Bifurcations of Periodic Solutions to Hamiltonian Systems with Different Types of Symmetries

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(joint work with Prof. A. Bruno)

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"...ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule bréche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

Les méthodes nouvelles de la méchanique céleste.
Vol I, Ch. III, § 36
Henri Poincaré

"Faute d'avoir ce qu'on aime, il faut aimer ce qu'on a."

Essais Michel Montaigne

Question

How do resonance and phase-space symmetries determine the number, period and symmetry of bifurcating periodic-orbit families?

Periodic orbits, reduced normal forms and bifurcations

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$$\longrightarrow \text{stationary germs} \longrightarrow \text{periodic-orbit families}$$

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Below this technique is applied to a Hamiltonian autonomous system with 2DOF

Talk structure

1. Reduced normal form (RNF)
2. Symmetry
3. Hill 3-Body Problem and some results

Normal form near a periodic solution

Let $\mathcal{M}$ be a periodic solution of period T. In local canonical coordinates

$$(u, v, R, \psi), \qquad \mathcal{M}: \ u = v = R = 0$$

the complex normal form is

$$\boxed{h(u, v, R, \psi) = h_{20} + h_{01} + \sum h_{pq\ell m}\, u^p v^q R^\ell e^{im\psi}} \qquad (*)$$

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If the nontrivial multiplier is

$$\nu = e^{2\pi\lambda}, \qquad \operatorname{Im}\lambda = \alpha = \frac{\mathfrak{r}}{\mathfrak{s}}, \qquad \gcd(\mathfrak{r}, \mathfrak{s}) = 1$$

then its monomials satisfy the **resonant equation**

$$\boxed{(p - q)\mathfrak{r} + m\mathfrak{s} = 0} \qquad (**)$$

Reduction to a Hamiltonian system with 1DOF

Twisting canonical transformation

$$\boxed{u = Ue^{i\alpha\varphi}, \quad v = Ve^{-i\alpha\varphi}, \quad R = \rho + i\alpha UV, \quad \psi = \varphi} \qquad (***)$$

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For every resonant monomial in (*), the phase factor cancels:

$$u^p v^q R^\ell e^{im\psi} \longmapsto U^p V^q \big(\rho + i\alpha UV\big)^\ell, \qquad m + \alpha(p-q) = 0$$

and for $\ell = 0$ there are almost no quadratic terms in U, V

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Families of **periodic solutions** (PS) correspond to **stationary points** (SP) $h_U(\rho) = h_V(\rho) = 0$.

Resonance support and phase reconstruction

Homogeneous resonant form

$$F_k^{(\mathfrak{s})}(U,V) = \sum_{p+q=k} a_{pq} U^p V^q, \quad (p-q)\mathfrak{r} + \mathfrak{s}m = 0.$$

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The exponent lattice in (U,V,ρ) is defined mainly by denominator $\mathfrak{s}$.

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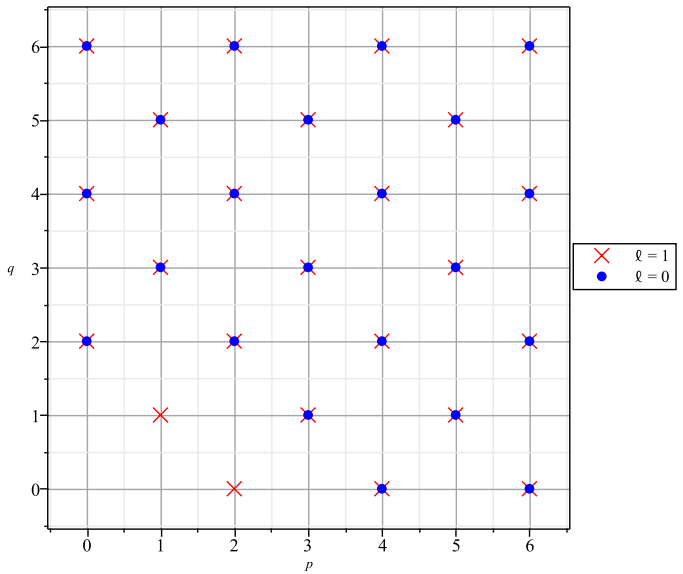
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Reconstruction

The inverse transformation contains $e^{\pm i\mathfrak{r}\varphi/\mathfrak{s}}$. The numerator $\mathfrak{r}$ controls the phase relation along the reconstructed orbit.

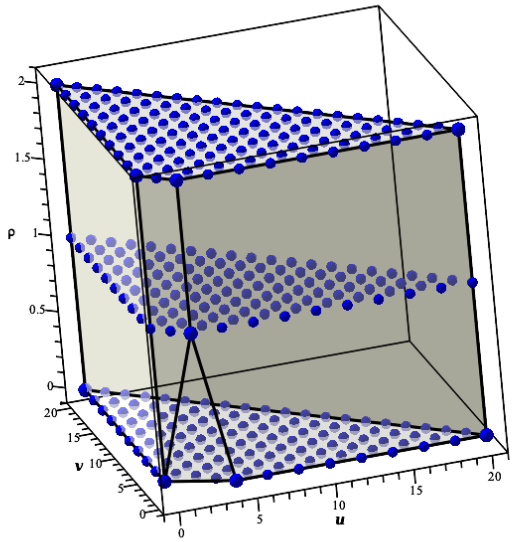
Power geometry of the stationary system

1. Compute the supports of $h(U, V, \rho)$ and then of SP system h_V and h_U



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$$N_1 = -(1, 1, 2),$$
$$N_2 = -(1, 3, 2)$$

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4. Solve the corresponding nontrivial truncations

$$\hat{\mathfrak{S}}_1 : \begin{cases} a_{310}u^3 + \sigma v + a_{111}u\rho = 0, \\ -4a_{400}u^3 - 2\rho a_{200}u = 0 \end{cases}$$

Power geometry of the stationary system

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For degenerate truncations we replace the stationary equations by suitable linear combinations.

Talk structure

1. Reduced normal form (RNF)
2. **Symmetry**
3. Hill 3-Body Problem and some results

Symmetry classes in the reduced phase plane

The group of automorphisms

We use the **dihedral** group

$$D_4 = \langle r, s \mid r^4 = s^2 = e,\ srs = r^{-1} \rangle,$$

with

$$s : (u, v) \mapsto (u, -v), \qquad r : (u, v) \mapsto (-v, u), \qquad r^2 : (u, v) \mapsto (-u, -v).$$

Relevant subgroups

$$V_4^{\mathrm{ax}} = \langle r^2, s \rangle = \{e, r^2, s, r^2 s\}, \quad V_4^{\mathrm{diag}} = \langle r^2, rs \rangle = \{e, r^2, rs, r^3 s\}.$$

The twisting transformation conjugates the symmetry action

Suppose the NF (*) before (***) is invariant under an axial action that keeps R and φ fixed.

$$S_v : (u, v, R) \mapsto (u, -v, R), \qquad \widehat{S}_v : (U, V, \rho) \mapsto (U, -V, \rho + 2i\alpha UV),$$
$$S_u : (u, v, R) \mapsto (-u, v, R), \qquad \widehat{S}_u : (U, V, \rho) \mapsto (-U, V, \rho + 2i\alpha UV),$$
$$C : (u, v, R) \mapsto (-u, -v, R), \qquad \widehat{C} : (U, V, \rho) \mapsto (-U, -V, \rho)$$

For $\alpha = 1/2$,

$$\boxed{h(U, -V, \rho + iUV) = h(U, V, \rho)}$$

Axial actions therefore mix the ρ-layers. Central symmetry remains linear.

