Radially anisotropic systems with rotation and their (numerical) stability









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Context

It is well known that:

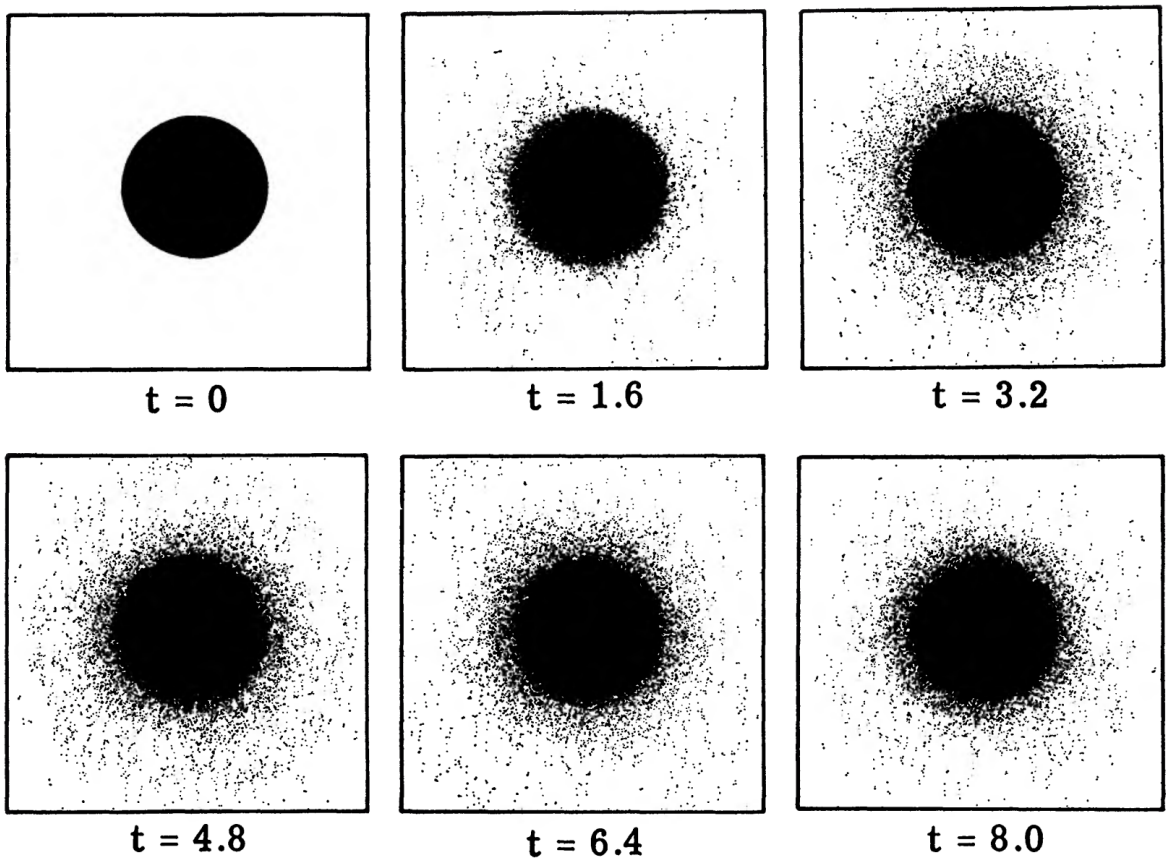
1. Spherical systems with a large fraction of kinetic energy stored in low angular momenutm orbits are unstable
2. Rotating systems are likewise asymptotically unstable

This is interesting as:

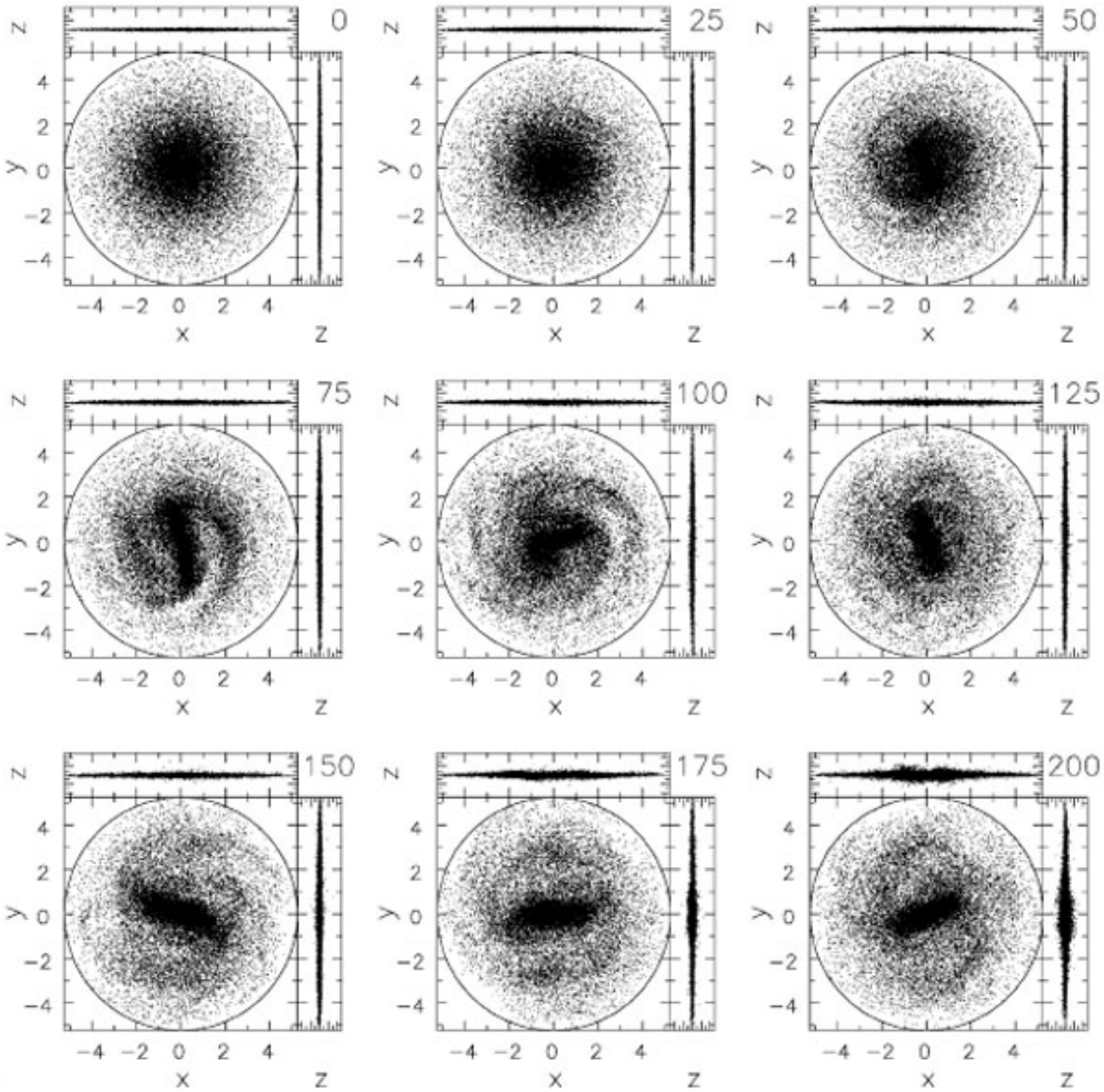
1. End products of numerical simulations are characterized by a considerable amount of radial anisotropy
2. Most star clusters have non zero J, high-z (proto)-galaxies show some degree of rotation...

