On the Sun-shadow dynamics

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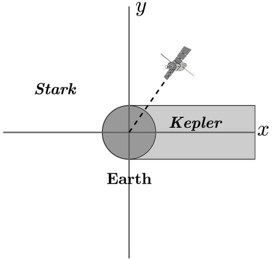
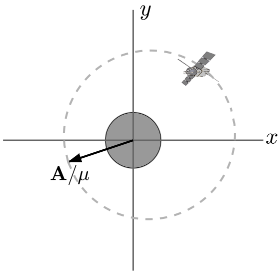


The Sun-shadow dynamics is a model to study the short-period evolution of an Earth satellite subjected to the solar radiation pressure which passes through the Earth shadow. It arises by patching two integrable dynamics: Kepler's and Stark's dynamics.

Kepler's problem

Hamilton's function: $H_k = \frac{1}{2}(p_x^2 + p_y^2) - \mu/\sqrt{x^2 + y^2}$

with p_x and p_y the momenta conjugated to the variables x and y and μ the gravitational parameter of the Earth



Other integrals of motion:

- Laplace-Lenz Vector:

$$\boldsymbol{A} = \begin{bmatrix} p_y(p_x y - p_y x) + \mu x/\sqrt{x^2 + y^2} \\ p_x(p_x y - p_y x) - \mu y/\sqrt{x^2 + y^2} \end{bmatrix}$$

- Angular momentum:

$$C_k = p_y x - p_x y$$

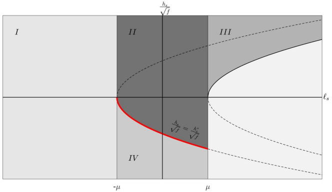
⚠ We denote by L_k the opposite of the x-component of $\boldsymbol{A}$

Stark's problem

Hamilton's function: $H_s = \frac{1}{2}(p_x^2 + p_y^2) - \mu/\sqrt{x^2+y^2} - fx$
with $f > 0$ constant acceleration

Other integral of motion: $L_s = p_y(p_x y - p_y x) + \mu x/\sqrt{x^2+y^2} - fy^2/2$

Depending on the values h_s, ℓ_s of H_s, L_s, there exist different **types of trajectories**:



- Region I: unbounded, self-intersecting, not encircling the center of attraction
- Region II : unbounded, self-intersecting, encircling the center of attraction
- Region III : unbounded, not self-intersecting
- Region IV: bounded + unbounded, self-intersecting, not encircling the center of attraction

There exists a **family of periodic orbits of brake type** (i.e. developing between two zero velocity points) at the boundary between regions II and IV, see red curve.

Parabolic Coordinates

There exists a suitable change of coordinates which separates the variables in the Hamilton-Jacobi equations of both Kepler's and Stark's problems:

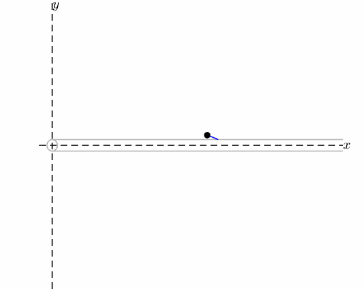
$$x = \frac{u^2 - v^2}{2}, \quad y = uv, \quad p_x = \frac{up_u - vp_v}{u^2 + v^2}, \quad p_y = \frac{vp_u + up_v}{u^2 + v^2}, \quad \frac{d\tau}{dt} = \frac{1}{u^2 + v^2}$$

Kepler's dynamics	Stark's dynamics
Hamilton-Jacobi equation:	Hamilton-Jacobi equation:
$\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2 = 2(h_k(u^2+v^2) + 2\mu)$	$\left(\frac{\partial W}{\partial u}\right)^2 + \left(\frac{\partial W}{\partial v}\right)^2 = 2(h_s(u^2+v^2) + 2\mu) + f(u^4 - v^4)$
$\downarrow$	$\downarrow$
$\begin{cases} p_u^2 = 2h_k u^2 + 2(\mu + \ell_k) \\ p_v^2 = 2h_k v^2 + 2(\mu - \ell_k) \end{cases}$	$\begin{cases} p_u^2 = 2h_s u^2 + 2(\mu + \ell_s) + fu^4 \\ p_v^2 = 2h_s v^2 + 2(\mu - \ell_s) - fv^4 \end{cases}$

with ℓ_s, h_s, ℓ_k, h_k the values of L_s, H_s, L_k and H_k

Sun-shadow dynamics

The solar radiation pressure can become the main perturbation when the *area-to-mass* ratio of the satellite is large, but it has no effect inside the shadow region of the Earth. During short periods of time, the relative motion between the Earth and the Sun can be neglected and the solar radiation pressure can be considered constant.



