Zero Velocity Surfaces in General Three Body Problem

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07.09.2022, CelMecVIII, Rome, Italy

Consider the planar problem.
Reduce phase space by translation ($\mathbb{R}^6 \to \mathbb{R}^4$; Jacobie coordinatges)
Next reduce the space by rotation ($\mathbb{R}^4 \to \mathbb{R}^3$: $\mathcal{S}^1 \hookrightarrow \mathcal{S}^3 \to \mathcal{S}^2$; Hopf mapping):

$$\begin{aligned}
\xi_1 &= \mu_1|\mathbf{Q}_1|^2 - \mu_2|\mathbf{Q}_2|^2, \\
\xi_2 &= 2\sqrt{\mu_1\mu_2}\ \mathbf{Q}_1 \cdot \mathbf{Q}_2, \\
\xi_3 &= 2\sqrt{\mu_1\mu_2}\ \mathbf{Q}_1 \times \mathbf{Q}_2,
\end{aligned}$$

$$I = \mu_1|\mathbf{Q}_1|^2 + \mu_2|\mathbf{Q}_2|^2 = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$

Here, $\mathbf{Q}_1$, $\mathbf{Q}_2$ – Jacobie coordinates, $\mu_1 = m_1m_2/(m_1+m_2)$, $\mu_2 = m_3(m_1+m_2)/(m_1+m_2+m_3)$.

$$T - U = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h.$$

Energy integral, Sundman inequality and zero velocity surface

$$\frac{J^2}{2I} - U(\xi_1, \xi_2, \xi_3) - h \leq$$
$$\frac{J^2}{2I} + \dot{I}^2/(8I) - U(\xi_1, \xi_2, \xi_3) - h \quad \leq$$
$$\frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) - h = 0,$$

$$\frac{J^2}{2\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h,$$

$$U(\xi_1, \xi_2, \xi_3) = \frac{1}{\sqrt[4]{\xi_1^2 + \xi_2^2 + \xi_3^2}}\, D(\varphi, \theta).$$

If $\mathbf{r}_i(t)$, $i = 1, \ldots, N$, is solution of N-body problem, the followimg expression gives the solution as well

$$\begin{aligned} \boldsymbol{\rho}_i(t) &= \lambda \mathbf{r}_i(\lambda^{-3/2} t) \\ \dot{\boldsymbol{\rho}}_i(t) &= \lambda^{-1/2} \mathbf{v}_i(\lambda^{-3/2} t) \\ h' &= h/\lambda \end{aligned}$$

Expressions for mutual distances

$$
\begin{aligned}
r_{12}^2 &= \frac{m_1+m_2}{2m_1m_2}\left(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1\right) \\
r_{13}^2 &= \frac{m_1+m_3}{2m_1m_3}\sqrt{\xi_1^2+\xi_2^2+\xi_3^2} \\
&\quad + \frac{m_2m_3-m_1(m_1+m_2+m_3)}{2m_1m_3(m_1+m_2)}\xi_1 + \frac{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}}{m_1m_3(m_1+m_2)}\xi_2 \\
r_{23}^2 &= \frac{m_2+m_3}{2m_2m_3}\sqrt{\xi_1^2+\xi_2^2+\xi_3^2} \\
&\quad + \frac{m_1m_3-m_2(m_1+m_2+m_3)}{2m_2m_3(m_1+m_2)}\xi_1 - \frac{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}}{m_2m_3(m_1+m_2)}\xi_2
\end{aligned}
$$

$m_1 = m_2 = m_3 = 1$: $O\xi_3$ and $O\xi_1$

$$J = 0: \quad U(0,0,\xi_3) = \quad 3/\sqrt{\xi_3} = -h = \frac{1}{2}, \quad \rightarrow \quad \xi_3 = 36$$

$$U(\xi_1,0,0) = \quad 1/\sqrt{2\xi_1} + 2/\sqrt{\xi_1/2}, \quad \rightarrow \quad \xi_1 = 50.$$

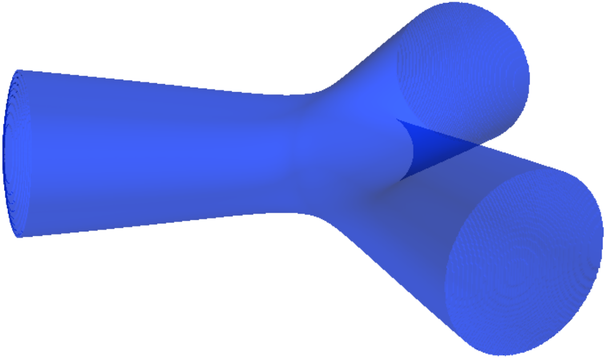
$$J \neq 0: \qquad U(0,0,\xi_3) - \frac{J^2}{2\xi_3} = 3/\sqrt{\xi_3} - \frac{J^2}{2\xi_3} = \frac{1}{2} \quad \rightarrow$$

$$\xi_3 \in \left[\left(3 - \sqrt{9 - J^2}\right)^2, \left(3 + \sqrt{9 - J^2}\right)^2 \right].$$

$$U(\xi_1,0,0) - \frac{J^2}{2\xi_1} = 1/\sqrt{2\xi_1} + 2/\sqrt{\xi_1/2} - \frac{J^2}{2\xi_1} = \frac{1}{2} \quad \rightarrow$$

