On the graph transform method to compute invariant manifolds attached to whiskered invariant tori on Poincaré Maps

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Introduction

- Invariant tori and their associated manifolds organize the long term behavior of quasi-periodically forced systems. In celestial mechanics, they can provide opportunities for bounded motion and transport between different regions.

- Our goal is to develop a numerical method to compute high-order parameterizations of invariant manifolds attached to **reducible invariant tori of skew-products** associated to Poincaré maps.

- We focus on center manifolds and we include the computation of the reduced dynamics to the invariant manifold.

- In particular, we decide to represent the manifold as a graph over a selected spectral subspace and compute its Fourier–Taylor coefficients recursively using jet transport and solving a set of cohomological equations.

- We illustrate the construction with a quasi-periodically forced pendulum and discuss its application to an oscillating solar sail in the Earth–Moon system.

The setting

- Consider a **quasi-periodically forced differential equation**

$$\dot{x} = X(x,t) = \widetilde{X}(x, \tilde{\omega} t), \qquad x \in \mathbb{R}^n,$$

where $\widetilde{X}$ is 1-periodic in each angular variable and $\tilde{\omega} \in \mathbb{R}^{d+1}$ has rationally independent components. Introducing the forcing phases gives the autonomous system

$$\dot{x} = \widetilde{X}(x, \tilde{\theta}), \qquad \dot{\tilde{\theta}} = \tilde{\omega}, \qquad \tilde{\theta} \in \mathbb{T}^{d+1}.$$

- Assume $\tilde{\omega}_0 > 0$ and write $\tilde{\theta} = (\tilde{\theta}_0, \theta)$. The **Poincaré section** $\Sigma = \{\tilde{\theta}_0 = 0\}$ has constant return time $T = 1/\tilde{\omega}_0$. If Φ_t denotes the extended flow, define

$$P(x,\theta) := \pi_x \Phi_T\big(x, (0,\theta)\big), \qquad \rho = \frac{1}{\tilde{\omega}_0}(\tilde{\omega}_1, \ldots, \tilde{\omega}_d).$$

- The return map on Σ is a **quasi-periodic skew-product**

$$\begin{cases} \bar{x} = P(x,\theta), \\ \bar{\theta} = \theta + \rho \pmod 1, \end{cases} \qquad (x,\theta) \in \mathbb{R}^n \times \mathbb{T}^d,$$

with $\langle k, \rho \rangle \notin \mathbb{Z}$ for every $k \in \mathbb{Z}^d \setminus \{0\}$.

Invariant tori and reducibility

- A map $\varphi : \mathbb{T}^d \to \mathbb{R}^n$ satisfying

$$P(\varphi(\theta), \theta) = \varphi(\theta + \rho)$$

defines an invariant torus

$$\mathcal{T} = \{(\varphi(\theta), \theta) : \theta \in \mathbb{T}^d\},$$

whose internal dynamics is the rigid rotation $\theta \mapsto \theta + \rho$.

- Linearizing the fiber equation along $\mathcal{T}$ gives the normal variational dynamics

$$\begin{cases} \bar{v} = A(\theta)v, \\ \bar{\theta} = \theta + \rho, \end{cases} \qquad A(\theta) := \mathrm{D}_x P(\varphi(\theta), \theta).$$

This linear skew-product is the quasi-periodic cocycle associated with the torus.

- The cocycle is reducible if a smooth map $C : \mathbb{T}^d \to \mathrm{GL}(n, \mathbb{R})$ and a constant matrix $B \in \mathbb{R}^{n \times n}$ satisfy

$$A(\theta)C(\theta) = C(\theta + \rho)B.$$

Then, the Floquet transformation $v = C(\theta)u$ gives

$$\bar{u} = Bu, \qquad \bar{\theta} = \theta + \rho.$$

B is the **Floquet matrix**, and its eigenvalues are the **Floquet multipliers**.