What is the interpaly between rotation and radial anisotropy?

Rotational instability



Rotational instability



Radial orbit instability

1. Collisionless equilibrium systems with a significant fraction of the kinetic energy stored in low angular momentum orbits are violently unstable. The amount of radial orbits is quantified by introducing the Fridman-Polyachenko-Shukhman parameter

$$\xi = 2K_r/K_t,$$

   as function of the radial and tangential components of the (initial) kinetic energy T_r and T_t
2. For approximately $\xi > 1.5$ Newtonian systems appear to be unstable, leading to triaxial end-states.

Radial orbit instability

1. Analytical stability results exist for the isotropic case. It is also known that phase-space distribution functions with $\mathrm{d}f(\mathcal{E})/\mathrm{d}\mathcal{E} \leq 0$ correspond to stable systems (Antonov theorem)
2. ROI is triggered by particles with orbital frequencies close to satisfying the condition $\Omega_P \equiv 2\Omega_\nu - \Omega_r \simeq 0$, where Ω_ν is the azimuthal frequency, Ω_r the radial frequency and Ω_P the precession frequency
3. Once a small non-spherical density perturbation is formed in a system dominated by low Ω_P orbits, it will grow more and more, as more and more particles tend to accumulate to it.

Self consistent anisotropic equilibrium models: Phase-space distribution

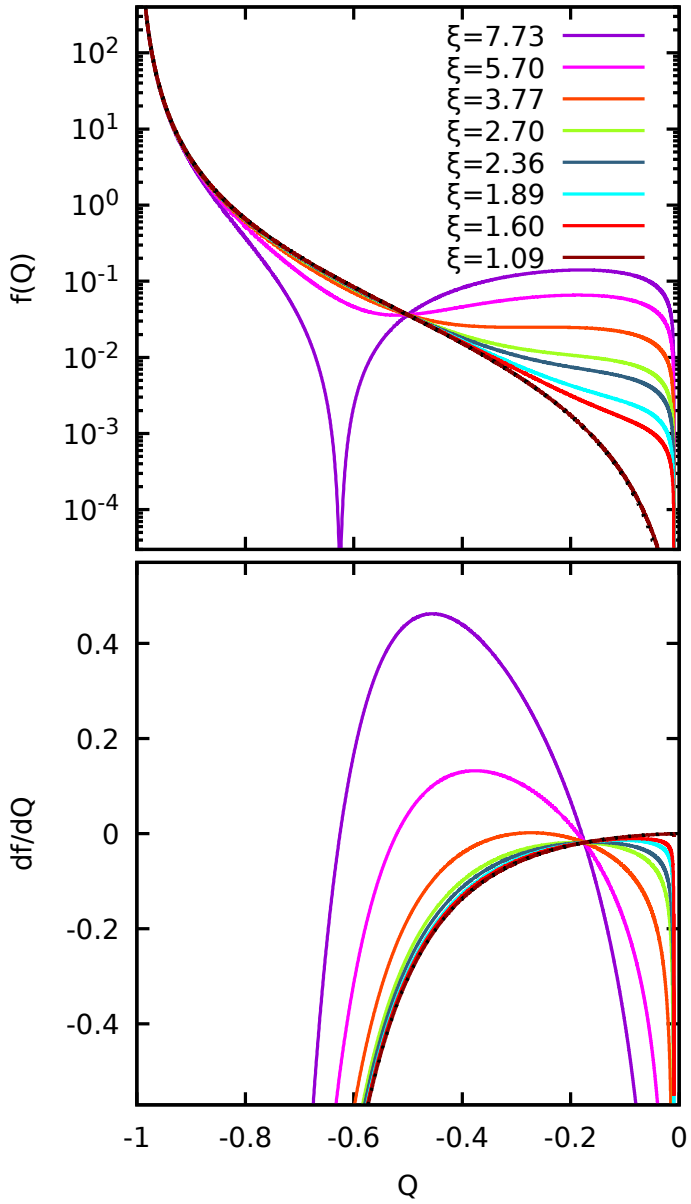
For a spherically symmetric, non-rotating continuum density profile $\rho(r)$ with anisotropy radius r_a and potential $\Phi(r)$, the anisotropic $f(Q)$ depends only on the quantity $Q = \mathcal{E} + J^2/2r_a^2$, where $\mathcal{E}$ is the energy, J is the angular momentum via the Osipkov (1979) - Merritt (1985) parametrization of the Eddington (1916) formula

$$f(Q) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}Q} \int_Q^{Q_{\mathrm{sup}}} \frac{\mathrm{d}\rho_a}{\mathrm{d}\Phi} \frac{\mathrm{d}\Phi}{\sqrt{\Phi - Q}}$$

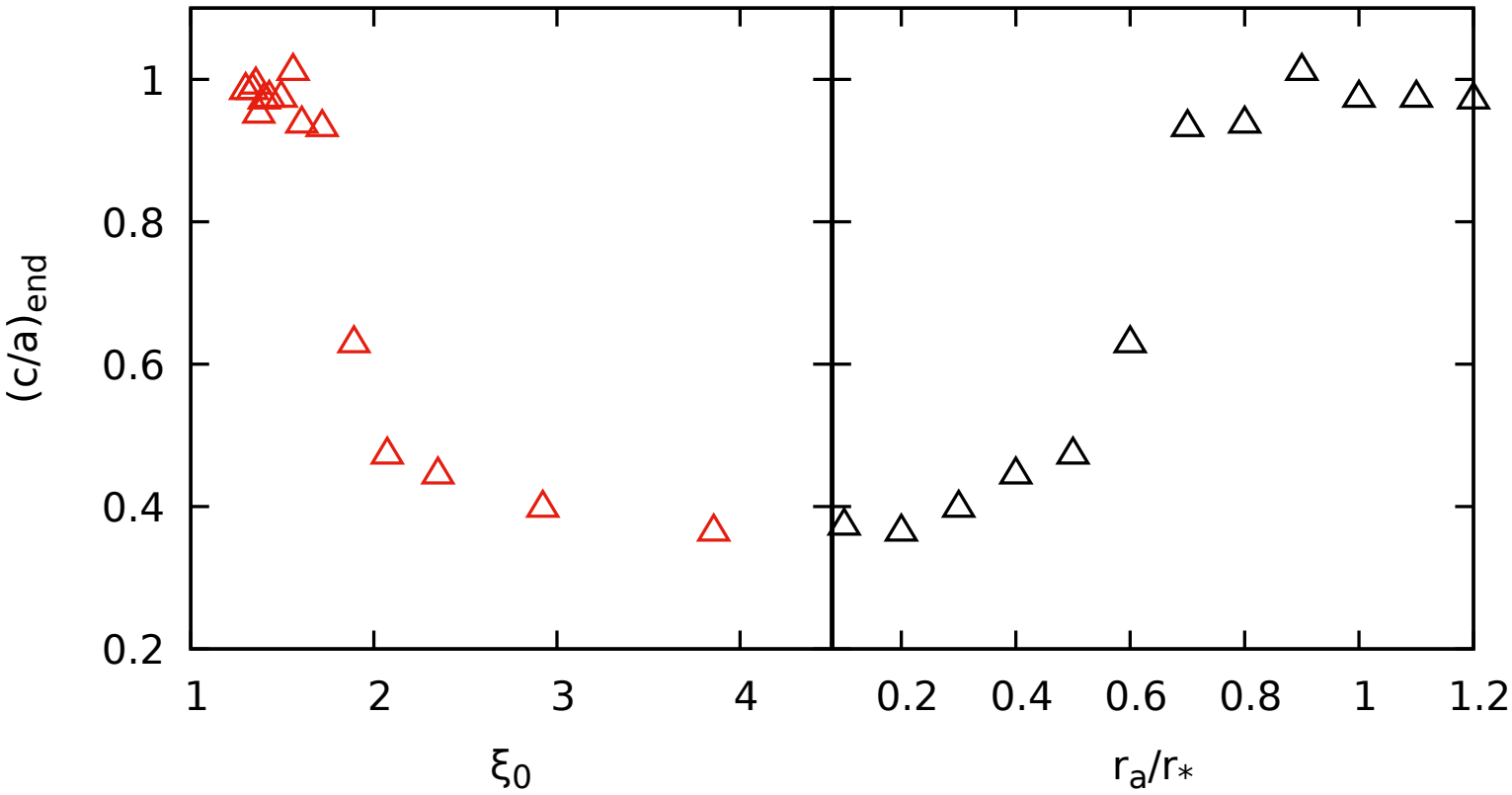
$$\rho_a(r) = \rho(r) \left(1 + \frac{r^2}{r_a^2}\right)$$

