KAM for planetary problems with double MMR: applications to the HD60532 extrasolar system

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CELMEC VIII — Rome, 7 September 2022

Model: extrasolar systems with a double libration in MMR

1. There are several extrasolar systems in MMR detected through:
   - *radial velocity method* (GJ876 c, b, e $(4:2:1)$, HD128311 c, b $(2:1?)$, etc.).
   - *transit method* (Kepler-80 d, e, b, c, g $(4:6:9:12:18)$, TOI-178 c, d, e, f, g $(2:4:6:9:12)$, etc.).
2. Extrasolar planets have different orbital characteristics w.r.t. the ones of our Solar System.
3. The classical Laplace-Lagrange secular theory has to be modified in order to take into account also the presence of the resonances.

Goal

Construct the **Kolmogorov normal form** for the secular Hamiltonian in MMR.

Application: HD60532 extrasolar system

Initial conditions (Laskar & Correia (2009), Sansottera & Libert (2019)):

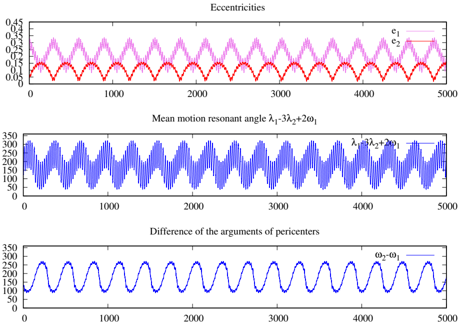
- Two giant planets $(m_1 \simeq 3.15\ M_J,\ m_2 \simeq 7.46\ M_J)$ in a $3:1$ **MMR**.
- **Co-planar motion** with an inclination $i = 20^\circ$ w.r.t. the plane of the sky.
- Semi-axes: $a_1 \simeq 0.76$ AU, $a_2 \simeq 1.59$ AU;
  Eccentricities: $e_1 \simeq 0.278$, $e_2 \simeq 0.038$.

Eccentricities are higher than the ones of the Jovian planets of our Solar System.

There are **two libration angles** with **wide amplitudes** of libration!

$\Downarrow$

The study of the system is trickier.



Planar planetary three-body problem: 4 degrees of freedom

Poincaré canonical variables:

$$(\Lambda_j, \lambda_j) = \left(\frac{m_0 m_j}{m_0 + m_j} \sqrt{\mathcal{G}(m_0 + m_j) a_j}, M_j + \omega_j \right) \quad \textbf{fast variables}$$

$$(\xi_j, \eta_j) = \sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2}} \, (\cos(\omega_j), -\sin(\omega_j)) \quad \textbf{secular variables}$$

with M_j the mean anomaly, ω_j the argument of the pericenter and $j = 1, 2$.

Expansion around the observed values a_j^* of the semi-axes

$$H(L, \lambda, \xi, \eta) = H^{(0)}(L) + \varepsilon H^{(1)}(L, \lambda, \xi, \eta) \ ,$$

with $L = \Lambda - \Lambda^*$ and $\varepsilon = \max\{m_1/m_0, m_2/m_0\} \simeq 5 \cdot 10^{-3}$.

Secular approximation at order one in the masses

Resonant Hamiltonian: consider all the terms related to the MMR.

Hamiltonian with two degrees of freedom

- Introduce action-angle variables $(\xi_j, \eta_j) = \sqrt{2I_j}\,(\cos(\omega_j), -\sin(\omega_j))$ and the resonant variables associated to the two libration angles:

$$\begin{cases} \sigma = \lambda_1 - 3\lambda_2 + 2\omega_1 \\ \delta = \omega_2 - \omega_1 \\ \phi = -\omega_2 \\ \theta = \lambda_2 \end{cases} \qquad \begin{cases} p_\sigma = L_1 \\ p_\delta = I_1 + 2L_1 \\ p_\phi = I_1 + I_2 + 2L_1 \\ p_\theta = L_2 + 3L_1 \end{cases}$$

Two constants of motion (conservation of the total angular momentum p_ϕ + average w.r.t. the fast angle θ): **reduction to two degrees of freedom**.

- There is an equilibrium point at the centre of the librations: $(\pi, \pi, p_\sigma^*, p_\delta^*)$.
- Translation at the equilibrium point + expansion of the Hamiltonian in series + diagonalization of the quadratic part.

$\Longrightarrow$ we get a polynomial **Hamiltonian** with **two degrees of freedom**:

$$H(X,Y) = \frac{\omega_1}{2}(X_1^2 + Y_1^2) + \frac{\omega_2}{2}(X_2^2 + Y_2^2) + \sum_{\ell \geq 1} H_\ell^{(0)}(X,Y)\ ,$$

where $H_\ell^{(0)}$ is a hom. pol. of degree $\ell + 2$ in the cartesian variables (X, Y).