Original-system symmetries and RNF-only automorphisms

Original-system symmetry in RNF coordinates

If A is an actual automorphism or reversor of the initial Hamiltonian, its RNF action is the conjugate $T^{-1}AT$, where T denotes (***). It must be used when periodic families are identified and counted.

RNF-only automorphism

Resonance or truncation may give the polynomial RNF an extra symmetry. Such an automorphism need not correspond to a symmetry of the original system.

Consequence

Symmetry restrictions must be imposed in the NF (*) before (***), after which the Hamiltonian and the symmetry action are transformed together.

Symmetry classes and admissible Jordan block cases

Symmetry of PO	Group information	Jordan block at $\nu = -1$
Asymmetric	trivial stabilizer	possible
One axial symmetry	one reversor or axial generator	possible
Central symmetry	half-period factorization of monodromy	impossible
Double symmetry	two generators; distinguish V_4^{ax} and V_4^{diag}	impossible

Reason for the last two rows

For $\mathfrak{s} = 2$, central or double symmetry gives $\mathcal{N} = A^2$ with a real half-period matrix A. A real 2×2 matrix square cannot have a nontrivial Jordan block at -1

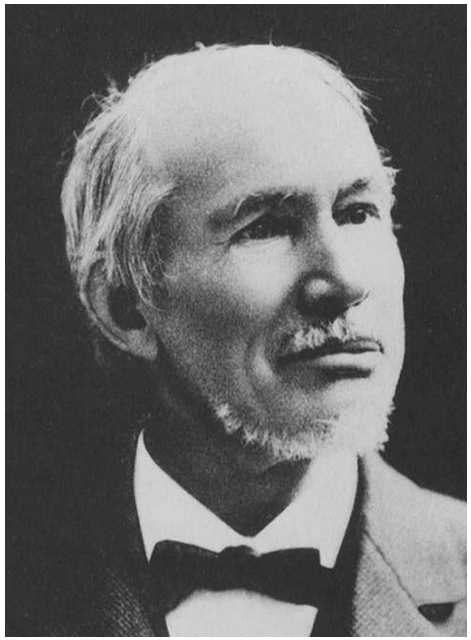
At $\mathfrak{s} = 1$ the resonant representative has $\alpha = 0$, so the twisting transformation (***) preserves both axial representations

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1. Reduced normal form (RNF)
2. Symmetry
3. **Hill 3-Body Problem and some results**

Planar circular Hill 3-body problem

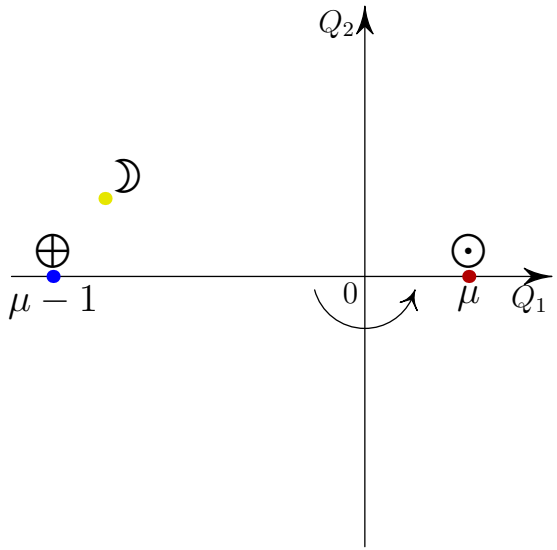
- Introduced by George Hill in lunar theory (1878), following Euler's approach



G. W. Hill

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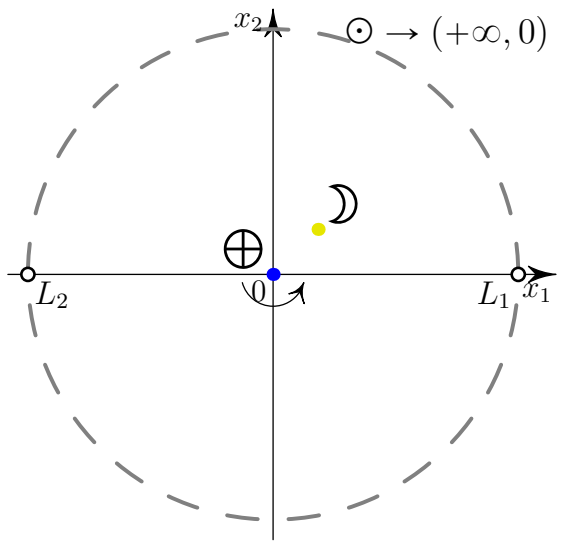
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Circular R3BP

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Hill coordinates

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- Major systematic studies of Hill periodic orbits were carried out by Michel Hénon



M. Hénon

Planar circular Hill 3-body problem

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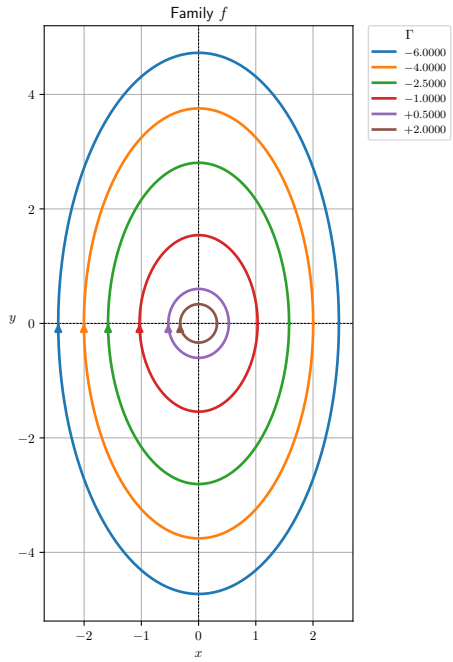
$$H = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{r}$$

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- Klein four-group: $\rho_{(a,b)} : (t, x_1, x_2, y_1, y_2) \mapsto (at, bx_1, abx_2, aby_1, by_2)$, $a, b = \pm 1$. Its periodic orbits form doubly symmetric, singly symmetric and asymmetric families



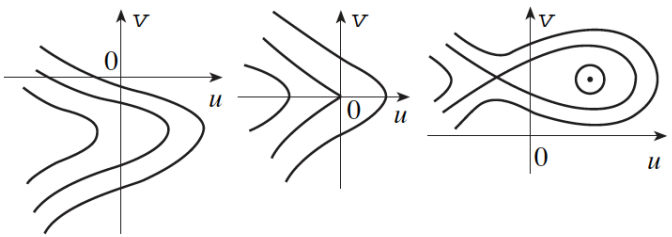
Case $\mathfrak{s} = 1$: two local mechanisms