This can be used to built, starting from a smooth ρ, $N-$body realizations via standard rejection method.

Radial orbit instability



Radial orbit instability



Introducing rotation

Orbits in models rotating with frequency ϖ around the $z-$axis in an inertial frame conserve the Jacobi constant

$$C_{\rm J} = 2\varpi(xv_y - yv_x) - 2\Phi(x,y,z) - (v_x^2 + v_y^2 + v_z^2) = 2\varpi J_z - 2\mathcal{E}.$$

The latter holds true for any spherically symmetric or axi-symmetric potential. Formally, using the OM parametrization yields the modified Jacobi constant $\tilde{C}_{\rm J} = 2\varpi J_z - 2Q$, that written in spherical coordinates becomes

$$\tilde{C}_{\rm J} = 2\varpi r \sin\vartheta v_\varphi - [v_r^2 + (1 + r^2/r_a^2)(v_\vartheta^2 + v_\varphi^2)] - 2\Phi(r),$$

where $r\cos\vartheta = R = \sqrt{x^2 + y^2}$ expresses the cylindrical radius and v_r and $v_t = \sqrt{v_\vartheta^2 + v_\varphi^2}$ are the radial and tangential components of the velocity vector.

Introducing rotation

starting from a parent OM distribution $f_0(Q)$, we introduce the explicit dependence on J_z by augmenting it as

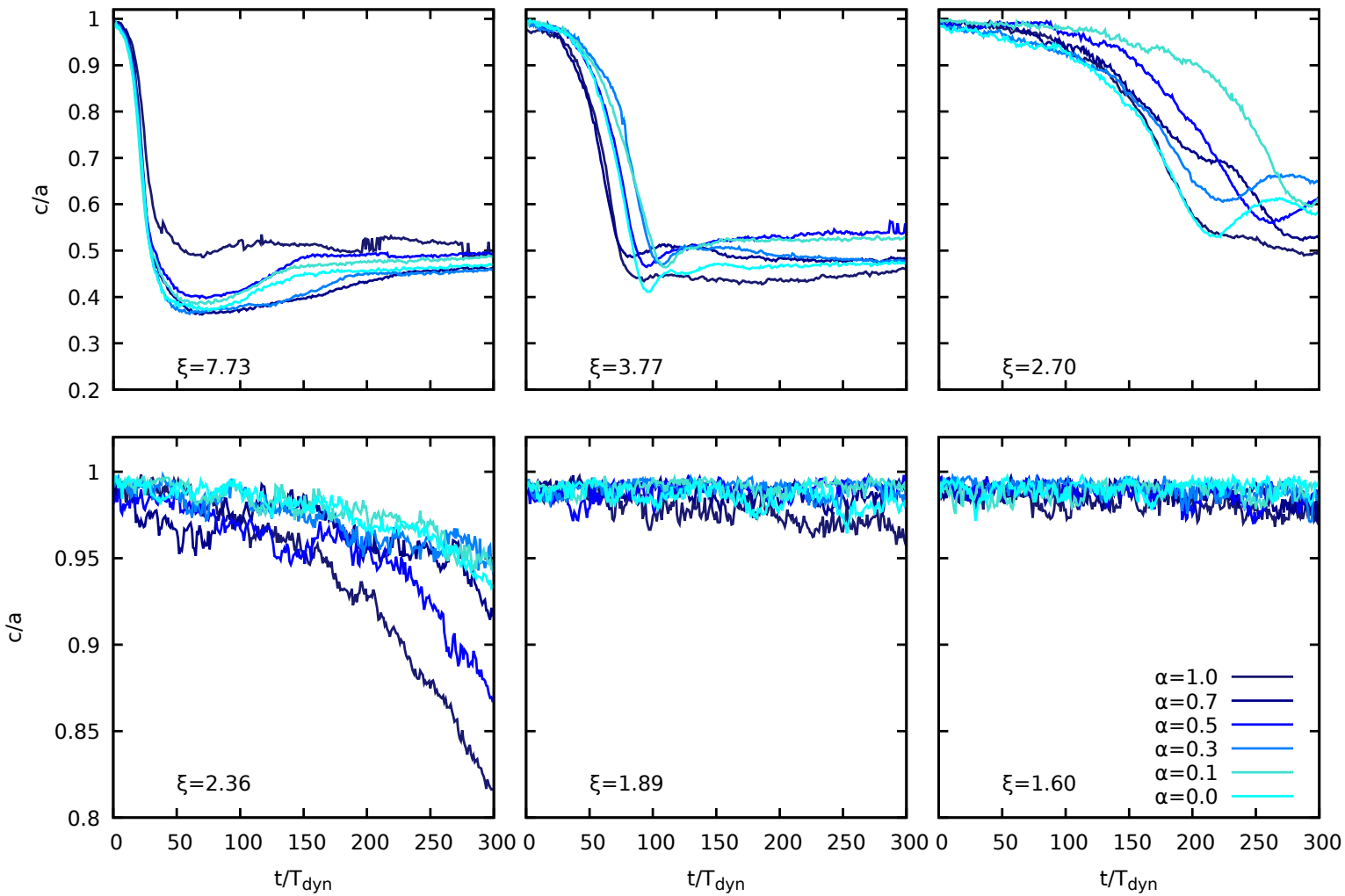
$$f(Q,\alpha) = f_0(Q) + \alpha f_0(Q)\mathrm{Sign}(J_z),$$

where $|\alpha| \leq 1$ is the rotation control parameter and $\mathrm{Sign}(x)$ is the usual sign function. Obviously, by construction one has that