Invariant tori and reducibility

- Reducibility allows the linear normal behavior of the torus to be studied using the standard approach for fixed points.
- Each B-invariant subspace $E \subset \mathbb{C}^n$ defines an invariant bundle $C(\theta)E$ along the torus. In particular,

$$\mathcal{E}^\star(\theta) = C(\theta)E_B^\star, \qquad \star \in \{s, c, u\},$$

where E_B^s, E_B^c, and E_B^u are the spectral subspaces associated with Floquet multipliers of modulus < 1, $= 1$, and > 1, respectively.

Statement of the problem

- Given a reducible invariant torus parameterized by φ, we seek a local invariant manifold attached to it and tangent to a selected invariant bundle $C(\theta)E_L$, where E_L is a B-invariant subspace of dimension n_L.

- Let $\mathcal{D} \subset \mathbb{C}^{n_L}$ be a neighborhood of the origin. We seek a **parameterization** and an **internal map**

$$K : \mathcal{D} \times \mathbb{T}^d \to \mathbb{C}^n, \qquad f : \mathcal{D} \times \mathbb{T}^d \to \mathbb{C}^{n_L},$$

satisfying the **invariance equation**

$$\boxed{P\big(K(s,\theta),\theta\big) = K\big(f(s,\theta),\theta+\rho\big).}$$

The induced dynamics in the reduced coordinates is

$$\bar{s} = f(s,\theta), \qquad \bar{\theta} = \theta + \rho.$$

- The manifold $K(\mathcal{D} \times \mathbb{T}^d)$ is attached to the torus and tangent to the selected bundle if

$$K(0,\theta) = \varphi(\theta), \qquad f(0,\theta) = 0, \qquad \mathrm{D}_s K(0,\theta) = C(\theta)E_L.$$

- The corresponding manifold in the extended phase space is

$$\mathcal{K} = \{(K(s,\theta),\theta) : (s,\theta) \in \mathcal{D} \times \mathbb{T}^d\}.$$

Adapted coordinates and spectral selection

- Choose a B-**invariant splitting**

$$\mathbb{C}^n = E_L \oplus E_N, \qquad \dim E_L = n_L, \quad \dim E_N = n_N.$$

With bases V_L, V_N of these subspaces,

$$V = [V_L \mid V_N], \qquad V^{-1} B V = \Lambda = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_N \end{pmatrix}.$$

- E_L selects the directions tangent to the desired manifold, while E_N contains the complementary directions. For example:

Manifold	E_L	E_N
Stable	E_B^s	$E_B^c \oplus E_B^u$
Unstable	E_B^u	$E_B^s \oplus E_B^c$
Center	E_B^c	$E_B^s \oplus E_B^u$

- The adapted frame along the reducible invariant torus is

$$Q(\theta) := C(\theta) V = [Q_L(\theta) \mid Q_N(\theta)],$$

that reduces the cocycle to

$$A(\theta) Q(\theta) = Q(\theta + \rho) \Lambda.$$

Graph representation and normalization

- Introduce the affine change of coordinates

$$x = \Psi_\theta(s, r) := \varphi(\theta) + Q_L(\theta)s + Q_N(\theta)r.$$

The fiber dynamics of P in these coordinates is

$$\Upsilon(s, r, \theta) := Q(\theta + \rho)^{-1}\big[P(\Psi_\theta(s, r), \theta) - \varphi(\theta + \rho)\big].$$

- From torus invariance and frame reducibility

$$\Upsilon(0, 0, \theta) = 0, \quad \mathrm{D}\Upsilon(0, 0, \theta) = \Lambda,$$

and splitting in components

$$\begin{cases} \bar{s} = \Upsilon_L(s, r, \theta) = \Lambda_L s + F(s, r, \theta), \\ \bar{r} = \Upsilon_N(s, r, \theta) = \Lambda_N r + G(s, r, \theta), \\ \bar{\theta} = \theta + \rho, \end{cases} \qquad F, G = \mathcal{O}(\|(s, r)\|^2).$$

Graph representation and normalization

- We represent the desidered invariant manifold as a **graph**

$$r = W(s, \theta), \qquad W : \mathcal{D} \times \mathbb{T}^d \to \mathbb{C}^{n_N}.$$

Thus, the corresponding parameterization in the original fiber variables is

$$K(s, \theta) = \varphi(\theta) + Q_L(\theta)s + Q_N(\theta)W(s, \theta),$$

and the internal dynamics is given by

$$f(s, \theta) := \Upsilon_L(s, W(s, \theta), \theta) = \Lambda_L s + F(s, W(s, \theta), \theta).$$

