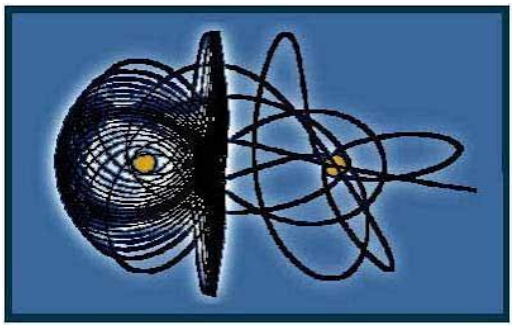
Study of resonant regimes in Rings' dynamics

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In collaboration with **Alessandra Celletti** and **Sara Di Ruzza**





CELMEC IX – September 14 2026

1. Introduction
2. The Hamiltonian model
3. Multi - scale KAM theorem
4. Analysis of the resonant regime

Outline

1. Introduction
2. The Hamiltonian model
3. Multi - scale KAM theorem
4. Analysis of the resonant regime

A centuries-long story: Galileo anagram to Kepler

A year has now passed since Galileo wrote to Prague that he had observed something new in the heavens in addition to what had previously been discovered. And lest there be someone who, out of a spirit of disparagement, would claim to have made the same observation earlier, he allowed a period of time for anyone to divulge whatever new things they might have seen; meanwhile, he himself described what he had found with letters transposed in the following manner:

S m a i s m r m i l m e p o e t a l e u m i b u n e n u g t t a u i r a s.

From these letters, I composed a semi-barbaric verse, which I included in my Narrative in the month of September of the previous year:

Be greeted, twinned middle-held, offspring of Mars.[142]
[Salve umbistineum geminatum Martia proles.]

But I strayed very far from the meaning of the letters, which contained nothing about Mars. And lest I detain you, reader, behold the revelation of the riddle, in the words of its author, Galileo himself:[143]



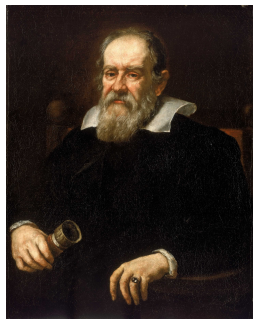
J. Kepler, preface to *Dioptrice*, 1611

But I now come to the next point, regarding the letters which I had transposed and sent to Your Most Illustrious Lordship,[144] *and which were printed by Mr. Kepler in his recent Narrative. As it has also been indicated to me that His Majesty desires to be informed about the meaning of the letters, behold, I am transmitting it to Your Most Illustrious Lordship in order to communicate it to His Majesty and to Mr. Kepler and to whomever you wish.*

The meaning of the letters, connected in the correct order, is this:

I have observed the highest planet triple.
[Altissimum planetam tergeminum observavi]

With the utmost astonishment, I have discovered, beyond any doubt, that Saturn is not one single star but three stars very close to each other, so close indeed that they almost touch each other. They are completely immobile with respect to each other and arranged in this manner: **oOo.** *The middle*



G. Galilei, letter to Julian de Medici, 13 November 1610

"Perhaps Saturn devoured its own children?"

A centuries-long history: Huygens anagram



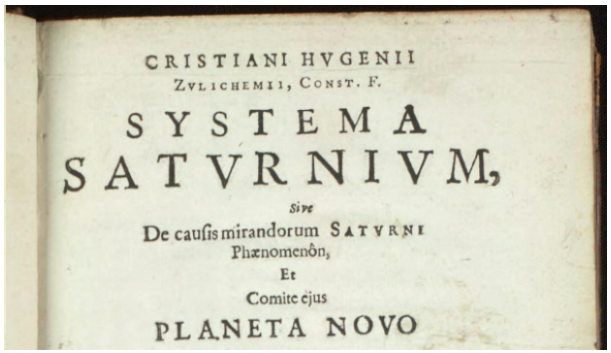
62 LIBER SECUNDUS

CHRISTIANI HUGENII DE SATURNI LUNA OBSERVATIO NONA.

ANNO millesimo sexcentesimo quinquagesimo quinto, Mensis Martii die quinto & vicesimo, Saturni planetam per tubum dioptricum aspectans, animadverti præter ansas sive brachia quæ utrimque illi cohærent, stellulam quandam ab occasu adstantem, tribus circiter scrupulis remotam, eratque dispositа secundum eam quæ per utraque brachia ducta fuisset rectam. Et cum subdubitarem nunquid fortasse planeta esset ejus generis, quales circa Jovem quatuor circumferuntur, locum Saturni stellulæque & positum utriusque ad aliam quandam quæ tantumdem ferè, sed in contrarias partes, à Saturno diffita erat annotavi; ex inerrantium numero hanc potiùs quam illam fore ratus, quod ab ea quam dixi recta linea deflectebat. Neque me

De Saturni luna observatio nova, 1656

" AAAAAAA CCCCC D EEEEE G H IIIIIII LLLL MM NNNNNNNNN OOOO PP Q RR S TTTTT UUUUU "



CRISTIANI HVGENII ZVLICHEMII, CONST. F.