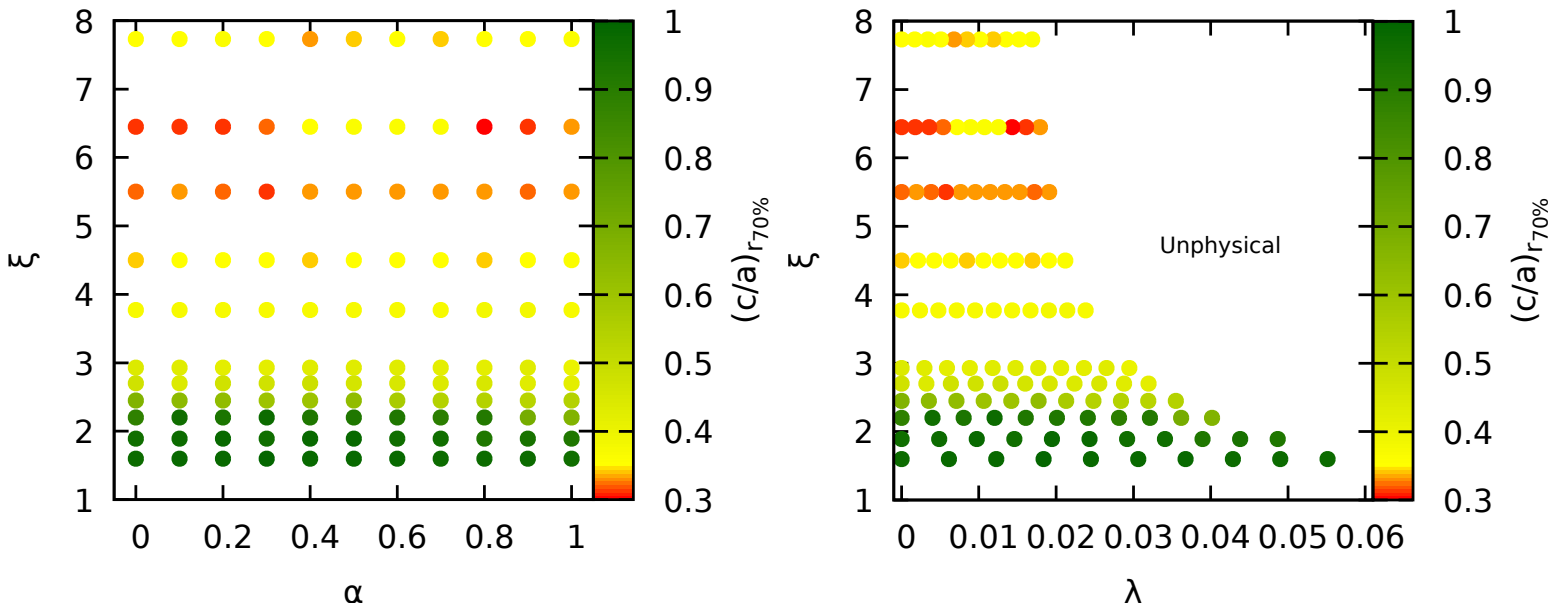
$$\int_{-J}^{+J} \mathrm{Sign}(J_z)\mathrm{d}J_z = 0.$$

The redefinition of f does not alter the degree of anisotropy with respect to the OM model $f_0(Q)$ but for an orbit corresponding to a certain value of Q, the probability of having $J_z > 0$ (< 0), is multiplied by $1 + \alpha$ $(1 - \alpha)$ with respect to the non rotating case.

