



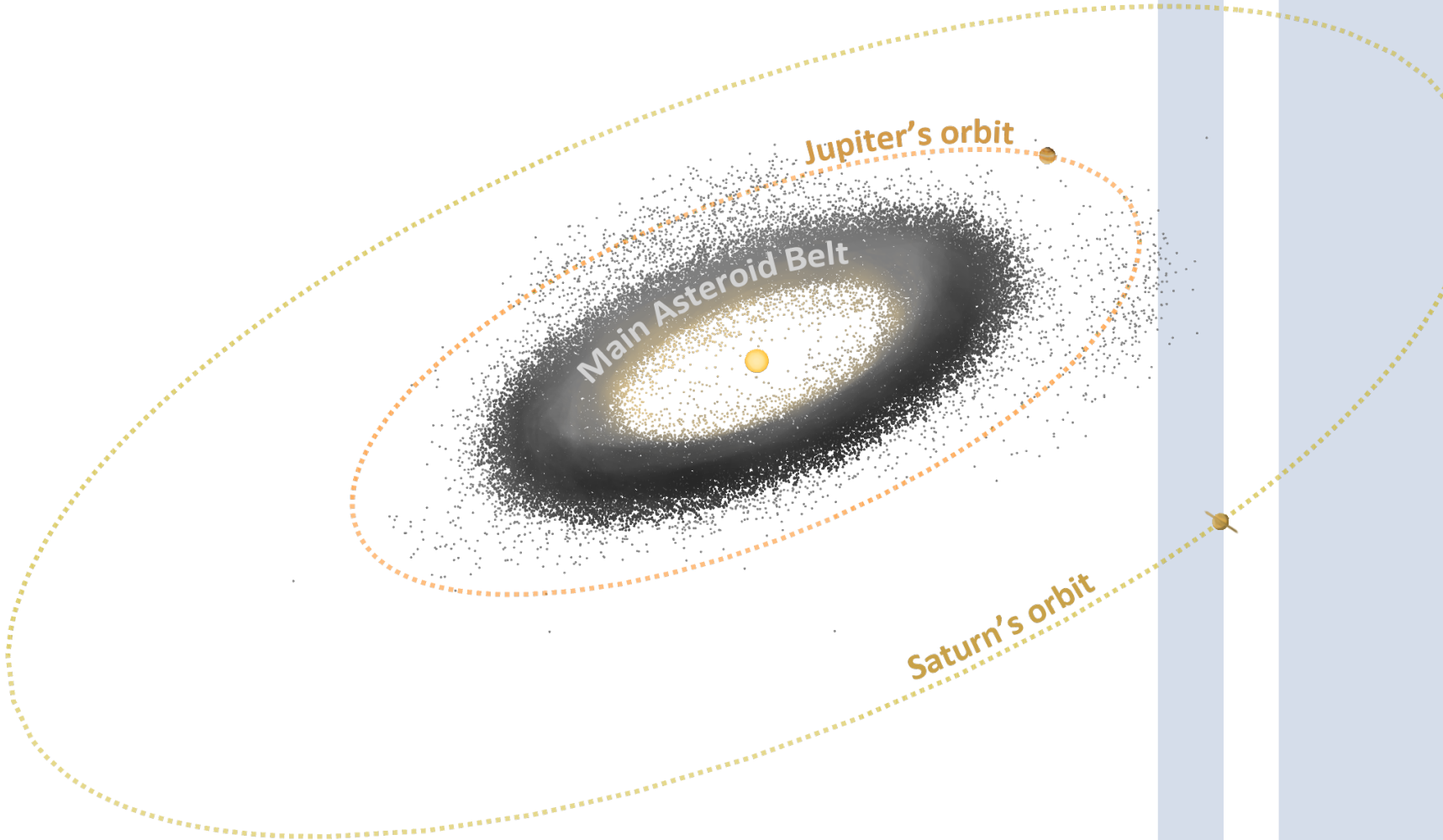
Generalized Le Verrier's development and proper elements of asteroids

Anargyros Dogkas & Giulio Baù

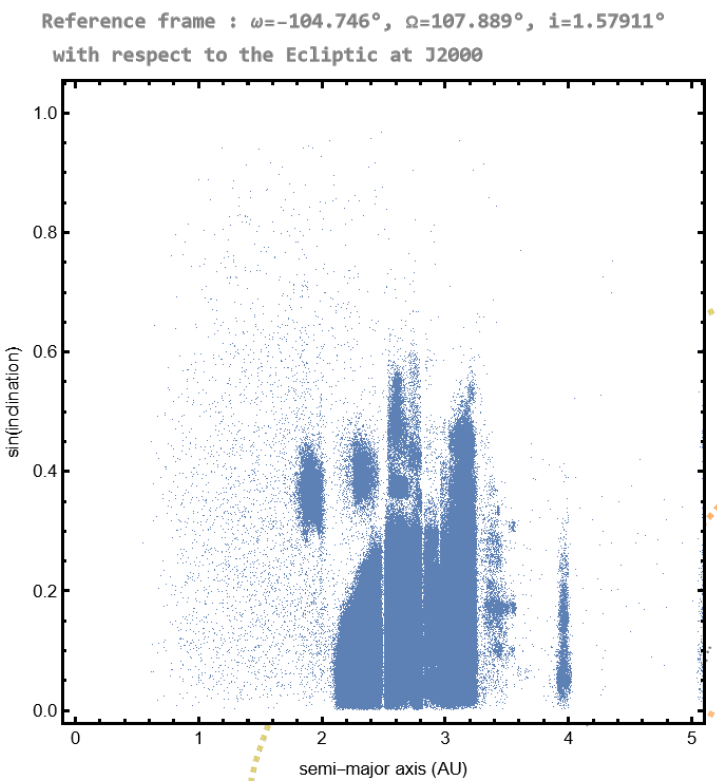
▶ **Introduction**
- Asteroid families in the main belt

▶ **Development of the perturbing function**
- Historical developments
- Le Verrier's development
- Generalized Yuasa-Tisserand corrections
- Preliminary convergence results

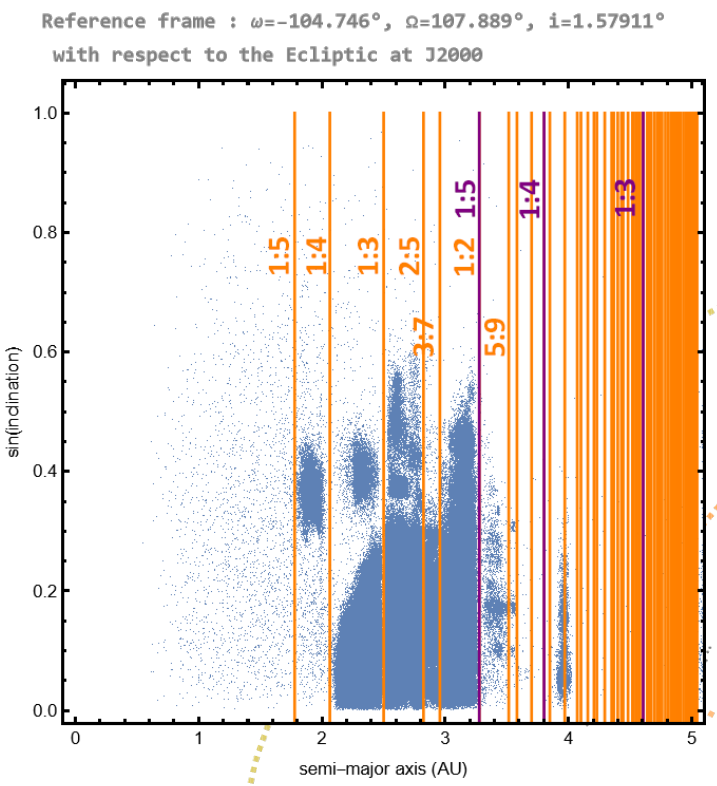
▶ **Proper Elements of Asteroids**
- Resonant domains
- High-degree analytical proper elements