SYSTEMA SATVRNIVM,

Sive

De causis mirandorum SATVRNI Phænomenôn,

Et

Comite ejus

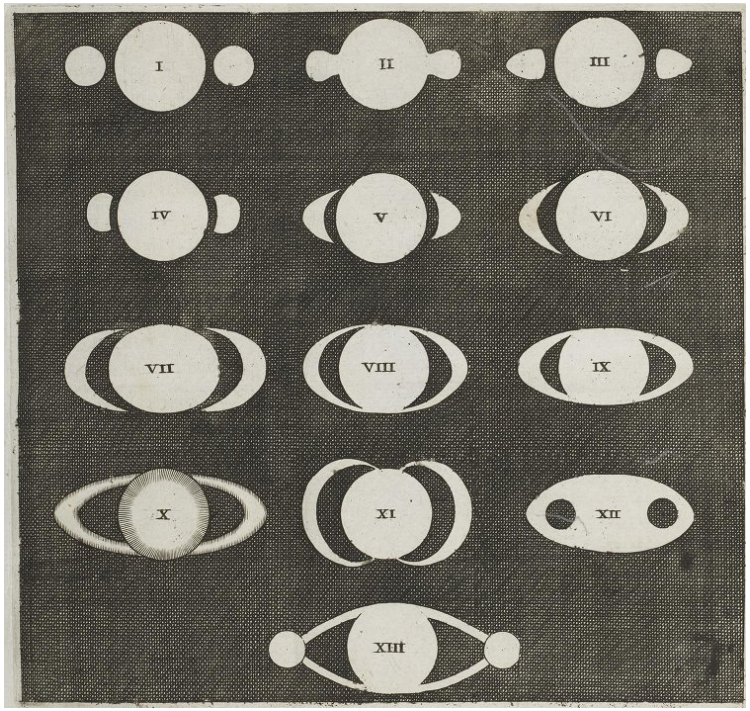
PLANETA NOVO

Systema Saturnium, 1659

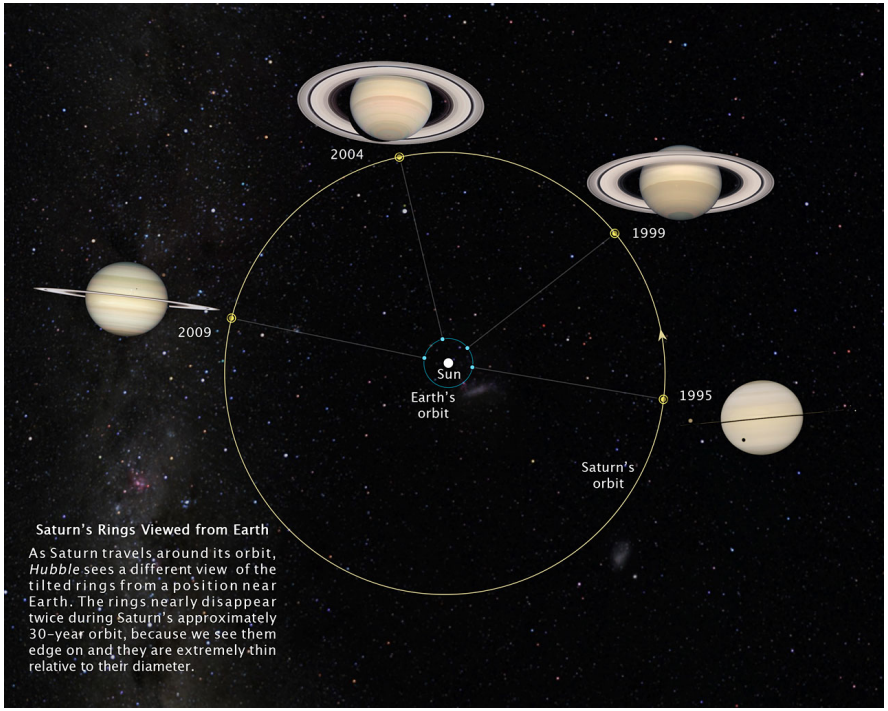
"Annulo cingitur tenui plano nusquam cohaerente ad eclipticam inclinato"

"Saturn is surrounded by a tiny and flat ring, not touching the planet ad inclined with respect to the ecliptic"

A centuries-long history: Huygens and Hubble



Christian Huygens representation of Saturn's rings between 1610 and 1659.



Saturn's rings viewed from Earth.

From Huygens to Maxwell

Huygens (1655): **first formulation** of (solid) rings around Saturn.

Maxwell (1859): ring composed of several **independent particles**, using dispersion relations and Fourier analysis (first example of a **collective process** emerging from the interaction of single particles).

Methods for studying rings:

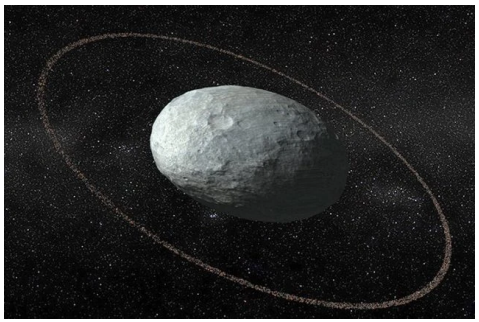
▷ **simulation** of N ring's particles, difficult since $N \sim 10^{15}$;

▷ **statistical approach** based on kinetic theory, though more difficult than gas theory, since particles' collisions are inelastic and distribution velocity is anisotropic;

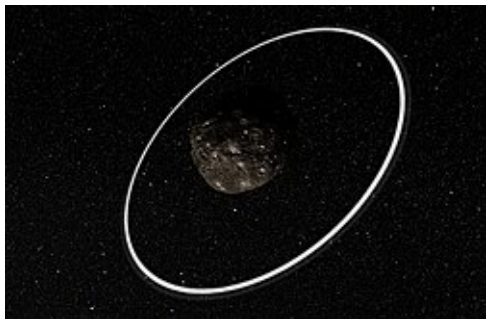
▷ **dynamical system approach** through perturbation and bifurcation theories.