Flattening

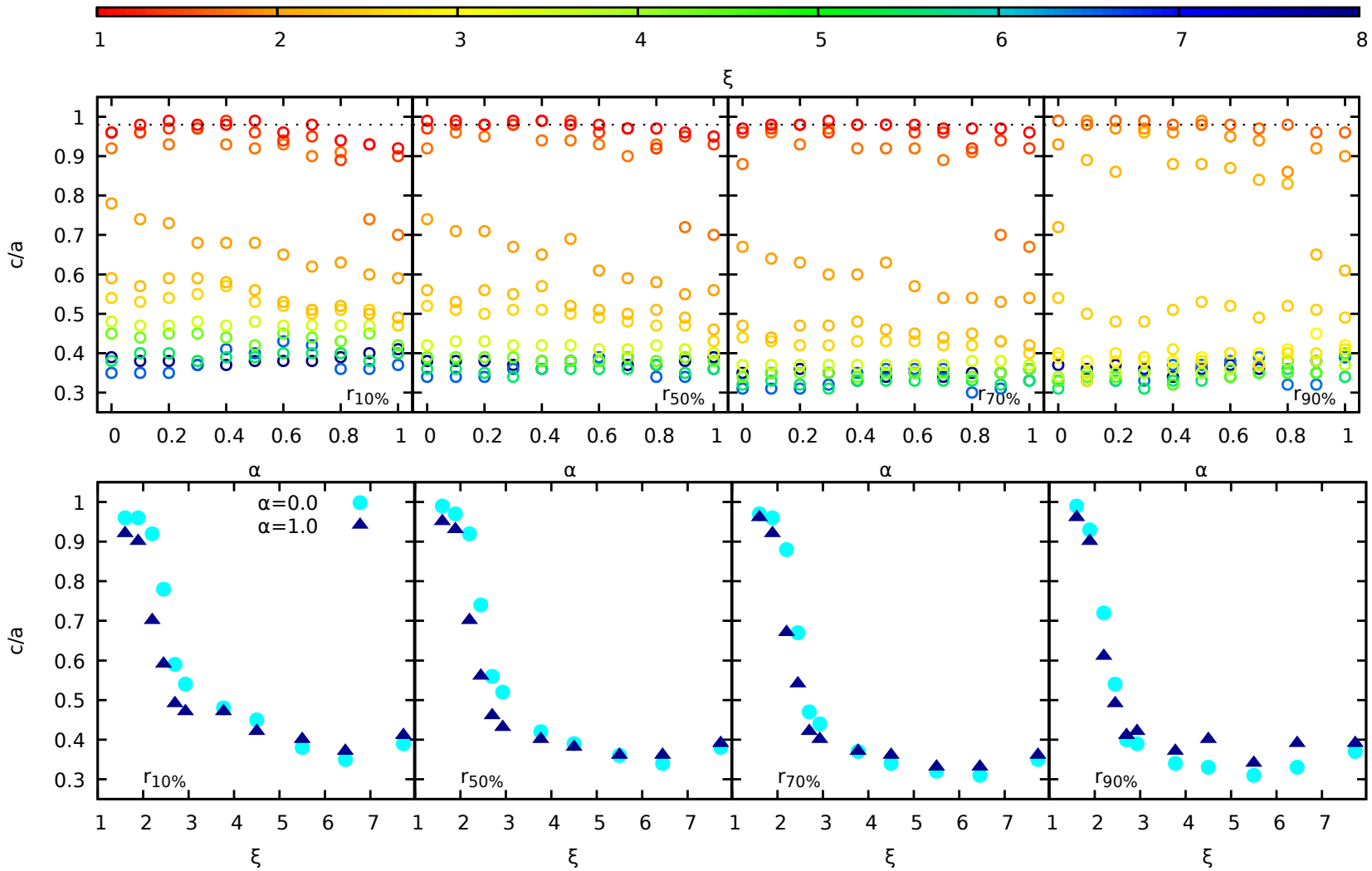


Flattening

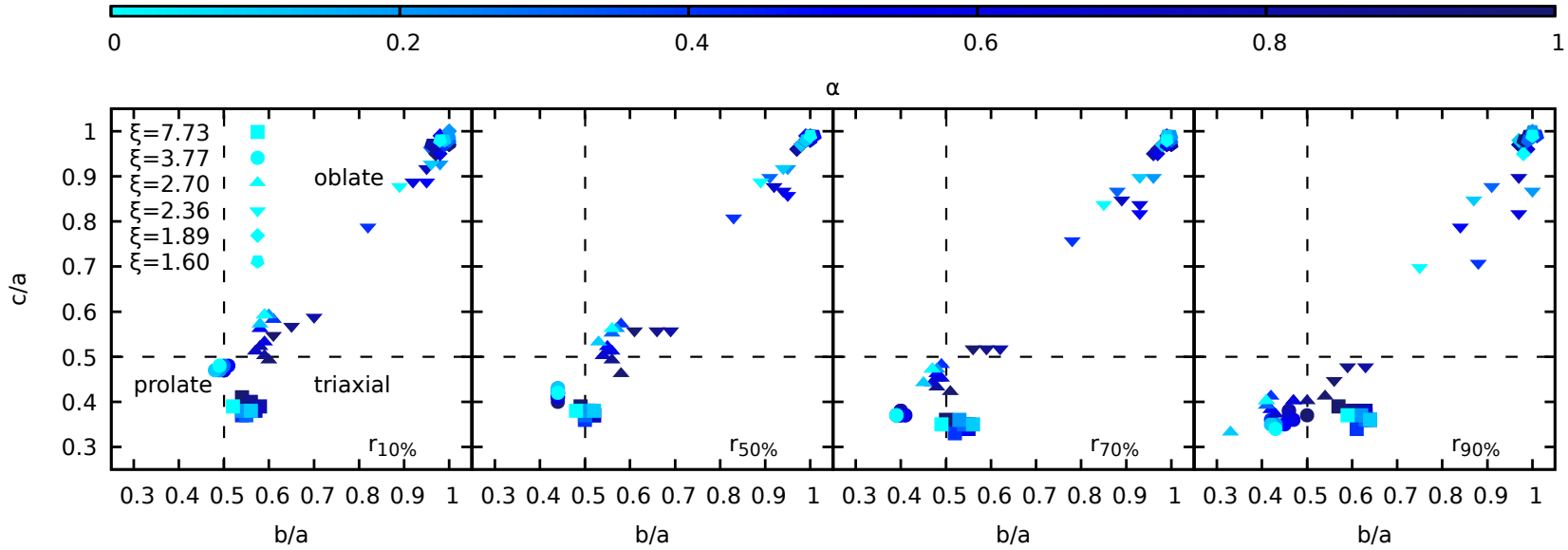


$$\lambda = \frac{J_{\rm tot}\sqrt{-\mathcal{E}_{\rm tot}}}{GM^{5/2}}.$$

Flattening

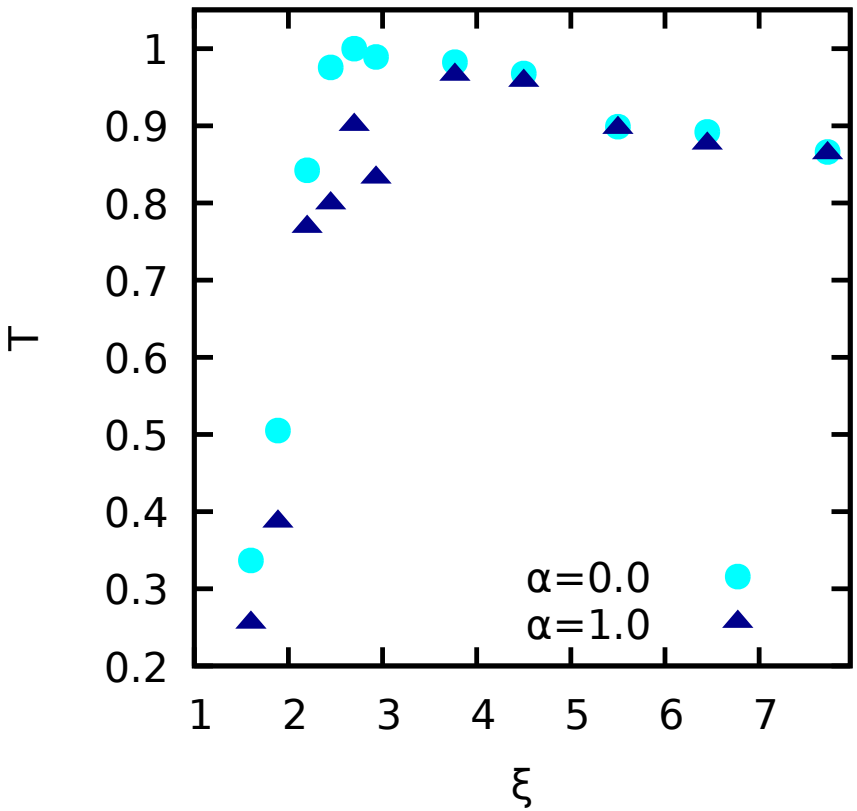


Triaxiality

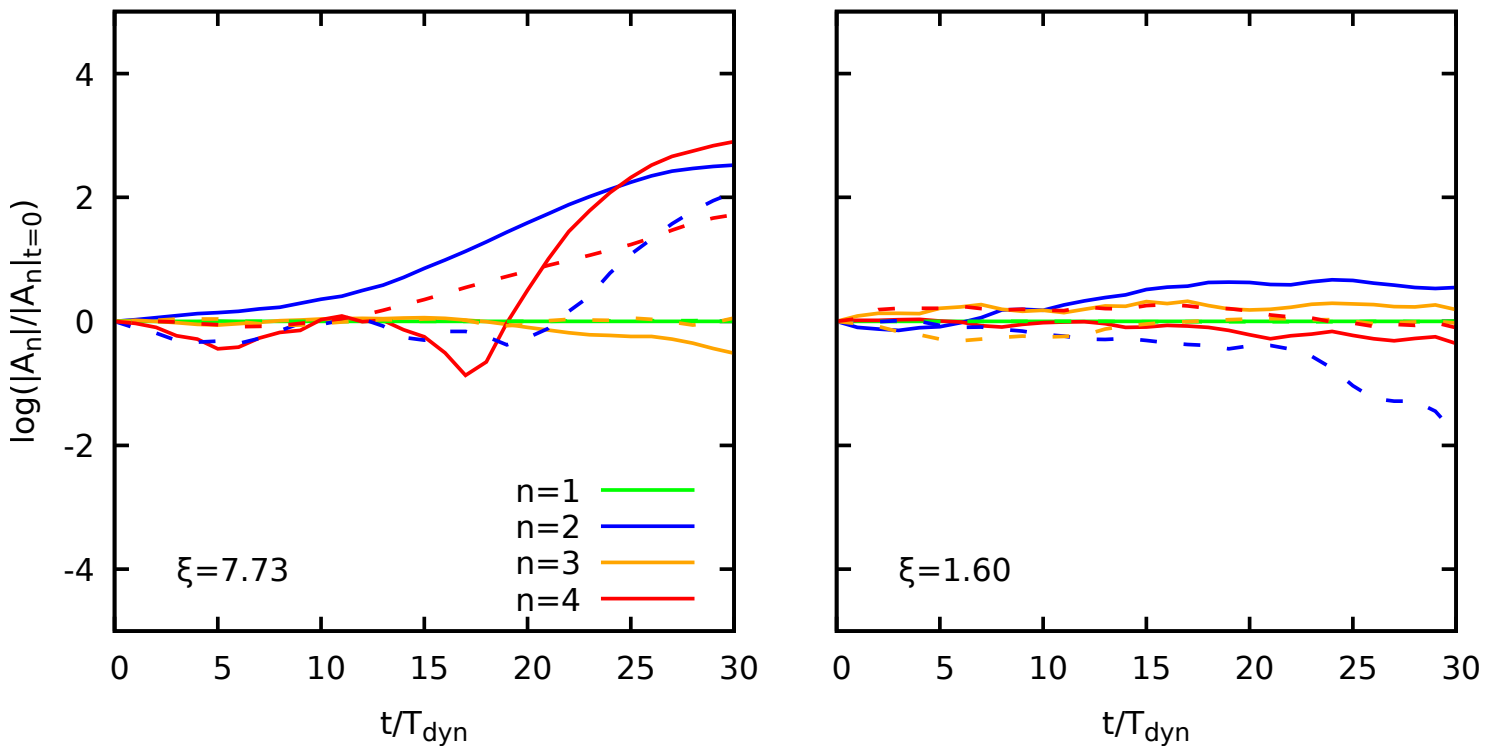


Triaxiality

$$T = \frac{a^2 - b^2}{a^2 - c^2}. \tag{1}$$



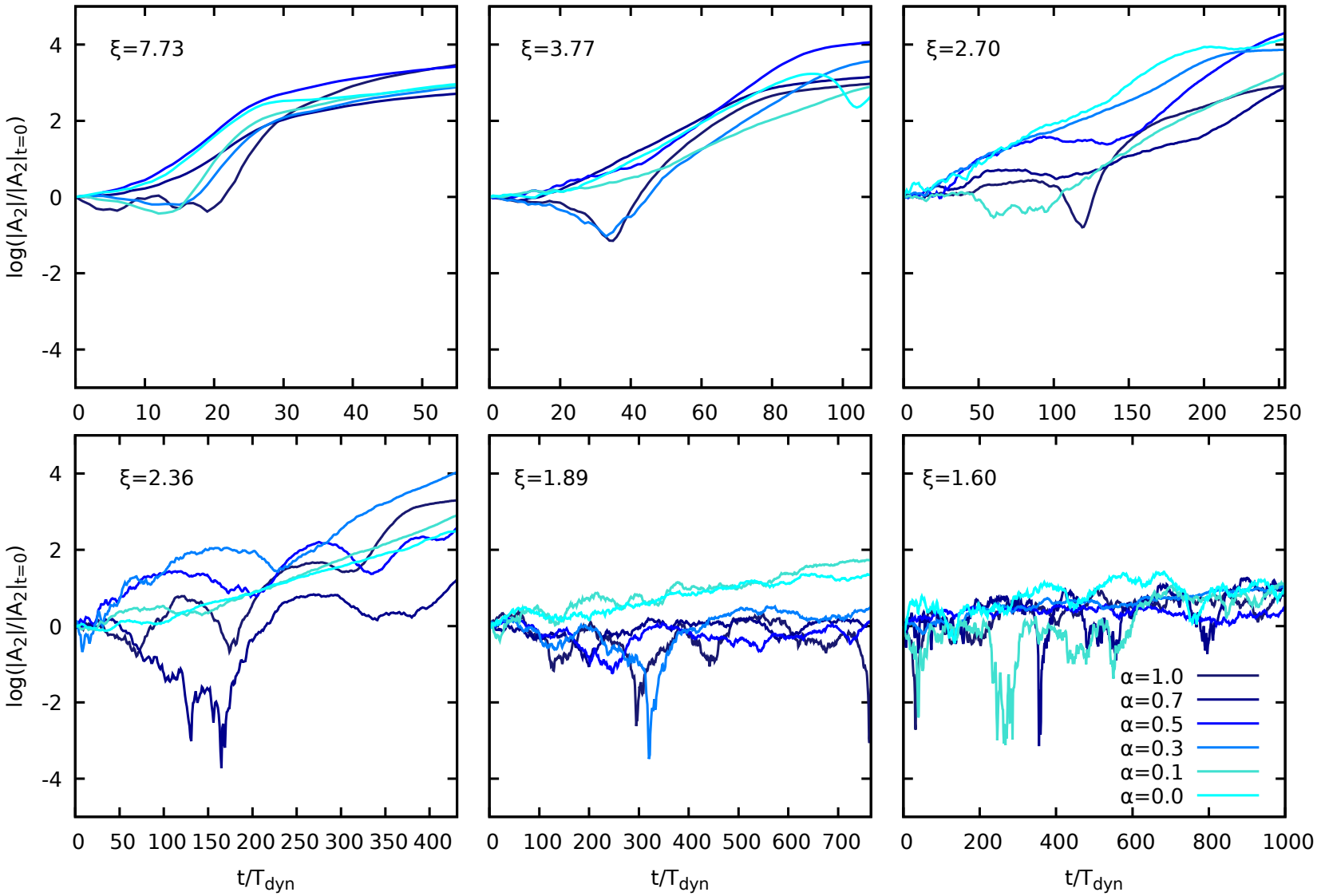
