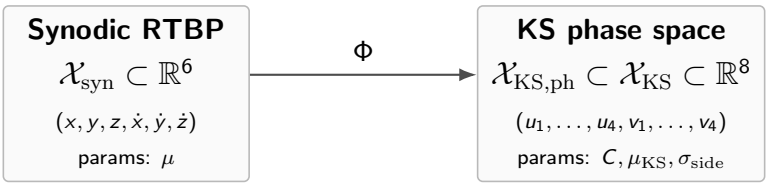
$\mathcal{X}_{\mathrm{syn}} \subset \mathbb{R}^6$

$(x, y, z, \dot{x}, \dot{y}, \dot{z})$

params: $\mu$

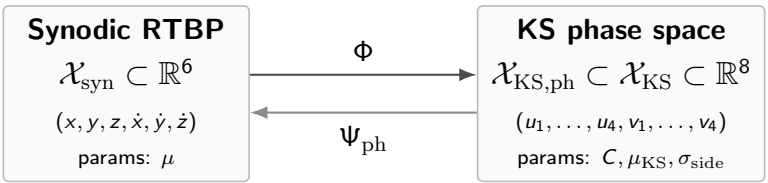
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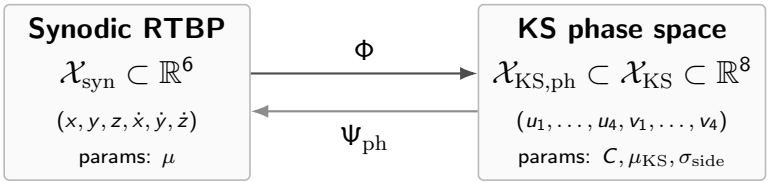
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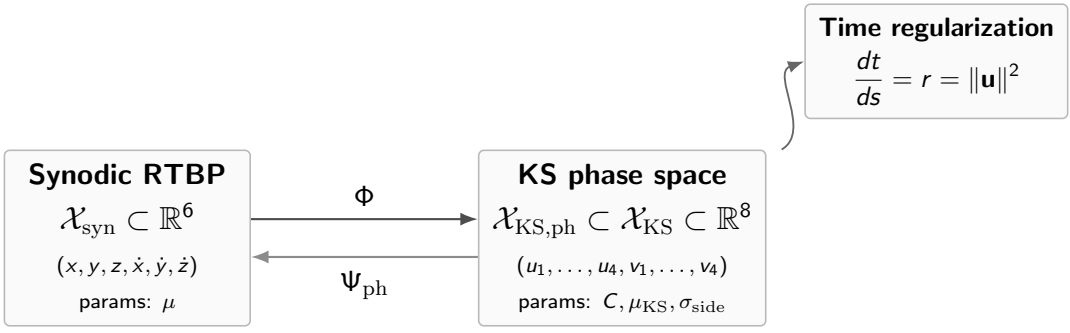



$$\Psi_{\rm ph} \circ \Phi = \mathrm{id}_{\mathcal{X}_{\rm syn}} \quad \text{on the physical submanifold.}$$

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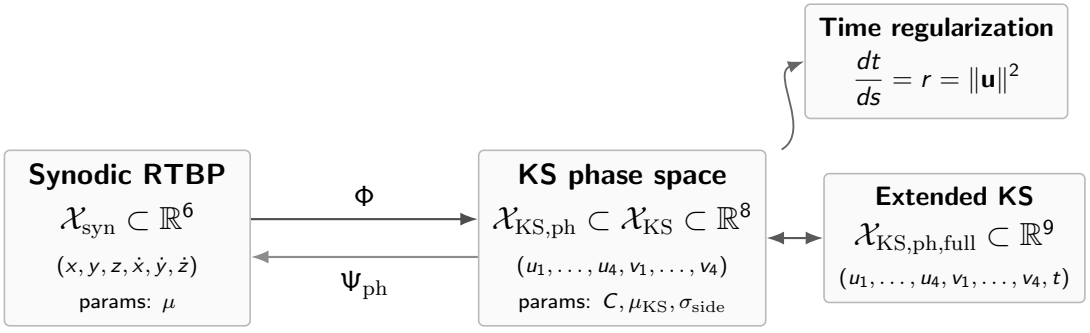



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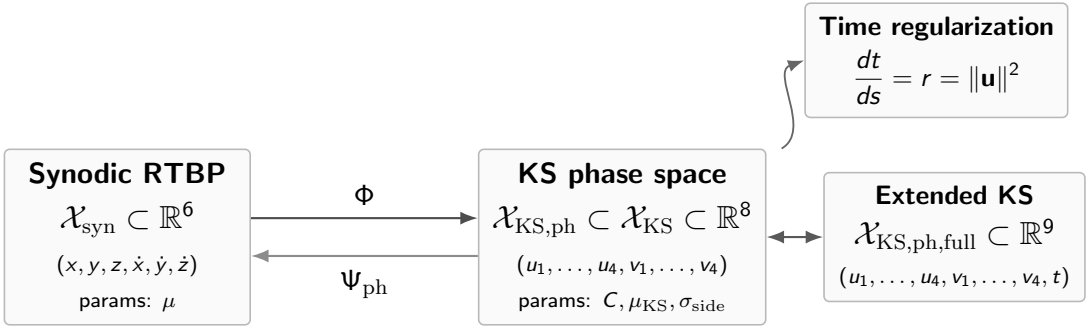



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$\Psi_{\text{ph}} \circ \Phi = \text{id}_{\mathcal{X}_{\text{syn}}}$ on the physical submanifold.

