Birkhoff Orbital Elements in Restricted 3- and 4-Body Problems

CELMEC VIII

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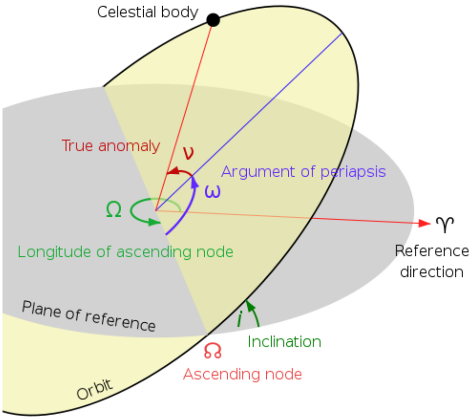
- Relative 2-Body Problem is an **integrable** dynamical system
- Emits the Keplerian orbital elements via integrals of motion
- Perturbations (J_2, SRP, etc.) yield time-varying or *osculating* elements
- In N-body problems, $N \geq 3$, the Keplerian orbital elements are **no longer well-defined**

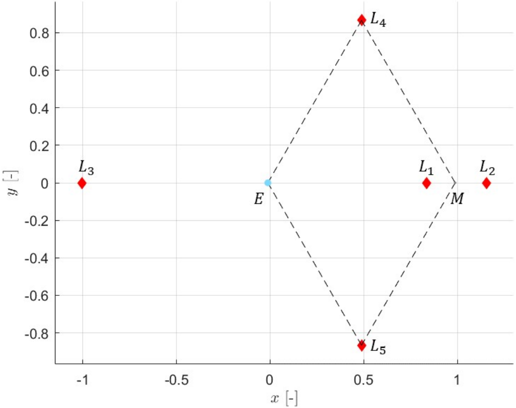


Keplerian Orbital Elements

- Restricted 3-Body Problem (R3BP) is an autonomous, **nearly integrable** Hamiltonian dynamical system

$$H = \frac{1}{2}(p_x^2+p_y^2+p_z^2)+yp_x-xp_y-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}$$

- R3BP emits special solutions:
  - Equilibira
  - Periodic orbits
  - Quasi-periodic orbits
  - Hyperbolic invariant manifolds



R3BP synodical rotating frame

Preliminaries

Birkhoff orbital elements are a **local analogue** to orbital elements in nearly integrable Hamiltonian dynamical systems, e.g. R3BP, defined using **action-angle coordinates** in a **Birkhoff normal form** about a bounded invariant manifold.

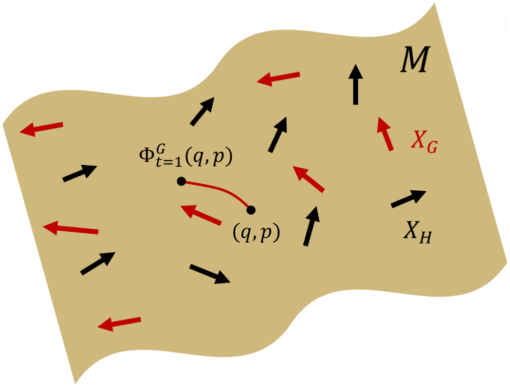
In order to compute Birkhoff orbital elements, we must first:

- Re-center about bounded invariant manifold
  - Eq. point, periodic orbit, reducible torus
- Diagonalize quadratic terms H_2, e.g. $H_2 = \lambda q_1 p_1 + \frac{\omega_p}{2}(q_2^2 + p_2^2) + \frac{\omega_v}{2}(q_3^2 + p_3^2)$
- Complexify H_2, e.g. $H_2 = \lambda q_1 p_1 + i\omega_p q_2 p_2 + i\omega_v q_3 p_3$
- Apply Lie series method (Jorba 1997, Rosales 2021)

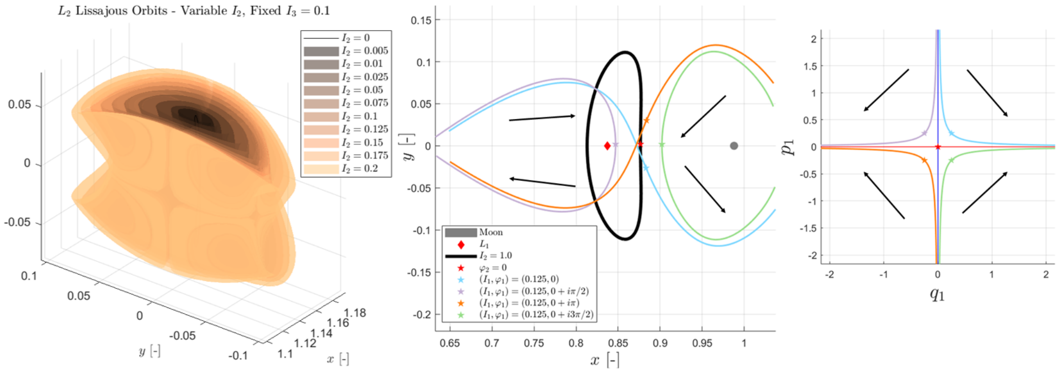
Lie series method

- **Maintain integrability** to higher order via Lie series
- Uses Lie transforms, i.e., the time one flow of X_G, defined by a generating function G
- Obtain Birkhoff orbital elements via **action-angle** transformations in a **Birkhoff normal form**