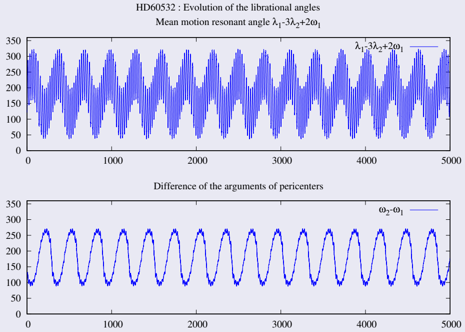
How to get the **stability of the system**?
How to **reconstruct the quasi-periodic motion**?

What we tried to do

1. The frequencies ω_1 and ω_2 have the same sign: **Lyapunov confinement**.
2. **Birkhoff normalization algorithm**.
3. Translation of the actions + direct application of the **KAM algorithm**.
4. Finite number of steps of the **Birkhoff normalization algorithm** + translation of the actions + **KAM algorithm**.

With these attempts the algorithm does not converge
or **we do not get the stability!**

Idea: there is a fast frequency and a slow one!



The MMR angle highlights the presence of the fast period!

$\Downarrow$

we average over the fast angle of libration.

Average over the fast angle of libration

Introduce action-angle variables $(X, Y) = \sqrt{2J}(\sin(\vartheta), \cos(\vartheta))$.

Starting Hamiltonian

$$H^{(0)}(J, \vartheta) = \omega \cdot J + H_1^{(0)}(J, \vartheta) + H_2^{(0)}(J, \vartheta) + H_3^{(0)}(J, \vartheta) + \ldots$$

$H_\ell^{(0)}$ is a hom. pol. of degree $\ell + 2$ in $\sqrt{J}$ and a trig. pol. in the angles ϑ .

Homological equation

The generating function χ_r averages the function $H_r^{(r-1)}$ w.r.t. the fast angle ϑ_2 (associated to the MMR angle) and is determined by solving

$$L_{\chi_r}(\omega \cdot J) + H_r^{(r-1)} = \langle H_r^{(r-1)} \rangle_{\vartheta_2} \ .$$

A non-resonance condition on the frequencies ω has to be verified.

Hamiltonian in normal form up to order r

After a finite number r of normalization steps, we get

$$H^{(r)}(J, \vartheta) = \underbrace{\omega \cdot J + \sum_{\ell=1}^{r} \langle H_\ell^{(\ell-1)} \rangle_{\vartheta_2}}_{integrable\ approximation\ \mathcal{Z}^{(r)}} + \underbrace{\mathcal{R}^{(r+1)}(J, \vartheta)}_{remainder} ,$$

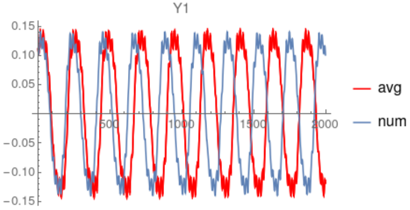
Comparison between numerical and semi-analytic solution

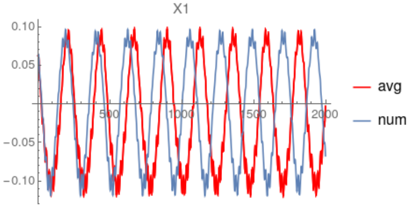
Numerical integration of the Hamiltonian in MMR

vs

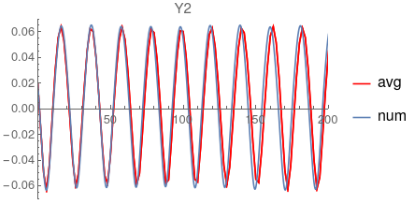
Semi-analytic solution of the averaged Hamiltonian $\mathcal{Z}^{(6)}$

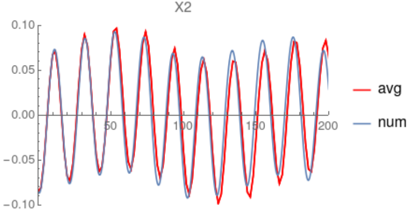
Slow variables





Fast variables





Another essential preliminary transformation

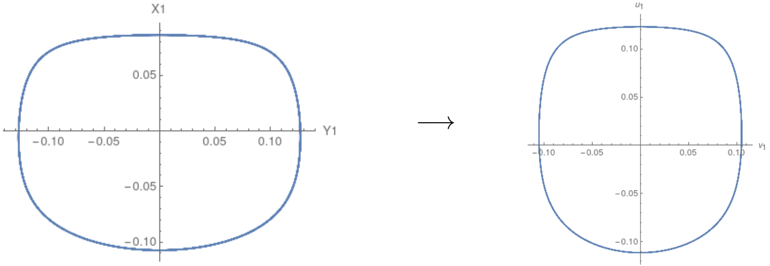
Action-angle variables adapted to the integrable approximation $\mathcal{Z}^{(r)}$

Fast variables: J_2 is a constant of motion for $\mathcal{Z}^{(r)} \Rightarrow$ circular orbit in (X_2, Y_2).
Slow variables: J_1 is not a constant of motion $\Rightarrow$ non-circular orbit in (X_1, Y_1).