$\mathcal{X}_{\text{KS,ph}} \not\ni 0 \in \mathcal{X}_{\text{KS}}$

# Chart convention: $\mu_{\mathrm{KS}}$ and $\sigma_{\mathrm{side}}$

I use one formula for both primary charts. The table fixes the convention.

| Chart | regularized primary | location | $\sigma_{\mathrm{side}}$ | $\mu_{\mathrm{KS}}$ | $m_{\mathrm{reg}} = 1 - \mu_{\mathrm{KS}}$ |
|---|---|---|---|---|---|
| **primary 1** | large primary | $+\mu$ | $+1$ | $\mu$ | $1 - \mu$ |
| **primary 2** | small primary | $\mu - 1$ | $-1$ | $1 - \mu$ | $\mu$ |

**Earth–Moon values**

$\mu = 0.0121505856093$

primary 1 (Earth): $\mu_{\mathrm{KS}} = 0.01215$, $m_{\mathrm{reg}} = 0.98785$

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**$\mu_{\mathrm{KS}}$ is the mass of the other primary in this notation.**

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$$\boxed{x_{\mathrm{syn}} = x_{\mathrm{KS}} + \sigma_{\mathrm{side}}\,\mu_{\mathrm{KS}}}$$

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# Deterministic KS chart

For a given SYN position, choose the representative associated with the larger of

$$s_A = r + x_{\mathrm{KS}}, \qquad s_B = r - x_{\mathrm{KS}}, \qquad r = \sqrt{x_{\mathrm{KS}}^2 + y^2 + z^2}$$

**Chart A:** $s_A > 0$

$$u_1 = \sqrt{\frac{s_A}{2}}, \qquad u_2 = \frac{y}{2u_1},$$

$$u_3 = \frac{z}{2u_1}, \qquad u_4 = 0.$$

Choose A when $\boxed{s_A \geq s_B}$.

**Chart B:** $s_B > 0$

$$u_2 = \sqrt{\frac{s_B}{2}}, \qquad u_1 = \frac{y}{2u_2},$$

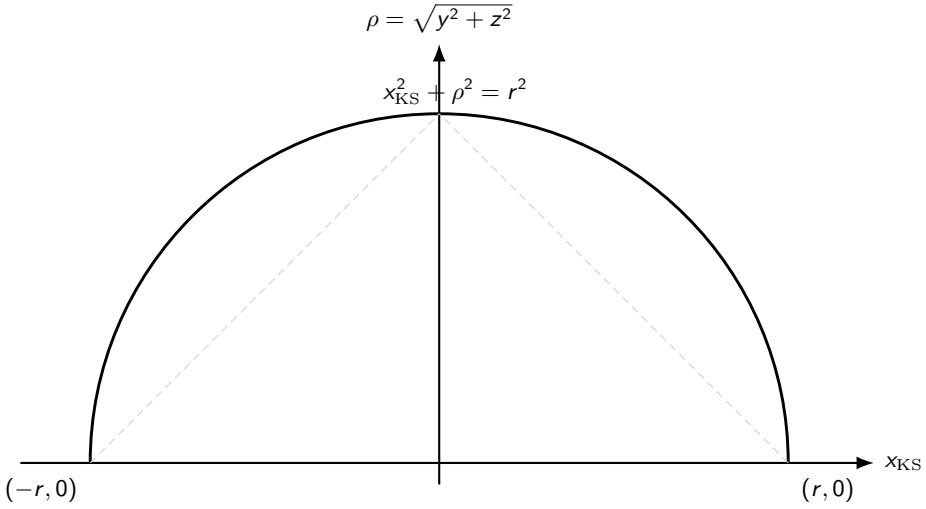
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Choose B when $\boxed{s_B > s_A}$.