With a Jordan block

The quadratic term selects one direction and gives the typical leading balance

$$U \sim \rho^{1/2}, \qquad V \sim \rho$$

This produces a fold



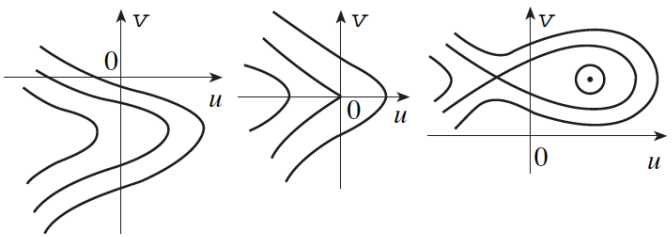
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Without a Jordan block

The leading terms start at higher order. A singly symmetric parent family can lose its remaining symmetry and generate an asymmetric family

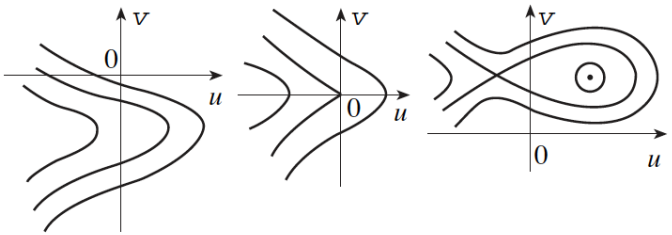
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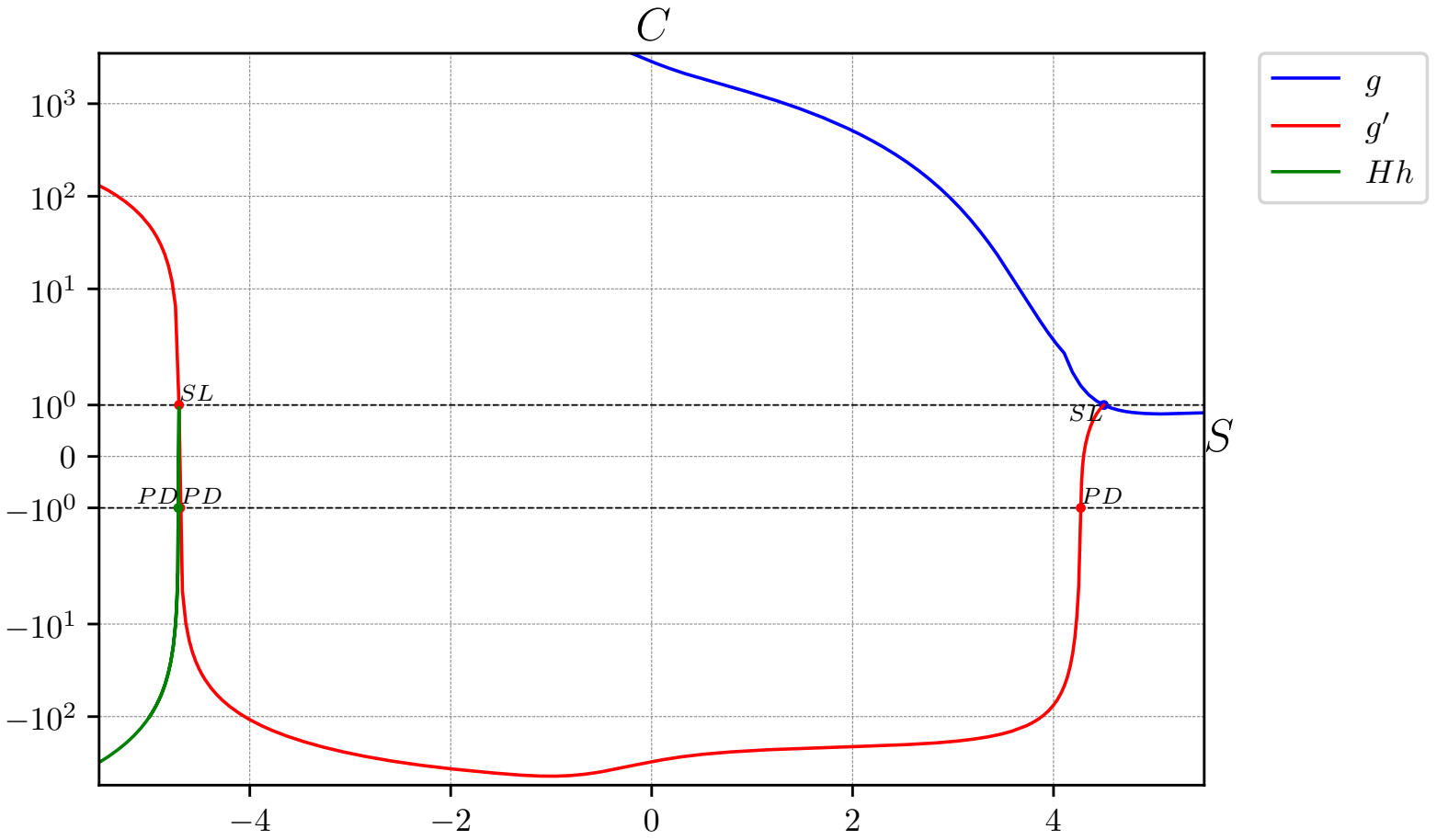
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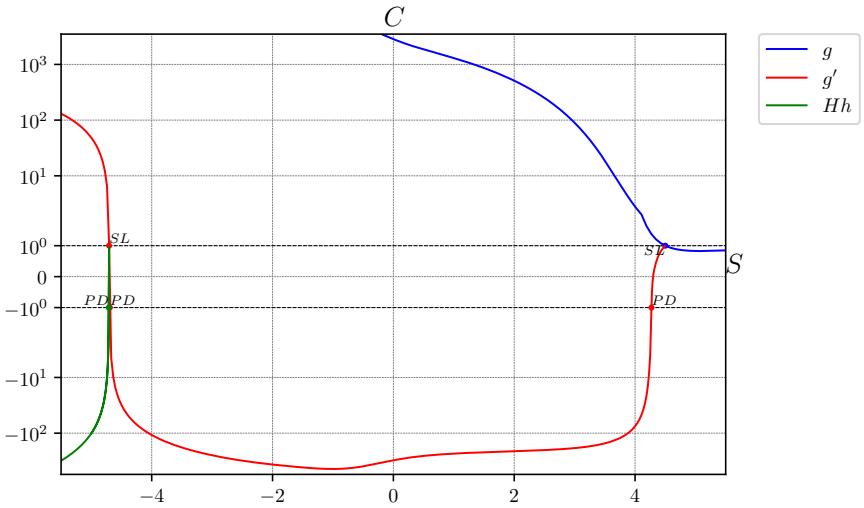
Conclusion