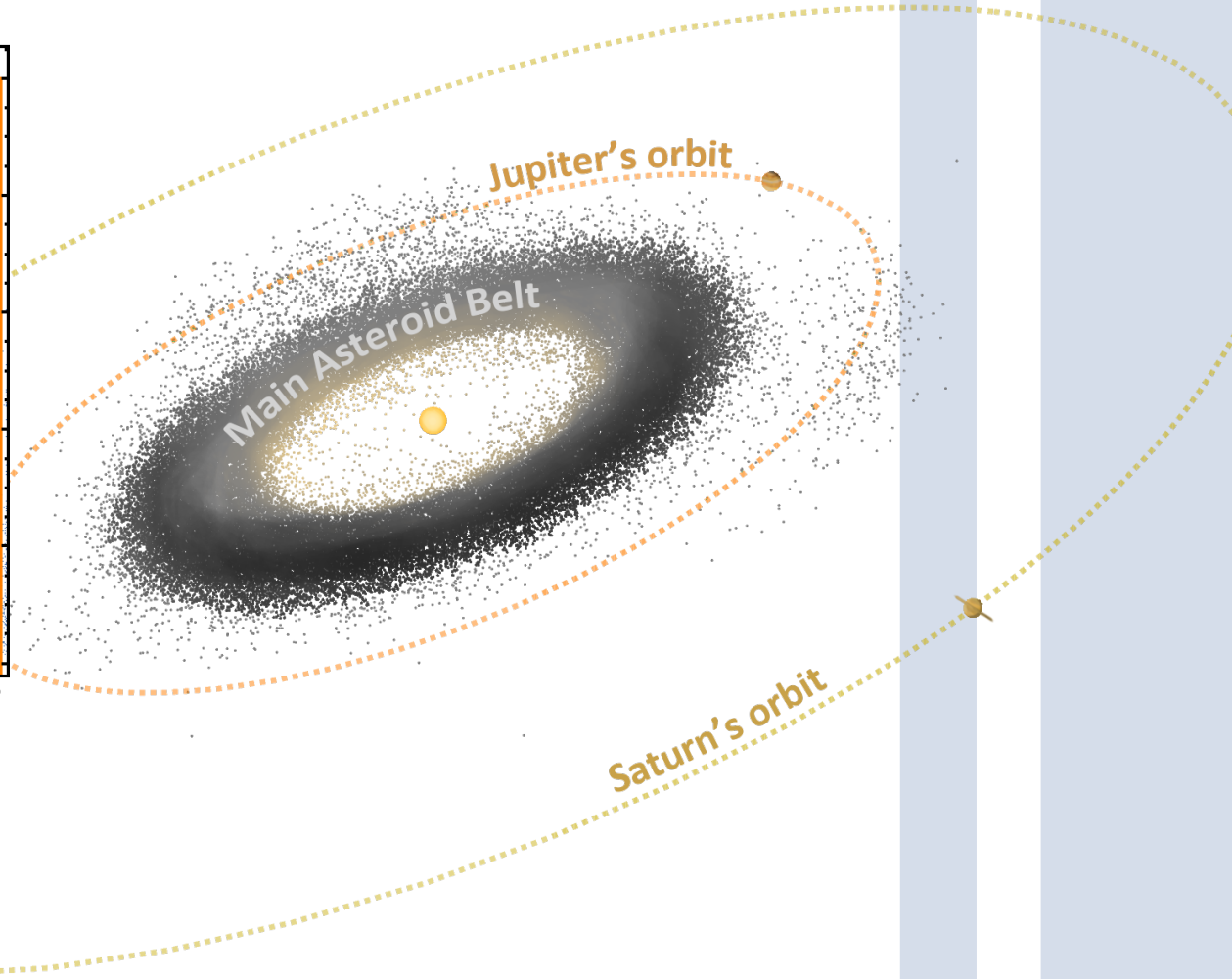

Jupiter's orbit
Main Asteroid Belt
Saturn's orbit

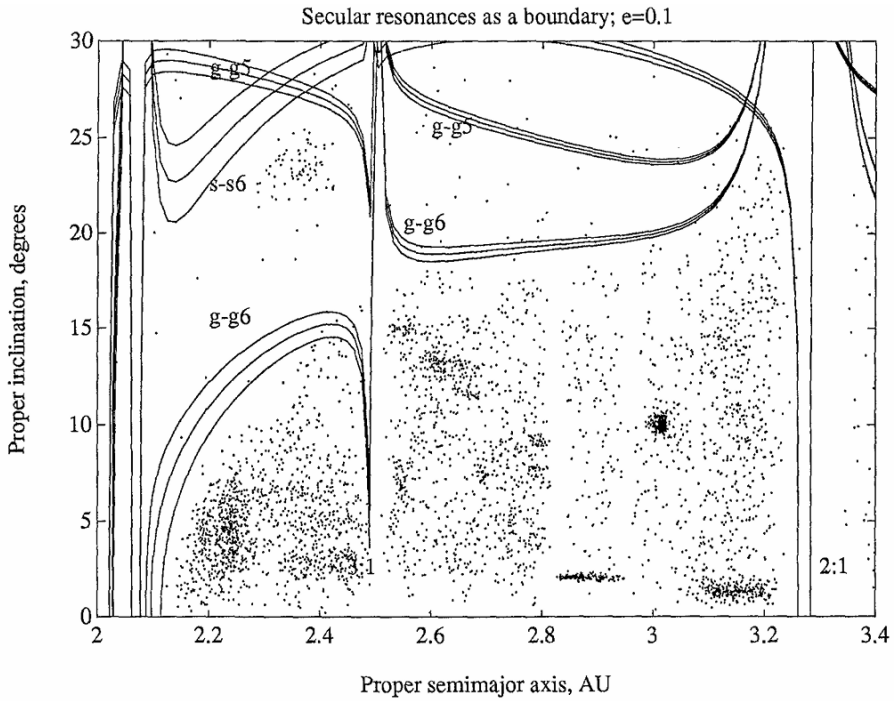
A. Dogkas & G. Baù

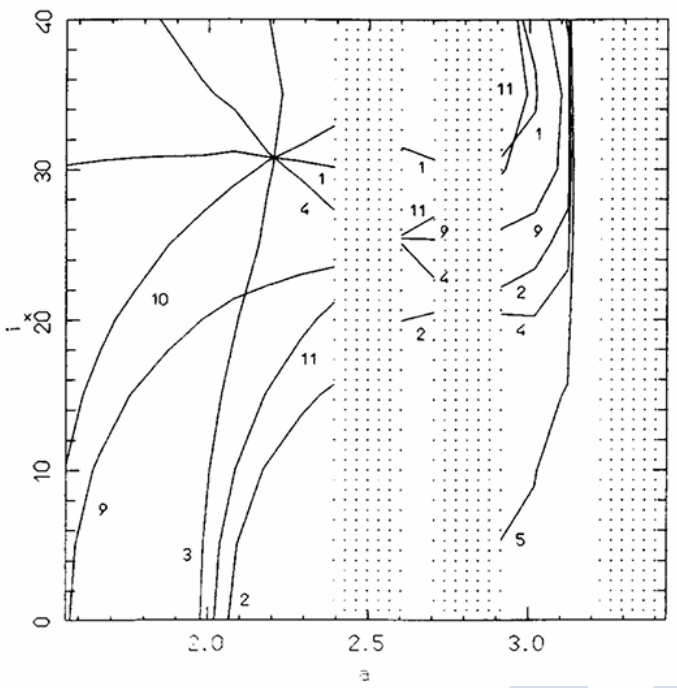


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Source : (Left) Milani, A. & Knezevic, Z. Secular perturbation theory and computation of asteroid proper elements. *Celestial Mechanics and Dynamical Astronomy* **49,** 347–411 (Dec. 1990)

(Right) Morbidelli, A. & Henrard, J. Secular resonances in the asteroid belt: Theoretical perturbation approach and the problem of their location. *Celestial Mechanics and Dynamical Astronomy* **51,** 131–167 (June 1991)

