Existence of KAM tori in the realistic planar three-body problem

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CELMEC IX



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We consider the planar planetary 3-body problem, with masses and orbital parameters taken from observed exoplanetary systems.

$$H_{\mathrm{3BP}} = H_{\mathrm{KEP}} + \varepsilon H_{\mathrm{PERT}},$$

where ε is the planet-to-star mass ratio.

What happens to the Keplerian orbits under perturbation?

KAM theory studies the persistence of quasi-periodic motions under small perturbations. However, its application to the planetary 3-body problem is highly non-trivial. In particular:

- **Smallness of ε:** to apply the theorem the perturbation must be sufficiently small.
  $\Rightarrow$ Classical estimates require unrealistically small planetary masses and eccentricities (many different results by, e.g., Arnold, Chierchia, Fejoz, Laskar, Pinzari, Robutel...).

Why extrasolar systems?

Extrasolar systems often have moderate masses and eccentricities, beyond the regime covered by classical analytical KAM results.

A key tool: computer-assisted approaches

Computer-assisted proofs have extended KAM results to more realistic planetary models. Some examples:

- **Secular or restricted problems**
  Celletti&Chierchia(1997); Giorgilli&Locatelli(2000); C.,Locatelli,Sansottera&Volpi(2022); Danesi,Locatelli&Sansottera(2023); Mastroianni&Locatelli(2024);
- **Planetary 3-body problems (3 d.o.f.)**
  verified computations: Locatelli&Giorgilli(2005), Figueras&Haro(2024);

A computer-assisted proof based on a normal form approach is available (Locatelli, 2023; Mendeley Data).

When does the computer-assisted proof have chances to succeed?

The Hamiltonian must be

- expressed in **action-angle coordinates**
- sufficiently close to **Kolmogorov normal form**.

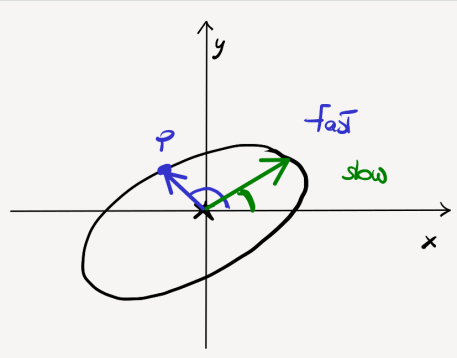
Poincaré Variables for the Planar Three-Body Problem: 4 d.o.f.

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{(m_0 + m_j)a_j}, \quad j = 1, 2$$

$$\lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) = \sqrt{2G_j}\cos(g_j)$$

$$\eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j) = \sqrt{2G_j}\sin(g_j)$$



Expansion around the average values of the semi-major axes

$$L_1 = \Lambda_1 - \Lambda_1^*, \quad L_2 = \Lambda_2 - \Lambda_2^*$$

Reduction 4 d.o.f. $\rightarrow$ 3 d.o.f. : invariance under rotations

$$p_\delta = G_1 - L_1, \qquad \delta = \omega_2 - \omega_1,$$

$$p_\Phi = G_2 - L_2 + G_1 - L_1, \qquad \Phi = -\omega_2.$$

The Hamiltonian is **independent of** Φ.

- p_Φ is therefore a **constant of motion**.
- $\delta = \omega_2 - \omega_1$ describes the *relative orientation* of the two ellipses.

$$H = H(p_1, p_2, p_\delta, q_1, q_2, \delta)$$

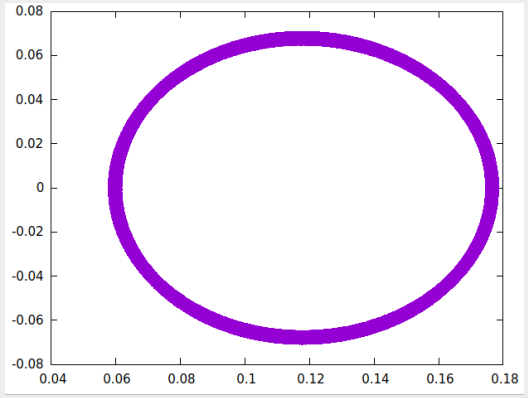
- q_1, q_2: *fast* angles
- δ: secular (*slow*) angle

Secular dynamics

We consider the evolution of the secular variables

$$\begin{cases} x = \sqrt{2p_\delta}\cos\delta, \\ y = \sqrt{2p_\delta}\sin\delta, \end{cases}$$

whose trajectories form an ellipse.



