An improved result about the change of time in the equations of the dynamics

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Dedicated to the memory of
Àngel Jorba, Ken Meyer and Christian Marchal

Abstract. The planar harmonic oscillator may be mapped onto the Kepler problem in such a way that the centered ellipses are sent onto Keplerian ellipses. The body makes two turns on the Keplerian orbit with a non-Keplerian law of time. This map has many surprising features.

At the end of the 19th century, a vaguely stated conjecture emerged from this example and from Appell's projective dynamics (see [1]). Thomas [2] published a proof in 1946. His conclusion is stated in a technical way. I will reformulate and prove the conjecture in the following form.

If there is a map from a "natural" mechanical system onto another natural mechanical system where the time is changed from s to t, then ds/dt is a function of the position optionally multiplied or divided by the square root of a quadratic first integral.

My "natural" class includes the systems with a quadratic kinetic energy and a potential. It is wider than the one chosen by Thomas, and gives a simpler proof.

[1] Painlevé P., Sur les transformations des équations de la dynamique, Comptes Rendus, 1896, vol. 123, pp. 392-395.

[2] Thomas, T.Y. On the Transformation of the Equations of Dynamics, Journal of math. and phys., 1946, vol. 25, pp. 191-208.

Note on my abstract. Indeed pages 1 to 9 of Thomas' paper give the proof of the generalization I am presenting here. My proof is shorter, but it is essentially the same. The main point of my talk is to give the statement of this generalization (which Thomas never does) and to observe that it fits with simple classes of examples.

The connection-force systems as generalized natural systems. The ingredients are a linear torsion free connection ∇ on the configuration space and a force field f (which depends on the position only). If the Γ^k_{ij} are the Christoffel symbols of the connection, q^i the coordinates of the position q, f^i the coordinates of the force field f, the system of differential equations is

$$\ddot{q}^k = \sum_{ij} \Gamma^k_{ij} \dot{q}^i \dot{q}^j + f^k,$$

which we write with a traditional abusive notation

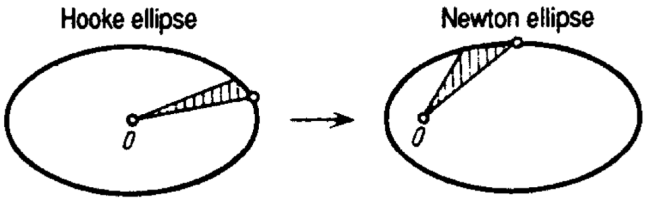
$$\nabla_{\dot{q}} \dot{q} = f(q).$$

We call the *pre-trajectories* the unparametrized curves described by the "body" q in the configuration space.

Theorem. If two connection-force systems define the same pre-trajectories in the same configuration space, if moreover the parallel transport of each connection is volume preserving, then the change of time from s to t is defined by a ds/dt which is a function of the position only, or is a function of the position multiplied by the square root of a quadratic first integral of the second system.

Remarks. The conjecture which became Thomas' theorem is about the case where each parallel transport preserve a Riemannian form. It is a stronger hypothesis. I always found the conjecture strange since the chosen equation uses a Riemannian structure on the left hand side but not on the right hand side (the force is not derived from a potential).

Example. The 2D linear oscillator and Kepler. Following Levi-Civita, this map is used as a regularization of the binary collisions in the planar n-body problem.



Complex map $z \mapsto z^2$

Let me choose a 2D-linear oscillator with period 4π. The image of the 2D-linear oscillator is a natural system. Its orbits are the Keplerian orbits with a change of time. The new time is the eccentric anomaly u. All the orbits have period 2π.

Recall that the Kepler equation $a^{-3/2}(t - t_0) = u - e \sin u$ together with the formula $r = a(1 - e\cos u)$ gives

$$\frac{du}{dt} = \frac{1}{r\sqrt{a}} = \frac{\sqrt{-2H}}{r}.$$

A more general example. Denote by q the position in a configuration space, and v the velocity vector. Let $T(q, v)$ be a "natural" kinetic energy: a positive definite quadratic form for any fixed q. Let $U(q)$ be a nonzero potential. The following theorem defines **the Darboux inverses** of the natural Lagrangian $T + U$.