$$\hat{H} = \sum_{k=2}^{N} \hat{H}_k(I) + R_N(q,p)$$



Birkhoff OEs: $(x, y, z, \dot{x}, \dot{y}, \dot{z}) \mapsto (I_s, \varphi_s, I_p, \varphi_p, I_v, \varphi_v)$



Left: Lissajous orbits at a fixed vertical action, I_v

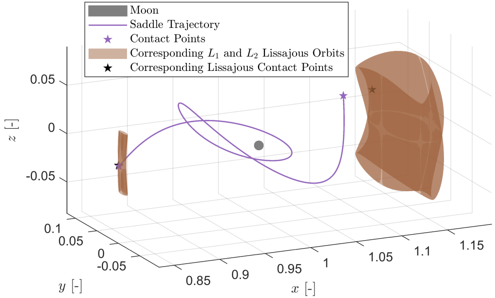
Right: Saddle trajectory about a periodic orbit with representation in saddle plane.

An Existence Theorem

Theorem

For each point (q, p) in the phase space of the Restricted 3-Body Problem, there exists a time $t^* \in \mathbb{R}$ such that $\phi_{t^*}(q, p)$ has a Birkhoff orbital element representation.

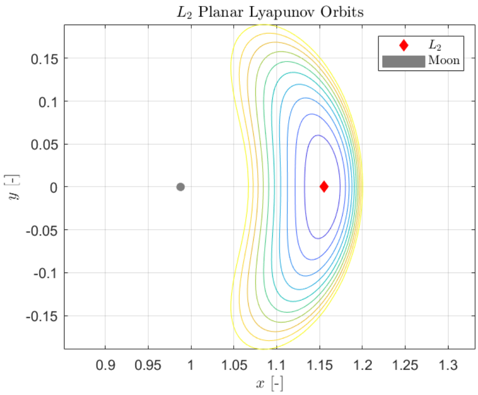
We say that **each point has a Birkhoff orbital element representation** via the Birkhoff normal form flow backwards time $-t^*$. However, this representation is **non-unique**, as showed in the figure...



Example of non-unique representation in Birkhoff orbital elements in R3BP

Periodic Orbits in R3BP

- Libration point periodic orbits have Birkhoff orbital element representations
- We can also re-center our coordinates about an orbit outside this region
  - Halo orbits, DROs, etc.
- We can apply this same ideology to periodic orbits in R4BPs
  - HR4BP, BCP, and QBCP
  - Dynamically equivalents of equilibria
  - Resonant orbits



L_2 Planar Lyapunov Orbits via Birkhoff Orbital Elements

Guaranteeing Symplectic Re-centering

Re-centering about periodic orbits in the R3BP and R4BP requires:

Theorem (Symplectic Floquet Theorem)

Let

$$H(\theta, X, Y) = X^T A(\theta) X + X^T B(\theta) Y + Y^T C(\theta) Y$$

be an n-d.o.f. Hamiltonian, where $A(\theta), B(\theta), C(\theta) \in \mathcal{C}^0(\mathbb{T}^1, M_n\mathbb{C})$*, and* $A(\theta), C(\theta)$ *are symmetric for all* θ*. Then,* $\exists$ *a symplectic, linear and periodic time-dependent change of variables that transforms* H *into a new autonomous Hamiltonian* H_F *on variables* (x, y)*:*

$$H_F(x, y) = x^T \hat{A} x + x^T \hat{B} y + y^T \hat{C} y$$

and with $\hat{A}, \hat{B}, \hat{C} \in M_n\mathbb{C}$*, and* $\hat{A}, \hat{C}$ *symmetric matrices.*

Summary and Future Work

Summary

- **Birkhoff orbital elements** are a **local analogue** to orbital elements in the R3BP and HR4BP (resp. BCP, QBCP) defined using **action-angle coordinates** in a **Birkhoff normal form** centered about a bounded invariant manifold
- Every point in phase space can be represented by *at least* one set of Birkhoff orbital elements in the R3BP and R4BP

Future Work

- Finish building computational tool for periodic perturbation
- Study reducibility of tori to apply to quasi-periodic invariant tori