Control of Keplerian collision orbits

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1. Motivations: the controlled Kepler problem
2. Singularities
3. Uniform bounds: existence of optimal solutions
4. Extremal analysis

Motivations: the controlled Kepler problem

$$\ddot{q} + \frac{q}{|q|^3} = u(t), \quad q \in \mathbb{R}^2, \ \sqrt{u_1^2 + u_2^2} \leq 1$$

describes a simplified model of a spacecraft orbiting a celestial body

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Goal: $t_f \to \min$ given two endpoints.

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> PROPOSITION 4.7. *Consider the 2D-orbital transfer problem. Then, for each pair of points x_0, x_1, in the elliptic domain, there exists a trajectory transferring x_0 to x_1. If r_0 is the distance to collision of this trajectory, there exists a time minimal trajectory such that $r \geq r_0$. Each optimal arc not meeting the boundary $r = r_0$ is bang-bang with maximal thrust and is a concatenation of order zero arcs, the switchings being Π-singularities.*

Figure: From *Geometric optimal control of elliptic Keplerian orbits*, B. Bonnard, J.-B. Caillau, E. Trélat

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We will allow solutions with collisions to improve the Proposition.

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Singularities of the perturbed Kepler problem

Proposition (Sperling, 1969): Let $q :]t_0, t_1[\to \mathbb{R}^2$ be a maximal solution of $\ddot{q} + \frac{q}{|q|^3} = u(t)$ with $t_1 < +\infty$, then, as $t \to t_1^-$

- $\lim q(t) = 0$, *there is a collision* at t_1
- The mechanical energy $h(t) = \frac{1}{2}\dot{q}^2 - \frac{1}{q}$ has finite limit
- $q(t) \sim \alpha(t_1 - t)^{2/3}, \quad \dot{q}(t) \sim \beta(t_1 - t)^{-1/3}$

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We look for optimal solutions in the sense of this definition.

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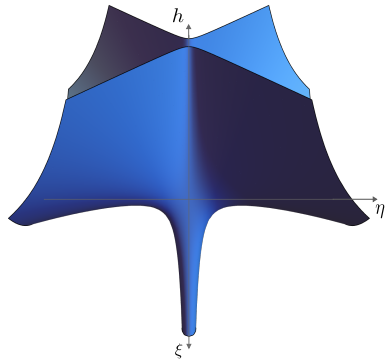
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After the regularization $q = \xi^2$, $dt = |q|d\tau$, the dynamics is a flow on the manifold $\longrightarrow$

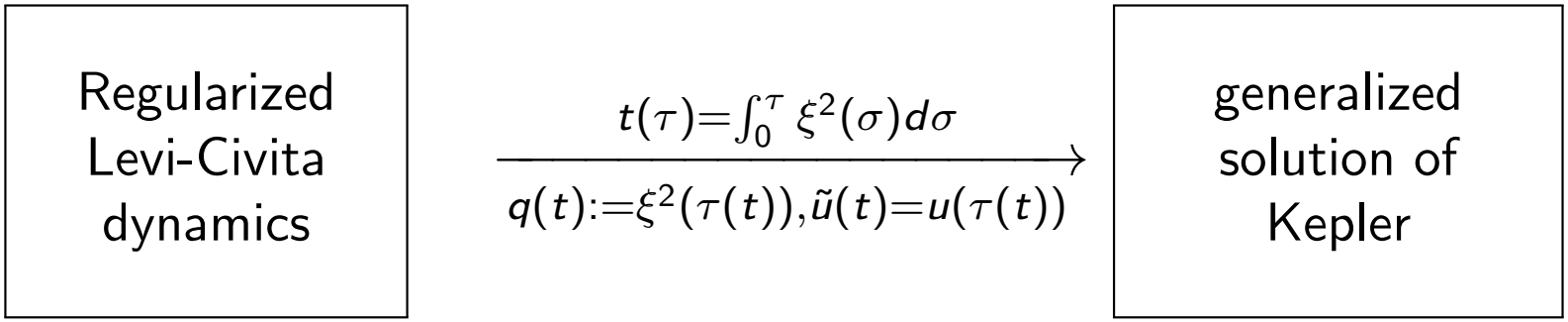


• Sperling, H.J. The collision singularity in a perturbed two-body problem. In: Celestial mechanics 1(2), 1969 • Ortega, R. Linear motions in a periodically forced Kepler problem, In: Port. Math. 68, 2011.

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Regularized Levi-Civita dynamics	$\xrightarrow[q(t):=\xi^2(\tau(t)),\tilde{u}(t)=u(\tau(t))]{t(\tau)=\int_0^\tau \xi^2(\sigma)d\sigma}$	generalized solution of Kepler

It is necessary to show that the dynamics only allows a uniformly bounded amount of collisions within a compact interval $[0, t_f]$.

This bound is closely linked to the bound on the energy, and to the existence of an optimal min-time trajectory in the generalized sense by the *Filippov's theorem*.

We need uniform bounds on ξ, ξ', h, τ.