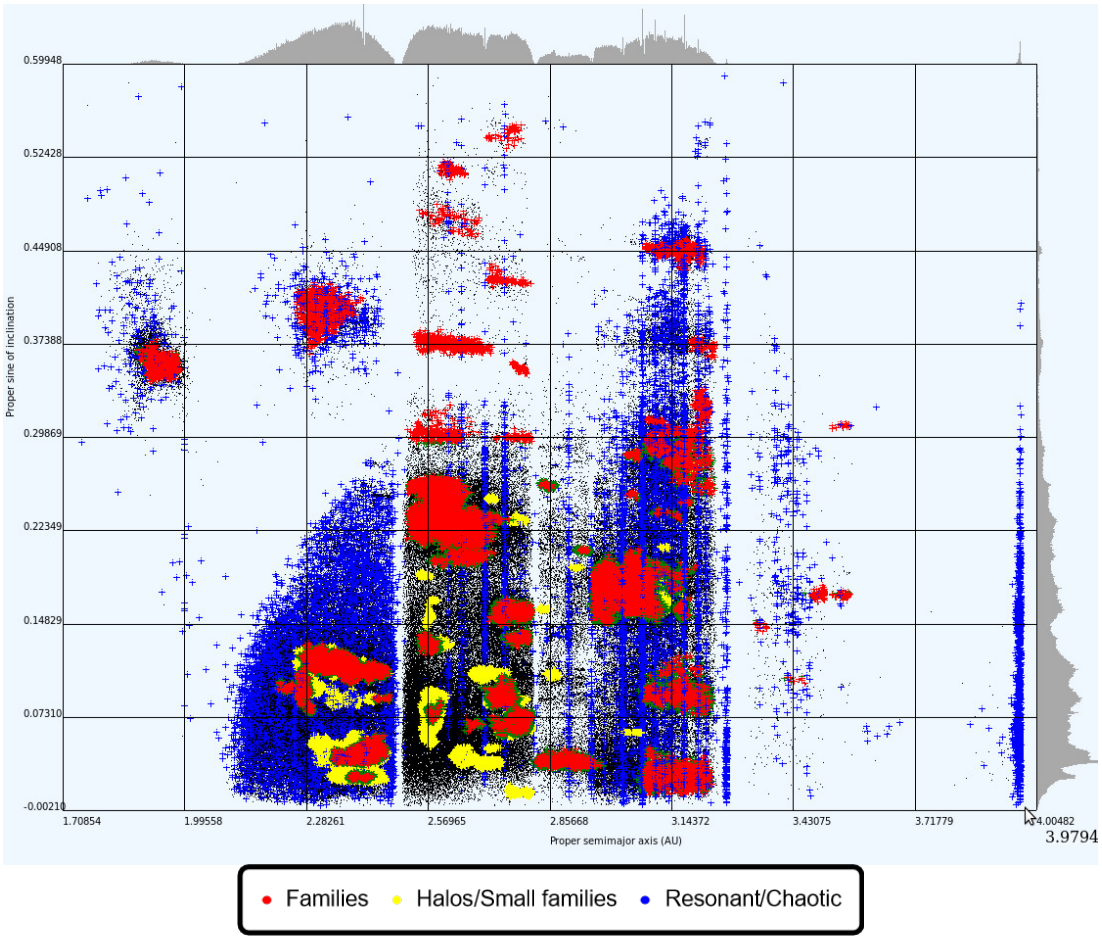
In a completely non-resonant domain, there exists a transformation $\exp(\mathcal{L}_\chi)$ such that if $\mathcal{H}^{(1)} = \exp(\mathcal{L}_\chi)\mathcal{H}$, then

$$\mathcal{H}(\boldsymbol{q},\boldsymbol{p}) = \underbrace{h(\boldsymbol{p})}_{\mathcal{O}(1)} + \underbrace{\mathcal{R}_\mathcal{K}(\boldsymbol{q},\boldsymbol{p})}_{\mathcal{O}(\varepsilon)} \qquad \mathcal{H}^{(1)}(\boldsymbol{q},\boldsymbol{p}) = \underbrace{h(\boldsymbol{p}) + \langle \mathcal{R}_\mathcal{K} \rangle_{\boldsymbol{q}} (\boldsymbol{p})}_{\mathcal{O}(1)+\mathcal{O}(\varepsilon)} + \underbrace{\mathcal{R}^{(1)}(\boldsymbol{q},\boldsymbol{p})}_{\mathcal{O}(\varepsilon^2)}$$

and as a result, the new actions $\boldsymbol{p}^{(1)}$ and the associated orbital elements

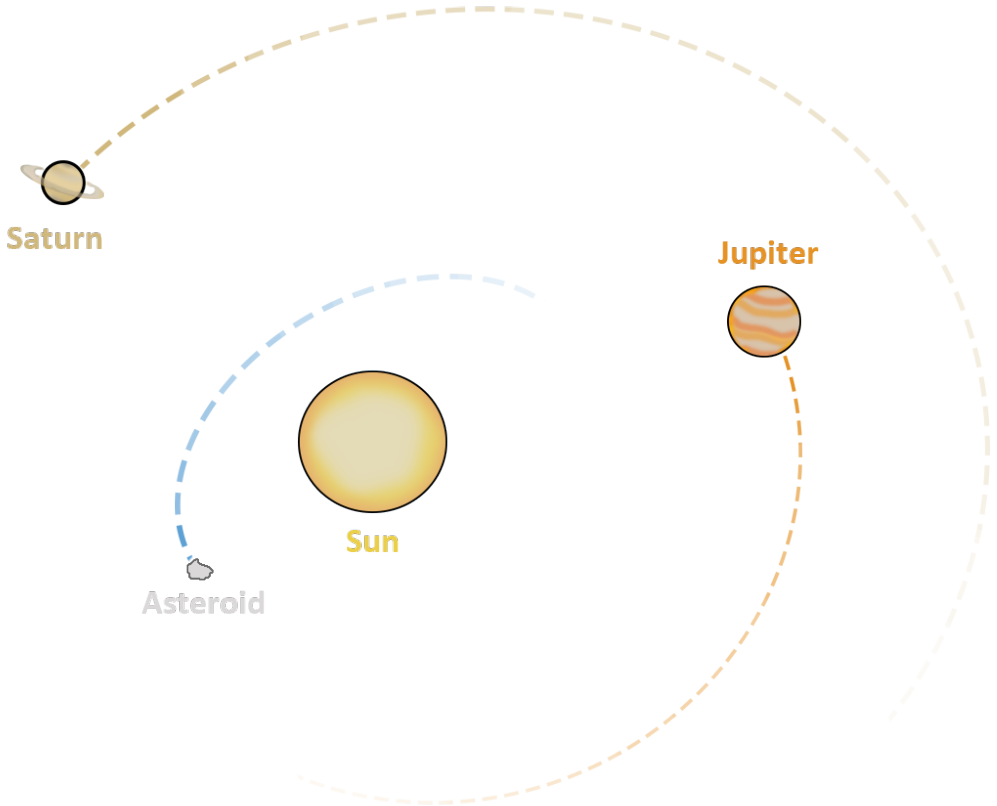
$$\begin{cases} a^{(1)} = \exp(\mathcal{L}_\chi) a \simeq a + \{a, \chi\} \\ e^{(1)} = \exp(\mathcal{L}_\chi) e \simeq e + \{e, \chi\} \\ i^{(1)} = \exp(\mathcal{L}_\chi) i \simeq i + \{i, \chi\} \end{cases}$$

are almost constant, and they are called **proper elements**.



Source : SpaceDys website, https://newton.spacedys.com/

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Saturn
Jupiter
Sun
Asteroid

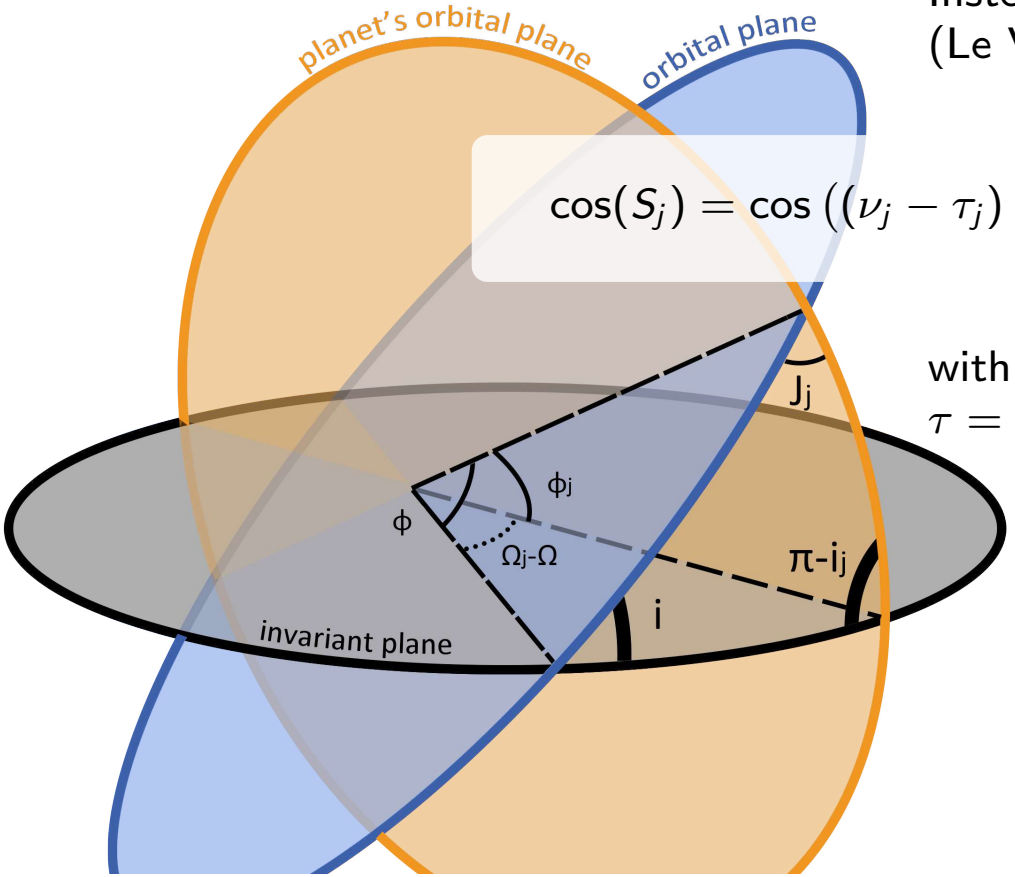
The motion of an asteroid is approximated by a restricted $N+1$-body problem