We introduce action-angle variables (p_3, q_3) by rescaling the ellipse to a circle and shifting its centre:

$$\begin{cases} \xi = x^* + \sqrt{2p_3}\,\alpha\cos q_3, \\ \eta = \dfrac{\sqrt{2p_3}}{\alpha}\sin q_3, \end{cases} \qquad \alpha^2 = \frac{a}{b},$$

where a, b are the semi-axes of the ellipse.

The Hamiltonian is now written as a truncated Poisson series:

$$\boxed{H_0 = \boldsymbol{\omega}\cdot\mathbf{p} + \sum_{\ell=0}^{3} f_\ell(\mathbf{p},\mathbf{q})} \quad \text{with} \quad f_\ell = \sum_{\boldsymbol{\ell},\mathbf{k}} c_{\boldsymbol{\ell},\mathbf{k}}\,\mathbf{p}^{\boldsymbol{\ell}} e^{i\mathbf{k}\cdot\mathbf{q}}.$$

$$H_0 = \boldsymbol{\omega} \cdot \mathbf{p} + \sum_{\boldsymbol{\ell},\mathbf{k}} \mathbf{p}^{\boldsymbol{\ell}} e^{i\mathbf{k}\cdot\mathbf{q}} \text{ with } |\boldsymbol{\ell}| \leq 3,\ |\mathbf{k}| \lesssim 100.$$

KAM as in [Stefanelli&Locatelli(2012)]

If we can define constants $\rho, \sigma, \varepsilon, E$ such that H_0 satisfies

(a) the frequency vector $\boldsymbol{\omega}$ *satisfies the Diophantine condition*

$$\left|\boldsymbol{k}\cdot\boldsymbol{\omega}\right| \geq \frac{\gamma}{|\boldsymbol{k}|^{\tau}} \qquad \forall\ \boldsymbol{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\ , \tag{42}$$

where γ *and* τ *are real positive parameters with* $\tau \geq n-1$ *;*

(b) the Hessian matrix related to the angular average of the quadratic term $f_2^{(0)} \in \mathfrak{P}_2$ *is non-degenerate, i.e.,*

$$\left|\mathcal{C}^{(0)}\boldsymbol{v}\right| \geq m|\boldsymbol{v}| \qquad \forall\ \boldsymbol{v} \in \mathbb{R}^{n_1}\ , \tag{43}$$

where the $n_1 \times n_1$ *matrix* $\mathcal{C}^{(0)}$ *is defined so that* $\frac{1}{2}\,(p_1, \ldots, p_{n_1})\cdot\mathcal{C}^{(0)}(p_1, \ldots, p_{n_1})^T = \left\langle f_2^{(0)} \right\rangle$ *and* $m > 0$ *;*

$$H_0 = \boldsymbol{\omega} \cdot \mathbf{p} + \sum_{\boldsymbol{\ell},\mathbf{k}} \mathbf{p}^{\boldsymbol{\ell}} e^{i\mathbf{k}\cdot\mathbf{q}} \text{ with } |\boldsymbol{\ell}| \leq 3,\ |\mathbf{k}| \lesssim 100.$$

KAM as in [Stefanelli&Locatelli(2012)]

If we can define constants $\rho, \sigma, \varepsilon, E$ such that H_0 satisfies