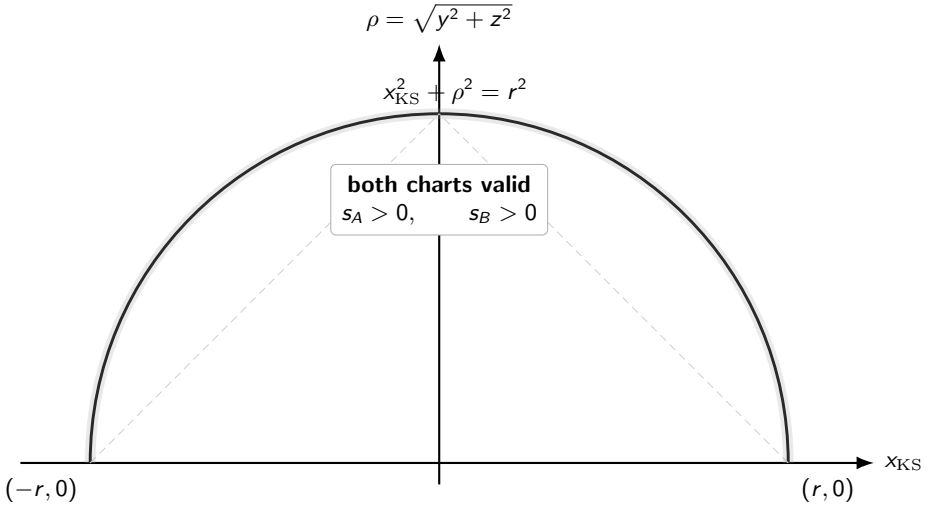
$$\boxed{r = 0 \iff s_A = s_B = 0}$$

Both charts are valid on their overlap; the rule char fixes a **unique representative**.

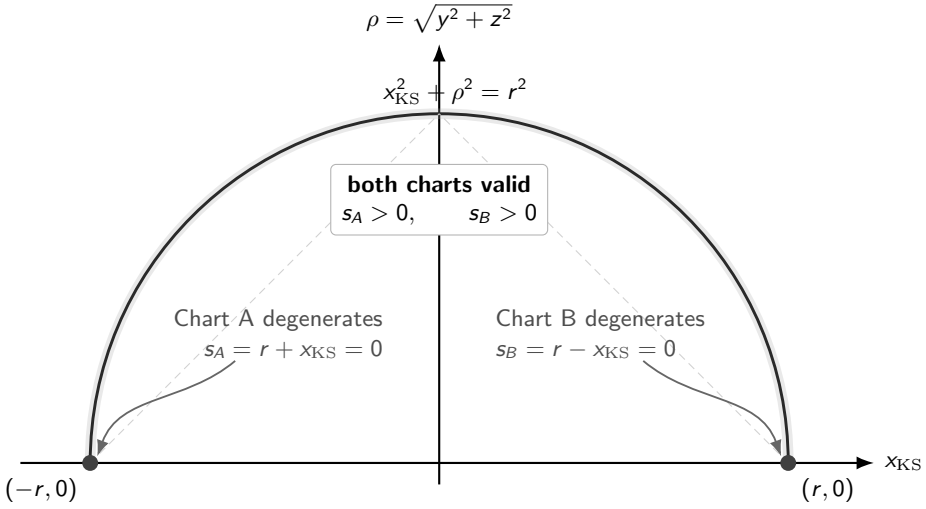
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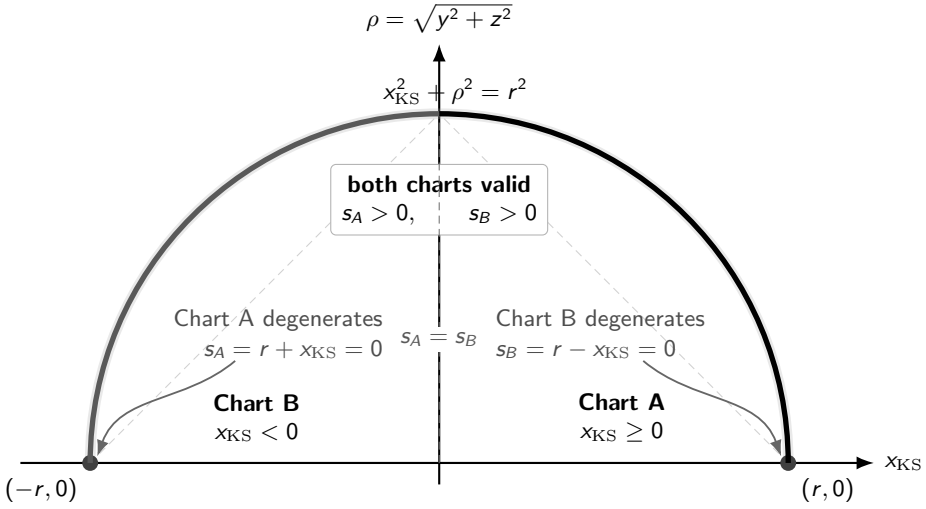
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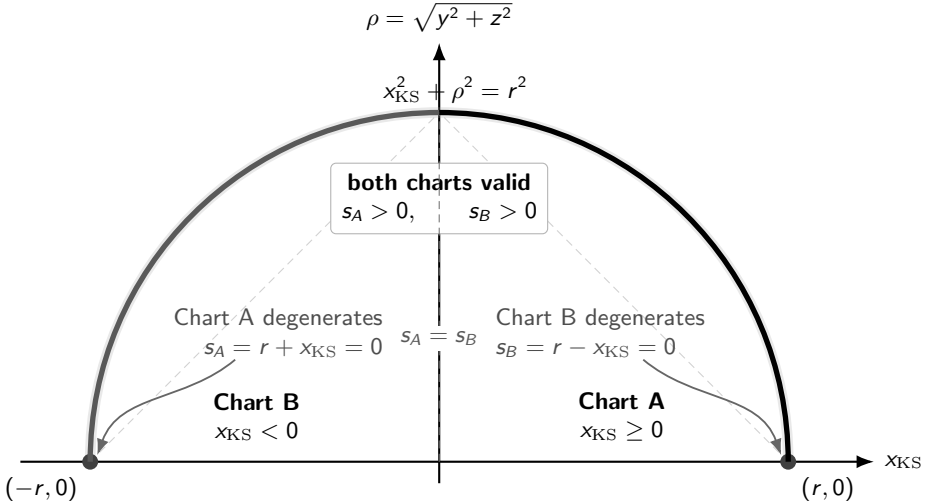
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$$s_A - s_B = 2x_{\mathrm{KS}} \quad \Longrightarrow \quad \boxed{A: \; x_{\mathrm{KS}} \geq 0, \qquad B: \; x_{\mathrm{KS}} < 0}$$