$$\begin{cases} \mathcal{H}_{AS}(\boldsymbol{x},\boldsymbol{y},t) = \dfrac{\|\boldsymbol{y}\|^2}{2} - \dfrac{\mu}{\|\boldsymbol{x}\|} - \mu \displaystyle\sum_{j=1}^{N} \dfrac{m_j}{m_0} \left(\dfrac{1}{\|\boldsymbol{x}-\boldsymbol{x}_j(t)\|} - \dfrac{\boldsymbol{x}\cdot\boldsymbol{x}_j(t)}{\|\boldsymbol{x}_j(t)\|^3} \right) \\ \begin{cases} \dot{\boldsymbol{x}}_j = \dfrac{\boldsymbol{y}_j}{m_j} \\ \dot{\boldsymbol{y}}_j = \mathcal{G} m_j \left(-\dfrac{(m_0+m_j)}{\|\boldsymbol{x}_j\|^3}\boldsymbol{x}_j + \displaystyle\sum_{k\neq j,\, k\neq 0} m_k \left(\dfrac{(\boldsymbol{x}_k-\boldsymbol{x}_j)}{\|\boldsymbol{x}_j-\boldsymbol{x}_k\|^3} - \dfrac{\boldsymbol{x}_k}{\|\boldsymbol{x}_k\|^3} \right) \right) \end{cases} \end{cases}$$

with $\mu = \mathcal{G} m_0$, and $\mathcal{G} \simeq 4\pi^2\ (\mathrm{AU})^3 (M_\odot)^{-1} (\mathrm{yr})^{-2}$ the gravitational constant.

$$\frac{1}{\|\boldsymbol{x}-\boldsymbol{x}_j\|} = \begin{cases} \text{(\underline{\textbf{Laplace's development}})} \\ \dfrac{1}{x_j}\displaystyle\sum_{s=0}^{\infty}\binom{-1/2}{s} A^{-s-1/2} V^s \quad , \quad A = \|\boldsymbol{x}-\boldsymbol{x}_j\|^2 / x_j^2, \quad \text{when } e=0,\ i=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V = \|\boldsymbol{x}-\boldsymbol{x}_j\|^2 / x_j^2 - A \\ \text{(Laskar \& Robutel 1995 based on Laplace 1785)} \\ \\ \text{(\underline{\textbf{Hansen-Tisserand development}})} \\ \dfrac{1}{x_j}\displaystyle\sum_{s=0}^{\infty}\mathcal{P}_s\left(\cos(S_j)\right)\left(\frac{x}{x_j}\right)^s \quad , \quad \cos(S_j) = \frac{\boldsymbol{x}\cdot\boldsymbol{x}_j}{x\ x_j} \\ \text{(Hansen 1855, Tisserand 1889, Laskar \& Boué 2010)} \end{cases}$$

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Instead, considering the mutual inclination J_j, (Le Verrier 1855)

$$\cos(S_j) = \cos\left((\nu_j - \tau_j) - (\nu - \tau)\right) + 2\eta_j^2 \sin(\nu_j - \tau_j)\sin(\nu - \tau),$$

with $\eta_j = \sin(J_j/2)$, ν the true longitude, and $\tau = \phi + \Omega$.

$$\frac{1}{\|\boldsymbol{x}-\boldsymbol{x}_j\|} = \sum_{k=0}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{\xi=0}^{\infty}\sum_{z=0}^{k}\sum_{\zeta=0}^{k}\frac{(-1)^{k+z+\zeta}}{2}\binom{-1/2}{k}\binom{k}{z}\binom{k}{\zeta}$$

$$\left[\frac{1}{a_j}\left(\frac{a}{a_j}\right)^{\xi}\mathcal{B}_k^{s,\xi}\left(a/a_j\right)\right]\exp(i(2\zeta-k+s)\omega_j+i(2z-k-s)\omega)$$

$$\left[\frac{1}{\delta_j}\left(\frac{\delta}{\delta_j}-1\right)^{\xi}\exp(i(2\zeta-k+s)f_j+i(2z-k-s)f)\right]$$

$$\left[\eta_j^{2k}\exp(-i(2\zeta-k+s)\phi_j+i(2z-k-s)\phi)\right],$$

with $\delta = r/a$, $f =$ true anomaly, $\omega =$ argument of pericenter.

$$\frac{1}{\|\boldsymbol{x}-\boldsymbol{x}_j\|} = \sum_{k=0}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{\xi=0}^{\infty}\sum_{z=0}^{k}\sum_{\zeta=0}^{k}\frac{(-1)^{k+z+\zeta}}{2}\binom{-1/2}{k}\binom{k}{z}\binom{k}{\zeta}$$

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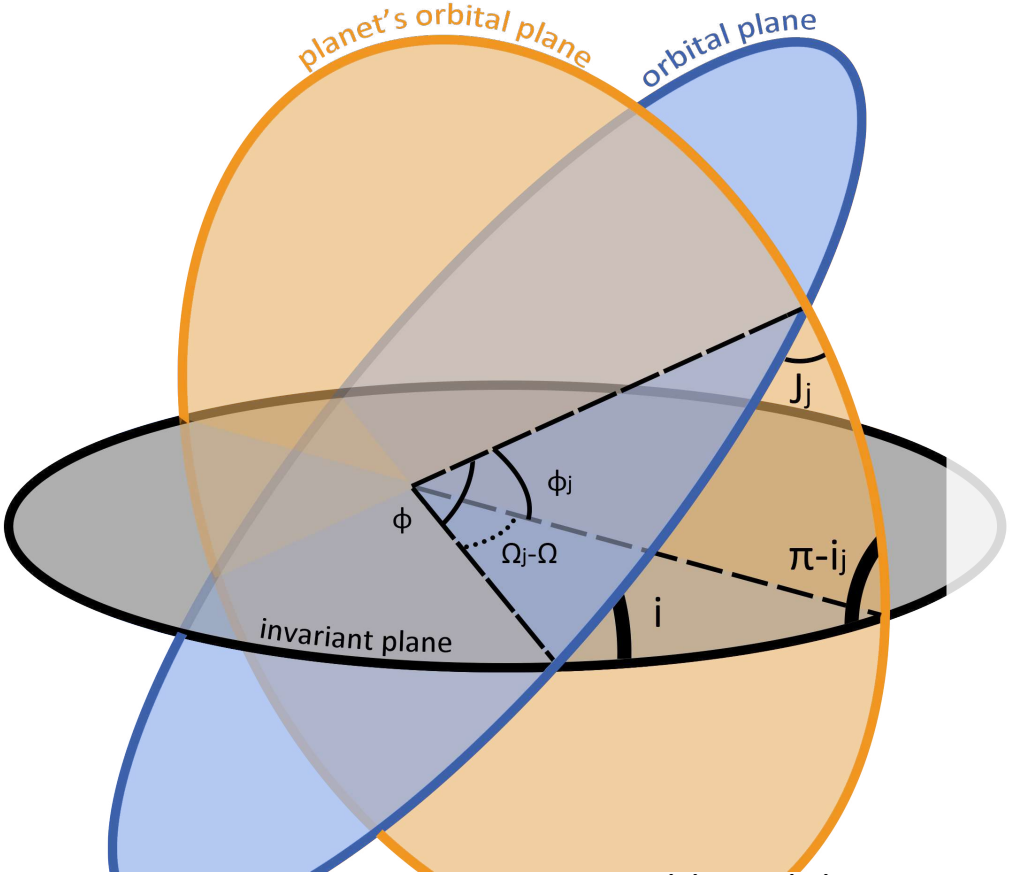
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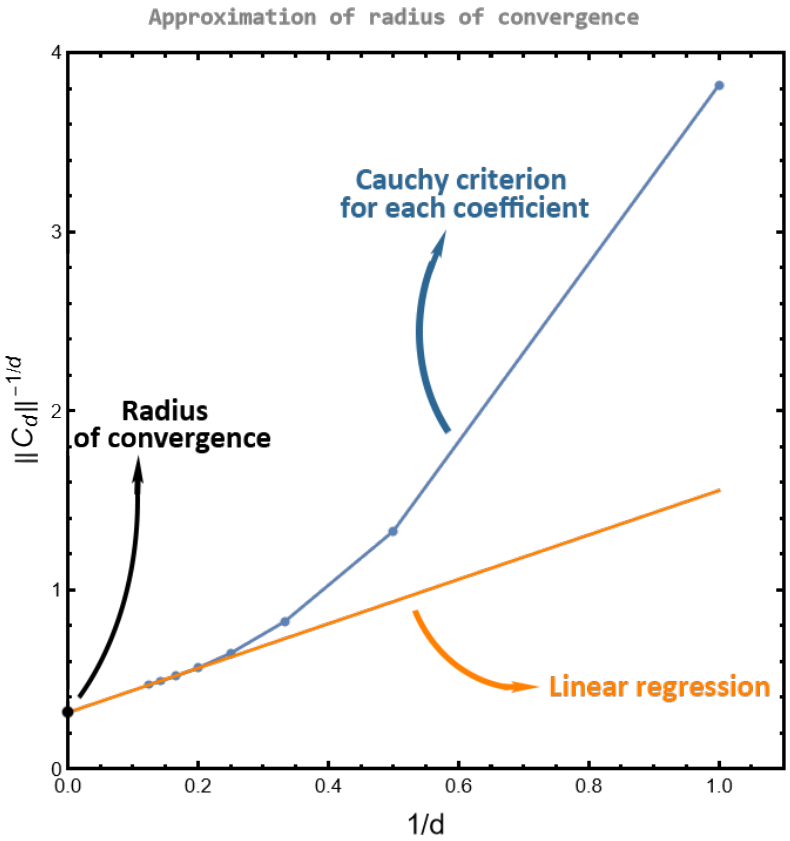
Solving for the Euler angles associated with the asteroid's and planet's planes