Rings around small bodies

Occultation of stars revealed that some minor bodies have rings!



136108 Haumea, dwarf planet beyond Neptune's orbit.



10199 Chariklo, Centaur asteroid in the outer Solar System.



50000 Quaoar, dwarf planet in the Kuiper belt.

They are **small bodies** with an **irregular shape**.

Rings around small bodies

Rings of these minor bodies are usually located close to the **1:3 Lindblad resonance** $\rightarrow$ a ring particle completes **one orbit** every **three full rotations** of the central body about its spin axis.

Questions:

(Q1) Why the 1:3 resonance and not other resonances (1:1, 1:2,..)?

(Q2) Can we implement a (suitable) KAM theorem to detect invariant tori?

(Q3) Which are the mathematical differences between the analysis of (almost round) planets and (irregular) minor bodies?

(Q4) Are there bifurcations around resonances?

Outline

1 Introduction

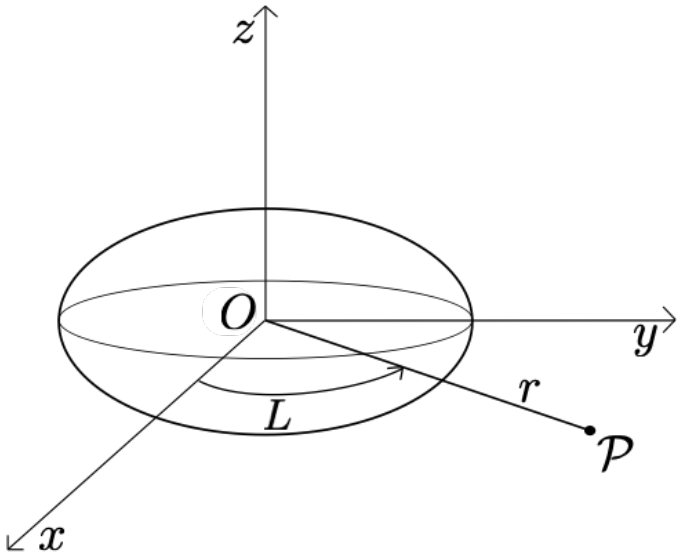
2 The Hamiltonian model

3 Multi - scale KAM theorem

4 Analysis of the resonant regime

The model

A particle $\mathcal{P}$ moves in the **equatorial plane** of an **ellipsoid** with semi–axes $a > b > c > 0$. The ellipsoid **rotates** around the shortest physical axis with constant angular velocity Ω_P.

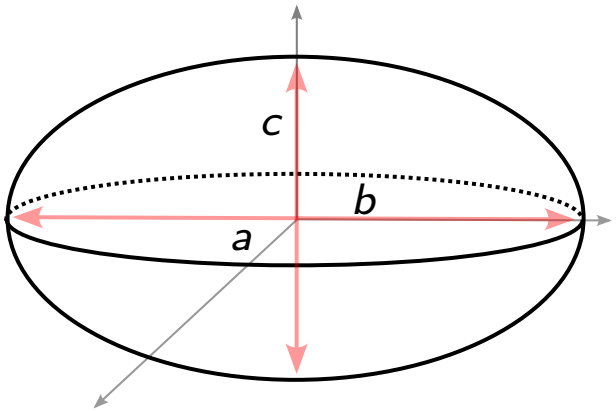


The **coordinates** of $\mathcal{P}$ are denoted as (r, L). In a synodic reference frame rotating with angular velocity Ω_P, we introduce the angle θ defined as

$$\theta = L - \Omega_P t \ .$$

The model: the ellipsoid's shape

The ellipsoid can be described via two shape parameters:



- **Elongation ($\mathcal{E}_\ell$):** measures how much the equatorial cross-section is stretched in one horizontal direction;
- **Oblateness ($\mathcal{O}_b$):** measures how much the ellipsoid is flattened along the shortest axis c, relative to the overall size.

$$\mathcal{E}_\ell = \frac{a^2 - b^2}{2R^2} \qquad \mathcal{O}_b = \frac{a^2 + b^2 - 2c^2}{4R^2}$$

where $R = \sqrt{3}(a^{-2} + b^{-2} + c^{-2})^{-1/2}$ = reference radius.

The model: gravitational potential

Potential generated by the ellipsoid at a distance r from its barycenter on a particle in the equatorial plane at angular distance θ in the synodic frame rotating with frequency Ω_P:

$$U(r,\theta) = -\frac{\mathcal{G}M_P}{r} \sum_{p=0}^{\infty} \cos(2p\theta) \sum_{\ell=p}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell,2p} P_{2\ell,2p}(0)$$

where $\mathcal{G}$ is the gravitational constant, M_P is the mass of the central body, $P_{2\ell,2p}$ are the associated Legendre polynomial, $C_{2\ell,2p}$ are real coefficients (depending on a, b, c).

$$U(r,\theta) = U_s(r) + U_{ns}(r,\theta)$$

$U_s =$ **axisymmetric** term, $U_{ns} =$ **non-axisymmetric** term.