Theorem (Darboux 1889). The natural Lagrangian

$$T(q, v) + U(q)$$

has on the domain of positive energy the same pre-trajectories as the Lagrangian

$$U(q)T(q, v) + \frac{1}{U(q)},$$

and on the domain of negative energy the same pre-trajectories as the Lagrangian

$$-U(q)T(q,v)+\frac{1}{U(q)}.$$

Example. The planar Kepler problem and its Darboux inverse for negative energy are respectively defined by the Lagrangians

$$\frac{\dot x^2+\dot y^2}{2}+\frac{1}{r}, \qquad -\frac{\dot x^2+\dot y^2}{2r}+r.$$

The second Lagrangian appears to be the image of the 2D linear oscillator by the $z\mapsto z^2$ map (for some choice of the coefficients).

Proof by the Jacobi-Maupertuis metric (Darboux).

Let $(\mathcal{M},g,U)$ be a natural mechanical system. The energy is $H=\langle g,\dot q\otimes\dot q\rangle/2-U$. The unparametrized trajectories of energy $H=h$ are pre-geodesics of the metric:

$$(U+h)g. \tag{1}$$

We can change g into λg, where λ is a conformal factor depending on position, while keeping the metric (1). Then, the pre-geodesics are unchanged. The idea is very simple. Write:

$$(1+\frac{h}{U})(Ug). \tag{2}$$

So, the pre-trajectories are the same as in the natural mechanical system $(\mathcal{M},Ug,h/U)$ with energy 1. We can divide by a constant factor, and write:

$$-(\frac{1}{h}+\frac{1}{U})(Ug). \tag{3}$$

So the system is now $(\mathcal{M},Ug,-1/U)$, the energy value is $-1/h$. QED

Nersessian and Pogosyan found in 2001 elegant examples of Darboux inversions which are not flat. With my co-author Lei Zhao we considered the Darboux inverses of the curved Kepler problem. We found that they give all the analytic surfaces of revolution endowed with a rotationally invariant potential with an open set filled up with periodic orbits. The same class has an interesting property: a "Lambert like" theorem.

[3] Albouy, A., Zhao L., *Darboux inversions of the Kepler problem*, Regular and Chaotic Dynamics, 2022, vol. 27, pp. 253-280.

[4] Nersessian, A., Pogosyan, G., *On the Relation of the Oscillator and Coulomb Systems on (Pseudo)Spheres*, Phys. Rev. A, 2001, vol. 63, 020103, 4 pp.

A forgotten theory. The theory of the "Transformation of the equations of dynamics" was concluded by Painlevé in 1896 and Thomas in 1946. The subject is obviously important: to classify the transformations that send a "natural" mechanical system onto another one, the changes of time being allowed. But this theory was soon misunderstood: Whittaker in his *Treatise* presents it as an exercise, of which he gives the solution... which is obviously wrong. The theory was then universally forgotten. We claim that the reasons for this bad reception are:

- A hesitation about the hypothesis: "natural" forces are they derived from a potential?
- A too complicated conclusion, unrelated with the classical examples, which are not even recalled.
- The lack of examples.

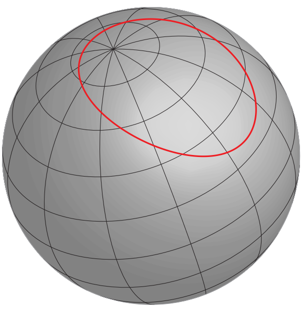
We improve the situation by discussing a 3rd hypothesis, by relating the conclusion to the examples and by giving new examples.

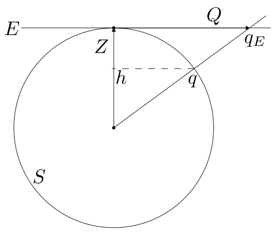
[5] Whittaker, E. T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge, 1937, p. 261.

First class of examples

The first example is this pair: *Kepler problem in the plane* and *Kepler problem on the sphere*. They are related by *Appell's central projection*, which sends orbit onto orbit with a change of time depending on position only.

Charles Graves (1812–1899) and Wilhelm Killing (1847–1923) discovered impressive analogies between the Kepler problem in constant curvature spaces and the usual Kepler problem.





Second class of examples

The second example is this pair: *Hooke problem (harmonic oscillator in the plane)* and *Kepler problem in the plane.* They are related by the conformal map, written $z\mapsto z^2$ in the complex coordinate z of the plane. The change of time depends on the position *and on the energy*. The energy of Kepler is restricted to be negative. Energy k of Hooke is sent on energy $h=-1/k$ of Kepler. Here in the figure from V.I. Arnold's book of 1990.

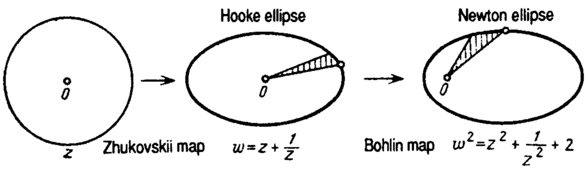


Fig. 35.
The ellipses of Hooke and Newton

The velocity of the image is the initial velocity such that the chosen curve will be described. It is essentially unique, except for a zero force field (geodesic flow).

