MULTIPOLE MOMENT EXPANSION FOR SPINNING ASTROPHYSICAL BODIES

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Abstract

In this work, we present a new technique for the analytical derivation of approximate gravitational potentials including the rotation (spin) of the astrophysical object. The series expansion is made using the Ernst potentials of axisymmetric relativistic distributions of mass for well-known compact objects, such that in the limit of weak-fields and low velocities the resulting expressions conserve not only the mass, quadrupole moment, octupole moment, charge, or magnetic dipole, but also the spin of the body. An example using the Kerr metric is presented, giving place to the approximate gravitational potential of a non-spherical spinning object. Some applications to n-body systems composed of rotating and non-spherical primaries are introduced.

Approximate potentials

Let us begin considering the Ernst potentials $\mathcal{E}$ and Φ [1], which are related to the electromagnetic and gravitational potentials of a given source through the following relations

$$\mathcal{E} = \frac{1-\xi}{1+\xi}, \quad \Phi = \frac{\varsigma}{1+\xi}.$$

With the introduction of these new expressions, the Einstein–Maxwell field equations reduce to a system of two complex equations

$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma)\cdot\nabla\xi,$$
$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\varsigma = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma)\cdot\nabla\varsigma.$$

where

$$\xi = \phi_M + i\phi_J, \quad \varsigma = \phi_E + i\phi_H,$$

For simplicity, in all what follows we shall refer to axial symmetric astrophysical bodies, then it is natural to use the cylindrical coordinates (r, φ, z).

In order to measure the moments of an asymptotically flat spacetime, according to the Geroch-Hansen procedure [2], we map the initial 3-metric onto a conformal one

$$h_{ij} \rightarrow \tilde{h}_{ij} = \Omega^2 h_{ij}.$$

The conformal factor Ω should satisfy the following conditions:

$$\Omega|_\Lambda = \tilde{D}_i\Omega|_\Lambda = 0 \text{ and } \tilde{D}_i\tilde{D}_j\Omega|_\Lambda = 2h_{ij}|_\Lambda$$

where Λ is the point added to the initial manifold that represents infinity.

The factor Ω transforms the potentials ξ and ς into $\tilde{\xi} = \Omega^{-1/2}\,\xi$ and $\tilde{\varsigma} = \Omega^{-1/2}\,\varsigma$.

The conformal factor is given by $\Omega = r^2 = \rho^2 + z^2$, and the transformation relation between the barred and unbarred variables is given as

$$\bar{\rho} = \frac{\rho}{\rho^2+z^2}, \qquad \bar{z} = \frac{z}{\rho^2+z^2}, \quad \bar{\phi} = \phi,$$

which brings infinity at the origin of the axes $(\rho, z) = (0,0)$. The new potentials $\tilde{\xi}$ and $\tilde{\varsigma}$ can be written in a power series expansion of ρ and z as

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij}\bar{\rho}^i\bar{z}^j, \qquad \tilde{\varsigma} = \sum_{i,j=0}^{\infty} b_{ij}\bar{\rho}^i\bar{z}^j.$$

Due to the analyticity of the potentials at the axis of symmetry, a_{ij} and b_{ij} must vanish when i is odd [3]. The coefficients in the above power series can be calculated by the relation

$$\begin{aligned}(r+2)^2 a_{r+2,s} =& -(s+2)(s+1)a_{r,s+2}\\ &+ \sum_{k,l,m,n,p,g}(a_{kl}a^*_{mn} - b_{kl}b^*_{mn})\\ &\times [a_{pg}(p^2+g^2-4p-5g-2pk-2gl-2)\\ &+ a_{p+2,g-2}(p+2)(p+2-2k) + a_{p-2,g+2}\\ &\times (g+2)(g+1-2l)]\end{aligned}$$

and

$$\begin{aligned}(r+2)^2 b_{r+2,s} =& -(s+2)(s+1)b_{r,s+2}\\ &+ \sum_{k,l,m,n,p,g}(a_{kl}a^*_{mn} - b_{kl}b^*_{mn})\\ &\times [b_{pg}(p^2+g^2-4p-5g-2pk-2gl-2)\\ &+ b_{p+2,g-2}(p+2)(p+2-2k) + b_{p-2,g+2}\\ &\times (g+2)(g+1-2l)],\end{aligned}$$

where

$$\begin{aligned}&m = r-k-p,\\ &0 \le k \le r,\\ &0 \le p \le r-k,\end{aligned}$$

with k and p even, and

$$\begin{aligned}&n = s-l-g,\\ &0 \le l \le s+1,\\ &-1 \le g \le s-l\end{aligned}$$

These recursive relations could build the whole power series of $\tilde{\xi}$ and $\tilde{\varsigma}$ from their values on the axis of symmetry

$$\tilde{\xi}(\bar{\rho}=0) = \sum_{i=0}^{\infty} m_i\bar{z}^i, \quad \tilde{\varsigma}(\bar{\rho}=0) = \sum_{i=0}^{\infty} q_i\bar{z}^i.$$

The values of the multipolar moments of the spacetime are determined in terms of the coefficients in the series expansion. The relations between a_{ij} and b_{ij} can be used in order to express the moments in terms of $q_i \equiv b_{0i}$ and $m_i \equiv a_{0i}$.