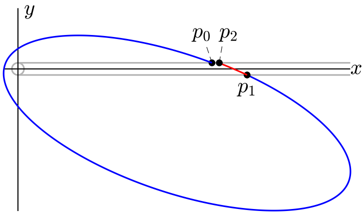
Kepler's regime: red - Stark's regime : blue

↓

The Sun-shadow problem arises by patching Kepler's dynamics, in the shadow region, to Stark's dynamics, in the out-of-shadow region.

Sun-shadow dynamics

Each time the satellite crosses the boundary of the shadow region, there is a leap in energy from h_s to h_k, or vice versa. A similar leap occurs from ℓ_s to ℓ_k, or vice versa. When the satellite goes back to Stark's regime, the value of L_s is the same as before crossing the shadow; on the other hand, the energy usually changes unless the orbit is symmetric with respect to the x-axis.



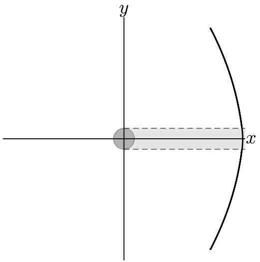
at point p_0: ℓ_{s_0}
at point p_1: $\ell_{k_1} = \ell_{s_0} + fR^2/2$
at point p_2: $\ell_{s_2} = \ell_{k_1} - fR^2/2$
where R is the Earth's radius
$\Downarrow$
$\ell_{s_0} = \ell_{s_2}$

Periodic orbit of brake type

There exists a family of periodic orbits of brake type, which are close to the brake-type periodic orbits of Stark's problem, for $\ell_s \in I$,

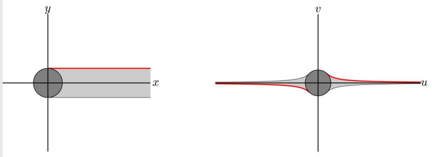
$$I = [\ell_s{}^-, \ell_s{}^+] \in (-\mu, \mu), \quad \ell_s^{\pm} = -5fR^2/4 \pm \sqrt{\mu^2 + 9f^2R^4/16 - 5fR^2\mu/2}$$

Idea of the proof:
For a fixed value $\ell_s \in I$, we search for a point $(x_0, 0)$ in Kepler's regime allowing us to reach a zero velocity point in Stark's regime, taking advantage of the features of Stark's periodic orbits and Stark's regions II and IV.



Sun-shadow map

To study the Sun-shadow dynamics, we consider a Poincaré map



$$\mathfrak{S} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, p_u) \mapsto (u', p'_u),$$

with Poincaré section Σ,

$$\Sigma = \{(p_u, p_v, u, v) : \ |u| \geq \sqrt{R}, \ uv = R, \ up_v > \max(0, -p_u v), \ L_s = \ell_s\}.$$

The map is differentiable and non area-preserving.

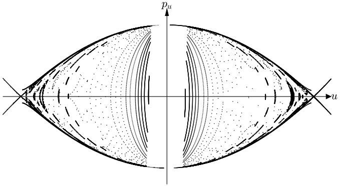
The periodic orbit of brake type corresponds to two hyperbolic points of the map.

⚠ Note that in the (u, v) plane the shadow region is doubled because the map $(u, v) \mapsto (x, y)$ doubles the values of the angles, like in Levi-Civita regularisation.

Sun-shadow map

Global picture of the map

In the central region, the plotted points show regular structures. In a neighbourhood of the hyperbolic points, along the stable and unstable branches of their invariant manifold, the regular behaviour seems to be lost.



Comparison with Stark's phase portrait

For the same value of L_s, we plot the level curves of Stark's problem on the (u, p_u) plane. The two hyperbolic points appearing here correspond to Stark's periodic orbit of brake type.

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The Sun-shadow problem can be seen as a perturbed Stark's dynamics