$$
\begin{cases}
\rho_1 = \alpha_1 \left(1 - \epsilon_1 \cos(\theta)\right) \\
\cos(\phi + \phi_j) = \dfrac{\alpha_1}{\rho_1}(\cos(\theta) - \epsilon_1) \\
\sin(\phi + \phi_j) = \dfrac{\alpha_1}{\rho_1}\sqrt{1 - \epsilon_1^2}\sin(\theta) \\
\rho_2 = \alpha_2 \left(1 - \epsilon_2 \cos(\theta)\right) \\
\cos(\phi - \phi_j) = \dfrac{\alpha_2}{\rho_2}\left(\cos(\theta) - \epsilon_2\right) \\
\sin(\phi - \phi_j) = \dfrac{\alpha_2}{\rho_2}\sqrt{1 - \epsilon_2^2}\sin(\theta)
\end{cases}
$$

$$
\theta = \Omega - \Omega_j, \quad
\begin{array}{lll}
\alpha_1 = 1 - \cos(i)\cos(i_j), & \epsilon_1 = \sin(i)\sin(i_j)/\alpha_1, & \rho_1 = (1 - \cos(J_j)) = 2\eta_j^2 \\
\alpha_2 = 1 + \cos(i)\cos(i_j), & \epsilon_2 = \sin(i)\sin(i_j)/\alpha_2, & \rho_2 = (1 + \cos(J_j))
\end{array}
$$

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Setting

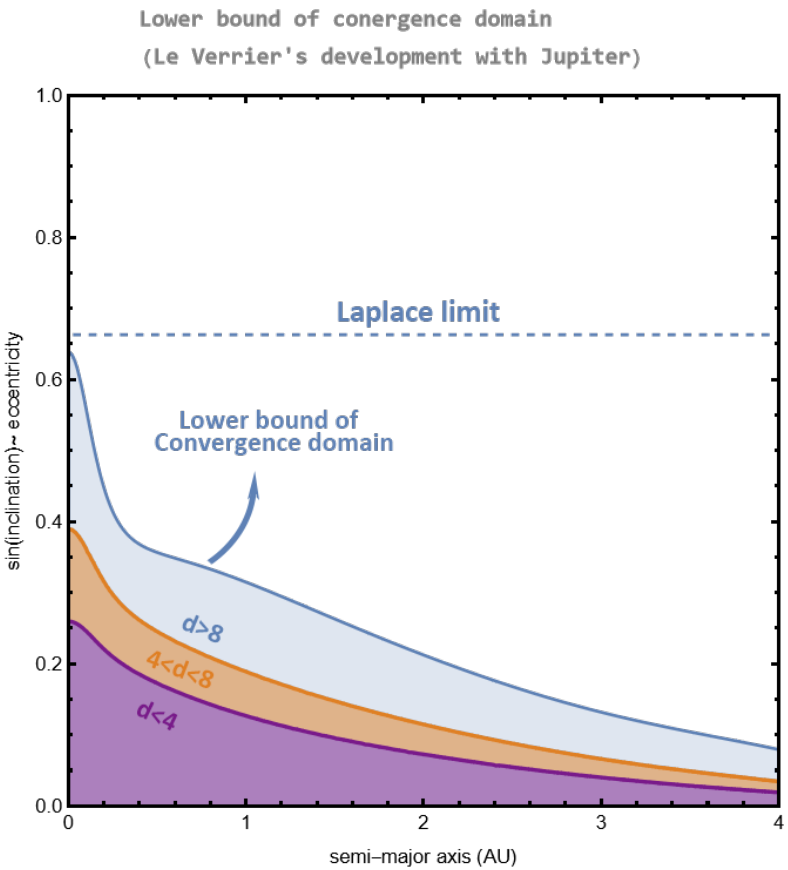
$$e \sim e_j \sim \sin(i) \sim \sin(i_j) \sim \varepsilon,$$

the convergence domain is bounded from below by the convergence domain of the majorant series

$$\frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_j\|} = \sum_{d=0}^{\infty} C_d \varepsilon^d \leq \sum_{d=0}^{\infty} \|C_d\| \, \varepsilon^d$$

and for a choice of a and a_j, the convergence radius is given by the linear regression on the tail of Cauchy's criterion for each coefficient.

Figure for $a = 1$ AU, $a_j = 5.20427$ AU (Jupiter at J2000), and $\|C_d\| = \left\| \sum_{j=0}^{N} c_n \exp(\boldsymbol{k}_n \cdot \boldsymbol{q}) \right\| = \sum_{j=0}^{N} |c_n|$.



We get, therefore a lower bound of the convergence domain. Then we can approximate the remainder because of d-degree truncation as

$$E_d = \left| \|C_d\| \, \varepsilon^d \right| \frac{\varepsilon/R}{1-\varepsilon/R}$$

with R the radius of convergence.

Setting a tolerance 10^{-3} we find the regions where $E_d \leq 10^{-3}$ for each degree d.

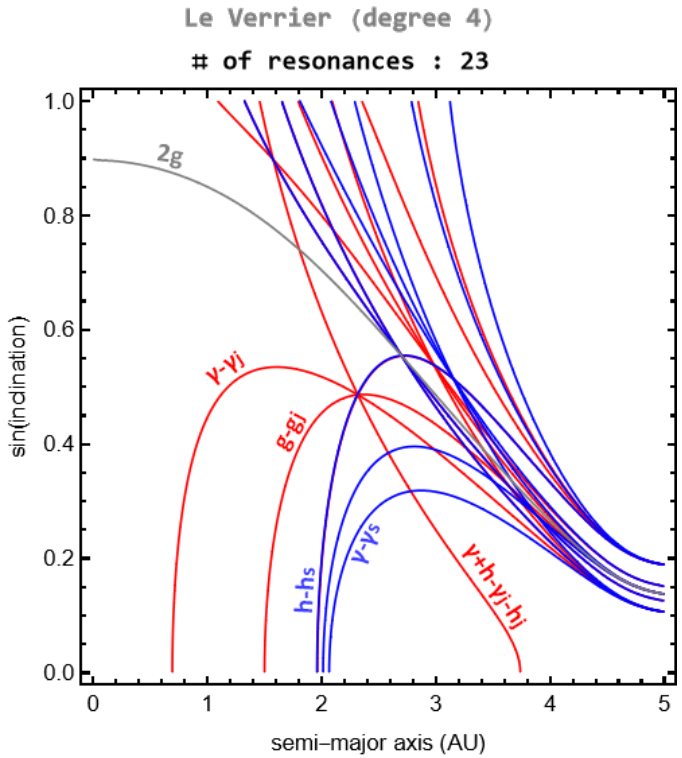
Figure for $a_j = 5.20427\ AU$ (Jupiter at J2000), and $\|C_d\| = \left\| \sum_{j=0}^{N} c_n \exp(\boldsymbol{k}_n \cdot \boldsymbol{q}) \right\| = \sum_{j=0}^{N} |c_n|$.