# $\Phi\colon \mathrm{SYN} \to \mathrm{KS}$ and its quasi-inverse

- The choice of charts A/B provide a uniquely transformation:

$$\Phi\colon \mathrm{SYN} \to \mathrm{KS}, \qquad (\mathbf{x}, \dot{\mathbf{x}}) \mapsto (\mathbf{u}, \mathbf{v}), \quad \mathbf{v} = \mathbf{u}', \quad t' = \|\mathbf{u}\|^2.$$

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- To define its inverse $\mathrm{KS} \to \mathrm{SYN}$, we introduce

$$L(\mathbf{u}) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_{\mathrm{KS}} \\ y \\ z \\ 0 \end{pmatrix} = L(\mathbf{u})\mathbf{u} \qquad \text{and} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \end{pmatrix} = \frac{2}{\|\mathbf{u}\|^2} L(\mathbf{u})\mathbf{v}.$$

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$\rightsquigarrow$ $u_4 u_1 - u_3 u_2 + u_2 u_3 - u_1 u_4 = 0$ and $\boxed{g_{\mathrm{KS}} \overset{\text{def}}{=} u_4 v_1 - u_3 v_2 + u_2 v_3 - u_1 v_4 = 0}$.

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# First integrals and KS manifold

**Proposition: physical constraint set**

$$g_{\mathrm{KS}} = u_4 v_1 - u_3 v_2 + u_2 v_3 - u_1 v_4,$$

$$E_{\mathrm{KS}} = \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 h_{\mathrm{KS}}(\mathbf{u}; C) - \frac{1-\mu_{\mathrm{KS}}}{2},$$

$$\mathcal{X}_{\mathrm{KS,ph}} = \{(\mathbf{u},\mathbf{v}) \in \mathcal{X}_{\mathrm{KS}} \colon g_{\mathrm{KS}} = 0,\ E_{\mathrm{KS}} = 0\}.$$

(depends on $C$, $\mu_{\mathrm{KS}}$)

**Flow invariance**

$$\frac{d}{ds} g_{\mathrm{KS}} = 0, \qquad \frac{d}{ds} E_{\mathrm{KS}} = 0.$$

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**Transformations for $\|\mathbf{u}\| \neq 0$**

$$\Phi \colon \mathcal{X}_{\mathrm{syn}} \to \mathcal{X}_{\mathrm{KS,ph}}$$

$$\Psi_{\mathrm{ph}} \colon \mathcal{X}_{\mathrm{KS,ph}} \to \mathcal{X}_{\mathrm{syn}}$$

# From the vector fields to the KS time-$h$ map

**Physical dynamics**

$$\dot{\mathbf{x}} = f_{\mathrm{syn}}(\mathbf{x}; \mu), \qquad \mathbf{x} \in \mathcal{X}_{\mathrm{syn}} \subset \mathbb{R}^6.$$

Flow:

$$\varphi_t^{\mathrm{syn}}(\mathbf{x}; \mu).$$

The Jacobi constant: $C = C(\mathbf{x})$.

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The Jacobi constant: $C = C(\mathbf{x})$.