- The graph is invariant when $\bar{r} = W(\bar{s}, \bar{\theta})$:

$$\Upsilon_N(s, W(s, \theta), \theta) = W(f(s, \theta), \theta + \rho).$$

Equivalently,

$$\boxed{\Lambda_N W(s, \theta) + G(s, W(s, \theta), \theta) = W(f(s, \theta), \theta + \rho).}$$

- The normalization of the graph is

$$W(0, \theta) = 0, \qquad \mathrm{D}_s W(0, \theta) = 0.$$

Solving the graph invariance equation

- Expand the graph in **homogeneous Taylor polynomials**:

$$W(s,\theta) = \sum_{l\geq 2} W_{(l)}(s,\theta), \qquad W_{(l)}(s,\theta) = \sum_{|\alpha|=l} w_{l,\alpha}(\theta) s^{\alpha}.$$

  Each coefficient has a Fourier expansion

$$w_{l,\alpha}(\theta) = \sum_{k\in\mathbb{Z}^d} \widehat{w}_{l,\alpha,k} e^{2\pi i\langle k,\theta\rangle}.$$

- The internal dynamics has the corresponding expansion

$$f(s,\theta) = \Lambda_L s + \sum_{l\geq 2} f_{(l)}(s,\theta).$$

- At degree l, assume that

$$W^{<l}(s,\theta) := \sum_{q=2}^{l-1} W_{(q)}(s,\theta)$$

  is known and satisfies graph invariance through degree $l-1$. Matching degree-l terms determines $W_{(l)}$ and $f_{(l)}$.

The recursive equation at degree l

- Evaluate the transformed map on the known graph:

$$\Upsilon^{<l}(s,\theta) := \Upsilon(s, W^{<l}(s,\theta),\theta).$$

Denote extraction of the homogeneous degree-l term by $[\cdot]_{(l)}$. The known mismatch is

$$Z_{(l)}(s,\theta) := \left[W^{<l}\left(\Upsilon_L^{<l}(s,\theta),\theta+\rho\right) - \Upsilon_N^{<l}(s,\theta)\right]_{(l)}.$$

- Since F, G start at degree two, the new term satisfies the **linear equation**

$$\boxed{\Lambda_N W_{(l)}(s,\theta) - W_{(l)}(\Lambda_L s,\theta+\rho) = Z_{(l)}(s,\theta).}$$

The internal dynamics at this degree is

$$f_{(l)}(s,\theta) = \left[\Upsilon_L^{<l}(s,\theta)\right]_{(l)}.$$

- All nonlinear terms on the right-hand side depend only on previously computed graph coefficients.

Sketch of the algorithm

Input: P, ρ, φ, the adapted frame Q, the spectral blocks Λ_L, Λ_N, and a truncation order J.

Initialize $W = 0$ and $f(s,\theta) = \Lambda_L s$. For $l = 2, \dots, J$:

1. **Jet transport and projection.** Form $K^{<l} = \varphi + Q_L s + Q_N W^{<l}$ and compute, through degree l,

$$Q(\theta+\rho)\Upsilon^{<l}(s,\theta) = P(K^{<l}(s,\theta),\theta) - \varphi(\theta+\rho).$$

2. **Composition.** Compute $Z_{(l)}$ from the shifted graph composition and the complementary component $\Upsilon_N^{<l}$.

3. **Cohomological solve.** Assume $\Lambda_L = \mathrm{diag}(\lambda_{L,1}, \dots, \lambda_{L,n_L})$. For each $|\alpha| = l$, solve

$$\Lambda_N w_{l,\alpha}(\theta) - \beta_\alpha w_{l,\alpha}(\theta+\rho) = z_{l,\alpha}(\theta), \qquad \beta_\alpha = \prod_{j=1}^{n_L} \lambda_{L,j}^{\alpha_j}.$$

4. **Update.** Append $W_{(l)}$ and $f_{(l)} = [\Upsilon_L^{<l}]_{(l)}$ to the computed expansions.