(c) the following upper bounds on the terms $f_l^{(0)} \in \mathfrak{P}_l$ *hold true:*

$$\|f_0^{(0)}\|_{(\varrho,\sigma)} \leq \varepsilon E\ , \qquad \|f_1^{(0)}\|_{(\varrho,\sigma)} \leq \frac{\varepsilon E}{2}\ , \qquad \|f_l^{(0)}\|_{(\varrho,\sigma)} \leq \frac{E}{2^l} \quad \forall\ l \geq 2\ , \tag{44}$$

where $\varepsilon \in \mathbb{R}^+$ *and* $E \in \mathbb{R}^+$ *;*

(d) the parameter ε*, ruling the size of the perturbing terms which are at most linear in* $\boldsymbol{p}$*, is so small that* $\varepsilon \leq \varepsilon^\star$*, with a threshold value*

$$\varepsilon^\star = \min\left\{ \frac{1}{20^{24+8\tau}\,\bar{\Theta}\,\bar{W}^2\,\bar{Z}^2}\ , 9\left(\frac{m\varrho^2}{n_1\bar{\mathcal{E}}}\right)^2 \right\}\ , \tag{45}$$

$\Rightarrow$ there exists a canonical change of coordinates bringing H_0 in **Kolmogorov normal form**:

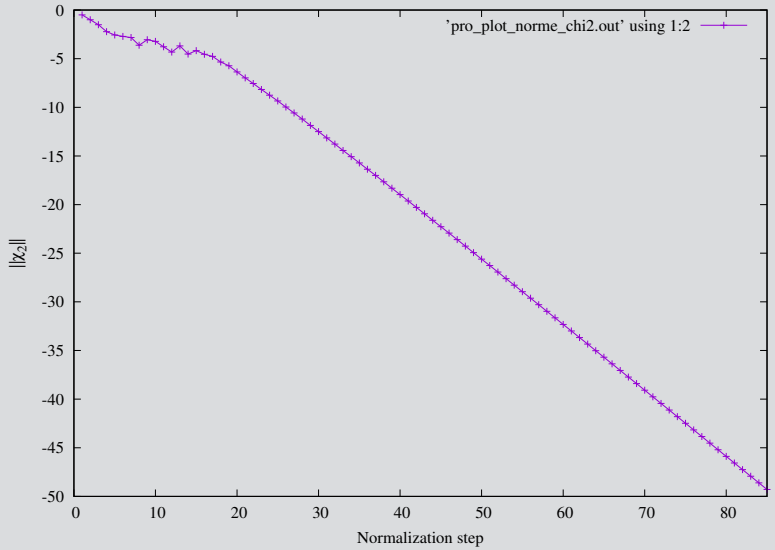
$$H(\mathbf{P},\mathbf{Q}) = \boldsymbol{\omega}\cdot\mathbf{P} + O(\|\mathbf{P}\|^2)$$

and $(\mathbf{P}(t),\mathbf{Q}(t)) = (0, \mathbf{Q}_0 + \boldsymbol{\omega} t)$ is an invariant torus.

We cannot directly apply the result to H_0, since ε is **not smaller** than ε^*

We decrease the perturbation ε by rigorously iterating an algorithm that constructs the Kolmogorov normal form.

- (1) We perform $R_1 \simeq 100$ normalization steps (normalizing the low-order terms).