**Regularized dynamics**

$\mathbf{z} = (\mathbf{u}, \mathbf{v}) \in \mathcal{X}_{\mathrm{KS}} \subset \mathbb{R}^8$,

$$\mathbf{z}' = f_{\mathrm{KS}}(\mathbf{z}; \mu_{\mathrm{KS}}, C), \qquad t' = \|\mathbf{u}\|^2.$$

Flow:

$$\varphi_s^{\mathrm{KS}}(\mathbf{z}; \mu_{\mathrm{KS}}, C).$$

$C$ **parameter** and

$\mathbf{z}(0) \in \mathcal{X}_{\mathrm{KS,ph}} \Longrightarrow \mathbf{z}(s) \in \mathcal{X}_{\mathrm{KS,ph}}$ for all $s$

# Collision: the physical target is not the ambient origin

**Collision in position**

$$x_{\mathrm{KS}}^2 + y^2 + z^2 = r^2, \qquad r = \|\mathbf{u}\|^2.$$

Hence

$$(x_{\mathrm{KS}}, y, z) = (0,0,0) \quad \Longleftrightarrow \quad r = 0$$

**Intersect with the physical manifold**

At $\mathbf{u} = 0$, $g_{\mathrm{KS}} = 0$, and

$$E_{\mathrm{KS}} = \|\mathbf{v}\|^2 - \frac{1-\mu_{\mathrm{KS}}}{2}.$$

Therefore

$$\boxed{\mathcal{C}_{\mathrm{col}}^{\mathrm{ph}} = \left\{ (0,\mathbf{v}) : \|\mathbf{v}\|^2 = \frac{1-\mu_{\mathrm{KS}}}{2} \right\} \simeq S^3.}$$

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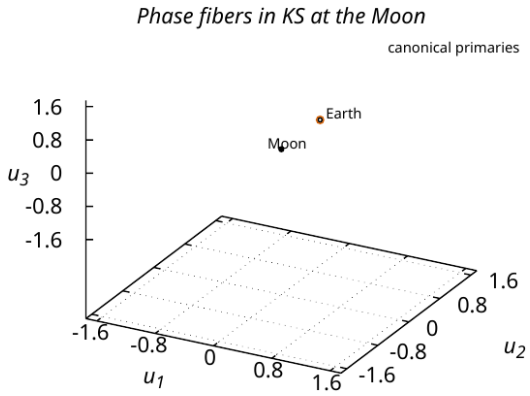
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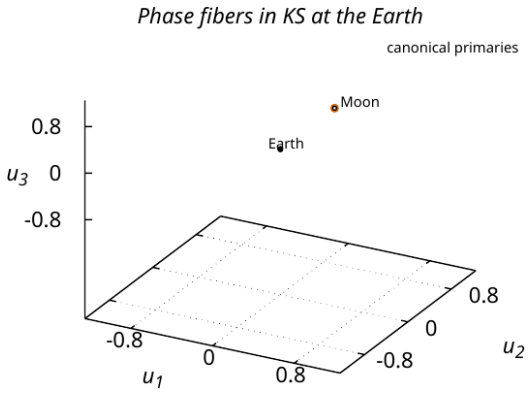
$$\boxed{\mathcal{C}_{\mathrm{col}}^{\mathrm{ph}} = \left\{(0,\mathbf{v}) : \|\mathbf{v}\|^2 = \frac{1-\mu_{\mathrm{KS}}}{2}\right\} \simeq S^3.}$$

Since $E_{\mathrm{KS}}(0,0) = -\frac{1-\mu_{\mathrm{KS}}}{2} \neq 0$,

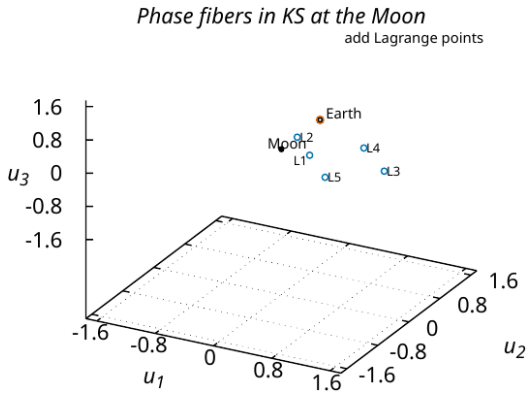
$(0,0)$ is an ambient KS equilibrium, but it is not a physical collision state.

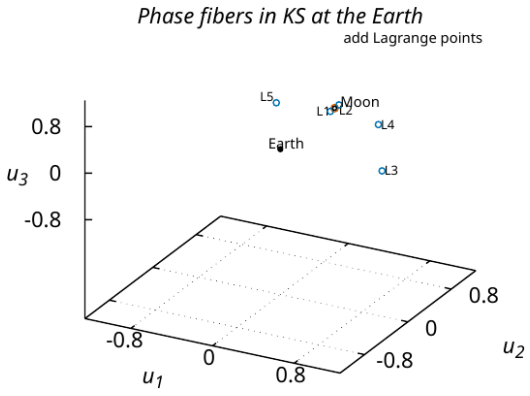
# Phase fibers via Stereographic Projection I