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We approximate the positions of the secular resonant planes, of degree d,

$$\boldsymbol{k} \cdot \nabla_{\boldsymbol{p}} \mathcal{N} = 0, \quad \boldsymbol{k} \in \mathcal{K}_d,$$

using the generalized Le Verrier development.

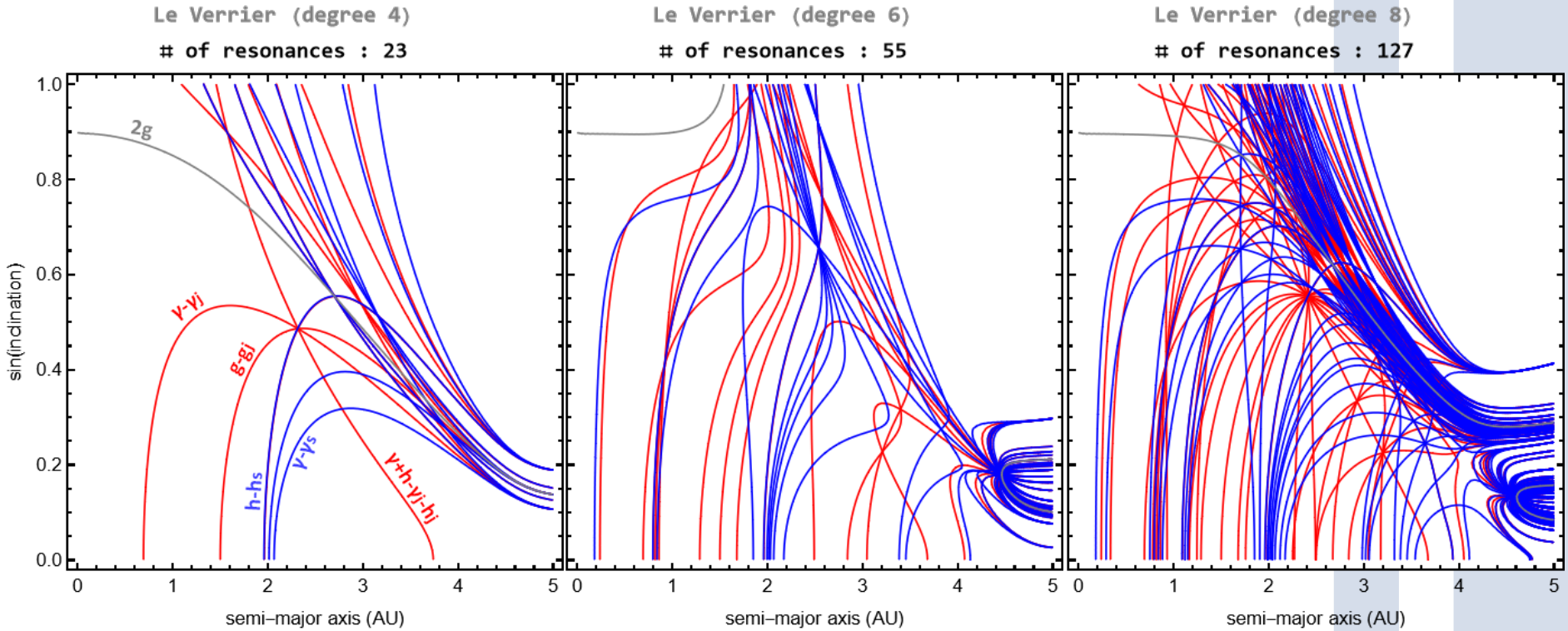


$e = 0.1$, $g =$ argument of pericenter, $\gamma =$ longitude of pericenter, $h =$ longitude of the node

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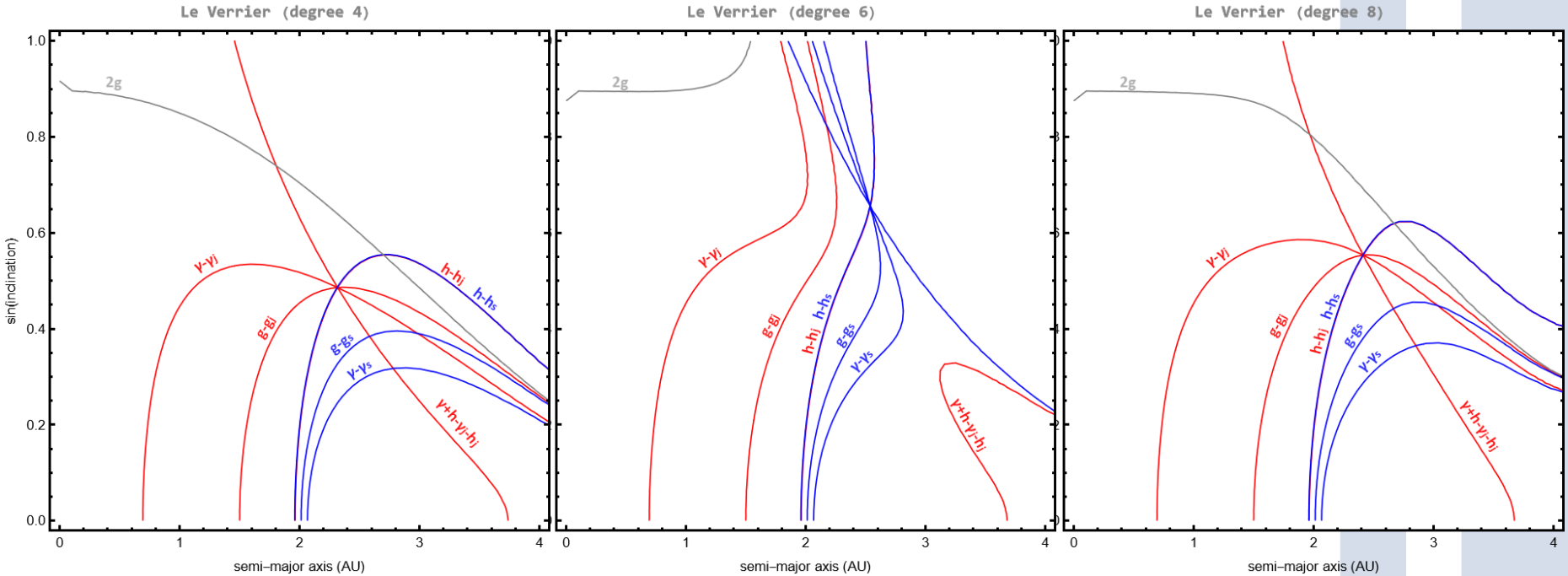


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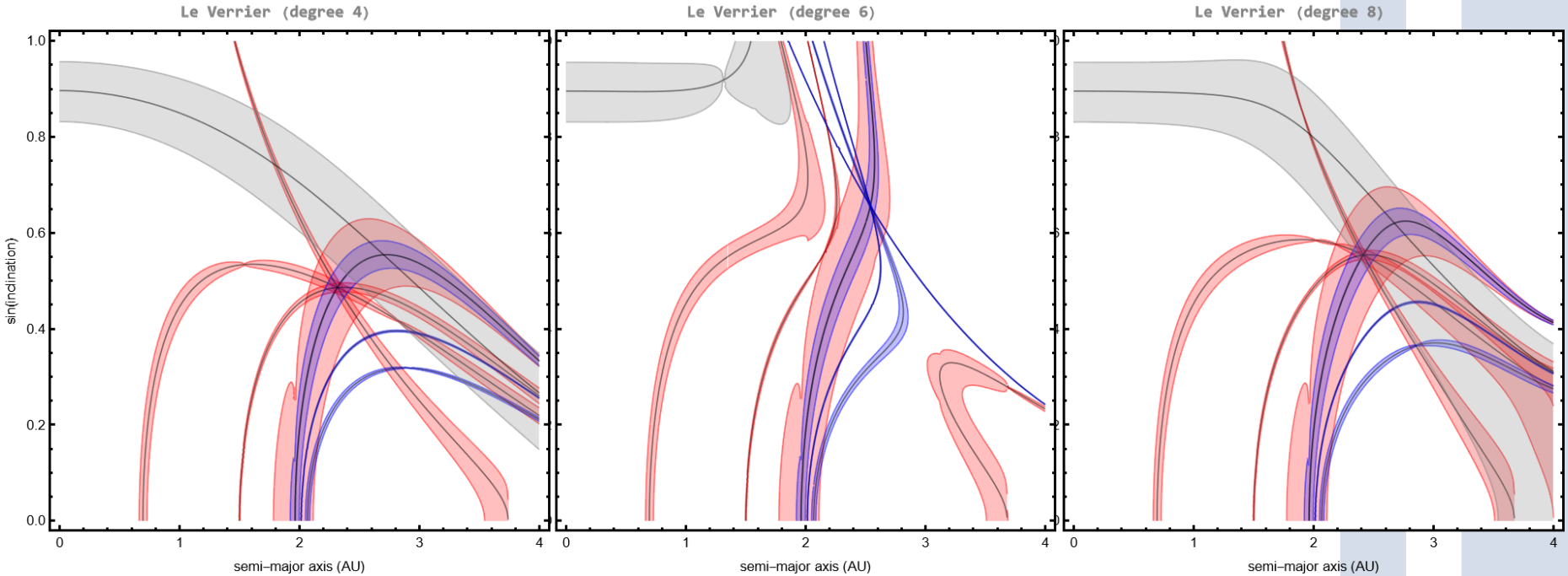