The model: frequencies

Hamiltonian: $\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \Omega_P\, p_\theta + U(r, \theta)$

Synodic frequency: $\boxed{n - \Omega_P,\ \ n^2(r) = \frac{1}{r}\ \frac{dU_s(r)}{dr}}$

angular frequency of the particle as seen in the frame rotating with the body.

Epicyclc frequency: $\boxed{\kappa = n - \dot{\omega},\ \ \kappa^2(r) = \frac{3}{r}\frac{dU_s(r)}{dr} + \frac{d^2U_s(r)}{dr^2}}$

(ω = argument of periastrum).

frequency of **small radial oscillations** around the reference circular orbit.

Model: epicyclic coordinates

Starting with the Hamiltonian

$$\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \Omega_P\, p_\theta + U(r, \theta)$$

fix a distance r_* : we can express the Hamiltonian $\mathcal{H}$ as a **Taylor expansion around r_***. More precisely, given the quantities

$$\rho = r - r_* \ , \qquad I = p_\theta - p_*$$

$$p_\rho = p_r \ , \qquad p_* = n_*\ r_*^2$$

we introduce the **epicyclic action-angle variables** (J, φ) related to (ρ, p_ρ) by the (Poincaré-like) canonical transformations:

$$\boxed{\rho = \sqrt{2J/|\kappa_*|}\ , \qquad p_\rho = \sqrt{2|\kappa_*|J}\ \cos\varphi\ .}$$

The model: the Hamiltonian in epicyclic coordinates

Proposition.

Given $\kappa_* = \kappa(r_*)$ and $n_* = n(r_*)$ for a given radius r_*, there exists a set of **canonical action–angle coordinates** (J, I, φ, θ), such that the particle's Hamiltonian, **locally** around r_*, can be expressed as

$$\mathcal{H}(J, I, \varphi, \theta) = |\kappa_*| J + \big(n_* - \Omega_P\big) I + \widetilde{F}_s(J, I, \varphi) + \widetilde{F}_{ns}(J, \varphi, \theta) \ ,$$

$\widetilde{F}_s$, $\widetilde{F}_{ns}$ = **axisymmetric** and **non–axisymmetric** parts of the potential in the new variables.

$\Rightarrow$ the synodic and epicyclic frequencies are now the **first-order frequencies** of a set of action-angle coordinates.

The model: the Hamiltonian in epicyclic variables

$$\mathcal{H}(J, I, \phi, \theta) = |\kappa_*| J + (n_* - \Omega_P) I + \widetilde{F}_s(J, I, \phi) + \widetilde{F}_{ns}(J, \phi, \theta)$$

- the non-axisymmetric part is linear in the parameter

$$\lambda = \mathcal{O}_b \left(R/r_*\right)^2$$

$\Rightarrow$ if $\mathcal{O}_b$ is small or r_* large, $\widetilde{F}_{ns}$ is a small perturbation of the axisymmetric part

- packing together the angles, the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}(J, I, \varphi, \theta) =& |\kappa_*| J + (n_* - \Omega_P) I \\ &+ \sum_{|i|=0}^{\infty} \sum_{j=0}^{\infty} \Big(P_{2i,j} \cos(2i\theta + j\varphi) + Q_{2i,j} \sin(2i\theta + j\varphi) \Big)\end{aligned}$$

with P, Q polynomials in $I, \sqrt{J}$.

The model: resonances

Fixed $j, m \in \mathbb{Z}\backslash\{0\}$, we can define:

Corotation resonance

$$\boxed{n_* = \Omega_P}$$

Lindblad resonance of order $m : (m - j)$

$$\boxed{j\kappa_* = m\big(n_* - \Omega_P\big)}$$

Note that for a Linblad resonance of order $m : (m - j)$ one has

$$\boxed{\frac{n - \dot{\omega}}{\Omega_P - \dot{\omega}} = \frac{m}{m - j}}$$

$\rightsquigarrow$ in a frame rotating at the particle precession rate $\dot{\omega}$, the particle completes ***m* revolutions**, while the central body completes ***m*–*j* rotations**.

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A nearly - integrable Hamiltonian

Problem: does the dynamics admit *invariant tori*?

Step I: find a **nearly-integrable** regime. Small parameters:

- **shape** parameters $\mathcal{O}_b$ and $\mathcal{E}_\ell$;
- **distance** parameters $\rho = r - r_*$ and $1/r_*$.

Normalization

After a normalization procedure based on Lie series transformations, the Hamiltonian in epicyclic variables can be expressed as a **near - integrable system** in the form

$$\mathcal{H}(J, I, \varphi, \theta) = h(J, I) + f(J, I, \varphi, \theta),$$

where $f(J, I, \varphi, \theta)$ is a **small perturbation** of $h(J, I)$.

A multi - scale integrable case

Different scales in the integrable Hamiltonian

The angle-independent part of $\mathcal{H}$ can be written as

$$h(J,I) = \underbrace{\eta\ k_1}_{\text{synodic freq.}}\ I + \underbrace{\frac{\eta^2}{r_*^{3/2}}\ k_2}_{\text{epicyclic freq.}}\ J + \frac{\eta^2}{r_*^4}\ k_3\ I^2 + \text{h.o.t.},$$

where $\eta \sim \rho$ is a small parameter.

$\Rightarrow$ the two frequencies have a **different scale**!

N.B.: multi-scale Hamiltonians are very frequent in Celestial Mechanics, where different dynamical processes naturally occur on **different time scales** (e.g. mean motion vs. secular precession, or fast orbital motion vs. slow resonant dynamics)

Multi - scale KAM theorem

Can we ensure the **persistence of KAM tori** for small values of η?