Modes analysis



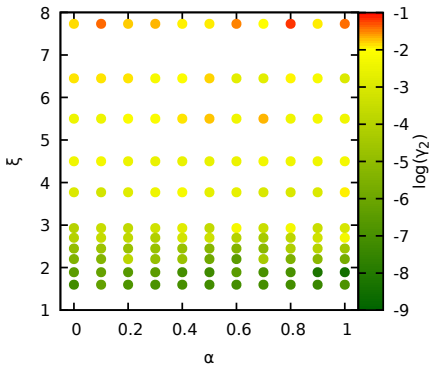
$$C_n(t) = \frac{1}{2\pi} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} \rho(R, z, \varphi, t) e^{-in\varphi} R \mathrm{d}\varphi \mathrm{d}z \mathrm{d}R.$$

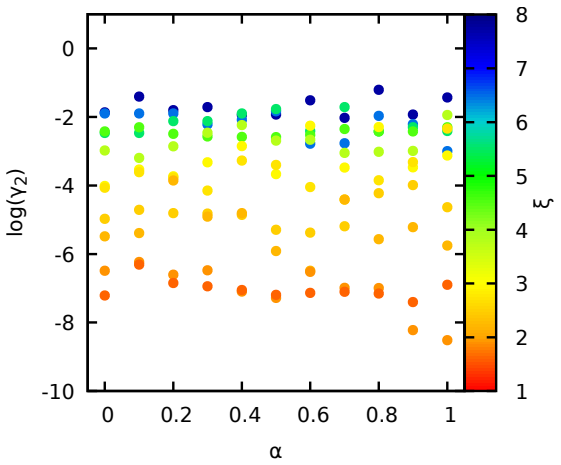
We define the quantities $A_n(t) = C_n(t)/C_0(t)$ whose norms $|A_n(s)|$ are proportional to the amplitude of the growing modes of ρ.