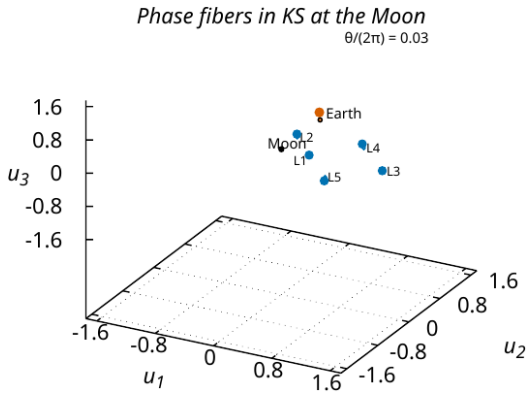



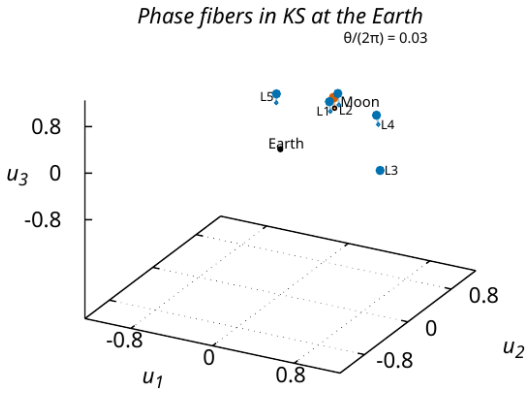
# Phase fibers via Stereographic Projection II



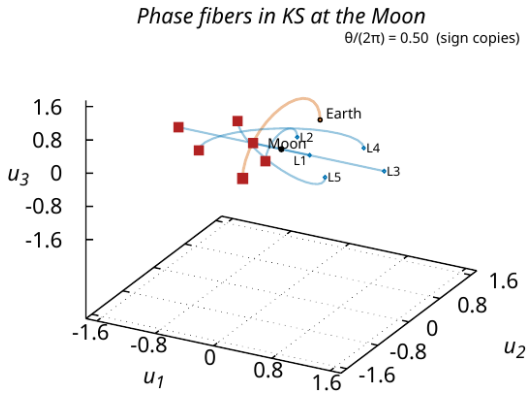



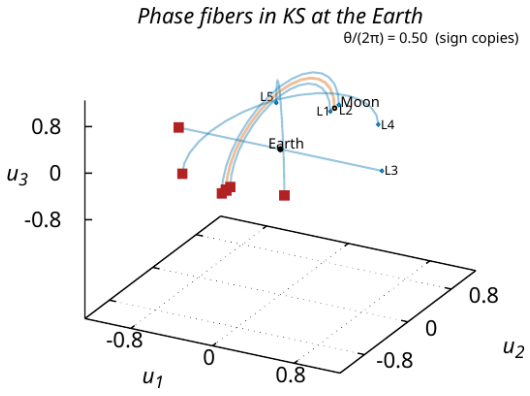
# Phase fibers via Stereographic Projection III



# Phase fibers via Stereographic Projection IV



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# The KS origin: a sharp spectral transition

**Proposition: origin linearization**
At $(\mathbf{u}, \mathbf{v}) = (0, 0)$,

$$\mathbf{u}' = \mathbf{v}, \qquad \mathbf{v}' = h_{\mathrm{KS}}(0)\,\mathbf{u},$$

with

$$h_{\mathrm{KS}}(0) = \frac{\mu_{\mathrm{KS}}}{2} + \frac{\mu_{\mathrm{KS}}^2}{4} - \frac{C}{4}.$$

Hence

$$A_0 = \begin{pmatrix} 0 & I_4 \\ h_{\mathrm{KS}}(0) I_4 & 0 \end{pmatrix}, \qquad \boxed{\lambda^2 = h_{\mathrm{KS}}(0)}.$$

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**One critical value**
Define

$$\boxed{C_{\mathrm{KS}}^* = 2\mu_{\mathrm{KS}} + \mu_{\mathrm{KS}}^2}$$

so that $h_{\mathrm{KS}}(0) = -\frac{C - C_{\mathrm{KS}}^*}{4}$.
Therefore

$$\begin{aligned} C < C_{\mathrm{KS}}^* &\implies \lambda = \pm\tfrac{1}{2}\sqrt{C_{\mathrm{KS}}^* - C}, \\ C = C_{\mathrm{KS}}^* &\implies \lambda = 0, \\ C > C_{\mathrm{KS}}^* &\implies \lambda = \pm \mathbf{i}\tfrac{1}{2}\sqrt{C - C_{\mathrm{KS}}^*}, \end{aligned}$$

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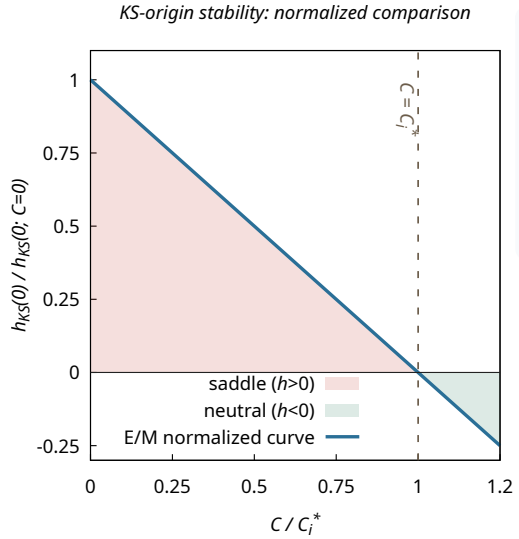
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$C > C_{\mathrm{KS}}^* \Rightarrow$ origin $1:1:1:1$ linear resonance.