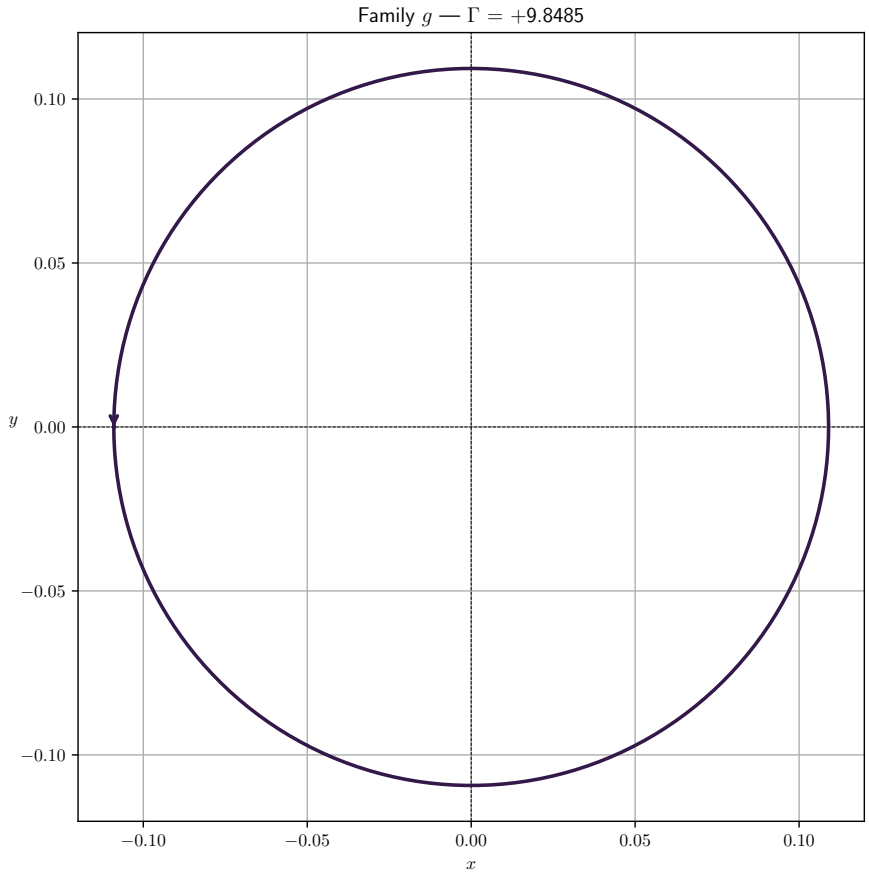
In the generic case $\mathfrak{s} = 1$ the only possible bifurcation for a family of asymmetric periodic solutions is a saddle-node bifurcation

Example of families bifurcations $g \to$ (SL) $g' \to$ (SL) Hh



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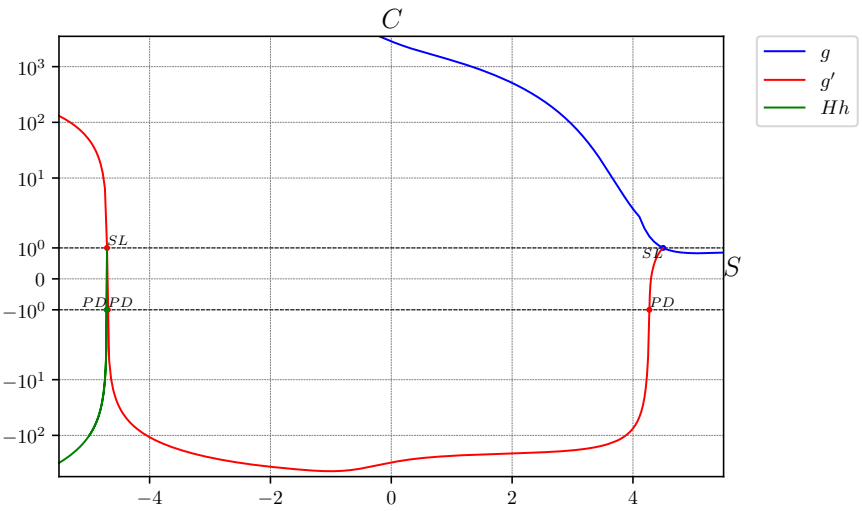