Uniform bounds: number of collisions

Let $C = \{t_0, t_1, \dots t_{N-1}, t_N\}$, $0 = t_0 < \cdots < t_N = t_f$ be the collision times set. We partition the interval $[0, t_f]$:

$$I_k := (t_k, t_{k+1}), \quad Q_k := \max_{t \in I_k} q(t), \quad k = 0, \dots, N-1$$

$$\Lambda := \{0 \leq i \leq N-1 : \ Q_i \geq 1\}, \quad \Gamma := \{0 \leq j \leq N-1 : \ Q_j < 1\}$$

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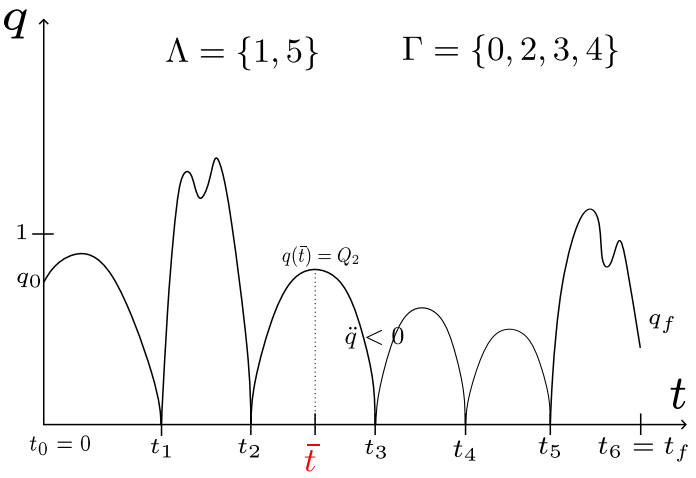
Lemma

$$i) \ |\Lambda| \le \lfloor \alpha \, t_f \rfloor$$

$$ii) \ |\Gamma| \le \lfloor (1 + \beta t_f)^4 \rfloor$$

The value 1 in $Q_j \lessgtr 1$ is the renormalization of

$$\left(\frac{\text{grav. parameter}}{\text{maximum size of the control}} \right)^{1/2}$$



$$|h(t) - h(t_i)| \le \int_{t_0}^{t} |\dot{q}| dt$$

$$|h(t) - h(t_i)| \le 2Q_i \text{ in } \Gamma$$

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- As a direct consequence of the number of collisions $h \in [-H, H]$, H uniform constant.

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- As a direct consequence of the number of collisions $h \in [-H, H]$, H uniform constant.
- Also the Sundman times τ_f are uniformly bounded, indeed

$$\tfrac{1}{2}\ddot{I} = \frac{1}{q} + 2h + q\,u, \quad I := q^2$$

(*generalized Lagrange-Jacobi equation*).
We integrate

$$\tau_f = \int_0^{t_f} \frac{1}{q} = \frac{\dot{I}(t_f) - \dot{I}(0)}{2} - 2\int_0^{t_f} h(t)dt - \int_0^{t_f} q \cdot u$$

all the quantities on the right are uniform.

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Pontryagin Maximum Principle on the line

After the change of time, we face with a Lagrange problem: $t_f = \int_0^{\tau_f} \xi^2(\tau) d\tau \to \min$. $x = (\xi, h)$, $p = (p_\xi, p_h)$. The pseudo-Hamiltonian $H = \langle p, F(x,u) \rangle + p_0 \xi^2$ is maximized for

$$u = sgn(p_h \xi) = \pm 1, \quad \textit{bang-bang control, no singular arcs}$$

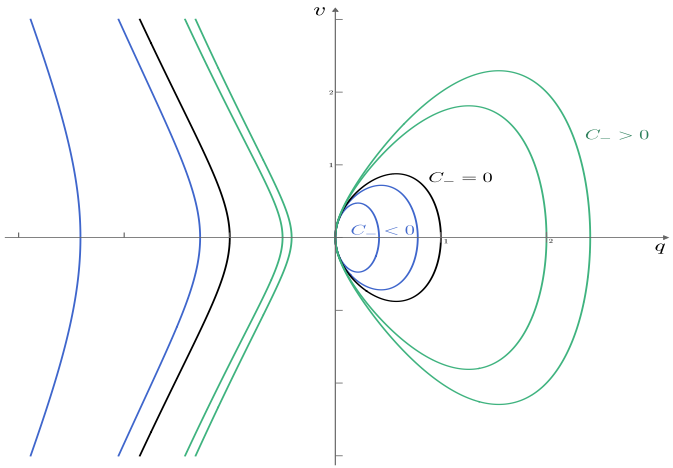
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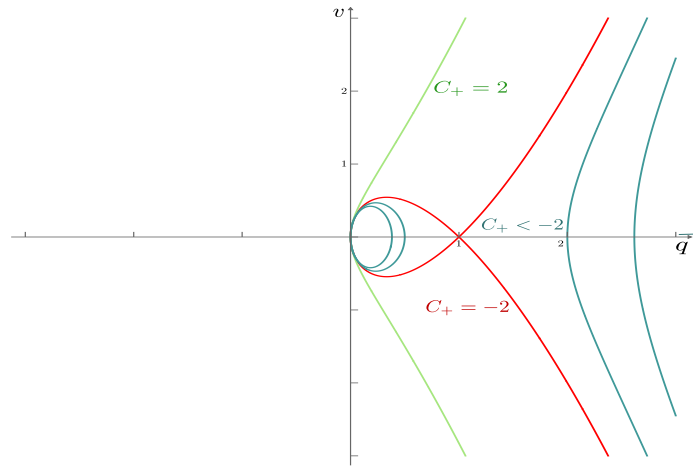
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The extremal curves are integrable thanks to the integrals of motion $C_\pm = \frac{1}{2}\dot{q}^2 - \frac{1}{q} \mp q$ and they can be expressed in terms of elliptic functions.

To explicitly find optimal curves given boundary conditions, we only need to appropriately join pieces of the extremal trajectories.



(a) $u = -1$



(b) $u = +1$

Some references

- Sperling, H.J. The collision singularity in a perturbed two-body problem, 1969
- Bonnard, B., Caillau, J.-B., and Trélat, E. Geometric optimal control of elliptic Keplerian orbits, 2005.
- Ortega, R. Linear motions in a periodically forced Kepler problem, 2011.
- Boscaggin, A., Ortega, R., Zhao, L., Periodic solutions and regularization of a Kepler problem with time-dependent perturbation, 2018.

Thank you for the attention!