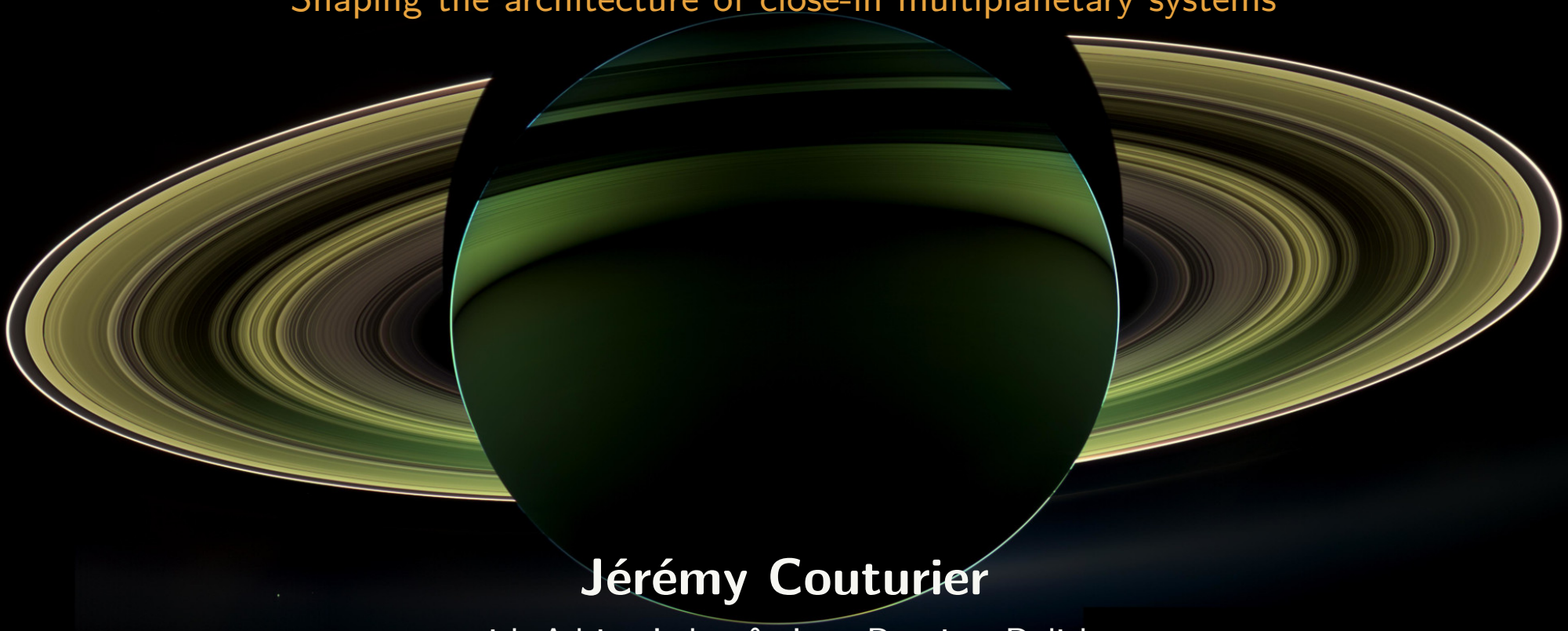
Dynamics of 3-planet zeroth-order resonances with tides