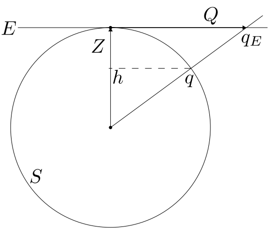
The question

The preferred class will be of *natural mechanical systems* which consist in triples $(\mathcal{M}, g, U)$ where: the manifold $\mathcal{M}$ is the configuration space, the Riemannian metric g defines the kinetic energy, the *force function* U defines the force field, which is the gradient of U with respect to the metric g. The expected result is as follows:

If two natural mechanical systems correspond with each other up to change of time, i.e. if there is a map (at least locally) such that all the trajectories of one are sent onto trajectories of the other after a change of time parameter, then we are in one of both classes.

Of course, we still have to describe the two classes. But, we should insist on one main distinction between them: in the first class, the change of time depends on the position only. In the second class, it depends *on the position and on a first integral*, the energy, which involves of course the velocities. It is important to think of the effect of this distinction when the velocity goes to infinity.

Appell's central projection.



Appell's fundamental result is not restricted to the framework of natural mechanical systems. The class is: systems defined on a configuration space by a force field and a connection ∇, which defines the second derivative of the position q. We only consider **torsion-free linear connections on the tangent space** of the configuration space. We say that such a force field (depending only on position) and such a connection define a **connection-force system**. It includes all the natural mechanical systems. It is written

$$\nabla_{\dot{q}}\dot{q} = f(q).$$

Examples of pairs of natural mechanical systems related by Appell's projection.

- Kepler on the plane and on the sphere
- Central force on the plane and on the sphere
- Two fixed centers problem on the plane and on the sphere
- Lagrange's extension and Kozlov's extension of the previous
- Neumann system and Braden system on the *n*-dim sphere

All the examples are integrable and indeed the correspondence induces a quasi-bi-Hamiltonian structure by exchanging the Hamiltonian structures of both natural mechanical systems.

Second class by successive extensions

- $z \mapsto z^2$ sending Hooke onto Kepler (Maclaurin 1742)
- $z \mapsto z^n$ sending a central force onto the dual one (Maclaurin)
- $z \mapsto f(z)$, general conformal map in the plane (Goursat 1889)
- Take $(\mathcal{M}, g, U)$, change g into Ug, U into $\frac{1}{U}$ (Darboux 1889)
- The same **out of** the framework of natural mechanical systems

Suppose that a **connection-force system** $(\mathcal{M}, \nabla, f)$, with equation $\nabla_{\dot{q}}\dot{q} = f(q)$, has a quadratic first integral, which is a quantity

$$G = \frac{1}{2}\langle k, \dot{q} \otimes \dot{q}\rangle - W,$$

where k is a scalar product (field of quadratic forms) and W a function of q. We have

$$\dot{G} = \frac{1}{2}\langle \nabla_{\dot{q}} k, \dot{q} \otimes \dot{q}\rangle + \langle k, f \otimes \dot{q}\rangle - \langle dW, \dot{q}\rangle = 0.$$

The *natural case* is when $\nabla k = 0$ and $f = \mathrm{grad}_k W$. Then G is the energy and W is the potential. But we begin our considerations with the general case.

Second class: the final extension

So, the equation is $\nabla_{\dot{q}}\dot{q} = f(q)$ and the quadratic first integral is

$$G = \frac{1}{2}\langle k, \dot{q} \otimes \dot{q}\rangle - W.$$

Suppose the convention of constant in G is such that the domain we will study is defined by $G < 0$. Consider the change of velocity vector $\dot{q} = \sqrt{-2G}q'$. Then G is expressed as:

$$G = -G\langle k, q' \otimes q'\rangle - W \quad \text{or} \quad G(1 + \langle k, q' \otimes q'\rangle) = -W$$

or

$$-\frac{1}{G} = \langle \frac{k}{W}, q' \otimes q'\rangle + \frac{1}{W}.$$

Consider now the equation of motion. It is

$$\nabla_{q'} q' = -\frac{f}{2G} = \langle \frac{k}{2W}, q' \otimes q'\rangle f + \frac{f}{2W}$$

This is again a connection-force problem with a quadratic first integral!

There are only these two classes

Suppose that systems $\nabla_v v = f(q)$ and $D_w w = \phi(q)$ have the same pre-trajectories.