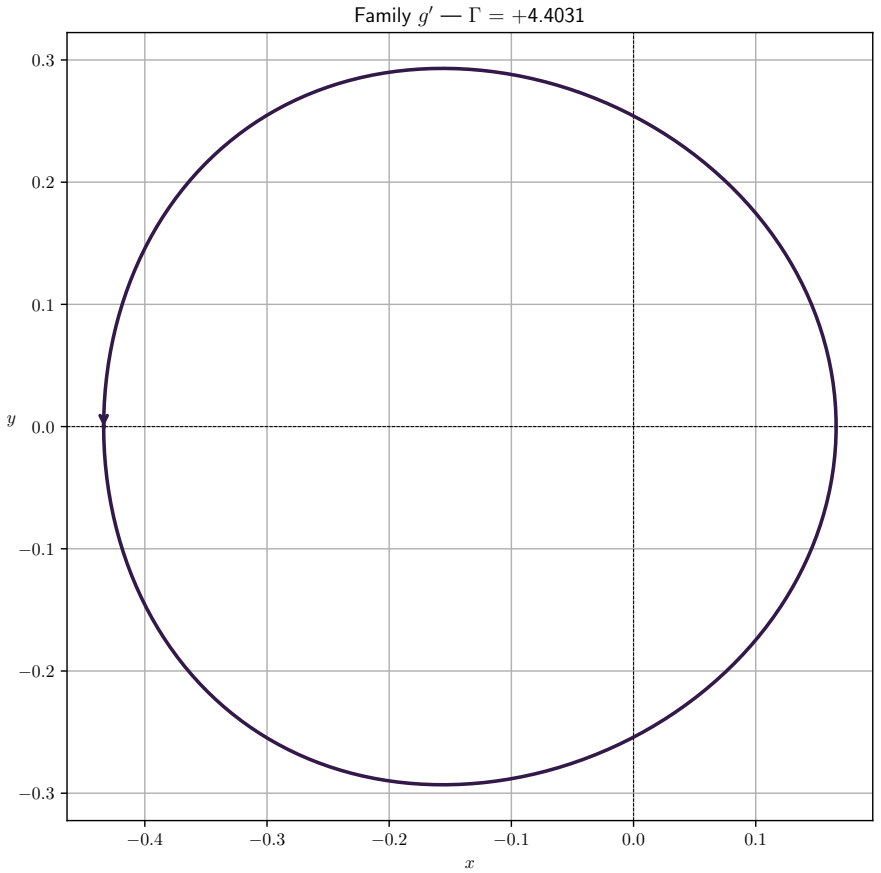


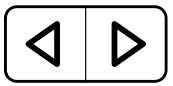
Stability vs Jacobi constant

Family g

Example of families bifurcations $g \to$ (SL) $g' \to$ (SL) Hh



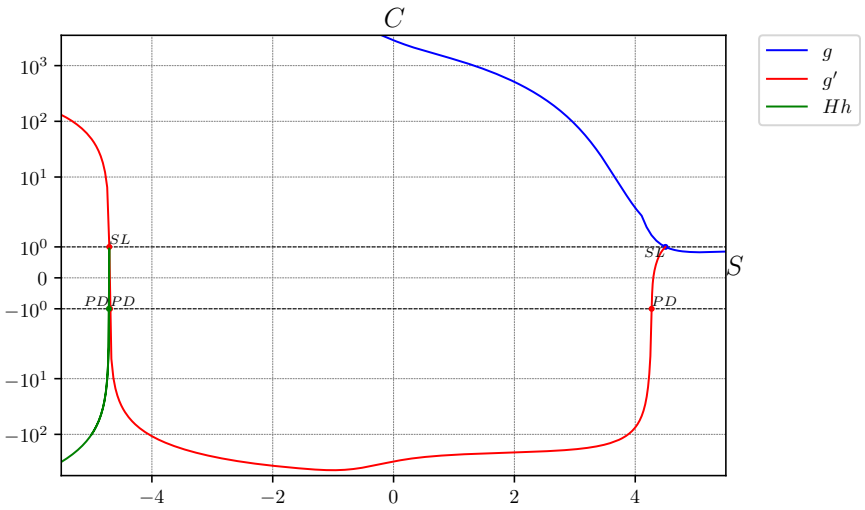




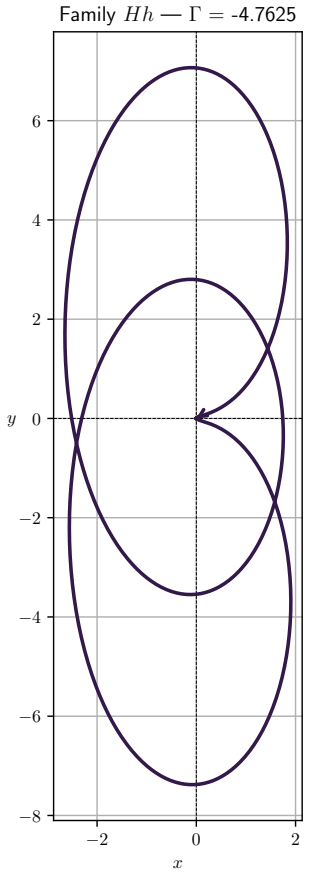
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Family g'

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Stability vs Jacobi constant



Family Hh

$\mathfrak{s} = 2$ with a Jordan block

The transformation (***) gives the leading truncation:

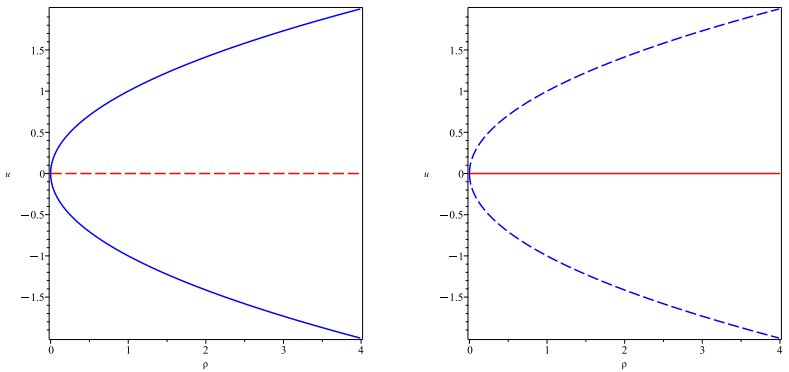
$$\widehat{H} = \frac{1}{2}\sigma V^2 + a_{400}U^4 + a_{201}U^2\rho$$

$$V = 0,\ U\left(4a_{400}U^2 + 2a_{201}\rho\right) = 0$$

Besides the parent branch $U = V = 0$, the generated branch satisfies

$$U = \pm\sqrt{-\frac{a_{201}}{2a_{400}}\rho}, \qquad V = 0$$

It represents a family with period $2T$.



The two real-side variants

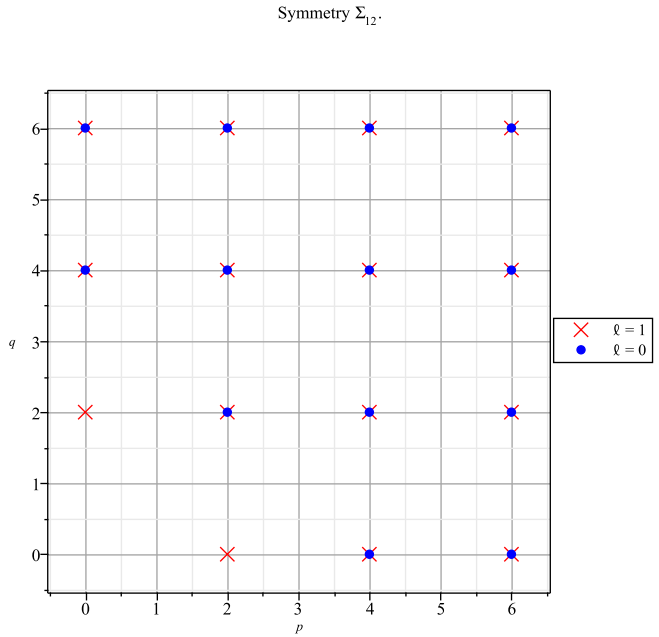
$\mathfrak{s} = 2$ without JB: double symmetry

Truncated Hamiltonian

$$\widehat{H} = \rho(AU^2 + BV^2) + CU^4 + DU^2V^2 + EV^4$$

Stationary equations

$$U(A\rho + 2CU^2 + DV^2) = 0,$$
$$V(B\rho + DU^2 + 2EV^2) = 0$$



Generated singly symmetric families

$$\mathcal{F}_U:\ V = 0, U^2 = -\frac{A}{2C}\rho, \qquad \mathcal{F}_V:\ U = 0, V^2 = -\frac{B}{2E}\rho$$

$\mathfrak{s} = 2$ without JB: two variants in DS case

Variant 1: simultaneous birth/death

If the two axial branches exist on the same side of $\rho = 0$, two singly symmetric double-period families are born or die simultaneously. Their stability depends on the types of extrema of $\widehat{H}$ along the branches.

Variant 2: branch replacement

If the two axial branches exist on opposite sides of $\rho = 0$, one observes the death of one singly symmetric double-period family and the birth of another family with the other axial symmetry.

$\mathfrak{s} = 3$: one triple-period family

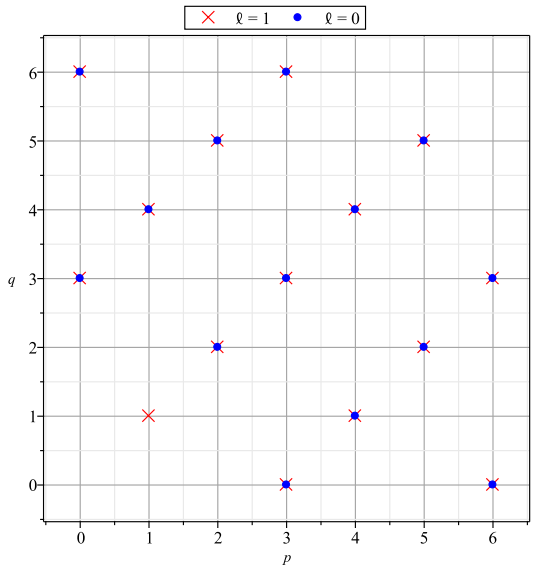
The leading asymmetric truncation is

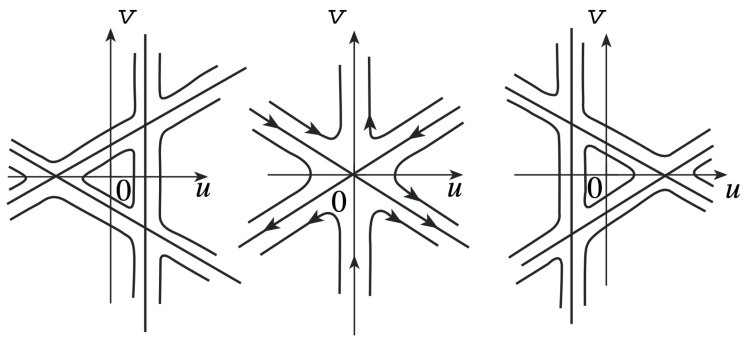
$$\widehat{H} = a_{111}\rho UV + a_{300}U^3 + a_{030}V^3$$

