Canonical transformation for multiple close encounters

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7^{th} September 2022

CELMEC VIII - University of Rome Tor Vergata

Introduction

Asteroids can have multiple close encounters with the same planet. For example, if the orbital period of the asteroid after the first encounter is in resonance with the orbital period of the planet, i.e.

$$\frac{T'_a}{T_\oplus} \sim \frac{k}{h} \qquad \Longrightarrow \qquad \frac{a'}{a_\oplus} \sim \left(\frac{k}{h}\right)^{2/3}, \qquad h, k \in \mathbb{N} \setminus \{0\}$$

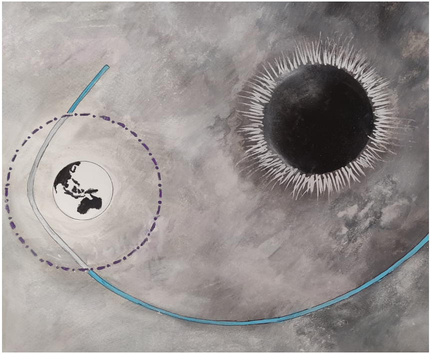
a second encounter with the same planet will happen after almost $kT_\oplus$. When this condition is satisfied we have a resonant return.

An approximate model to study resonant returns and to identify potential collisions already exists (theory of resonant returns and keyholes, see Valsecchi et al. (2003)), but the theory can be extended.

Goal: Build a map from the initial state of the asteroid before the first encounter to its state at the second encounter that:

- is canonical
- is not limited to resonant returns
- does not approximate encounters as instantaneous

The map we build is based on the patched-conic method, where the asteroid's three-body orbit is approximated by patching separate two-body solutions.



Sphere of influence

Given the nominal orbit, we select the most suitable radius for a sphere of influence of the planet by applying an optimization process with target function

$$f(r) = \sup_{t\in[t_0,t_1]} |\boldsymbol{x}^{\mathrm{3bp}}(t) - \boldsymbol{x}^{\mathrm{pc}}(t;r)| + |\boldsymbol{x}^{\mathrm{3bp}}(t_1) - \boldsymbol{x}^{\mathrm{pc}}(t_1;r)| \\ + |\boldsymbol{x}^{\mathrm{3bp}}(t_q) - \boldsymbol{x}^{\mathrm{pc}}(t_{q_P};r)|,$$

where $\boldsymbol{x}^{\mathrm{3bp}}(t)$ is the three-body orbit, $\boldsymbol{x}^{\mathrm{pc}}(t)$ is the patched-conic one, t_0 and t_1 are the initial and final times and t_q and t_{q_P} are the times of minimum planetocentric distance along $\boldsymbol{x}^{\mathrm{3bp}}(t)$ and $\boldsymbol{x}^{\mathrm{pc}}(t)$ respectively.

The chosen radius of the sphere is r_{soi} such that $f(r_{soi}) = \min_r f(r)$.

Nominal orbit

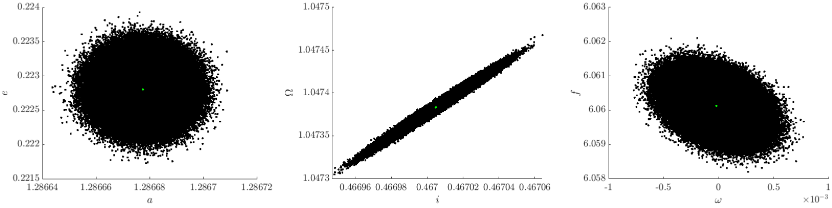
Take as initial condition the state of the nominal orbit well before the first encounter so that at the initial time the simple heliocentric elliptic orbit approximates well the three-body one.

Define:

- $t_{\mathrm{ell},1}$ = time spent on the heliocentric elliptic orbit until the distance from the planet is equal to r_{soi}.
- t_{hyp} = time on the planetocentric hyperbolic orbit inside the sphere of influence.
- $t_{\mathrm{ell},2}$ = time on the post-encounter heliocentric elliptic orbit until the nominal solution reaches again the planet's sphere of influence for the second encounter.

Initial conditions

Using a normal probability distribution derived from the covariance matrix associated to the nominal orbit, we take virtual asteroids around the nominal initial condition.



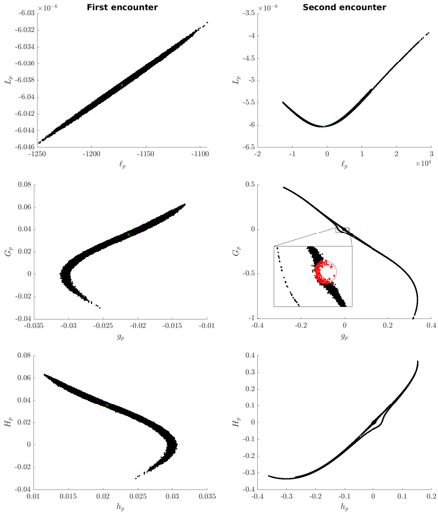
Canonical transformation

The map is made up of the following sequence of canonical transformations:

- Before the first encounter: two-body elliptic flow for a fixed time $t_{\mathrm{ell},1}$
- Change to planetocentric frame: translation of constant vector and rotation of constant angle
- During the first encounter: two-body hyperbolic flow for a fixed time t_{hyp}
- Change to heliocentric frame: translation of constant vector and rotation of constant angle
- After the first encounter and until the second one: two-body elliptic flow for a fixed time $t_{\mathrm{ell},2}$

At each encounter: canonical transformation from hyperbolic Delaunay elements to hyperbolic Poincaré-type elements

$$L_p = L$$
$$G_p = \sqrt{2(G-L)}\cos(g+h)$$
$$H_p = \sqrt{2(G-H)}\cos(h)$$
$$\ell_p = \ell + g + h$$
$$g_p = \sqrt{2(G-L)}\sin(g+h)$$
$$h_p = -\sqrt{2(G-H)}\sin(h)$$



In red: points that collide with the planet at the second encounter.

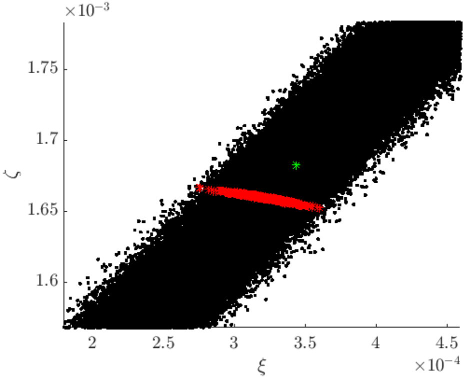
Collision probability

In the example of the previous slide the nominal three-body orbit collides with the planet at the second encounter. The patched-conic approximation of the nominal orbit does not collide. However, by sampling the confidence ellipsoid with virtual asteroids, collisions happen for other points:

- 0.031% of initial conditions collide with the planet at the second encounter with a three-body propagation
- 0.018% of points collide with the planet at the second encounter with the patched-conic approximation

The map reproduces the correct order of magnitude of the probability of collision at the second encounter.

Image of our map on the target plane (ξ, ζ) of the first encounter. In green, the image of the nominal orbit. In red, points that lead to collisions at the second encounter.



Using the patched-conic approach, we are able to save on computational time. Our map takes just 0.3% of the time required by the three-body computation.