Multi–scale KAM theorem (Han-Li-Yi, 2010)

Consider the Hamiltonian with $(y, x) \in \mathbb{R}^N \times \mathbb{T}^N$, $\varepsilon > 0$

$$\mathcal{H}(y, x; \varepsilon) = h_0(y^{n_0}) + \varepsilon^{m_1} h_1(y^{n_1}) + \cdots + \varepsilon^{m_a} h_a(y^{n_a}) + \varepsilon^{m_a+1} p(y, x; \varepsilon)$$

with the different integrable pieces depending on progressively more action variables. Define the **multi-scale frequency map**

$$\Omega(y) = (\nabla_{y^{n_0}} h_0, \ldots, \nabla_{y^{n_a}} h_a) .$$

If, in an open set $G \subset \mathbb{R}^N$, there exists $M \in \mathbb{N}$ such that

$$\mathbf{rank} \left\{ \partial_y^\alpha \Omega(y) : 0 \leq |\alpha| \leq M \right\} = N, \qquad \forall y \in G$$

then, for sufficiently small ε, the Hamiltonian admits a **Cantor set of invariant KAM tori**.

Multi - scale non - degeneracy condition

Take again our nearly - integrable Hamiltonian

$$\mathcal{H}(J, I, \varphi, \theta) = h(J, I) + f(J, I, \varphi, \theta),$$

$$h(J, I) = \underbrace{\eta\ k_0 I}_{h_0(I)} + \frac{\eta^2}{r_*^{3/2}} \underbrace{\left(k_{11} J + \frac{k_{12}}{r_*^{1/2}} I^2 + \frac{\eta k_{13}}{r_*^{1/2}} IJ + \frac{\eta k_{14}}{r_*} I^3 \right)}_{h_1(I,J)}$$

Proposition

If $\boldsymbol{\Omega_P - n_* \neq 0}$ and $\boldsymbol{|\kappa_*|^2 \neq 2\mathcal{G}M_P r_*^{-2}}$, the function $h(J, I)$ is **non–degenerate** in the sense of multi–scale KAM theorem.

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Resonant Hamiltonian

Fix j, k and let r_* be the **resonant radius**, which solves either

$$j\kappa(r) = m\big(n(r) - \Omega_P\big) \quad \text{or} \quad n(r) = \Omega_P \ .$$

The Hamiltonian can be written as:

$$\begin{aligned}\mathcal{H}(J, I, \varphi, \theta) =& |\kappa_*| J + (n_* - \Omega_P) I \\ &+ \sum_{|i|=0}^{\infty} \sum_{j=0}^{\infty} \Big(P_{2i,j} \cos(2i\theta + j\varphi) + Q_{2i,j} \sin(2i\theta + j\varphi) \Big)\end{aligned}$$

where $P_{2i,j}, Q_{2i,j}$ are polynomials in I and J.

- **Resonant angle** of order $m : (m - j)$: $\psi = -m\theta + j\varphi$.
- **Resonant Hamiltonian**: terms of $\mathcal{H}$ depending only on the actions and $\psi \rightsquigarrow \mathcal{H}_{res}(J, I, \psi)$. Close to the resonance, it is an approximation of the full motion.

The resonant Hamiltonian

Given the resonant Hamiltonian, we can introduce a **linear change of coordinates** $(J, I, \varphi, \theta) \mapsto (G, L, \psi, \mu)$:

$$\mathcal{H}_{res}(J, I, \varphi, \theta) \mapsto \mathcal{H}_{res}\Big(G, L, \psi, (\mu)\Big)$$

The angle μ is cyclic $\implies$ $L = L_0$ is a constant defined by the initial conditions (and in particular the particle's eccentricity)

$\implies$ $\mathcal{H}(G, L, \psi, \mu) = \mathcal{H}_{res}(G, \psi; L_0)$ is a 1-DOF Hamiltonian:

$$\begin{aligned}\mathcal{H}_{res}(G, \psi; L_0) =& \alpha_1(L_0)G + \alpha_2(L_0)G^2 \\ &+ \beta_1(G; L_0)cs(\psi) + \beta_2(G; L_0)\ cs(2\psi)\ + \dots\end{aligned}$$

with α_i, β_i algebraic functions in the action and L_0, cs either $\cos$ or $\sin$.

Equations of motion

$$\mathcal{H}_{res}(G,\psi;L_0) = \alpha_1(L_0)G + \alpha_2(L_0)G^2 + \beta_1(G;L_0)cs(\psi) + \beta_2(G;L_0)\ cs(2\psi)\ + \dots$$

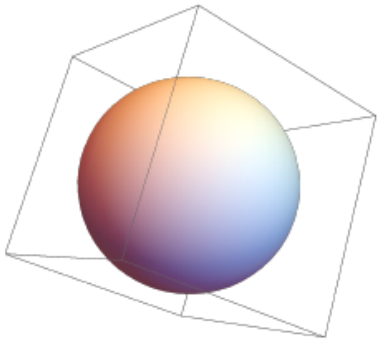
Equations of motion: $$\dot{\psi} = \frac{\partial \mathcal{H}_{res}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}_{res}}{\partial \psi}$$

We consider the **three resonances** corotation (1:1), 1:2, 1:3 and we analyse the associated 1 d.o.f. resonant dynamics in terms of :

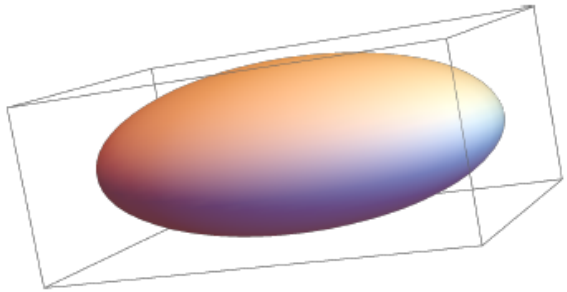
- existence and **stability** of the equilibria;
- amplitudes of **libration islands** associated to stable equilibria;
- occurrence of **bifurcations** as the initial particle's eccentricity changes.