Its stationary system has three nonzero complex stationary points with

$$U \sim \rho, \qquad V \sim \rho$$

The three points form one orbit of the phase shift and reconstruct one family of periodic solutions with period $3T$.
The truncated Hamiltonian satisfies $\widehat{H} \sim \rho^3$ along the family.





$\mathfrak{s} \geqslant 4$: a modified stationary system

The direct stationary truncations are degenerate. We use

$$g_1 = Vh_V - Uh_U, \qquad g_2 = Vh_V + Uh_U$$

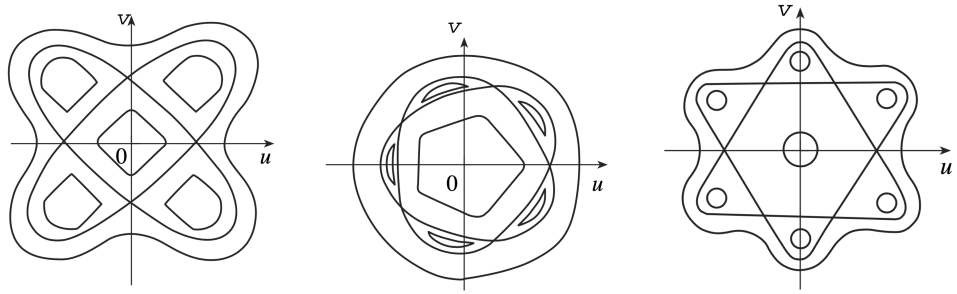
The leading nontrivial system has the structure

$$(2a_{220}UV + a_{111}\rho)UV = 0, \qquad -a_{\mathfrak{s}00}U^{\mathfrak{s}} + a_{0\mathfrak{s}0}V^{\mathfrak{s}} = 0$$

There are $2\mathfrak{s}$ nonzero stationary points with

$$U, V \sim \rho^{1/2}$$

They form two families with period $\mathfrak{s}T$; generically one is stable and the other unstable.



$\mathfrak{s} = 1$: symmetry classes without a Jordan block

Because $\alpha = 0$, the symmetry action is unchanged by (***).

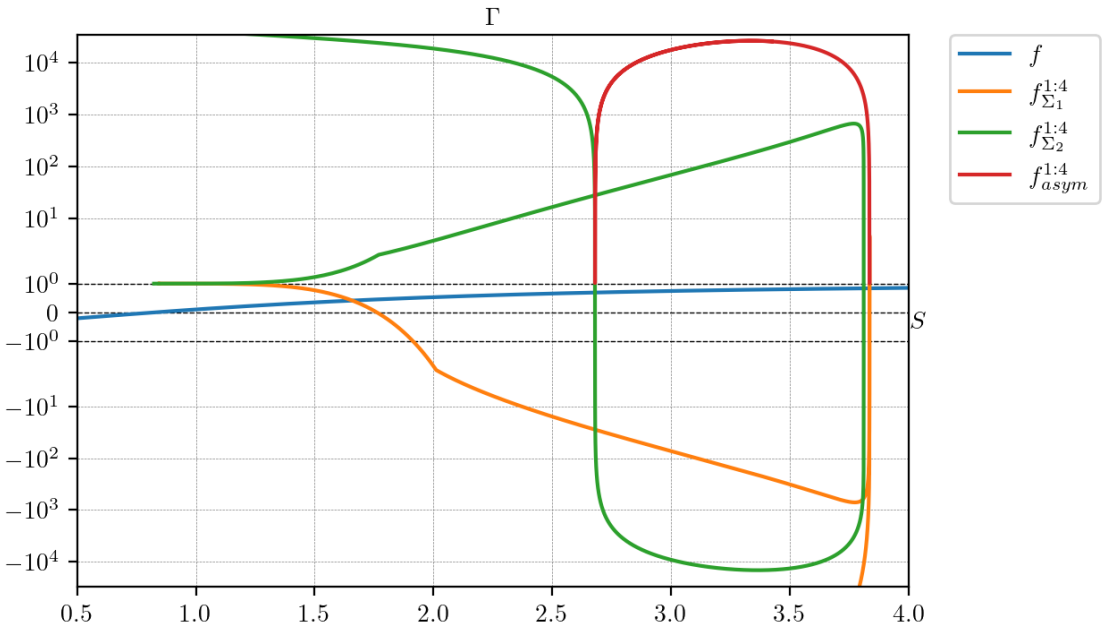
Class	Leading structure	Typical generated branches
One axial	invariant quadratic and quartic terms	fixed-axis and symmetry-losing branches
Central	$\rho Q_2(U,V) + Q_4(U,V)$; even total degree	projective branches, generally not fixed points
Double	even exponents for V_4^{ax}; coefficient pairing for V_4^{diag}	two axial branches; a possible mixed branch

Hill interpretation

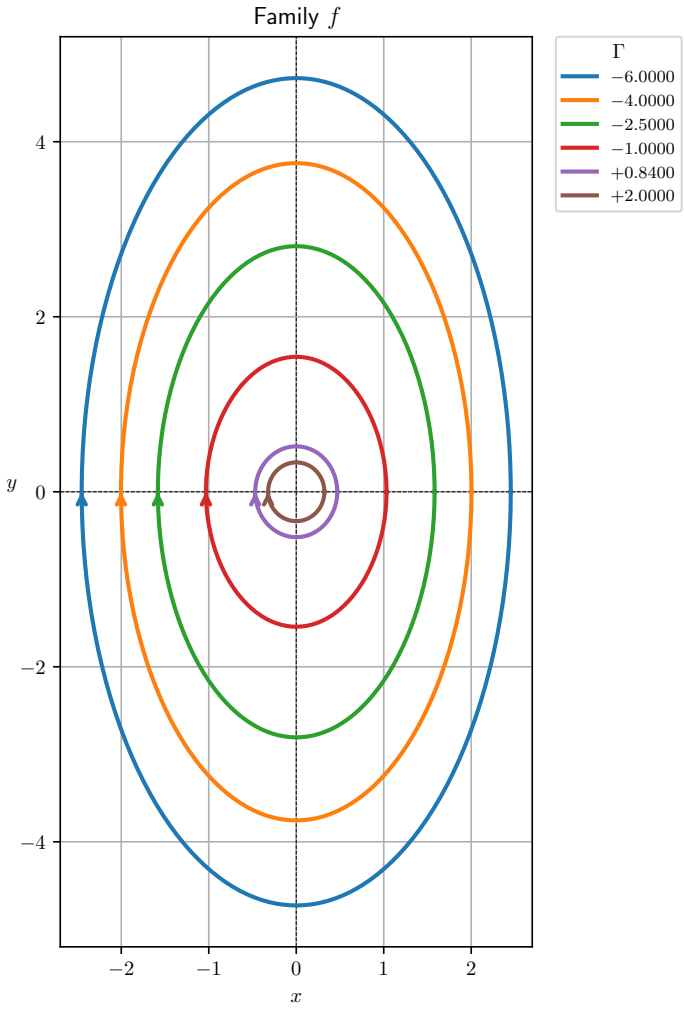
The bifurcation $g' \to Hh$ is the confirmed symmetry-losing example. No centrally symmetric Hill families have yet been found.