Shaping the architecture of close-in multiplanetary systems



Jérémy Couturier

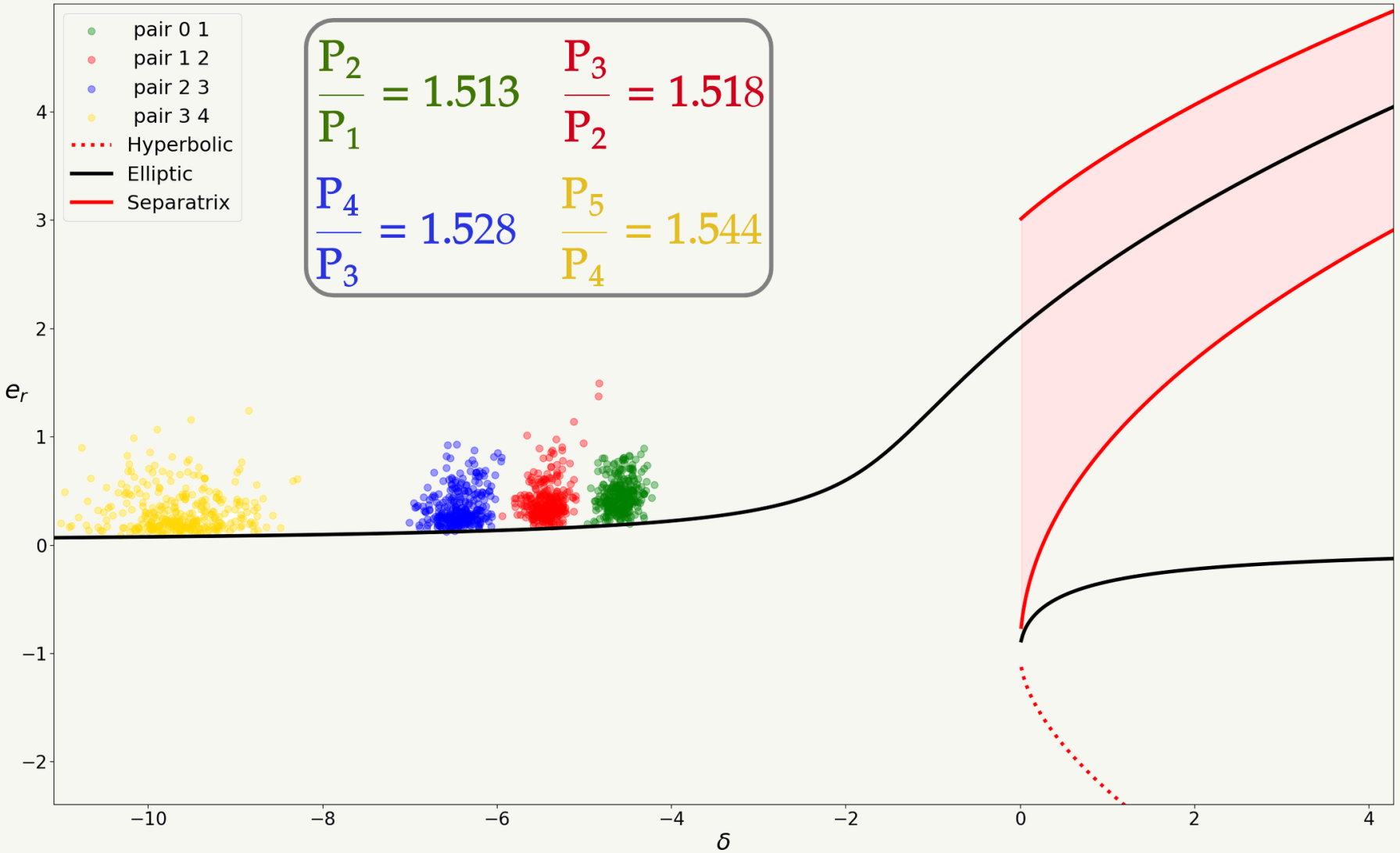
with Adrien Leleu & Jean-Baptiste Delisle