As did Thomas, we exclude the points where $f(q) = 0$ or $\phi(q) = 0$. Thomas also observes that $f(q) = 0$ implies $\phi(q) = 0$.

Then there is a smooth function $\alpha(q, v) > 0$ such that $w = \alpha v$, where w is the unique velocity generating the same pre-trajectory as v with the same orientation of motion (indeed, due to the reversibility of both systems, we can choose the orientation.)

Now, notice that $v = 0$ means that the moving body q is stationary: it goes to a point q_0 and then goes back on its path. At q_0, the trajectory goes in the direction of the force vector. This direction should be the same for the second system. Consequently, there is a function $\beta(q) > 0$ such that $\phi(q) = \beta(q)f(q)$. It is quite intuitive from the dynamics that $\beta(q) = \alpha(q, 0)^2$. We will now deduce this from a computation.

Write $D_v = \nabla_v + A_v$ where A is a tensor field.

$$\phi = \nabla_w w + A_w w = \alpha\nabla_v(\alpha v) + A_w w = \alpha\dot{\alpha}v + \alpha^2\nabla_v v + A_w w.$$

Here $\dot{\alpha}$ is the derivative of α along the flow of the first system. So

$$\frac{\phi}{\alpha^2} = \frac{\dot{\alpha}}{\alpha}v + f + A_v v$$

or

$$A_v v = (\frac{\beta}{\alpha^2} - 1)f - \frac{\dot{\alpha}}{\alpha}v. \qquad (*)$$

The quantities α and β are nonzero and smooth. But this formula gives rational expressions with a denominator which vanishes somewhere. So, the denominator should be in factor of the numerator.

Let us see these rational expressions. Consider each coordinate of

$$v \wedge A_v v = (\frac{\beta}{\alpha^2} - 1)v \wedge f.$$

This gives many expressions of $\beta/\alpha^2 - 1$ which are rational in v with homogeneous cubic numerator and linear denominator. So

$$\frac{\beta}{\alpha^2} - 1 = k(q, v)$$

is a quadratic form in v. In particular, at $v = 0$, this is zero and so $\beta(q) = \alpha(q, 0)^2$. Note the important particular case where k is identically zero. The other coefficient $\dot{\alpha}/\alpha$ should be linear. We get the simple and strong result:

$$\frac{1}{\alpha(q, v)^2} = \frac{k(q, v)}{\beta(q)} + \frac{1}{\beta(q)} \quad \text{and} \quad \frac{\dot{\alpha}}{\alpha} \text{ is a linear form in } v.$$

We have these two striking conclusions, but we want to obtain a stronger result. We want to prove that there exist an "integrating factor" $\gamma(q)$, ithat is, a function such that $\gamma(q)\alpha(q, v)$ is a first integral of the first system. This would give

$$\frac{\dot{\alpha}(q, v)}{\alpha(q, v)} + \frac{\dot{\gamma}(q)}{\gamma(q)} = 0.$$

So, we have to deduce that $\dot{\alpha}/\alpha$ is an exact linear form. This double condition (to be a linear form in v and to be exact) does not change if we change $\alpha(q, v)$ into $\lambda(q)\alpha(q, v)^m$ where λ is a nonzero function and $m \in \mathbb{R}$ an arbitrary exponent. So we will rather prove that if

$$\kappa(q, v) = k(q, v) + 1$$

is such that $\dot{\kappa}/\kappa$ is a 1-form in v, this form is closed. The lowest degree term of $\dot{\kappa}$ gives an expression for the 1-form. It is

$$v \mapsto \hat{k}|_q(f, v),$$

where $\hat{k}$ is a notation for the bilinear form associated to the quadratic form k, defined by the formula $\hat{k}|_q(v, v) = k(q, v)$

Let us denote $\dot{\alpha}/\alpha = 2h$. We know that $h|_q$ is a linear form in velocity but we want more. With the new notation k and h equation $(*)$ is

$$A_v v = kf - 2hv.$$

If instead of a quadratic form in v we write a symmetric bilinear form in (u, v)

$$A_u v = \hat{k}|_q(u, v)f|_q - h|_q(u)v - h|_q(v)u.$$

Let n be the dimension of the configuration space. Let Ω be the contraction of the tensor A. We have

$$\Omega(v) = \hat{k}(f, v) - (n + 1)h(v).$$

According to the previous paragraph, $\hat{k}(f, v)$ and $h(v)$ are proportional linear form. The hypothesis of volume preserving connections imply that the form Ω is exact. So the form h is exact. QED