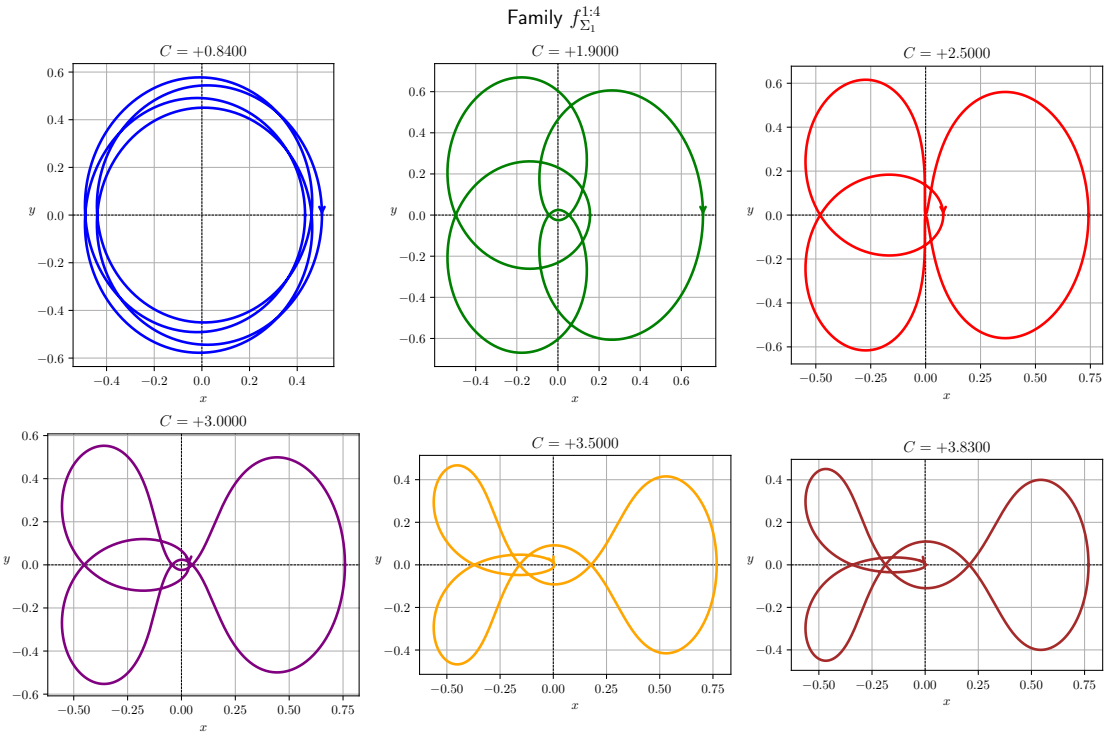
Example of asymmetric bridge between 4th covering of f

- Family f (DS) $\rightarrow$ 4th covering of two pairs of families $f^{1:4}_{\Sigma_1}$, $f^{1:4}_{\Sigma_2}$ of singly symmetric orbits
- These families are connected with a closed family of asymmetric orbits $f^{1:4}_{Asym}$