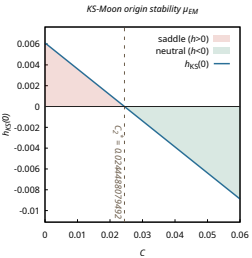
# KS origin spectral

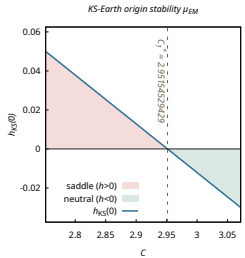



## E/M case

$$C^*_{\mathrm{Earth}} = 2.95154529429$$

$$C^*_{\mathrm{Moon}} = 0.0244488079492$$



# Overview




**KS lift**
charts, fiber,
regularized time

**physical subset**
first integrals
and constraints

**origin theorem**
critical value
and resonance

**normal form**
Hamiltonian
and time-h map

**next steps**

# KS time-$h$ map

**Time-$h$ map near the ambient KS origin**

Fix $h > 0$ and define

$$\boxed{F_{h,C} \stackrel{\text{def}}{=} \varphi_h^{\mathrm{KS}}(\,\cdot\,;\mu_{\mathrm{KS}}, C).}$$

Although $(0,0) \notin \mathcal{X}_{\mathrm{KS,ph}}$, it is an equilibrium of the ambient KS system. Hence

$$F_{h,C}(0) = 0, \qquad \boxed{M_{h,C} \stackrel{\text{def}}{=} \mathrm{D}F_{h,C}(0) = e^{hA_0(C)}.}$$

For $C > C^*_{\mathrm{KS}}$,

$$\boxed{\mathrm{spec}(M_{h,C}) = \{e^{\mathbf{i}\theta}, e^{-\mathbf{i}\theta}\}, \qquad \theta = \omega(C)h,} \qquad \omega(C) \stackrel{\text{def}}{=} \tfrac{1}{2}\sqrt{C - C^*_{\mathrm{KS}}}$$

each multiplier having multiplicity 4.

# Linear modes at the KS origin

$$A(C) = \begin{pmatrix} 0 & I_4 \\ h_{\mathrm{KS}}(0) I_4 & 0 \end{pmatrix},$$

$$\lambda^2 = h_{\mathrm{KS}}(0) = \frac{C^*_{\mathrm{KS}} - C}{4}.$$

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$$\begin{array}{c|cccc|cccc}
 & \lambda & & & & -\lambda & & & \\
u_1 & 1 & 1 & & & 1 & 1 & & \\
u_2 & 1 & -1 & & & 1 & -1 & & \\
u_3 & & & 1 & 1 & & & 1 & 1 \\
u_4 & & & 1 & -1 & & & 1 & -1 \\
\hline
v_1 & \lambda & \lambda & & & -\lambda & -\lambda & & \\
v_2 & \lambda & -\lambda & & & -\lambda & \lambda & & \\
v_3 & & & \lambda & \lambda & & & -\lambda & -\lambda \\
v_4 & & & \lambda & -\lambda & & & -\lambda & \lambda \\
 & s_+ & s_- & o_+ & o_- & s_+ & s_- & o_+ & o_-
\end{array}$$

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 & s_+ & s_- & o_+ & o_- & s_+ & s_- & o_+ & o_-
\end{array}$$

**Planar / LC**

$s_\pm$ : $u_3 = u_4 = v_3 = v_4 = 0.$

**Transverse to LC**

$o_\pm$ : $u_1 = u_2 = v_1 = v_2 = 0.$

# Periodic orbits near the elliptic KS origin

**Lyapunov center theorem**
If $\mathrm{D}H(0) = 0$, $\mathrm{D}^2H(0)$ has

$$\pm 2\pi \mathbf{i}\omega$$

simple imaginary pair and no other eigenvalue is an integer multiple of $2\pi\mathbf{i}\omega$.
Then a one-parameter family of periodic orbits bifurcates from 0, with period $1/\omega$.

**Not applicable here**
For $C > C^*_{\mathrm{KS}}$, $1:1:1:1$ Resonance.

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**Weinstein–Moser theorem ('70)**
$\mathrm{D}H(0) = 0$ and $\mathrm{D}^2H(0)$ is definite on the resonance space $E_\omega$.
Then, on every sufficiently small nearby energy level $H(0) + \varepsilon$, there are at least

$$\boxed{\frac{1}{2}\dim E_\omega}$$

geometrically distinct periodic orbits, with periods

$$\boxed{T_\varepsilon \longrightarrow T_0, \qquad T_0 = \frac{2\pi}{\omega}.}$$

in our case $E_\omega = \mathbb{R}^8 \Longrightarrow$ 4 p.o.