Therefore, the gravitational and electromagnetic multipole moments of the source can be written in terms of a_{ij} and b_{ij}, through the relations $m_i \equiv a_{0i}$ and $q_i \equiv b_{0i}$, since the multipole moments satisfy algebraic relations with m_i and q_i [4].

The gravitational moments as functions of the power series coefficients of $\tilde{\xi}$ along the symmetry axis are given by

$$\begin{aligned}&P_0 = m_0, \quad P_1 = m_1, \quad P_2 = m_2,\\ &P_3 = m_3, \quad P_4 = m_4 - \frac{1}{7}m_0^*M_{20} + \frac{1}{7}q_0^*S_{20} - \frac{3}{70}q_1^*S_{10}\\ &P_5 = m_5 - \frac{1}{3}m_0^*M_{30} - \frac{1}{21}m_1^*M_{20} + \frac{1}{3}q_0^*S_{30}\\ &+ \frac{4}{21}q_0^*S_{21} - \frac{1}{21}q_1^*S_{11} - \frac{1}{21}q_2^*S_{10}\end{aligned}$$

while the electromagnetic moments as functions of the power series coefficients of $\tilde{\varsigma}$ along the symmetry axis are given by

$$\begin{aligned}&Q_0 = q_0, \quad Q_1 = q_1, \quad Q_2 = q_2,\\ &Q_3 = q_3, \quad Q_4 = q_4 + \frac{1}{7}q_0^*Q_{20} - \frac{1}{7}m_0^*H_{20} + \frac{3}{70}m_1^*H_{10}\\ &Q_5 = q_5 + \frac{1}{3}q_0^*Q_{30} + \frac{1}{21}q_1^*Q_{20} - \frac{1}{3}m_0^*H_{30}\\ &- \frac{4}{21}m_0^*H_{21} + \frac{1}{21}m_1^*H_{11} + \frac{1}{21}m_2^*H_{10}\end{aligned}$$

where

$$M_{ij} = m_i m_j - m_{i-1}m_{j+1}, \quad Q_{ij} = q_i q_j - q_{i-1}q_{j+1},$$
$$S_{ij} = m_i q_j - m_{i-1}q_{j+1}, \quad H_{ij} = q_i m_j - q_{i-1}m_{j+1},$$

Results

The Kerr metric describes the geometry around a rotating black-hole, its multipole moments in the symmetry axis read as

$$m_n = m(ia)^n, \quad q_n = 0;$$

with m its total mass and a the total angular momentum per unit mass.

From the relation between the multipole moments and the coefficients (a_{0j}, b_{0j}), it appears that $m_j = a_{0j} = m(ia)^j$ for $0 \le j \le 5$ and $q_j = b_{0j} = 0$ for all j. Then, the rest of coefficients (a_{ij}, b_{ij}) can be calculated from the recursive relations. The first coefficients read as

$$\begin{aligned}&a_{00} = m,\\ &a_{01} = am,\\ &a_{02} = -am^2,\\ &a_{20} = \frac{1}{2}m(a-m)(a+m),\\ &a_{21} = \frac{1}{2}(3a^3m - 5am^3),\\ &a_{22} = -\frac{1}{2}m(6a^4 + a^2m^2 + m^4),\end{aligned}$$

The coefficients with i odd are equal to zero, $a_{1j} = a_{3j} = a_{5j} = 0$, and since $q_j = b_{0j} = 0$, then $b_{ij} = 0$ for all i and j.

Once the coefficients are known, we proceed to calculate the potentials ξ and ς, with the aid of $\tilde{\xi}$ and $\tilde{\varsigma}$ (see last equation of the first column), and the inverse transformations for the coordinates

$$\bar{\rho} = \frac{\rho}{\rho^2+z^2}, \qquad \bar{z} = \frac{z}{\rho^2+z^2}, \quad \bar{\phi} = \phi,$$

The unbarred approximate functions ξ and ς are then [4]

$$\xi = \frac{m}{\sqrt{\rho^2+z^2}} + \frac{amz}{(\rho^2+z^2)^{3/2}} + \frac{a^2mz^2}{(\rho^2+z^2)^{5/2}} + \frac{a^2m\rho^2}{2\,(\rho^2+z^2)^{5/2}} - \frac{m^3\rho^2}{2\,(\rho^2+z^2)^{5/2}} + \ldots$$

$$\varsigma = 0$$

Conclusions

In this work, via the Fodor–Hoenselaers–Perjés formalism, we propose a new approximate potential for the Kerr metric, directly derived from the multipole moments expansion of the source. Unlike previous works published in this line, our potential is fully consistent with its classical and relativistic limits. This approach allowed us to obtain new pseudo-Newtonian versions of the few-body problem where the spin of the primaries can be considered.

References

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