Numerical implementation details

- **Taylor-grid representation.** Store the Taylor coefficients on a uniform grid $\{\theta_i\}_{i=1}^{M} \subset \mathbb{T}^d$. At each phase θ_i, independently:
  - Form the polynomial jet $K^{<l}(s, \theta_i) = \varphi(\theta_i) + Q_L(\theta_i)s + Q_N(\theta_i)W^{<l}(s, \theta_i)$.
  - Propagate it through the flow defining P by **jet transport**, retaining terms through degree l.
  - Project onto the adapted frame and perform polynomial compositions to assemble $Z_{(l)}$.

- **Taylor–Fourier representation.** We us FFTs to convert grid values into Fourier coefficients. Rotations are performed in Fourier

$$w(\theta + \rho) \quad \longleftrightarrow \quad e^{2\pi i \langle k, \rho \rangle} \widehat{w}_k.$$

- **Cohomological solve.** For diagonal Λ_L, solve independently for each (α, k):

$$\left(\Lambda_N - \beta_\alpha e^{2\pi i \langle k, \rho \rangle} I\right) \widehat{w}_{l,\alpha,k} = \widehat{z}_{l,\alpha,k}, \qquad \beta_\alpha = \prod_j \lambda_{L,j}^{\alpha_j}.$$

If $\beta_\alpha \notin \mathrm{spec}(\Lambda_N)$ then no small divisors appear and W can be uniquely determined. For center-hyperbolic splitting, this condition is trivially satisfied.

Local domain of validity

Assume K and f are Taylor truncations up to degree J. **Goal:** to numerically estimate the accuracy of these expansions and to determine the domain of validity where they provide reliable approximations.

- Starting from $x_0 = K(s_0, \theta_0)$, compare the original and reduced dynamics:

$$x_{n+1} = P(x_n, \theta_n), \qquad s_{n+1} = f(s_n, \theta_n), \qquad \theta_n = \theta_0 + n\rho.$$

- The **invariance defect**:

$$e_I(s_0, \theta_0, n) := \|P(K(s_n, \theta_n), \theta_n) - K(f(s_n, \theta_n), \theta_n + \rho)\|_\infty .$$

  The **orbit discrepancy**:

$$e_O(s_0, \theta_0, n) := \|K(s_n, \theta_n) - x_n\|_\infty .$$

- For finite meshes $\mathcal{Q}_R \subset \{s : \|s\|_\infty \leq R\}$ and $\mathcal{G} \subset \mathbb{T}^d$, define

$$E_m(R) := \max_{\substack{s_0 \in \mathcal{Q}_R \\ \theta_0 \in \mathcal{G}}} \max_{0 \leq n \leq n_{\text{iter}}} e_m(s_0, \theta_0, n), \qquad m \in \{I, O\}.$$

  We use the largest tested R with $E_m(R) \leq \varepsilon_{\text{tol}}$ as an estimation of the practical domain of validity.

Example: quasi-periodically forced pendulum

- Consider a forced pendulum:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\alpha \sin x + \varepsilon\zeta(\tilde{\theta}), \\ \dot{z} = -\beta z + \varepsilon\eta(x, \tilde{\theta}), \\ \dot{\tilde{\theta}}_j = \omega_j, \qquad j = 0, 1, 2, \end{cases} \qquad \tilde{\theta} \in \mathbb{T}^3,$$

with $\zeta(\tilde{\theta}) = \left[5 + \sum_j \cos(2\pi\tilde{\theta}_j)\right]^{-1}$, $\eta(x, \tilde{\theta}) = (1 - \cos x)/\zeta(\tilde{\theta})$.