Observatoire de Genève, Université de Genève · LTE, Observatoire de Paris

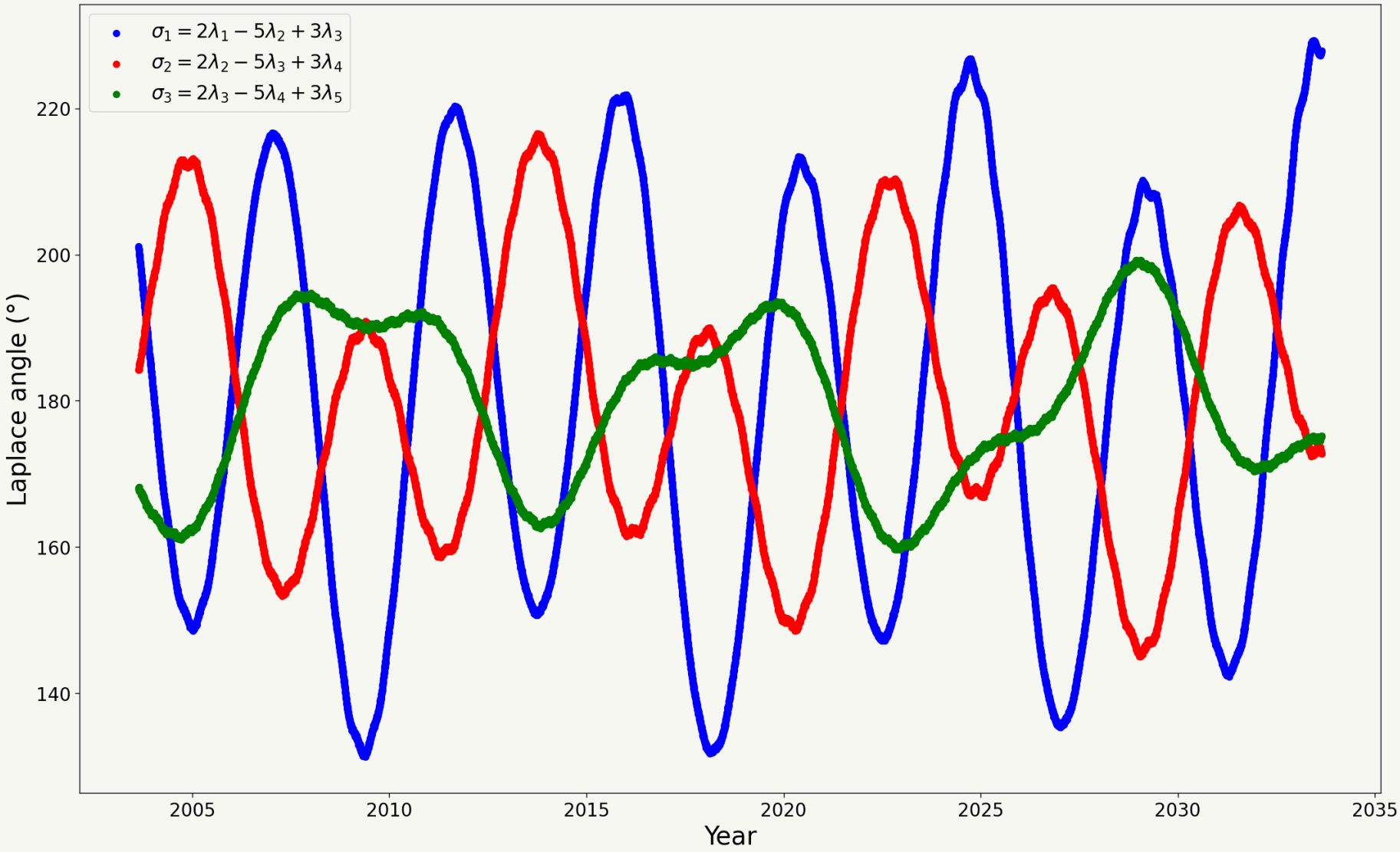
September 14th, 2026

No 2-planet first-order resonances (2P1R) in K2-138

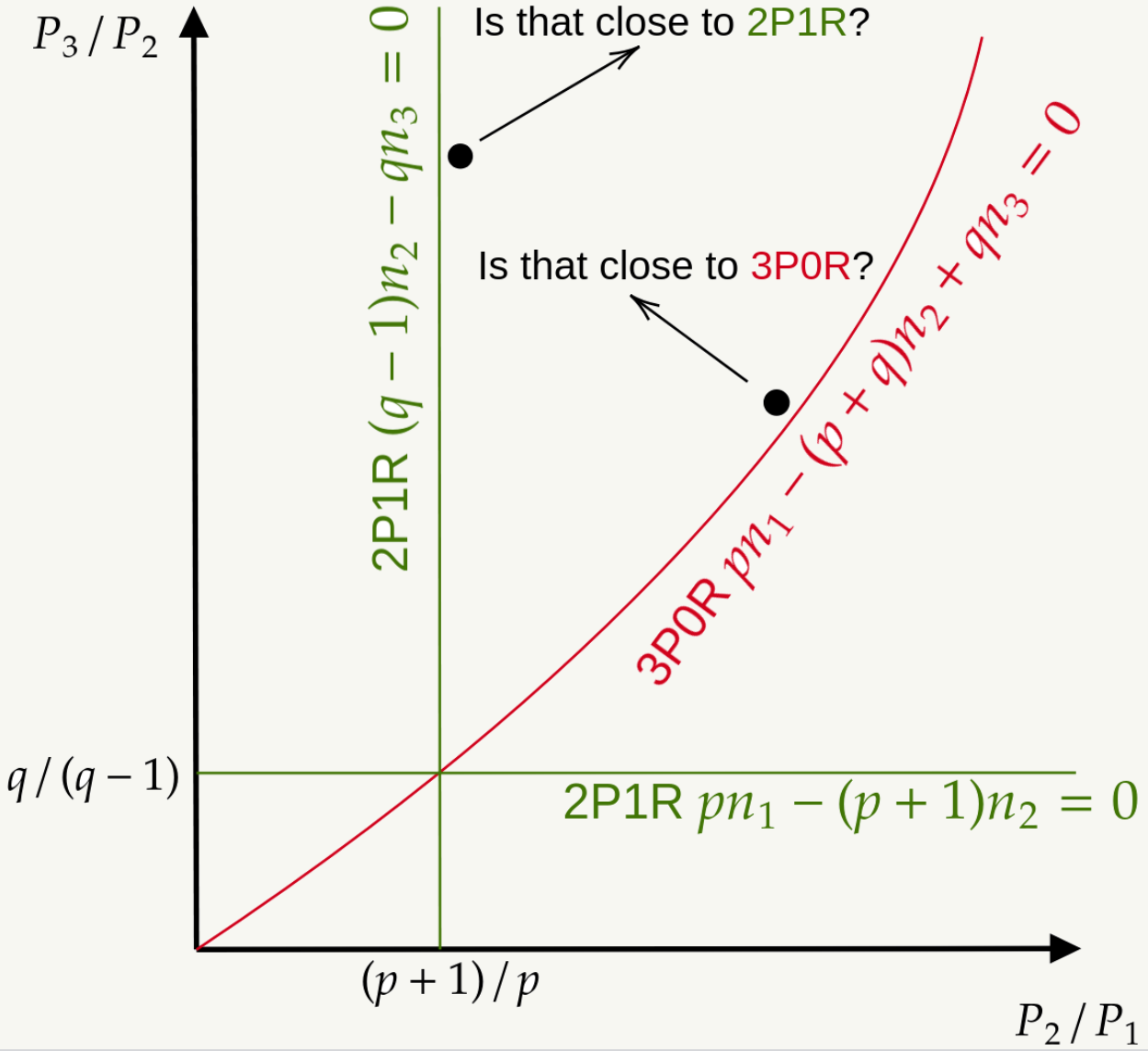
(Figure made using posterior analyses from Leleu et al. 2026a)