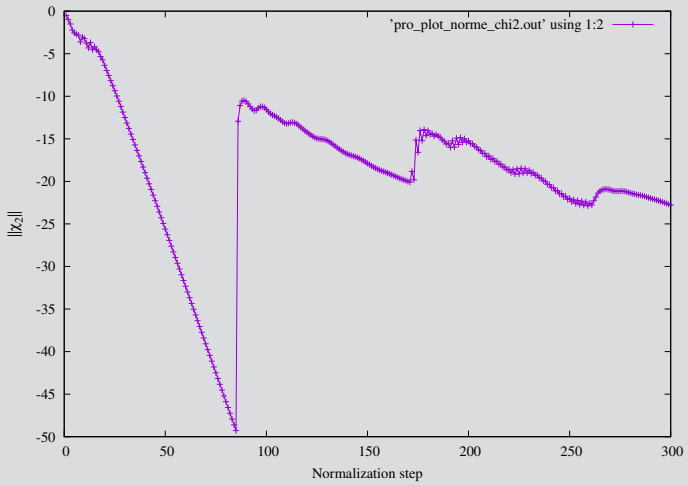


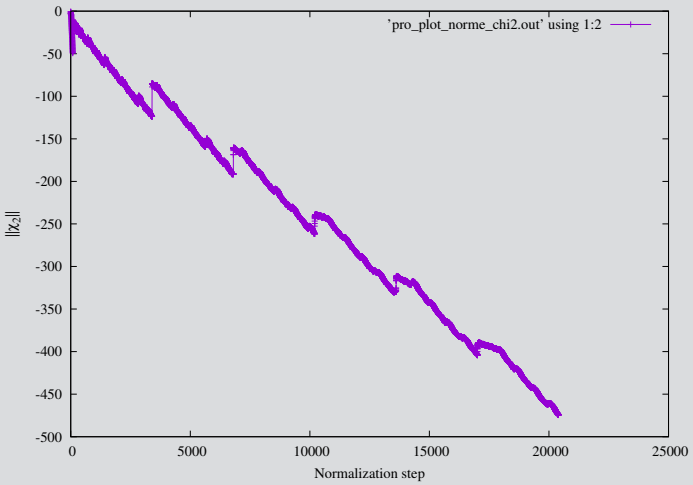
Error decrease

But this may still not be enough!

We cannot directly apply the result to H_0, since $\varepsilon > \varepsilon^*$

- (2)We rigorously estimate the norm of high-order terms up to $R_2 \simeq 10^4$ terms (analytic bound for the infinite tail).





Note: R_2 can be much larger than R_1, because we operate with numbers rather than functions.

We selected planetary systems with at least two planets, non-negligible eccentricities, and no mean-motion resonances.

HD4732

$$M_* = 1.74 M_S$$

	planet 1	planet 2
a	1.19 AU	4.6 AU
e	0.13	0.23
m	2.37 M_J	2.37 M_J

KAM-like theorem, computer-assisted

There is $\mathcal{W} \subset \mathcal{B} = \{\boldsymbol{\omega} : \max_j |\omega_j - \omega_j^*| \leq 10^{-14}\}$ with $\boldsymbol{\omega}^* = (6.372.., 0.844..., -8.92.10^{-5})$ such that

- the Lebesgue measure of $\mathcal{W}$ is more than 90% of the measure of $\mathcal{B}$;
- $\forall \boldsymbol{\omega} \in \mathcal{W}$ $\exists$ an invariant torus for the flow of $H^{(0)}$ carrying quasi-periodic motion with frequency $\boldsymbol{\omega}$.

We selected planetary systems with at least two planets, non-negligible eccentricities, and no mean-motion resonances.

HD11964

$$M_* = 1.12 M_S$$

	planet 1	planet 2
a	0.229 AU	3.16 AU
e	0.3	0.041
m	0.079 M_J	0.622 M_J

KAM-like theorem, computer-assisted

There is $\mathcal{W} \subset \mathcal{B} = \{\boldsymbol{\omega} : \max_j |\omega_j - \omega_j^*| \leq 10^{-14}\}$ with $\boldsymbol{\omega}^* = (60.79.., 1.187..., -8.53..10^{-6})$ such that

- the Lebesgue measure of $\mathcal{W}$ is more than 90% of the measure of $\mathcal{B}$;
- $\forall \boldsymbol{\omega} \in \mathcal{W}$ $\exists$ an invariant torus for the flow of $H^{(0)}$ carrying quasi-periodic motion with frequency $\boldsymbol{\omega}$.

We selected planetary systems with at least two planets, non-negligible eccentricities, and no mean-motion resonances.

HD142

$$M_* = 1.25 M_S$$

	planet 1	planet 2
a	1.02 AU	6.8 AU
e	0.17	0.21
m	1.25 M_J	5.3 M_J

KAM-like theorem, computer-assisted

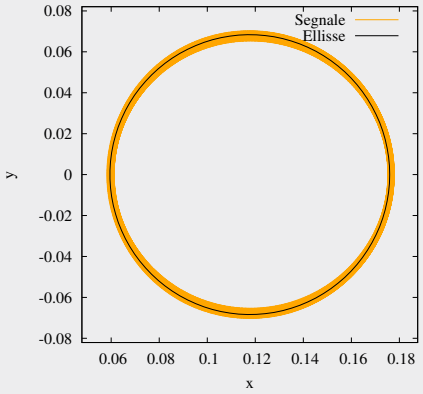
There is $\mathcal{W} \subset \mathcal{B} = \{\boldsymbol{\omega} : \max_j |\omega_j - \omega_j^*| \leq 10^{-14}\}$ with $\boldsymbol{\omega}^* = (6.382.., 0.374..., -7.55..10^{-5})$ such that