- We choose the parameters

$$\alpha = -\beta = \frac{1}{2}, \qquad \varepsilon = 0.015, \qquad (\omega_0, \omega_1, \omega_2) = (1, \sqrt{2}, \sqrt{3}).$$

- The **Poincaré map** on $\tilde{\theta}_0 = 0$ has return time

$$T = \frac{1}{\omega_0} = 1, \qquad \theta = (\tilde{\theta}_1, \tilde{\theta}_2), \qquad \rho = (\sqrt{2}, \sqrt{3}) \pmod 1.$$

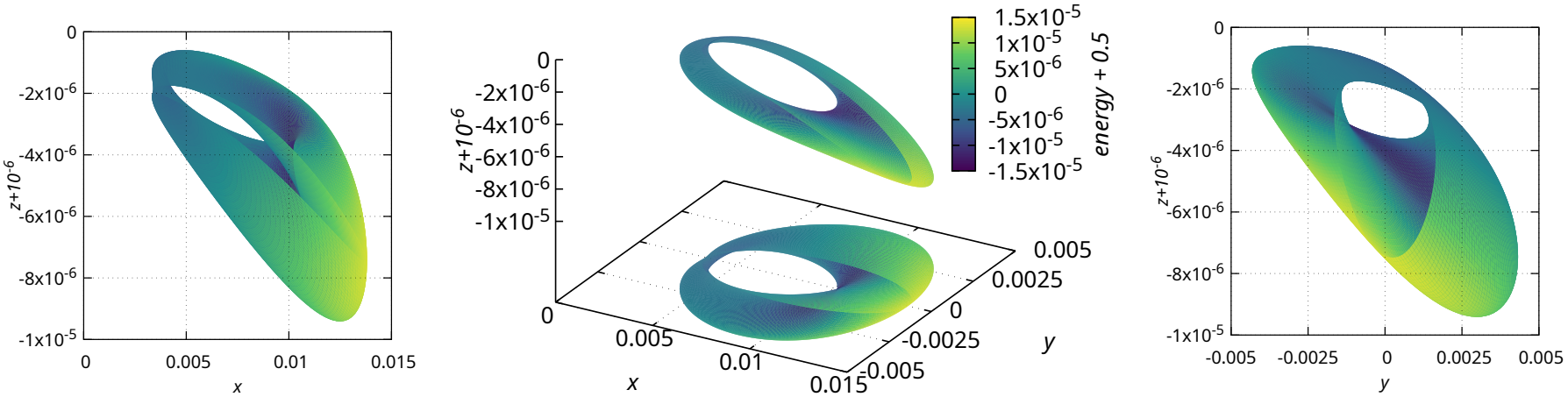
We seek the center manifold attached to the response 2-torus near the origin.

The response torus

- We compute the **response** 2-**torus** $\varphi : \mathbb{T}^2 \to \mathbb{R}^3$ near the origin, satisfying

$$P(\varphi(\theta), \theta) = \varphi(\theta + \rho), \qquad \rho = (\sqrt{2}, \sqrt{3}) \pmod 1.$$

- The computation uses 32 Fourier modes in each angular variable. The Floquet transformation $C(\theta)$ and the matrix B are computed simultaneously.



Projections of $\varphi(\theta)$ in the physical coordinates (x, y, z), colored by the energy-like function

$$H_\varepsilon(x, y; \tilde{\theta}) = \frac{y^2}{2} - \alpha \cos x - \varepsilon x\, \zeta(\tilde{\theta}), \qquad \tilde{\theta} = (0, \theta).$$

Application of the algorithm

- The spectrum of the Floquet matrix B consists of

$$\lambda_{1,2} = e^{\pm i\phi}, \qquad \lambda_3 \approx 23.1.$$

- We select the center directions:

$$E_L = E_B^c, \qquad E_N = E_B^u, \qquad n_L = 2, \quad n_N = 1,$$

such that,

$$\Lambda_L = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \qquad \Lambda_N = (\lambda_3).$$

- We compute a scalar graph $r = W(s_1, s_2, \theta)$ and the internal dynamics $f(s, \theta)$. Since $|\beta_\alpha| = 1$, the cohomological equations has no small divisors.

Computational performance

We report wall-clock times for computing the graph, keeping the Fourier resolution fixed.

Computational time versus Taylor order: 8 threads

Taylor order J	8	10	15	20
Time (s)	54.84	88.26	198.20	324.07

Computational time versus number of threads: order 16

Threads	1	2	4	8
Time (s)	472.72	365.17	267.55	185.34
Speedup	1.00	1.29	1.77	2.55

Jet transport and the rest of polynomial operations over the angular grid are parallelized.