3-planet zeroth-order resonances (3P0R) in K2-138



We need a 3P0R model, similar to the SFM for 2P1R



Averaging over fast angles

$$\boxed{\mathcal{H} = \mathcal{H}_K + \varepsilon \mathcal{H}_P}$$

- $\mathcal{H}_K$: Star–planet interactions
- $\varepsilon\mathcal{H}_P$: Planet–planet interactions of the form $\cos\left(k_1\lambda_1 + k_2\lambda_2\right)$

Far from any 2-planet resonances, $\cos\left(k_1\lambda_1 + k_2\lambda_2\right)$ is averaged out when $(k_1, k_2) \neq (0, 0) \rightarrow$ Lie serie expansion.

$$\boxed{\mathcal{H} = \mathcal{H}_K + \varepsilon \mathcal{H}_{\text{sec}} + \varepsilon^2 \mathcal{H}_2 + \mathcal{O}(\varepsilon^3)}$$

- $\varepsilon^2\mathcal{H}_2$: Planet–planet–planet interactions of the form $\cos\left(k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3\right)$

Hamiltonian at second order in mass

$$\varepsilon^2 \mathcal{H}_2 = \frac{1}{2} \{\chi, \mathcal{H}_P + \mathcal{H}_{\text{sec}}\}, \qquad \chi_{\mathbf{k}} = \frac{-\iota h_{\mathbf{k}}}{k_1 n_1 + k_2 n_2}$$

The Hamiltonian is truncated to degree zero in eccentricity

$$\varepsilon^2 \mathcal{H}_2 = \frac{m_1 m_2}{m_0^2} n_3 \Lambda_3 \sum_{p,q} \Xi_{p,q} \cos\left(p\lambda_1 - (p+q)\lambda_2 + q\lambda_3\right)$$

A 1-dof pendulum model of 3P0R was proposed by Quillen (2011) and Petit et al. (2020). However, the coefficients $\Xi_{p,q}$ are analytically hard to obtain and they neglected terms in the perturbing functions as well as Poisson brackets.

Exact coefficients at second order in mass

- For any 3-planet angles
- At any degree in eccentricity/inclination
- All planet pair (1,2), (1,3) and (2,3) are considered
- Indirect terms in $\mathcal{H}_P$ are considered
- Algebraic manipulator `celeries` in python (Delisle 2017), resembling Trip (Laskar, Gastineau)

> I implemented a module `PerHam3pla` in celeries for the exact computation of coefficients at second order in mass

- `Xi_k=h3.PerHam3pla(degree=n).angle((k1,k2,k3))` returns the coefficient in front of $e^{\imath(k_1\lambda_1+k_2\lambda_2+k_3\lambda_3)}$ in the Hamiltonian at degree n in eccentricity and inclination.
- An argument `ang2pla` allows some 2-planet angles not to be averaged upon

Reduction to 1 degree of freedom

- The secular part is eliminated by expanding in the vicinity of the Keplerian resonance + Averaging over fast 3-planet angles

$$\mathcal{H} = \mathcal{H}_K + \varepsilon^2 \sum_{k\in\mathbb{N}} \Xi_k \cos k\varphi', \qquad \varphi' = p'\lambda_1 - (p'+q')\,\lambda_2 + q'\lambda_3$$

Table: Exact values of Ξ_1 to Ξ_5 on one locus of the 3P0R.

Chain	p	q	$\Xi_1/2$	$\Xi_2/2$	$\Xi_3/2$	$\Xi_4/2$	$\Xi_5/2$
1:2 – 2:3	1	3	19.874	0.2740	0.0321	0.0054	0.0010
2:3 – 1:2	2	2	−0.253	42.946	0.8555	0.1833	0.0534
2:3 – 2:3	2	3	74.556	0.3617	0.0390	0.0059	0.0010
3:4 – 2:3	3	3	−8.316	−59.90	70.429	3.7389	1.0943
3:4 – 3:4	3	4	99.726	0.6333	0.0686	0.0104	0.0018
4:5 – 1:2	4	2	−1.752	36.366	1.0120	0.2152	0.0620

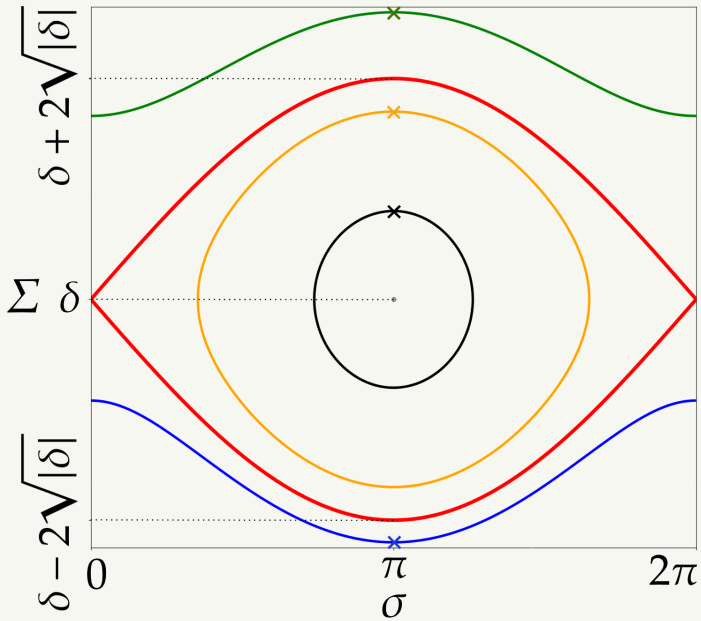
One harmonic generally dominates → Simplification to a pendulum form