$e = 0.1$, $g =$ argument of pericenter, $\gamma =$ longitude of pericenter, $h =$ longitude of the node

We approximate the widths of the secular resonances, of degree d,

$$\left|\boldsymbol{m}\cdot\nabla_{\boldsymbol{p}}\mathcal{N}\right| \leq 2\sqrt{\left\|\mathcal{R}_{\boldsymbol{k}\in[\boldsymbol{m}]}(\boldsymbol{p}_0)\right\|\left(\boldsymbol{m}\cdot\nabla_{\boldsymbol{p}_0}\right)^2\mathcal{N}(\boldsymbol{p}_0)}, \quad \boldsymbol{k}\in\mathcal{K}_d, \quad \boldsymbol{m}\cdot\nabla_{\boldsymbol{p}_0}\mathcal{N}(\boldsymbol{p}_0)=0,$$

using the generalized Le Verrier development.

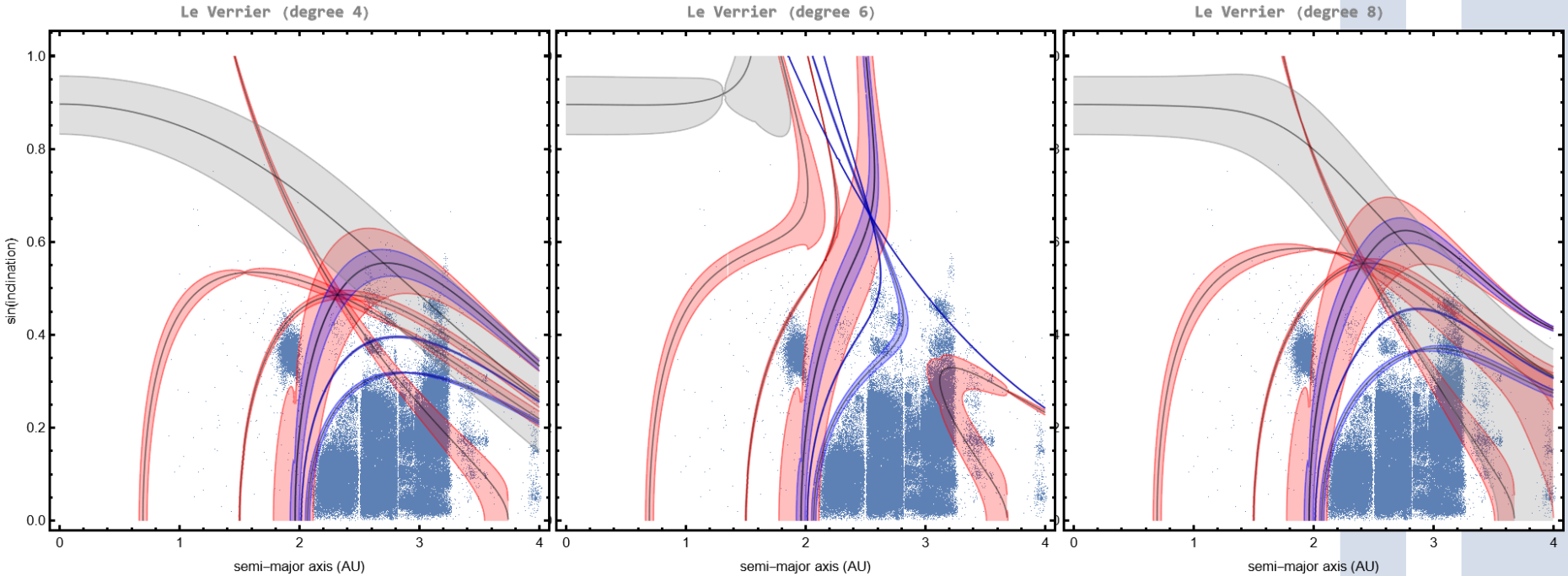


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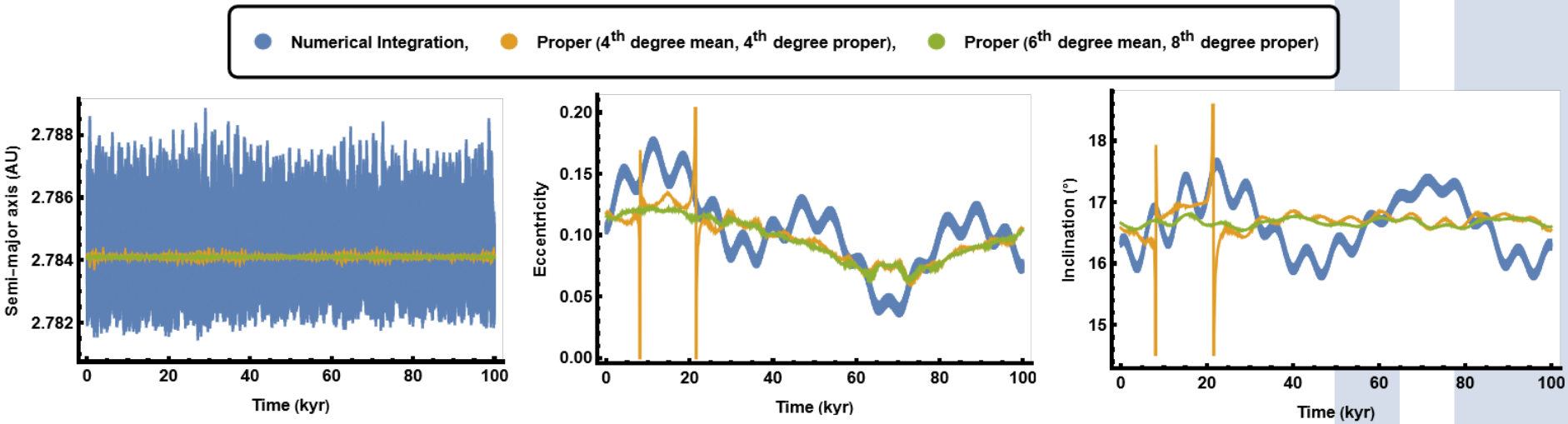
$$|\boldsymbol{m}\cdot\nabla_{\boldsymbol{p}}\mathcal{N}| \le 2\sqrt{\left\|\mathcal{R}_{\boldsymbol{k}\in[\boldsymbol{m}]}(\boldsymbol{p}_0)\right\|\left(\boldsymbol{m}\cdot\nabla_{\boldsymbol{p}_0}\right)^2\mathcal{N}(\boldsymbol{p}_0)}, \quad \boldsymbol{k}\in\mathcal{K}_d, \quad \boldsymbol{m}\cdot\nabla_{\boldsymbol{p}_0}\mathcal{N}(\boldsymbol{p}_0)=0,$$

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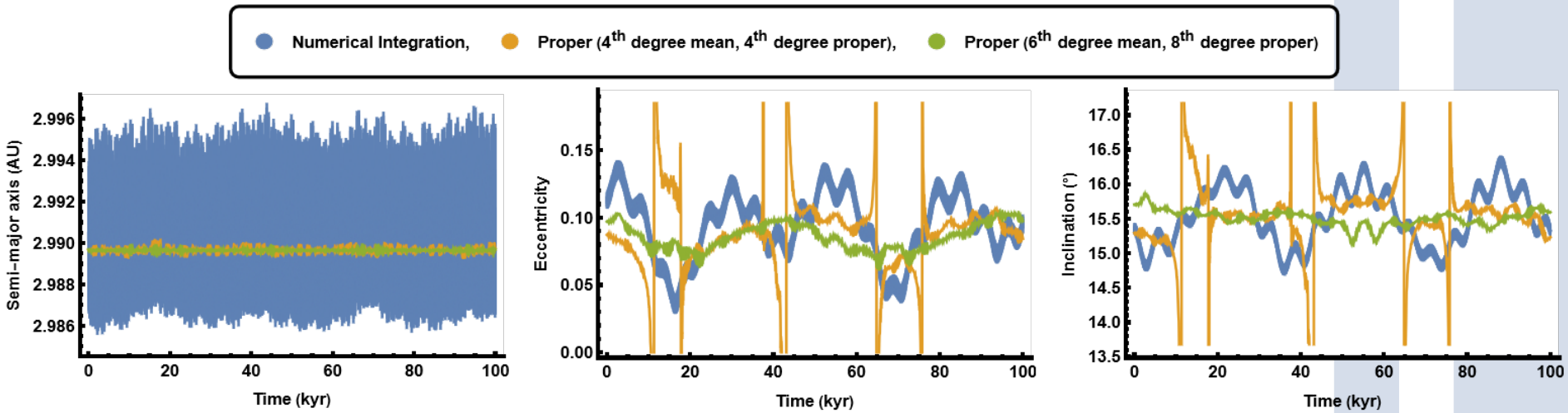