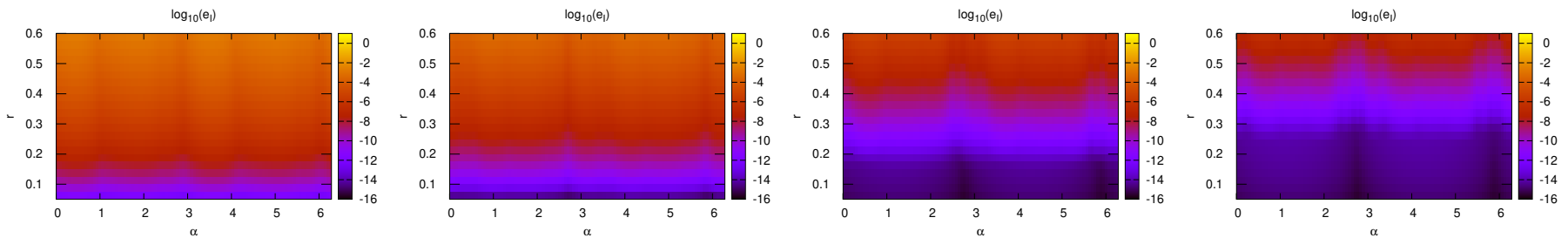
Fundamental domain

We estimate the **local domain of validity** using the one-step invariance defect of the order-J parameterization:

$$e_I(s, \theta, n) = \|P(K^{\leq J}(s, \theta), \theta) - K^{\leq J}(f^{\leq J}(s, \theta), \theta + \rho)\|_\infty,$$

with $n = 1$.
For the conjugate center coordinates, take

$$s(r, \alpha) = (re^{i\alpha}, re^{-i\alpha}), \quad r \in (0, 1], \quad \alpha \in [0, 2\pi) \quad \mathcal{E}_I(r, \alpha) := \max_{\theta \in \mathcal{G}} \|e_I(s, \theta, 1)\|_\infty.$$



$\log_{10} \mathcal{E}_I(r, \alpha)$ for $J = 8, 10, 15, 20$. Horizontal axis: α. Vertical axis: r.

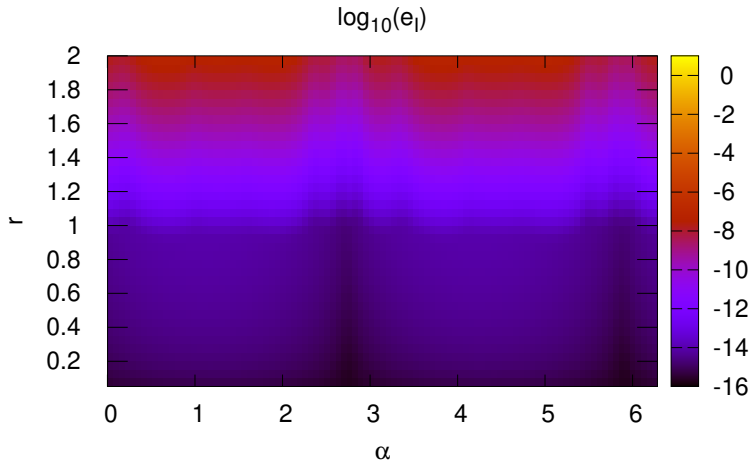
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$\log_{10} \mathcal{E}_I(r, \alpha)$ at order $J = 20$ after rescaling $V_L \to r \cdot V_L$.

The region where $\mathcal{E}_I(r, \alpha) \leq \varepsilon_{\text{tol}}$ gives a numerical estimate of the usable domain.