$$\mathcal{H} = \mathcal{H}_K + \varepsilon^2 \Xi_k \cos\left(p\lambda_1 - (p+q)\,\lambda_2 + q\lambda_3\right) = \boxed{\mathcal{H}_K + \varepsilon^2 \Xi_k \cos\varphi}$$

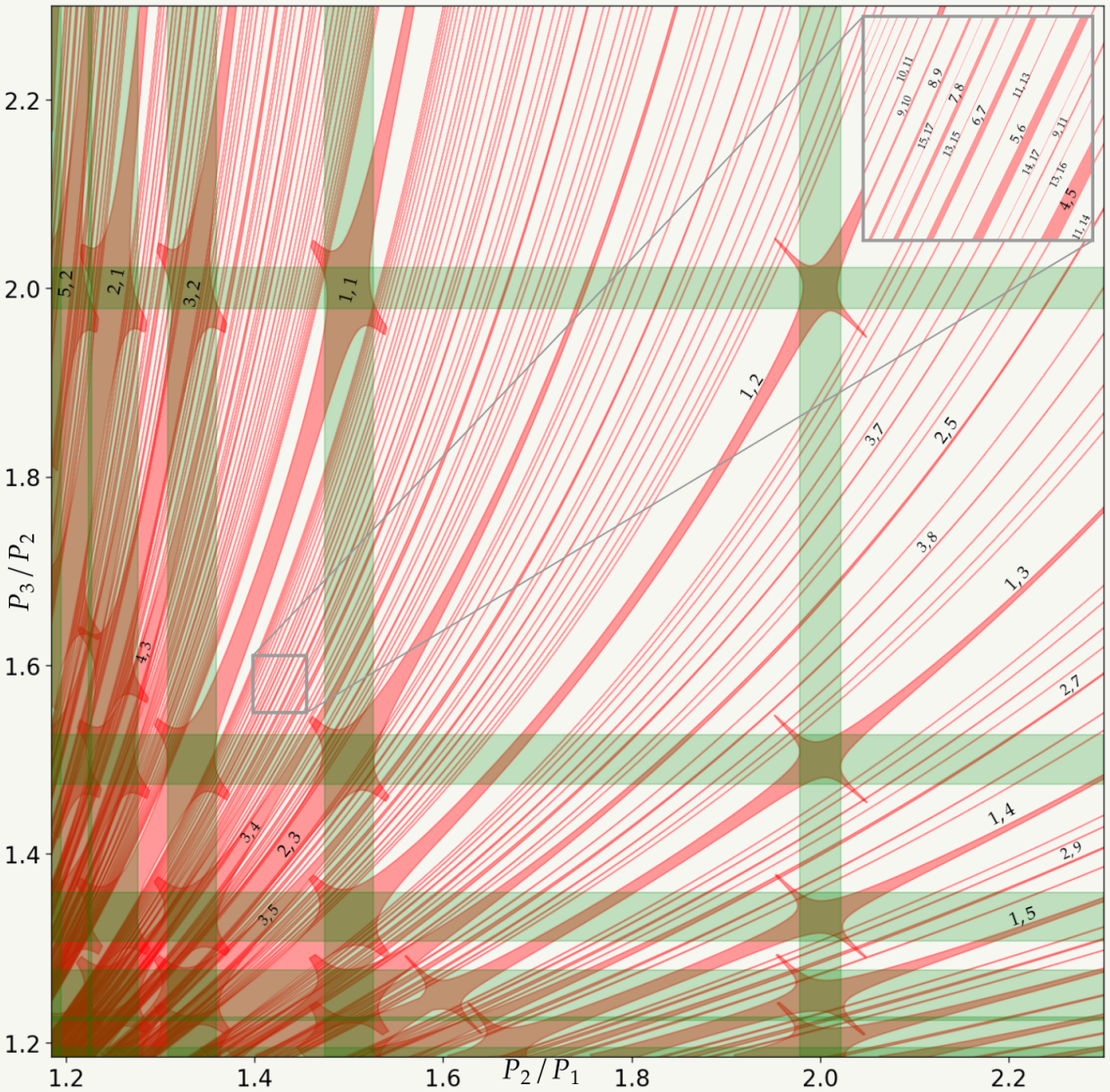
Reduction to one parameter

We use techniques described in (Henrard & Lemaitre, 1983) to reduce the Hamiltonian to one parameter

$$\boxed{\mathcal{H}(\Sigma;\sigma) = \frac{1}{2\delta}\left(\Sigma - \delta\right)^2 + \cos\sigma}, \qquad \sigma = p\lambda_1 - (p+q)\lambda_2 + q\lambda_3.$$