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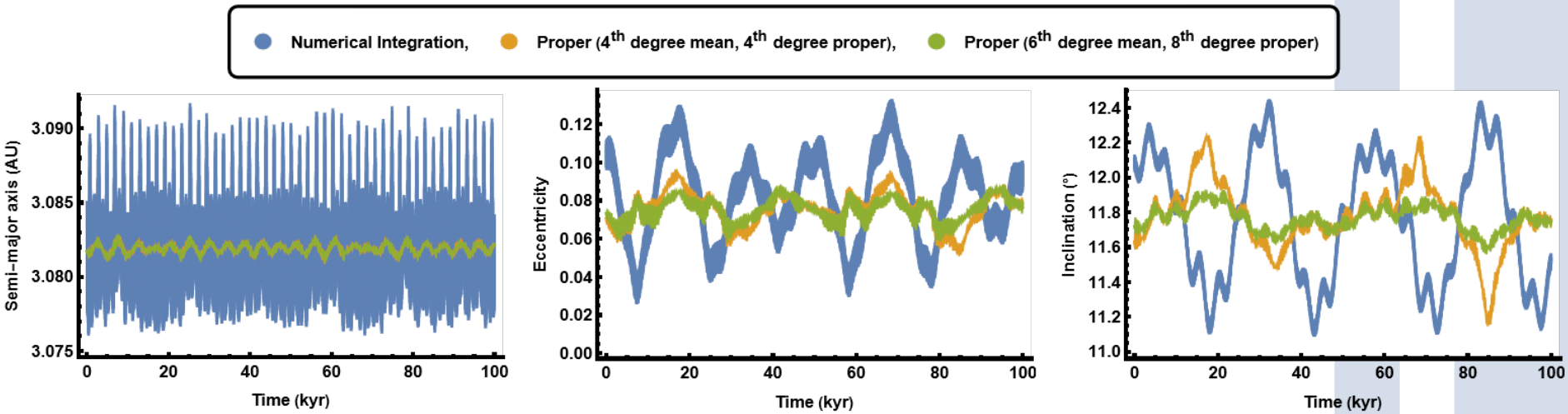
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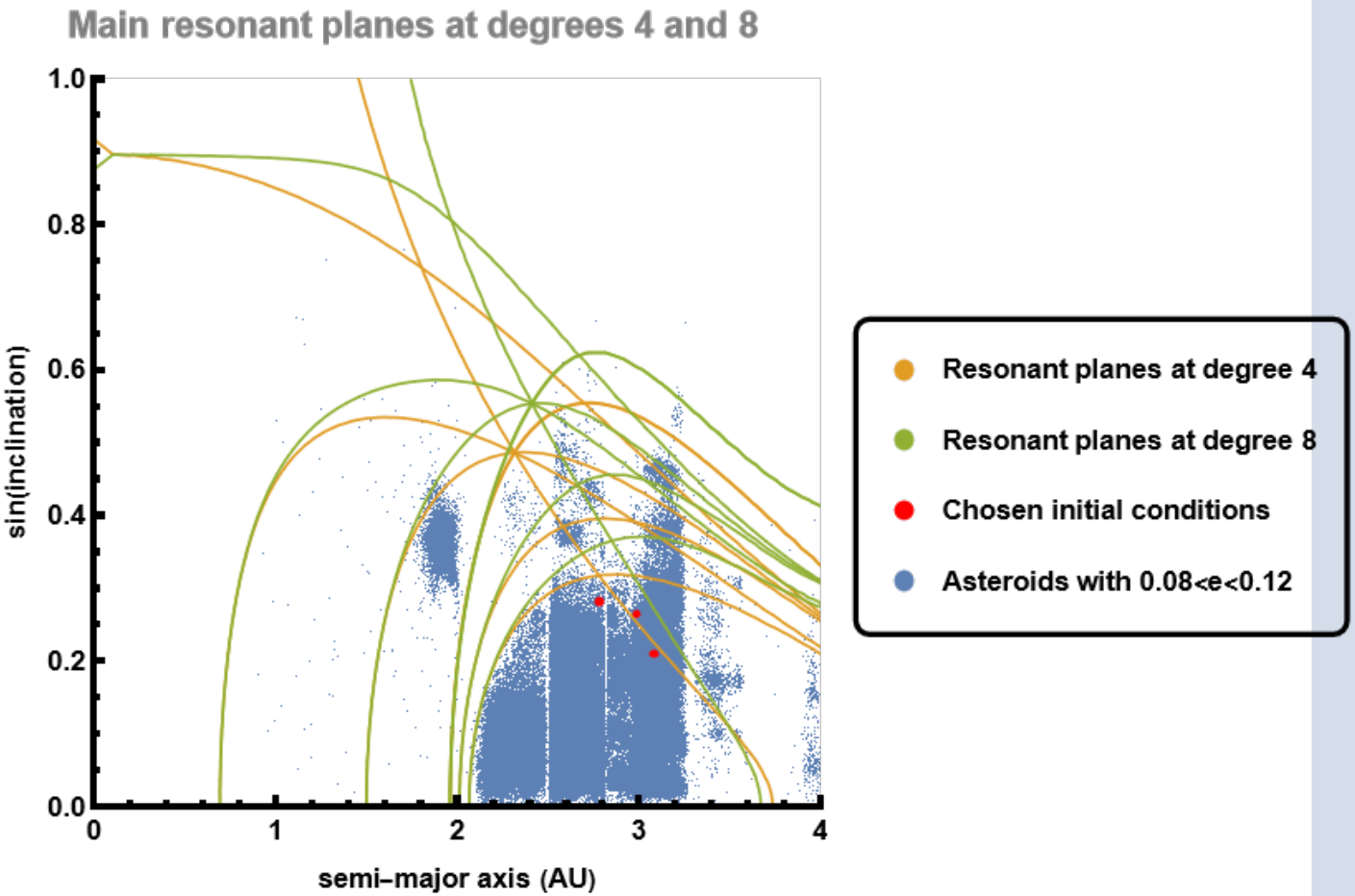
Initial Conditions : l_0=281.93 °, ω_0=175.77 °, Ω_0=111.11 °, a_0=2.7833 AU, e_0=0.10667 , i_0=16.332 °
Asteroid Number : '784378'
Date of given data : Mon 5 May 2025 03:00:00
Reference Frame : ω=−104.75 °, Ω=107.89 °, i=1.5791 ° with respect to the Ecliptic at J2000



Initial Conditions : l_0=31.266 °, ω_0=185.72 °, Ω_0=168.71 °, a_0=2.9905 AU, e_0=0.10932 , i_0=15.338 °
Asteroid Number : '325684'
Date of given data : Mon 5 May 2025 03:00:00
Reference Frame : ω=−104.75 °, Ω=107.89 °, i=1.5791 ° with respect to the Ecliptic at J2000



Initial Conditions : l_0=265.97 °, ω_0=33.613 °, Ω_0=2.3918 °, a_0=3.085 AU, e_0=0.098103 , i_0=12.082 °
Asteroid Number : '609173'
Date of given data : Mon 5 May 2025 03:00:00
Reference Frame : ω=−104.75 °, Ω=107.89 °, i=1.5791 ° with respect to the Ecliptic at J2000


Main resonant planes at degrees 4 and 8
sin(inclination)
semi–major axis (AU)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
Resonant planes at degree 4
Resonant planes at degree 8
Chosen initial conditions
Asteroids with 0.08<e<0.12

Future works and possible extensions

- Study in detail and refine the preliminary results on the convergence of this development.
- Study the dynamical properties and topology of the important secular resonances in the main asteroid belt.
- Use of resonant proper elements (Dogkas & Vartolomei 2026) for the improvement of the analytical classification of asteroid families.

References

This work

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Proper elements

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- Knežević, Z. & Milani, A. Synthetic Proper Elements for Outer Main Belt Asteroids. *Celestial Mechanics and Dynamical Astronomy* **78,** 17–46 (Sept. 2000)
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Dynamical Model

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