Two test cases: almost spherical and highly aspherical

(AS)



(HA)



	AS	HA
Rotation period T_{rot} (h)	8	8
Mass M_P (kg)	10^{21}	10^{21}
Semi–axes $a \times b \times c$ (km)	$1000 \times 980 \times 960$	$1000 \times 850 \times 750$
Reference radius R (km)	979	849
Elongation parameter	0.02	0.19
Oblateness parameter	0.03	0.21

Equilibria and bifurcations

We consider three resonances: **corotation (1:1), 1:2 and 1:3**, which represent, respectively, resonances at **close**, **intermediate** and **far** distances from the primary body.

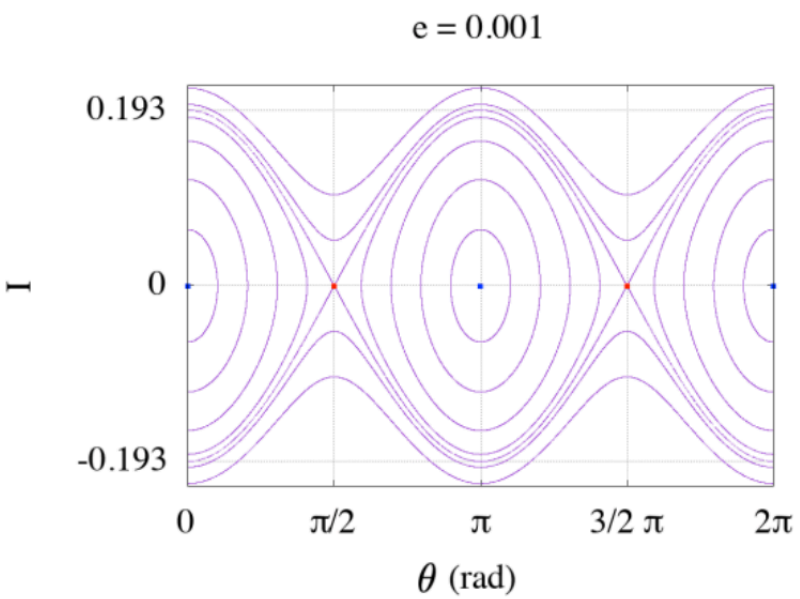
As an indication of the **strength** of the resonances, using perturbation and bifurcation theories, we compute:

- the **amplitude of libration**, as the distance between the separatrices around elliptic equilibria: **smaller amplitudes** are associated to **more stable regimes**, as resonance overlapping is more difficult;
- **bifurcation diagrams** at the equilibria: here, the **bifurcation parameter** is the initial **eccentricity** of the particle.

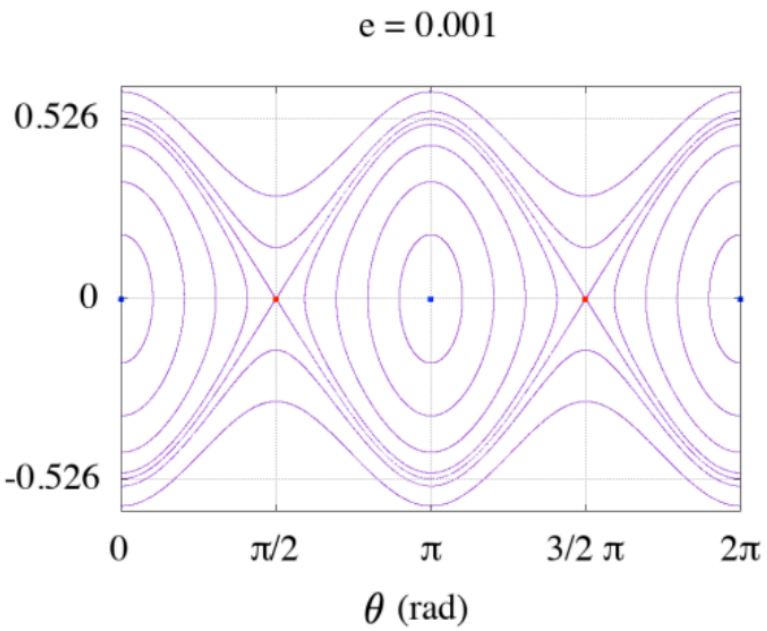
Corotation or 1:1 resonance: large amplitude

Corotation $\rightsquigarrow$ $r_* = 1124.6$ *km* in (AS), $r_* = 1148.08$ *km* in (HA)