3P0R in the period-ratio diagram



- 3P0R
  $pn_1 - (p+q)n_2 + qn_3 = 0$
  with $\max(p,q) \leq 17$, for
  $m_j = 2 \times 10^{-4} m_0$
- **Red**: inside the separatrices of a 3P0R
  Green: inside those of a 2P1R
- Exact width of 3P0R obtained from our pendulum model

> The width of 3P0R goes to zero exponentially quickly as p and q grow

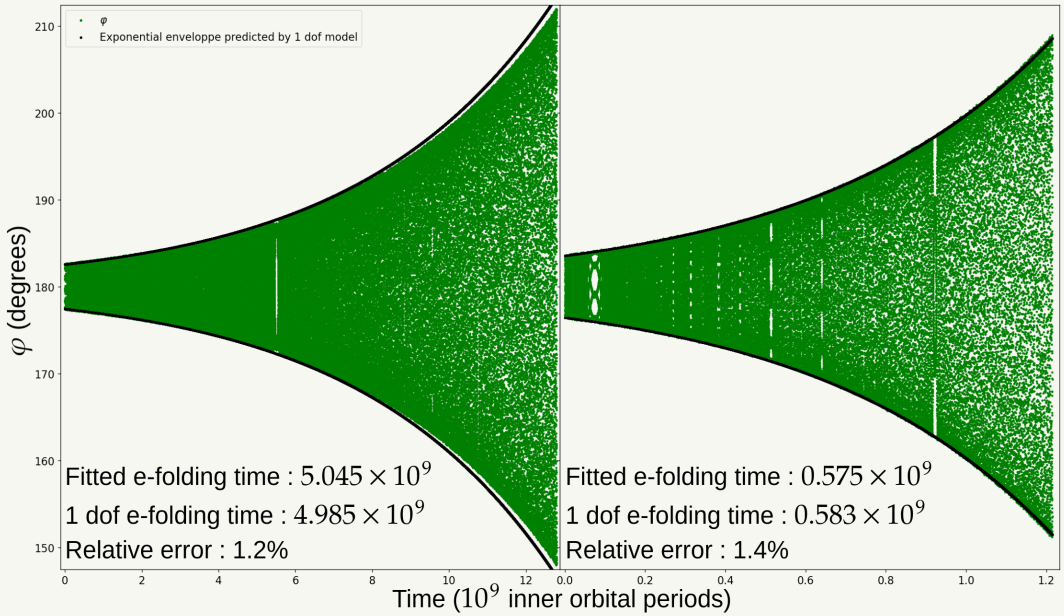
3P0R are always unstable to tides

The eigenvalue $\nleftrightarrow$ controlling (σ, Σ) near the 3P0R center is

$$\mathfrak{R}_{\mathrm{e}} \nleftrightarrow = \frac{9}{2} p^2 \frac{n_{1,0}^2}{K\Lambda_{1,0}} \frac{m_0}{m_1} \frac{q_1}{Q_1} + \frac{9}{2}(p+q)^2 \frac{n_{2,0}^2}{K\Lambda_{2,0}} \frac{m_0}{m_2} \frac{q_2}{Q_2} + \frac{9}{2} q^2 \frac{n_{3,0}^2}{K\Lambda_{3,0}} \frac{m_0}{m_3} \frac{q_3}{Q_3}$$

$\mathrm{Re} \nleftrightarrow > 0 \Rightarrow$ libration amplitude grows like $\mathcal{A}_\sigma(t) = \mathcal{A}_{\sigma,0} \exp(\mathrm{Re} \nleftrightarrow t)$.