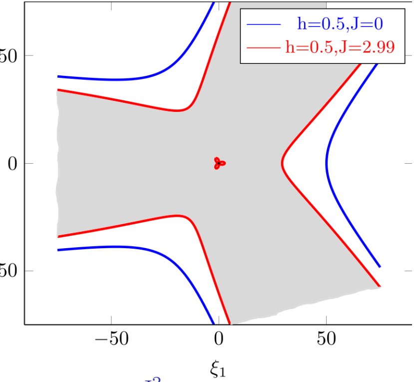
$$\xi_1 \in \left[\left(5 - \sqrt{25 - 2\,J^2}\right)^2/2, \left(5 + \sqrt{25 - 2\,J^2}\right)^2/2 \right].$$

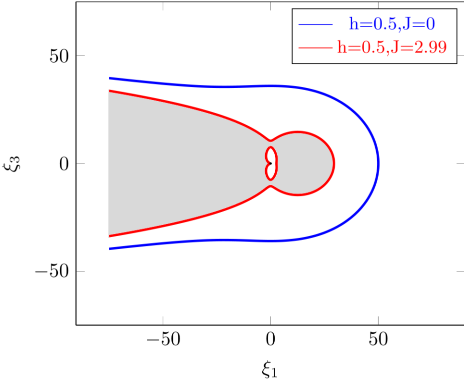
Zero velocity surface at $J = 0$ ($h = -\frac{1}{2}$)



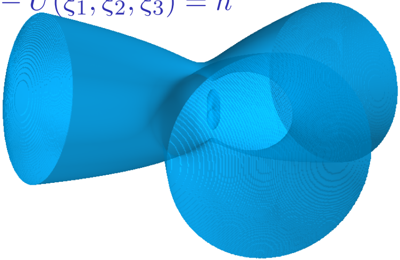
$$U(\xi_1, \xi_2, \xi_3) = -h$$

Zero velocity surface at $J = 2.99$

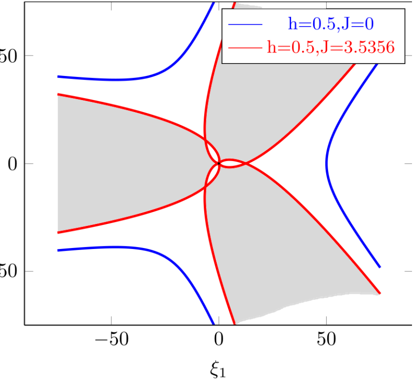


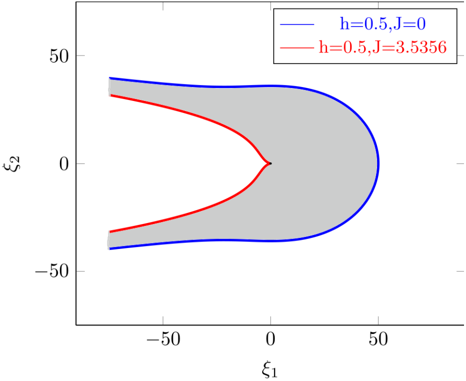


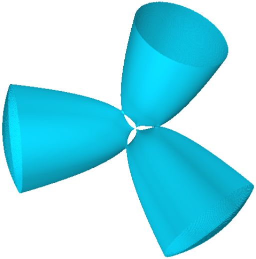
$$\frac{J^2}{2\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h$$