Aim: construct action-angle variables **circularizing the orbit of the slow motion** $\Rightarrow$ two changes of coordinates: translation and dilatation/contraction:

$$u_1 = \frac{X_1 - X_1^*}{\alpha}\ , \qquad v_1 = \alpha \cdot Y_1\ .$$

X_1^* and α can be determined by exploiting the frequency analysis method.



We get an action which is a quasi-constant of motion
$\Rightarrow$ we are closer to the torus we want to construct!

Construction of the Kolmogorov normal form

Introduce: $(v_1, u_1) = \sqrt{2I_1}\,(\cos(q_1), \sin(q_1))$, $(Y_2, X_2) = \sqrt{2J_2}\,(\cos(q_2), \sin(q_2))$.
$p_1 = I_1 - I_1^*$; I_1^* is the area enclosed by the orbit of the slow motion divided by 2π.
$p_2 = J_2 - J_2^*$; J_2^* is the mean value of the action J_2.

Starting Hamiltonian (expanded in Taylor-Fourier series)

$$H^{(0)}(p,q) = \omega^{(0)} \cdot p + \sum_{s\ge 1} \left(f_0^{(0,s)} + f_1^{(0,s)} \right) + \mathcal{O}(\|p\|^2) \ ,$$

$f_\ell^{(0,s)}$ is a hom. pol. of degree ℓ in p; trig. pol. of degree sK, with $K > 0$, in q.

Kolmogorov normal form we aim at

$$H(p,q) = \omega^* \cdot p + \mathcal{O}(\|p\|^2) \ ,$$

where ω_1^* is determined by means of the frequency analysis method.

Step r: the generating functions $\chi_0^{(r)}$ and $\chi_1^{(r)}$ remove $f_0^{(0,r)}$ and $f_1^{(0,r)}$, hence

$$H^{(r)}(p,q) = \omega^{(r)} \cdot p + \mathcal{O}(\|p\|^2) + \mathcal{R}^{(r+1)}(p,q) \ ,$$

where $\omega^{(r)}$ satisfies a **non-resonance condition**.

How can we get the wanted frequency ω_1^*?

If the translation I_1^* is accurate enough, then the slow frequency $\omega_1^{(r)} \simeq \omega_1^*$
$\Longrightarrow$ we **calibrate the initial translation with a Newton method**.

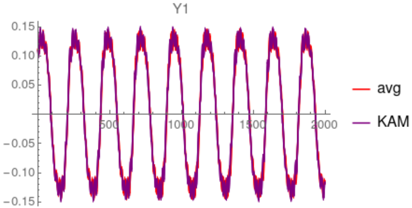
Comparison between semi-analytic solutions

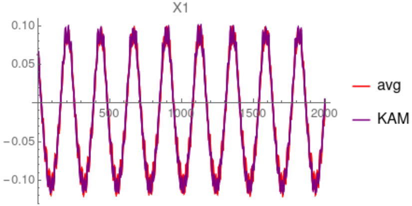
Semi-analytic solution of the averaged Hamiltonian $\mathcal{Z}^{(6)}$

vs

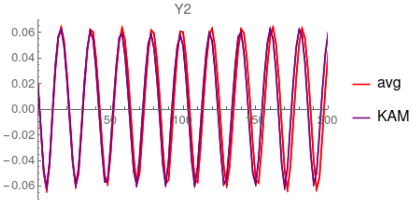
Semi-analytic solution of the Hamiltonian $H^{(5)}$ in Kolmogorov normal form up to order 5

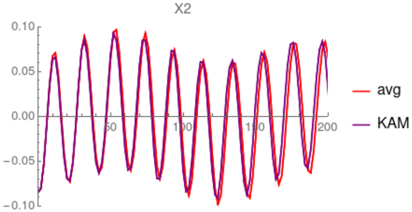
Slow variables





Fast variables





Conclusions

Model & Aim

Model: extrasolar system with a double libration in MMR.

Aim: to reconstruct the quasi-periodic motion that we see from the numerical investigations.

Problem

The construction of a corresponding normal form for the model is difficult because of the large width of the libration angles.

Strategy

Many preliminary steps including

- Average w.r.t. the fast angle of libration.
- Circularization of the orbit of the slow motion before the introduction of the final action-angle variables.

Final successful construction:

- Kolmogorov normal form: finite number of steps + computer assisted proof (following the approach in Valvo & Locatelli (2022)).

Future developments

Applications to other extrasolar systems in (multiple) MMR.