Almost spherical



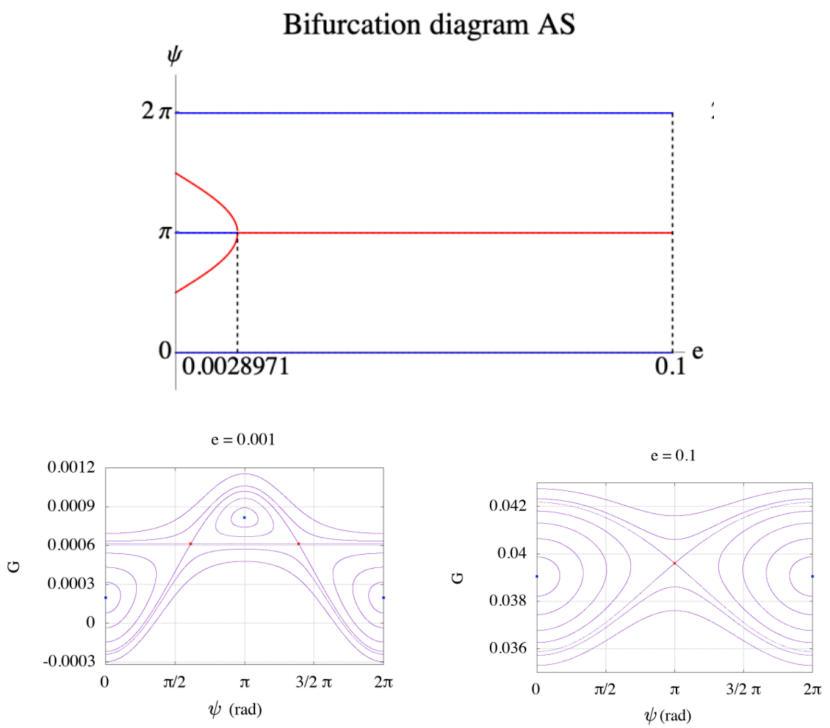
Highly aspherical



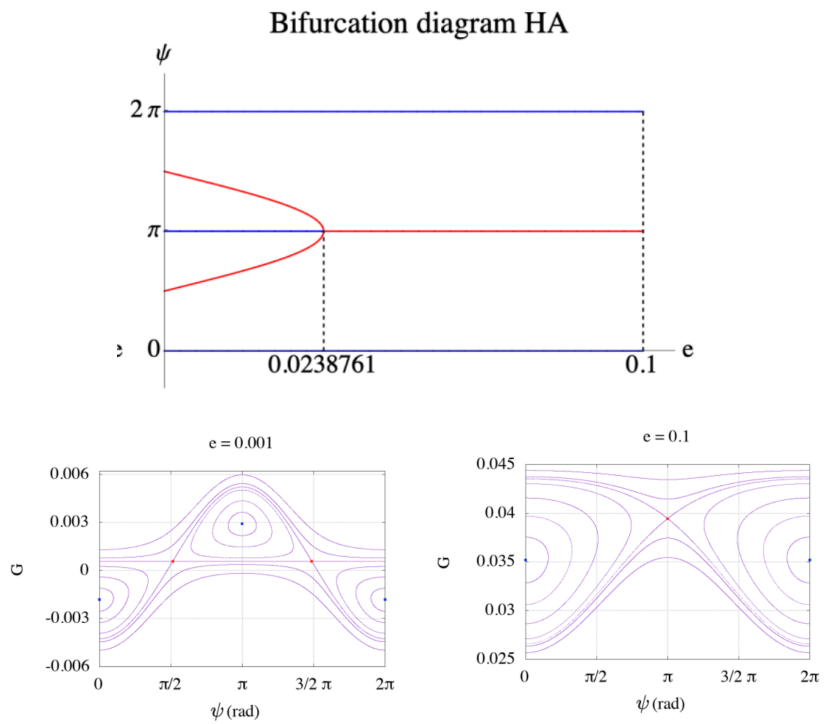
1:2 resonance: pitchfork bifurcations

1:2 resonance $\rightsquigarrow$ $r_* = 1776.73$ *km* in (AS), $r_* = 1775.51$ *km* in (HA).

Almost spherical



Highly aspherical

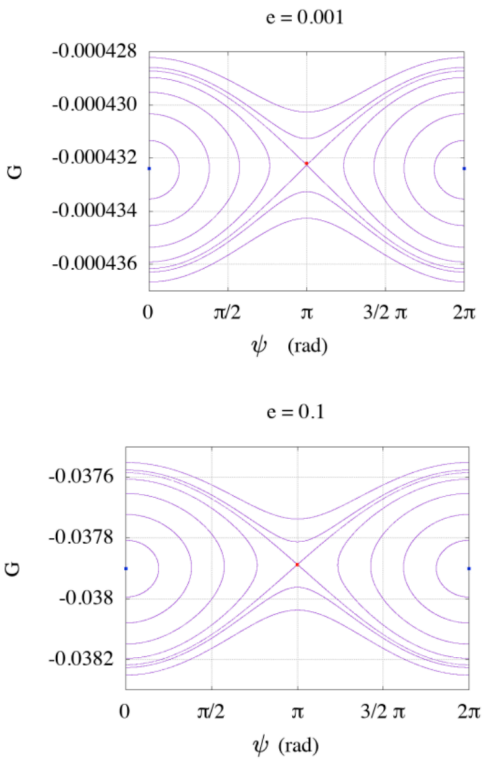


Pitchfork bifurcation in both cases!