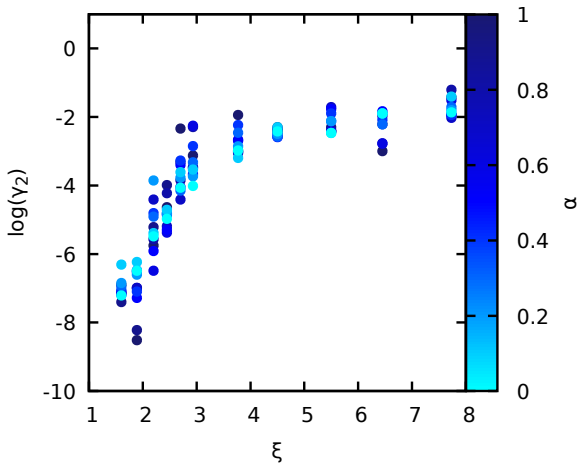
Modes analysis



Modes analysis

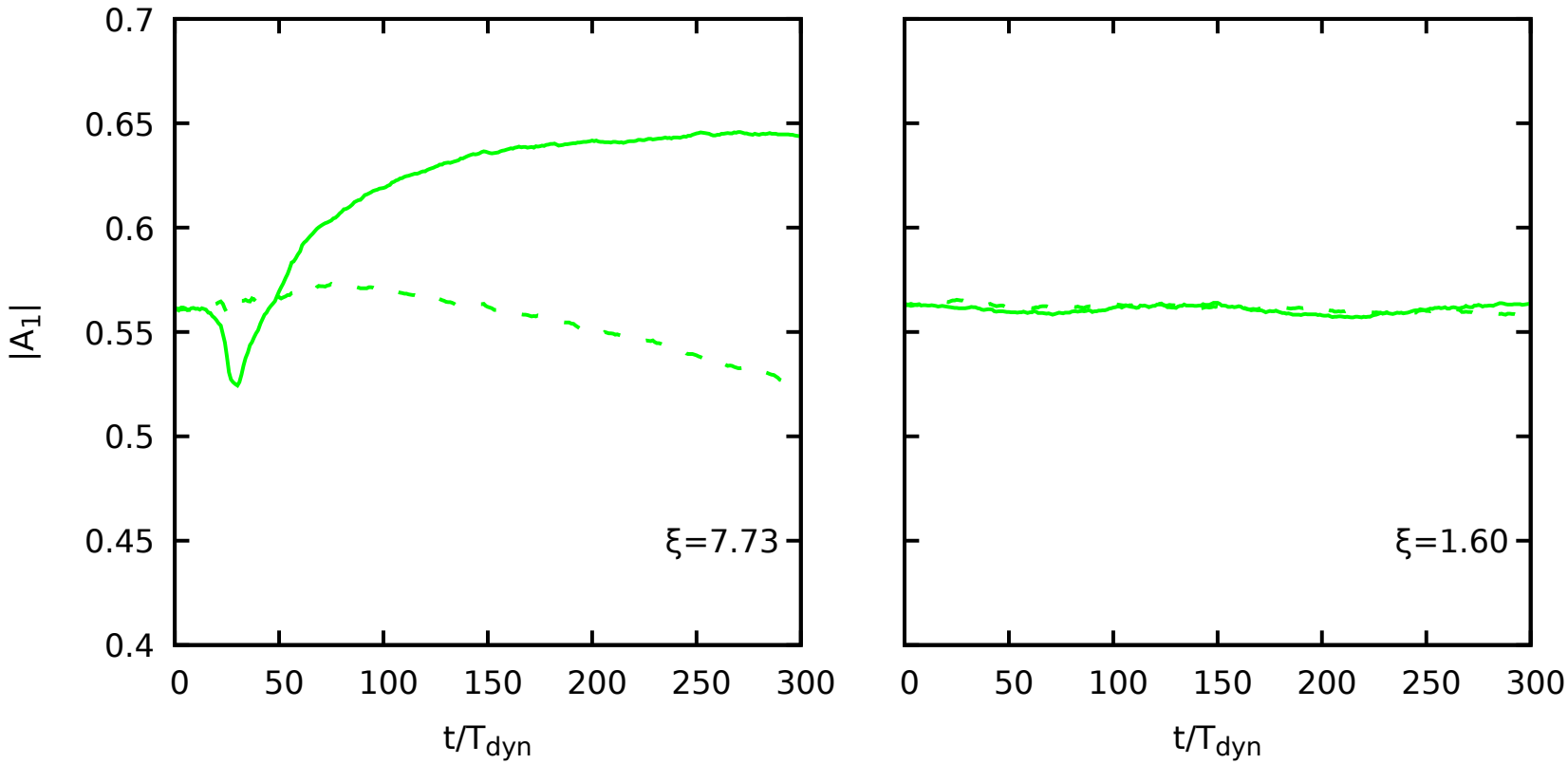




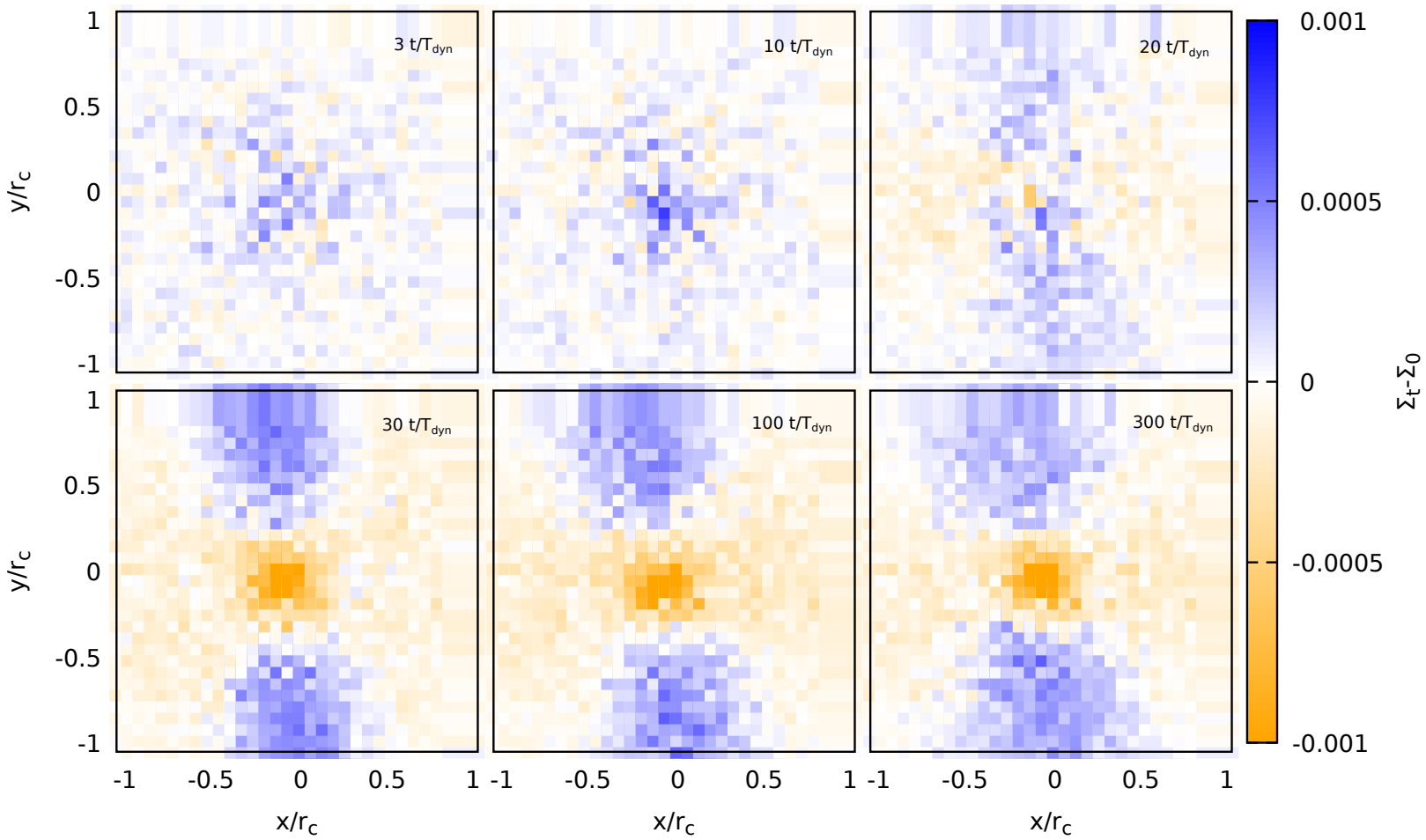


$$\gamma_n = \frac{\mathrm{d}\log|A_n(t)|}{\mathrm{d}t}$$

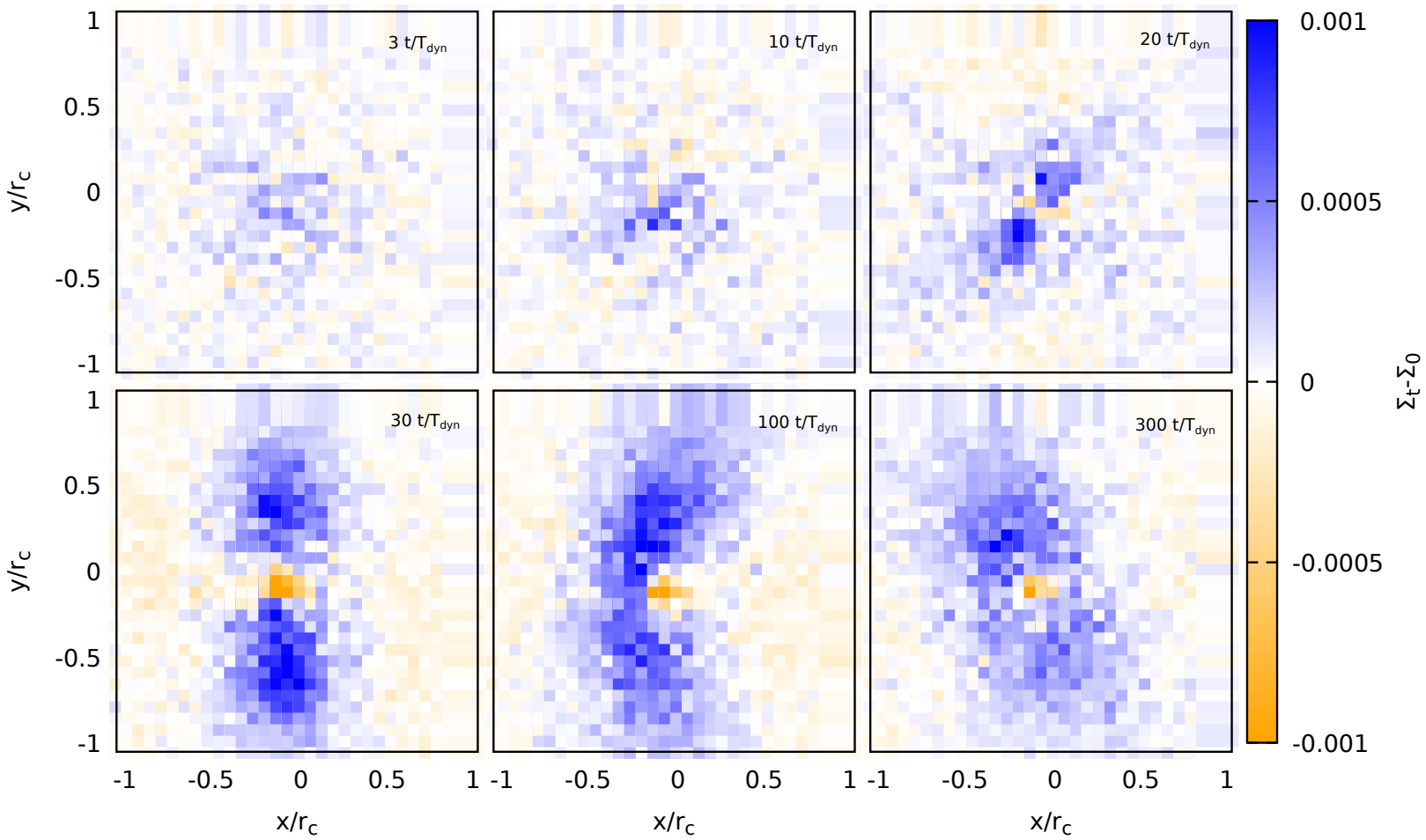
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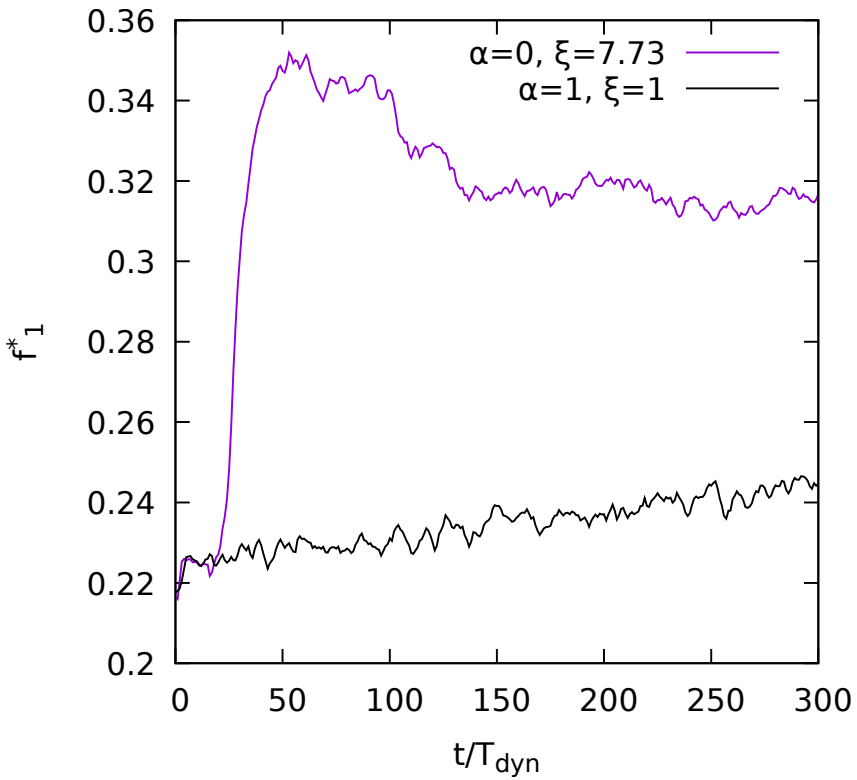
Modes analysis



Modes analysis



Phase space distribution function



$$f^*(\mathbf{k}, t) = \frac{1}{N} \sum_{k=1}^{N} e^{i(\mathbf{k}_r \cdot \mathrm{r}_i(t) + \mathbf{k}_v \cdot \mathrm{v}_i(t))}$$

Preliminary Conclusions

1. Net rotation mitigates ROI in high ξ models and enhances it for nearly stable ones.
2. In general, ROI is less violent for rotating models
3. Even mild rotation makes end states that are less prolate
4. The growth of the modes of f is nealry linear in rotation dominated models and exponential in ROI-prone ones