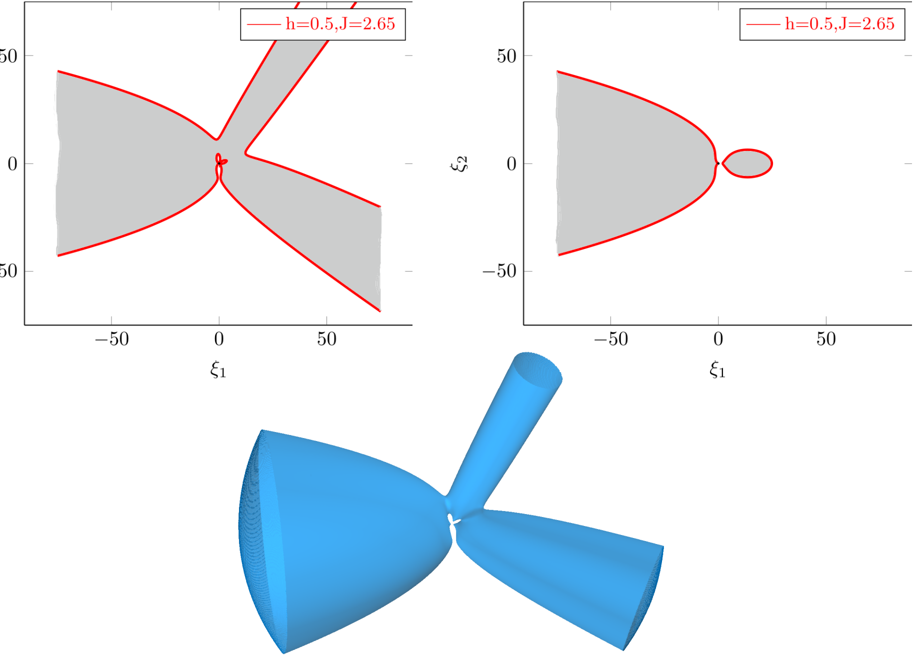
Zero velocity surface at $J = 3.53$



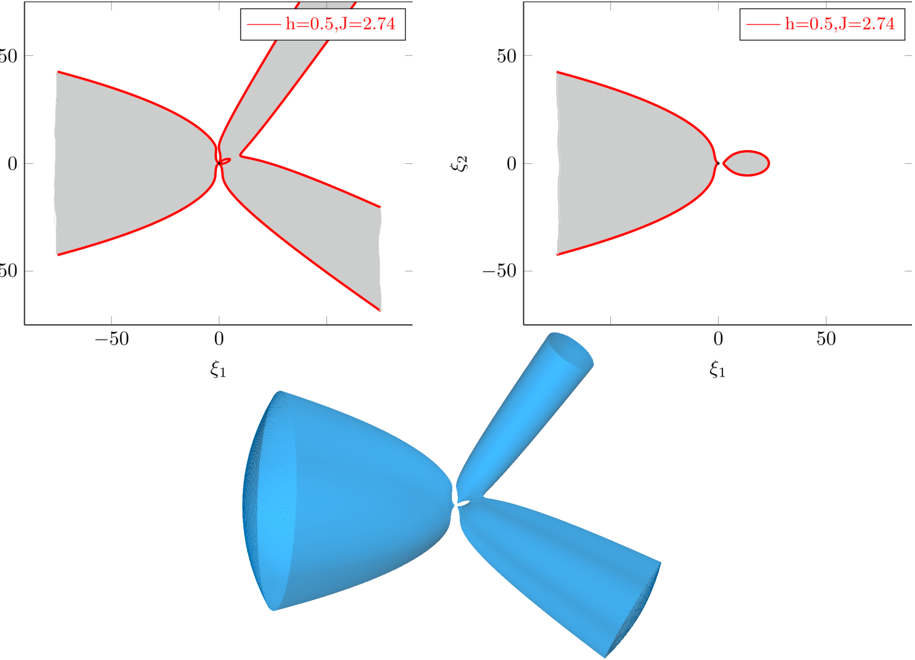




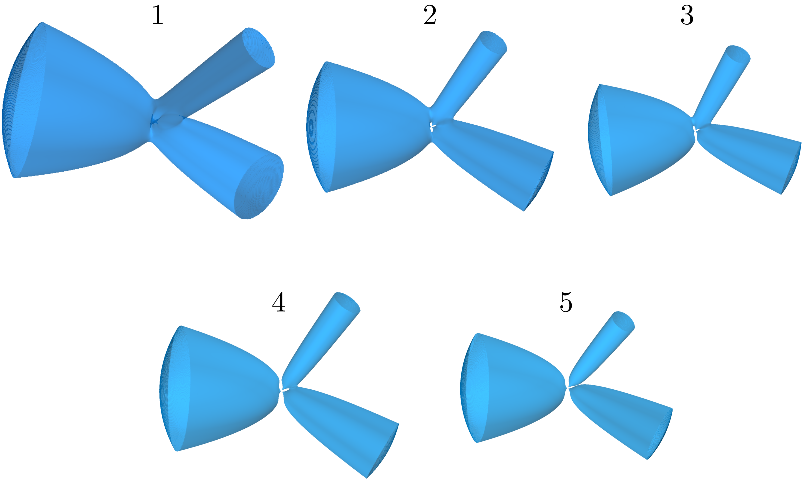
$m_1 = 2m_2 = 4m_3 = 12/7,\ J = 2.65$



$m_1 = 2m_2 = 4m_3 = 12/7$, $J = 2.74$



Zero velocity surface (general three-body problem)



Final motions, hierarchical systems