# Normal form of the KS time-$h$ map

For fixed $(h, C)$, expand near the origin: $F_{h,C}(z) = M_{h,C}z + F_2(z) + F_3(z) + \cdots$.

**Polynomial conjugacy**

Seek a near-identity transformation

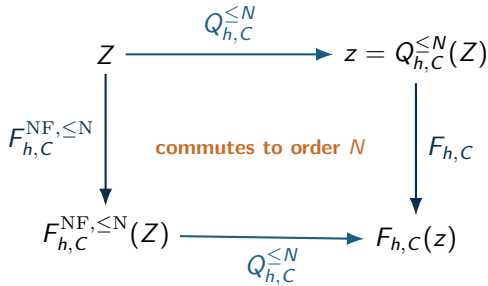
$$Q_{h,C}^{\leq N}(Z) = Z + \sum_{j=2}^{N} T_j(Z)$$

and a normal-form map

$$F_{h,C}^{\mathrm{NF},\leq \mathrm{N}}(Z) = M_{h,C}Z + \sum_{j=2}^{N} F_j^{\mathrm{new}}(Z)$$

such that

$$\boxed{F_{h,C} \circ Q_{h,C}^{\leq N} - T_{h,C}^{\leq N} \circ F_{h,C}^{\mathrm{NF},\leq \mathrm{N}} = \mathcal{O}(\|Z\|^{N+1}).}$$



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**But physicality must follow**

NOFO lives in $\mathcal{X}_{\mathrm{KS}}$

Physical constraints:

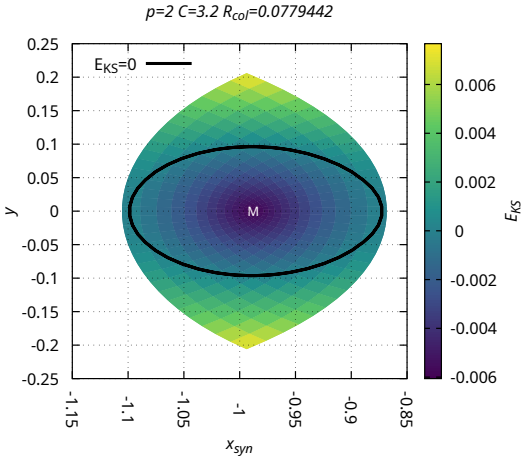
$$\widetilde{E}_{h,C}^{\leq N}(Z) = E_{\mathrm{KS}}\big(Q_{h,C}^{\leq N}(Z); C\big),$$

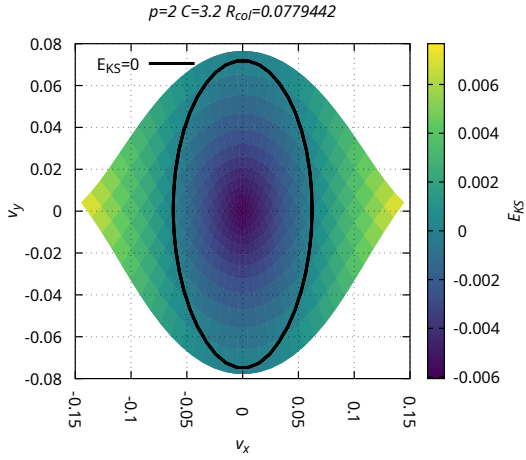
$$\widetilde{g}_{h,C}^{\leq N}(Z) = g_{\mathrm{KS}}\big(Q_{h,C}^{\leq N}(Z)\big).$$

Physical candidates satisfy

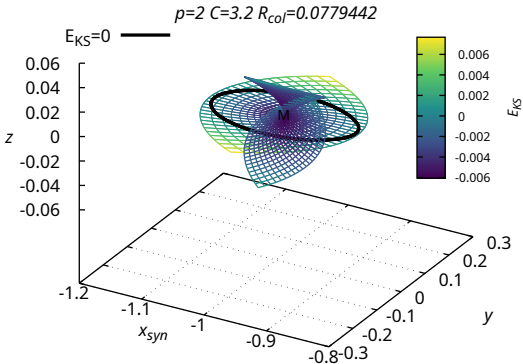
$$\boxed{\widetilde{E}_{h,C}^{\leq N} = 0, \qquad \widetilde{g}_{h,C}^{\leq N} = 0.}$$

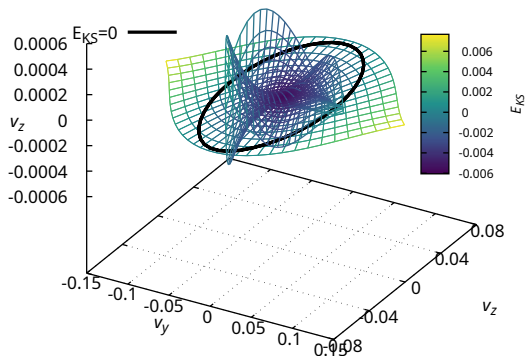
# Planar example



# Example



Thank you