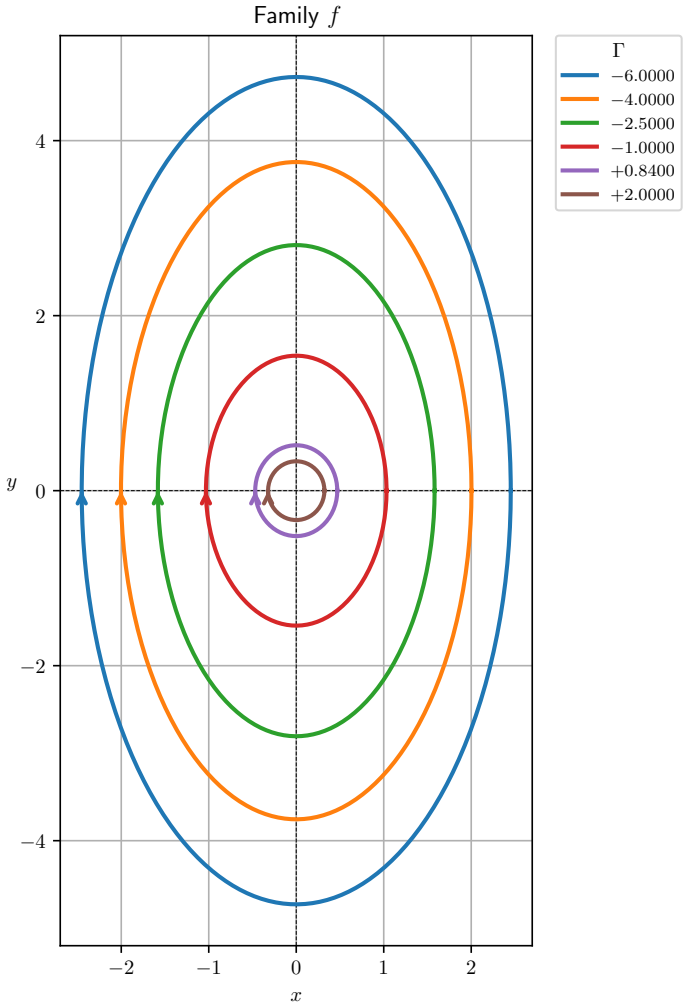


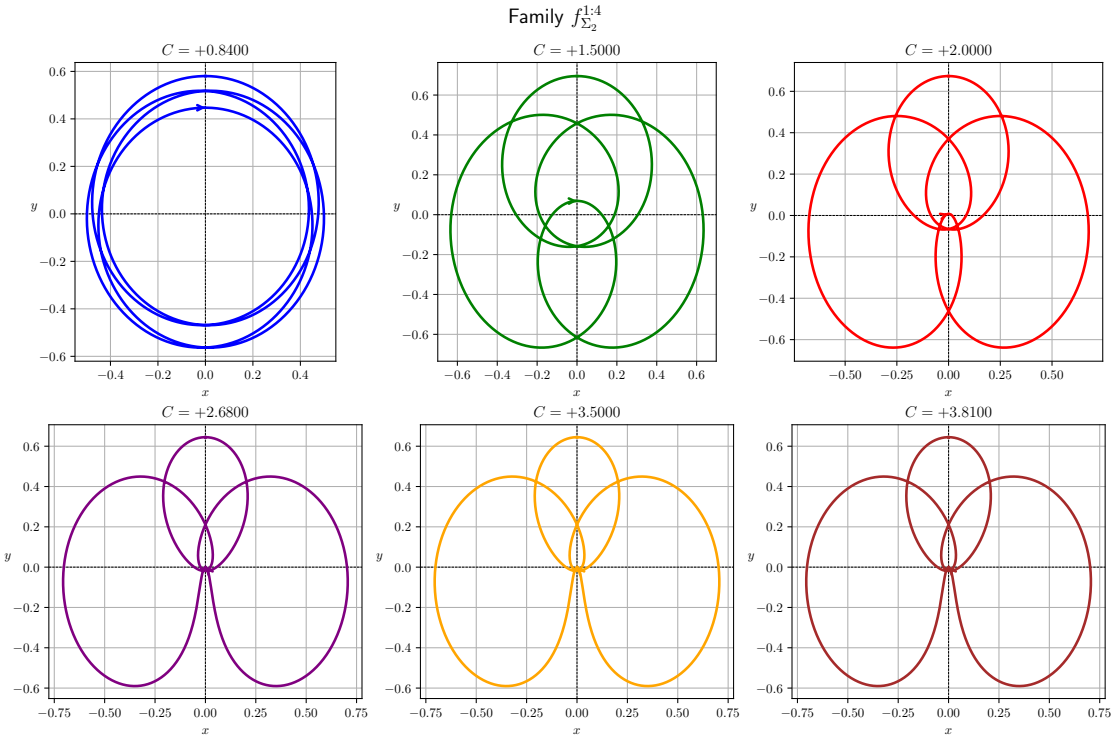
Example of asymmetric bridge between 4th covering of f





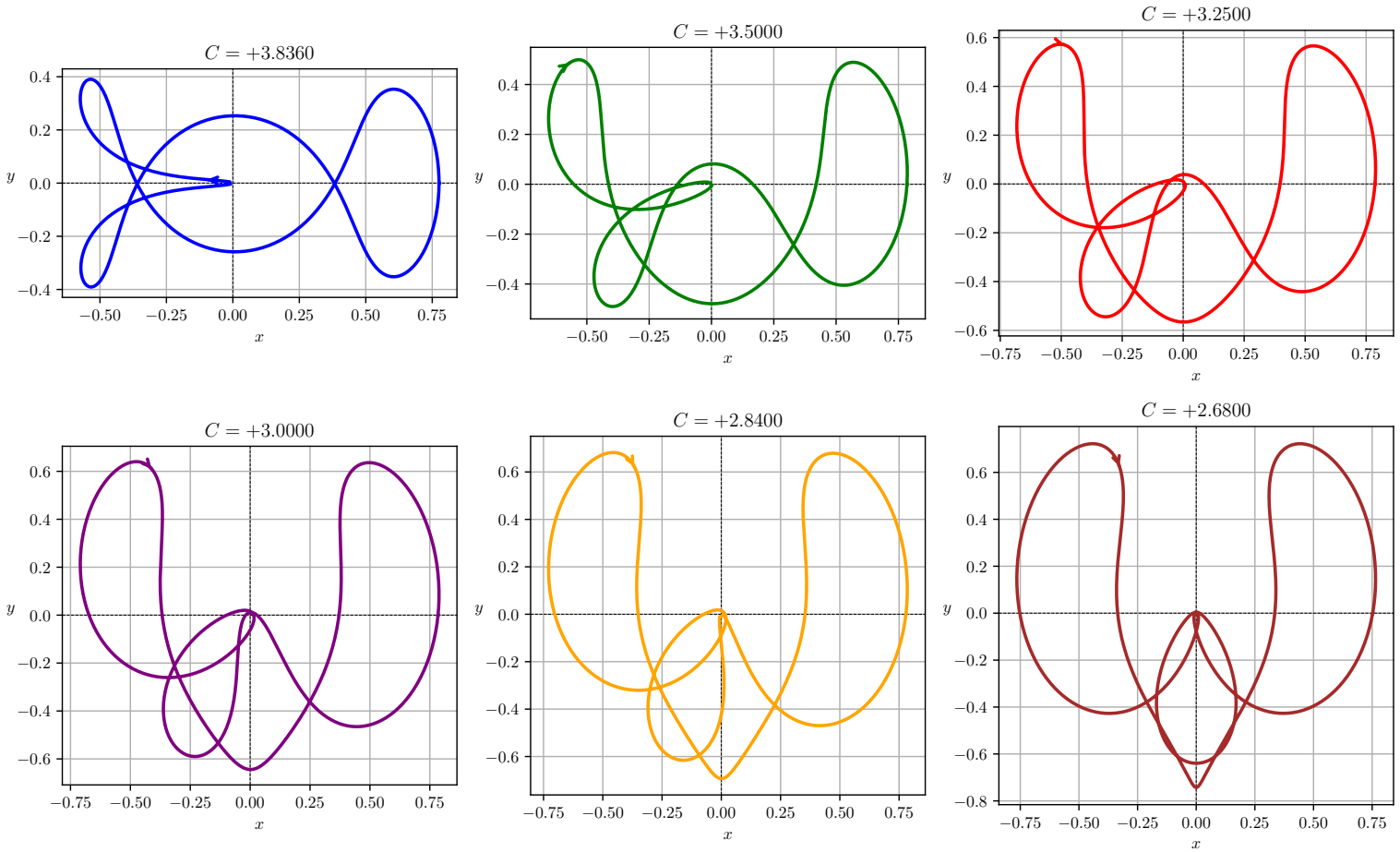
Example of asymmetric bridge between 4th covering of f





Example of asymmetric bridge between 4th covering of f

Family $f^{1:4}_{\Sigma_1} Asym$



Current bifurcation map

$\mathfrak{s}$	Linear case	Symmetry	Local outcome
1	JB	none or axial	square-root branch; fold or crossing
1	semisimple	axial	loss of symmetry, as in $g' \to Hh$
1	semisimple	central or double	axial and possible mixed branches
2	JB	none or one axial	period-doubling branch $U \sim \rho^{1/2}$
2	semisimple	central or double	support recomputation in progress
3	semisimple	asymmetric	one triple-period family
≥ 4	semisimple	asymmetric	two $\mathfrak{s}T$-periodic families

The inverse phase factor determines how the stationary points assemble into physical families and which symmetries survive.

Conclusions and next problems

Conclusions

- The corrected canonical transformation makes the resonant normal form autonomous
- The denominator $\mathfrak{s}$ selects the RNF support; the numerator $\mathfrak{r}$ returns in phase reconstruction
- Original-system symmetries must be conjugated through (***)
- Power geometry identifies the leading stationary germs and bifurcating periodic families

Next problems

Complete the classification for $\mathfrak{s} > 2$ and all symmetry classes. In the Hill problem, search for centrally symmetric and asymmetric planar families, then continue the search for asymmetric spatial solutions

THANKS FOR ATTENTION

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