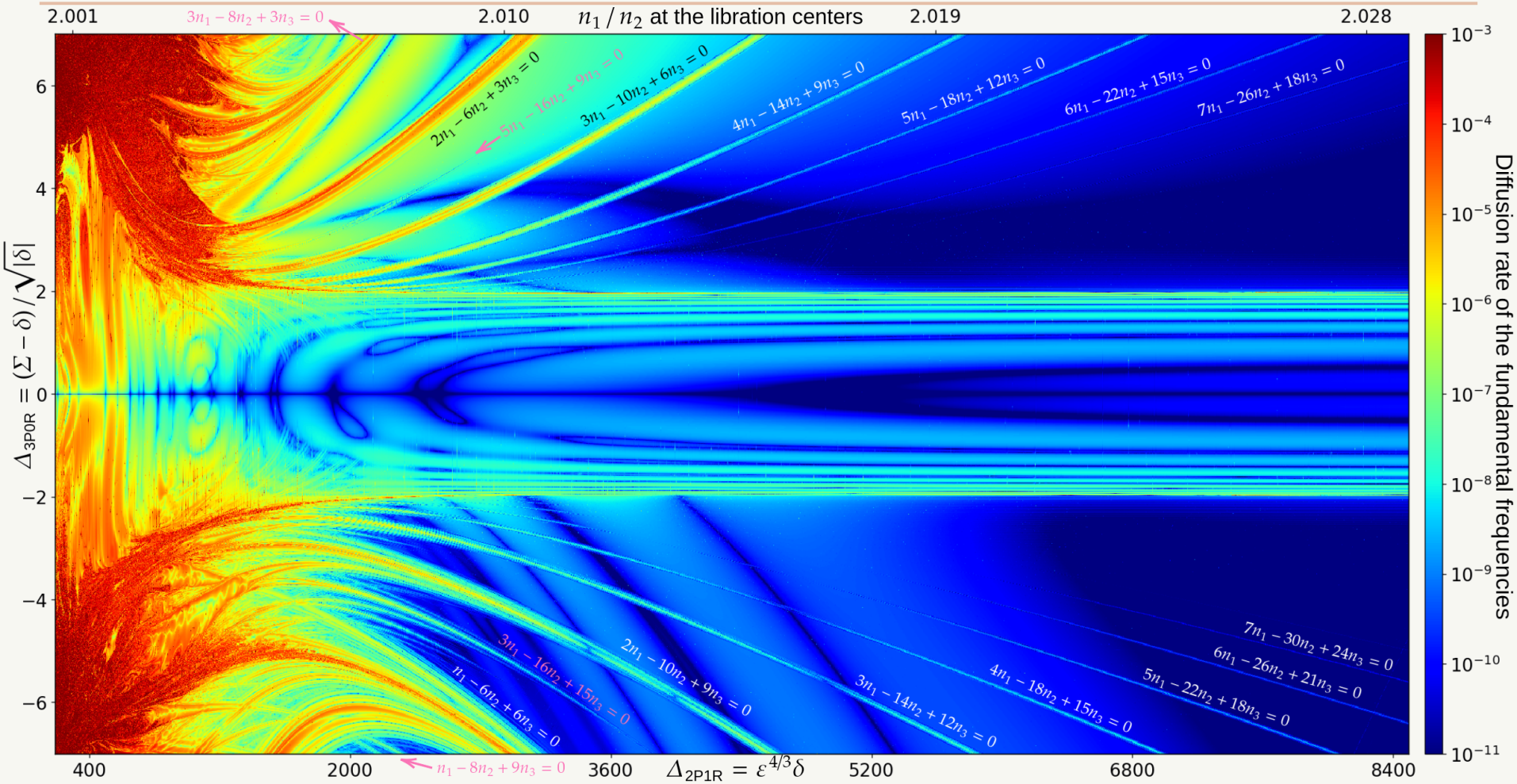
Green: N-body simulations with `Aptidal`. Black: 1-dof model



Similar to the co-orbital 2P0R (Couturier, Robutel, Correia, 2021)

Eventually, the libration amplitude reaches 2π and the 3P0R is exited.

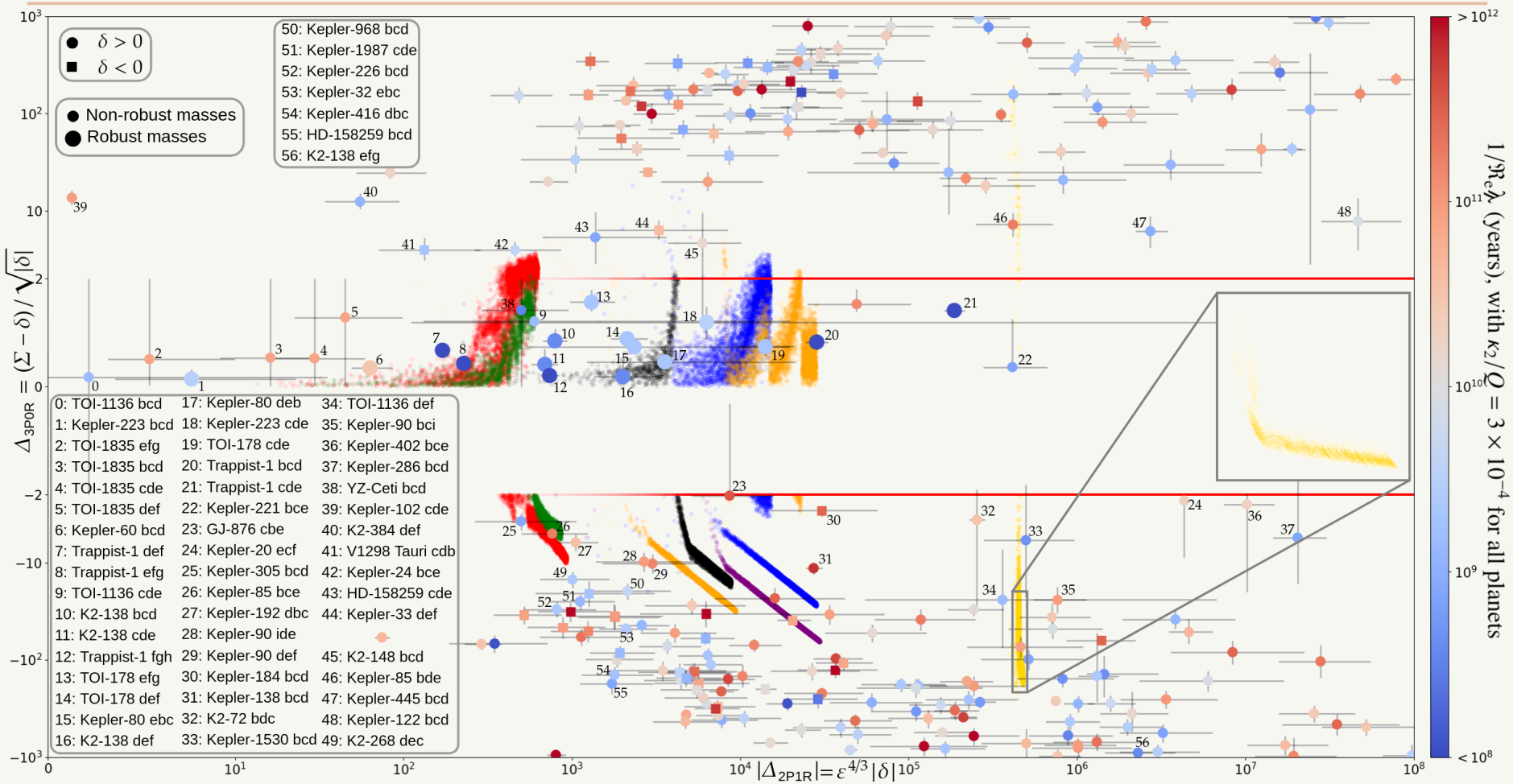
The 3P0R $n_1 - 4n_2 + 3n_3 \sim 0$ seen through chaos