- the Lebesgue measure of $\mathcal{W}$ is more than 90% of the measure of $\mathcal{B}$;
- $\forall \boldsymbol{\omega} \in \mathcal{W}$ $\exists$ an invariant torus for the flow of $H^{(0)}$ carrying quasi-periodic motion with frequency $\boldsymbol{\omega}$.

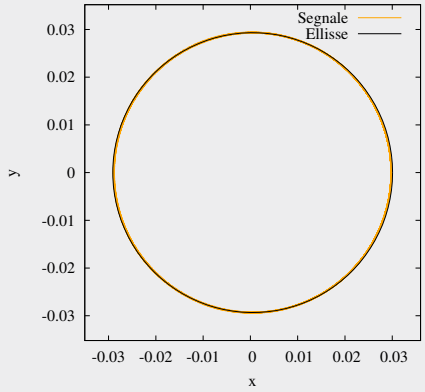
Success VS failure: evolution of the secular variables

Successful cases

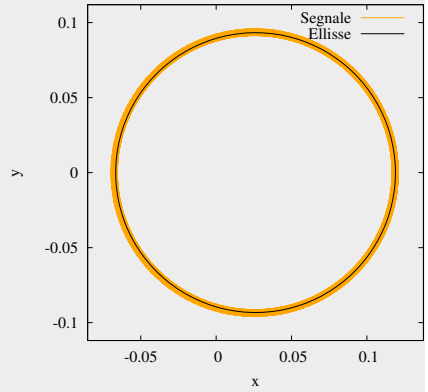
HD 4732



HD 11964

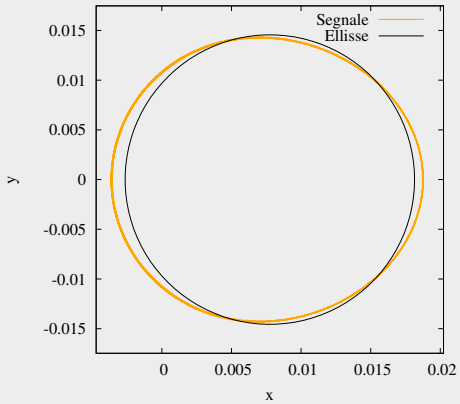


HD 142

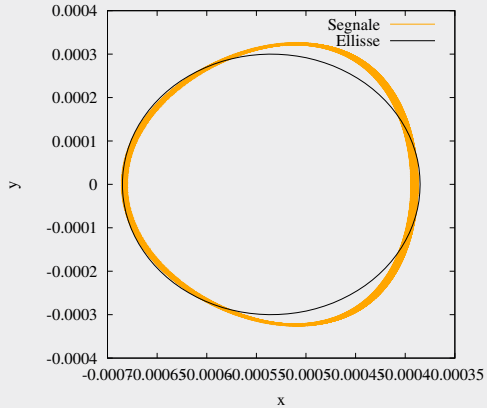


Failed cases

HD 109271



Teegarden



Here the secular variables do not evolve along an ellipse.

Note on the computational time

The CAP computation for a 3-DOF system can be very expensive: about one month of computation in our tests (using the "Ada" cluster at the University of Rome "Tor Vergata").

Therefore, it is important to develop a **fast preliminary criterion** to assess whether the CAP is likely to converge.

In the systems considered here, the CAP fails for the more compact configurations.

- **Successful cases:**
  $\frac{a_1}{a_2} \simeq 0.07,\ 0.15,\ 0.26$ (HD 11964, HD 142, HD 4732)
- **Failed cases:** $\frac{a_1}{a_2} \simeq 0.40,\ 0.57$ (HD 109271, Teegarden)

  In these cases an adapted preliminary work is needed.

Thank you for your attention!