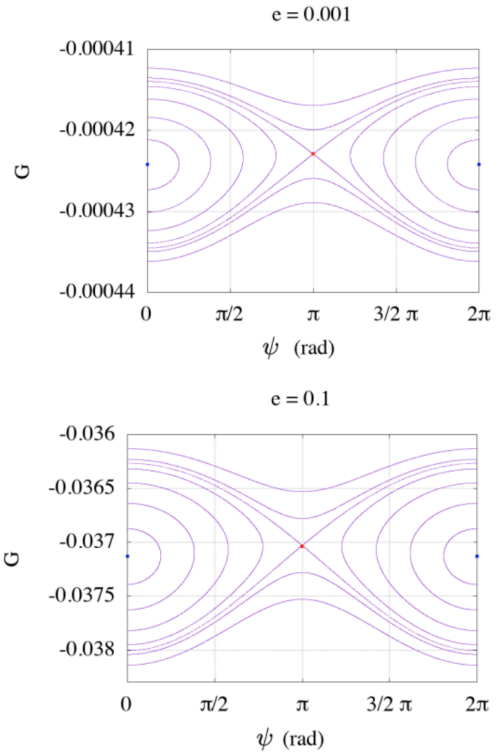
1:3 resonance: no bifurcations

1:3 resonance $\rightsquigarrow$ $r_* = 2327.36$ *km* in (AS), $r_* = 2323.11$ *km* in (HA)

Almost spherical



Highly aspherical



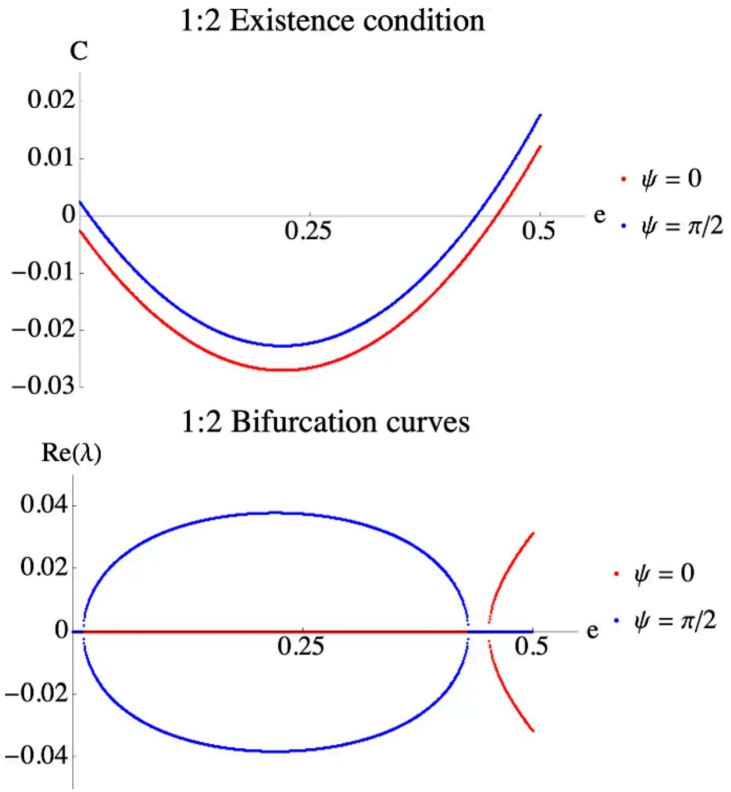
A bifurcation criterion

Proposition

- Take a resonance (n_1, n_2), with n_1, n_2 either both even or n_1 odd
- Let $G = aJ + bI$, $L = cJ + dI$ action coordinates in the resonant Hamiltonian
- Call $G_{eq}(e)$ the resonant action value at equilibrium for a given eccentricity; $L_0(e)$ the corresponding value of L

$\Longrightarrow$ The values of e at which the equilibrium bifurcates is a solution of

$$\boxed{dG_{eq}(e) - bL_0(e) = 0}$$



Amplitudes of the libration islands

	Almost spherical	**Highly Aspherical**
1:1	0.193	0.526
1:2	8.31687×10^{-4}	4.06825×10^{-4}
1:3	7.437×10^{-6}	2.058×10^{-5}

Amplitudes of the (smallest) resonant island for the three resonance in the same units of measure, taking an initial eccentricity $e = 10^{-3}$.

$\Longrightarrow$ resonance $1:3$ has always the smallest amplitude!

A zoo of resonance

m \ j	1	2	3	4	5	6	7
-1	Pitchfork	No bif.	No bif.	No bif.	No bif.	No bif.	Transcr.
-2	S-N		No eq.		No eq.		
-3	Transcr.	No bif.		No bif.	No bif.		
-4	S-N		$0 \to 3$ eq.				
-5	Transcr.	No bif.	Transcr.				
-6	S-N						
-7	Transcr.						

Bifurcations at low-order resonances for a test case with dimensions $a = 1000$ km, $b = 900$ km, $c = 800$ km, $M_P = 10^{21}$ kg, $\Omega_P = 2\pi/10$ h

Answers to our initial questions

(Q1) Why the 1:3 resonance?

1:3 resonance is associated to **smaller amplitude** of libration and **absence of bifurcations**.

(Q2) KAM theorem?

In principle, the **nondegeneracy** of **multi-scale KAM** theorem is satisfied almost everywhere.

(Q3) Difference between planets and minor bodies?

Shape factors impact the amplitude of libration and bifurcations.

(Q4) Bifurcations?

Close to resonances, many **different bifurcation** phenomena may happen.

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