2659×1007 initial conditions run with `Aptidal`. The x-axis is the 3P0R center. Moving vertically, only the initial Σ changes. The color is a measure of the diffusion of fundamental frequencies (Laskar 1989).

Separatrix measured at $\Delta_{3P0R} = \pm 1.97$, predicted at ± 2 $\Rightarrow$ the pendulum model works.

Classification of the exoplanet population



The color is the e-folding time $1/\mathrm{Re}\,\lambda$ for $k_2/Q = 3 \times 10^{-4}$. **Blue** systems ($\lesssim 10^{10}$ yr) are tidally shaped, unlike **red** systems. The clouds of dots are `Aptidal` simulations. Kepler-221 bce & YZ-Ceti are confirmed

Inside 3P0R, blue triplets are farther from the 2P1R than red triplets.

Exiting the 3P0R: capture in 3P1R

We use the metric $\Delta_{\text{3P1R}} = \min |k_1 n_1 + k_2 n_2 + k_3 n_3| / n_2$.

Our simulations (Couturier et al. 2026)

captured in 3P1R *after* exiting the 3P0R

Run	Triplet	(k_1, k_2, k_3)
pq13	Inner	$(2, -10, 9)$
746	Inner	$(1, -4, 4)$
33	Inner	$(1, -4, 4)$
446	Inner	$(1, -4, 4)$
703	Outer	$(3, -7, 5)$
K1987	Inner	$(4, -12, 9)$

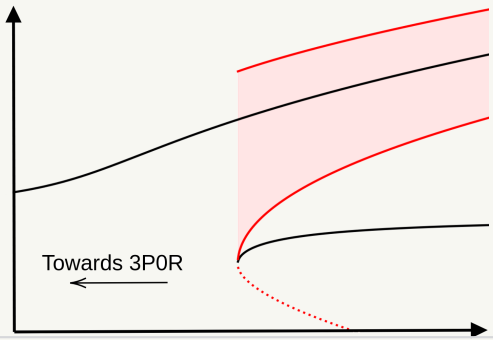
Observed population (Possibly in 3P1R)

Triplets with $\Delta_{\text{3P1R}} < 0.01$, where our simulations end

System	(k_1, k_2, k_3)
Kepler-305 bcd	$(1, -3, 3)$
Kepler-968 bcd	$(1, -6, 6)$
Kepler-226 bcd	$(1, -4, 4)$
K2-268 dec	$(2, -6, 5)$
Kepler-1987 cde	$(4, -12, 9)$
HD-158259 bcd	$(3, -10, 8)$

It is impossible to confirm if these triplets are actually in 3P1R

3P1R are represented by the SFM (Petit, 2021). They are oriented such that their openings face 3P0R (Couturier, Leleu, Delisle, 2026). Noticed by A. Leleu



Conclusion

- A **one-parameter**, **one-dof** model of 3P0R
- Two independent verifications (stability map + simulations)
- Resonance width computed exactly[(1)] analytically
- 3P0R are always unstable to tides
- 3P1R face 3P0R in such a way that capture in 3P1R is probable

[(1)] While the coefficients at second order in mass are exact in our work, we only consider the largest harmonic for each 3P0R. A more precise work could consider all harmonics (up to some truncation). The model would still be 1-dof, but more complicated, with more than one parameter and a phase space different than a pendulum. The ability to plot all systems in one figure would be lost.

Thank you for your attention!