Dynamics on the center manifold

- Choose initial conditions on the real slice of the center coordinates:

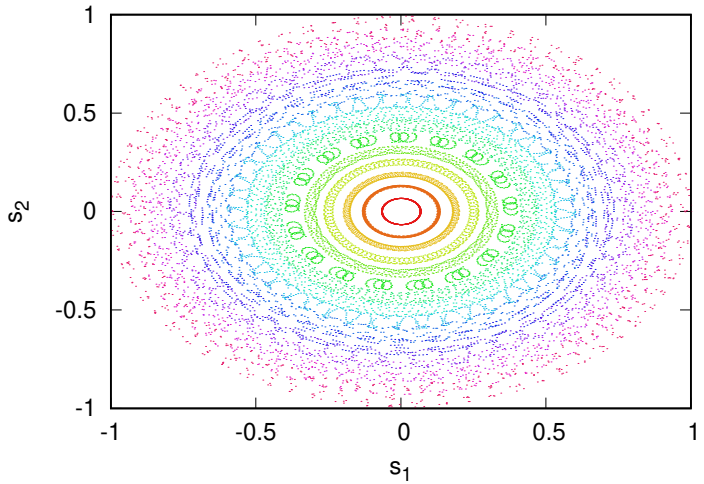
$$s^{(0)} = (re^{i\alpha}, re^{-i\alpha}), \qquad \theta^{(0)} = \theta_0.$$

- Iterate the computed internal map and lift the trajectories to the original coordinates:

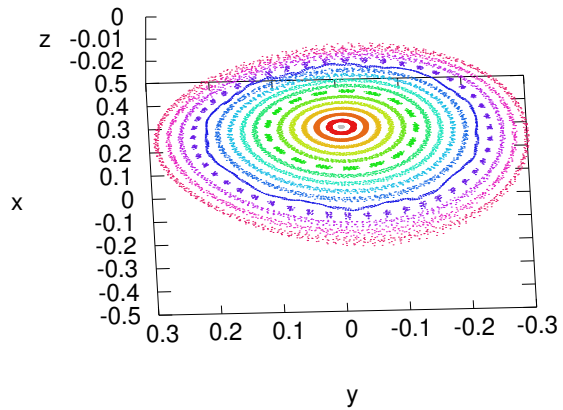
$$s^{(k+1)} = f(s^{(k)}, \theta^{(k)}), \qquad \theta^{(k+1)} = \theta^{(k)} + \rho, \qquad x^{(k)} = K(s^{(k)}, \theta^{(k)}).$$

The reduced orbit must remain within the estimated validity domain.

Reduced coordinates



Original coordinates (x, y, z)



20 initial conditions with $r_j = 0.02j$, $\alpha = 0$, $\theta_0 = (0, 0)$, and 1000 iterates. Color indicates the initial radius r_j.

Comments on celestial mechanics application

- Consider the **planar bicircular problem icnluding solar radiation pressure**. The sail orientation oscillates according to

$$\alpha(t) = A_{ss}\frac{\pi}{2}\sin(\omega_{ss}t + \psi_0).$$

The solar motion and sail oscillation introduce two forcing frequencies. If $\omega_{ss}/\omega_s \notin \mathbb{Q}$, the forcing is quasi-periodic.

- Taking the section at a fixed solar phase gives a **quasi-periodic Poincaré map**

$$\bar{x} = P(x, \theta), \qquad \bar{\theta} = \theta + \rho \pmod 1, \qquad T = \frac{2\pi}{\omega_s}, \quad \rho = \frac{\omega_{ss}}{\omega_s},$$

where $x = (x_1, x_2, v_1, v_2) \in \mathbb{R}^4$ and $\theta \in \mathbb{T}$ is the normalized sail phase.

- Obtain the dynamical equivalent of a collinear libration point now corresponding to a **response torus** together with its Floquet reduction. For a center–saddle spectrum, select

$$E_L = E_B^c, \qquad E_N = E_B^s \oplus E_B^u, \qquad n_L = n_N = 2.$$

- Apply the graph algorithm to obtain K and the reduced map f. These provide local coordinates to explore nearby bounded motion and its dependence on the sail parameters.

Next steps & Open questions

- Avoid complex arithmetic to exploit the conjugate structure of the eigenvectors to work entirely in $\mathbb{R}$ and saving memory space.
- Replace the single-period integration in the jet transport step by multiple-shooting integrations when $|\lambda_N| \gg 1$ or $|\lambda_N| \ll 1$.
- Considering the Bicircular Problem with oscillating solar sail: study how the center manifold geometry changes with sail parameters.
- Use these invariant objects to identify natural transport channels and design low-energy transfers near quasi-periodic orbits.