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Equations of secular perturbations of exoplanetary systems with variable masses

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Actuality of the topic: researching of dynamics of exoplanets [1] in the non-stationary stage of its formation also its evolution gives us the opportunity to determine further evolutionary tracks.

The purpose of this work: to research the dynamic evolution in the non-stationarity stage of a multi-planetary problem, when the masses of spherical bodies change isotropically with different velocities;
to obtain the evolution equations of a multi-planetary problem with variable masses.

Problem statement: A system of *n+1* bodies with variable masses $m_0 = m_0(t), \quad m_i = m_i(t), \quad i = (1, 2, ..., n)$, mutually gravitating according to Newton's law, is considered. Body masses vary isotropically at different rates

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_i}{m_i}, \quad \frac{\dot{m}_i}{m_i} \neq \frac{\dot{m}_j}{m_j}, \quad i, j = 1, 2, .., n, \quad i \neq j, \tag{1}$$

The mass of the central star varies according to the Eddington-Jeans law

$$m_i = m_i(t) = \left(\varepsilon_i (1 - n_i)(t - t_0) + m_{i0}^{1-n_i}\right)^{1/(1-n_i)}, \quad \varepsilon_i = const, \quad n_i = const \tag{2}$$

Let's assume that the masses of the planets also vary according to the Eddington-Jeans law.

[1] https://exoplanets.nasa.gov/

The equations of motion of n planets with isotropically varying masses in the relative coordinate system can be written as

$$\ddot{\vec{r}}_i = -f\frac{(m_0+m_i)}{r_i^3}\vec{r}_i + f\sum_{j=1}^{n}{}' m_j\left(\frac{\vec{r}_j-\vec{r}_i}{\Delta_{ij}^3} - \frac{\vec{r}_j}{r_j^3}\right), \quad (i=1,2,...,n), \quad (j=1,2,...,n) \tag{3}$$

where f is the gravitational constant, $m_0 = m_0(t)$ is the mass of the parent star, $m_i = m_i(t)$ is mass of the planet P_i, the sign « ' » in summing means that $i \neq j$, $\vec{r}_i(x_i, y_i, z_i)$ is the radius-vector of spherical bodies, Δ_{ij} is mutual distances of the center of spherical bodies

$$\Delta_{ij} = \sqrt{(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2} = \Delta_{ji} \tag{4}$$

The differential equations of motion of n+1 bodies in the relative coordinate system [2-3] are rewritten as

$$\ddot{\vec{r}}_i + f\frac{(m_0+m_i)}{r_i^3}\vec{r}_i - \frac{\ddot{\gamma}_i}{\gamma_i}\vec{r}_i = \vec{F}_i \qquad \gamma_i = \frac{m_0(t_0)+m_i(t_0)}{m_0(t)+m_i(t)} = \gamma_i(t) \tag{5}$$

here, γ_i is led function of mass, F_i is perturbing forces, t_0 is the initial moment of time. The following designations have been accepted

$$\vec{F}_i = grad_{\vec{r}_i} W_i \qquad W_i = W_{gi} + W_{ri} \tag{6}$$

W_i - is perturbing function

$$W_{gi} = f\sum_{j=1}^{n}{}' m_j\left(\frac{1}{\Delta_{ij}} - \frac{\vec{r}_i\cdot\vec{r}_j}{r_j^3}\right) \qquad W_{ri} = -\frac{\ddot{\gamma}_i}{2\gamma_i}r_i^2 \qquad \vec{\Delta}_{ij} = \vec{r}_j - \vec{r}_i, \ (i, j = 1,2,...,n) \tag{7}$$

When $W_i = 0$ equations (5)-(6) describe the aperiodic motion over a quasi-conic section [4]

$$\ddot{\vec{r}}_i + f\frac{(m_0+m_i)}{r_i^3}\vec{r}_i - \frac{\ddot{\gamma}_i}{\gamma_i}\vec{r}_i = 0 \tag{8}$$

[2] Minglibayev M.Zh., Kosherbayeva A.B. Differential equations of planetary systems // Reports of the National Academy of Sciences of the Republic of Kazakhstan, - 2020, Vol.2(330). P. 14-20, https://doi.org/10.32014/2020.2518-1483.26

[3] Minglibayev M.Zh., Kosherbayeva A.B. Equations of planetary systems motion // News of The national Academy of Sciences of the Republic of Kazakhstan. Physico-Mathematical Series, - 2020, Vol.6(334). P. 53 – 60, https://doi.org/10.32014/2020.2518-1726.97 .

[4] Minglibayev M.Zh. Dynamics of gravitating bodies with variable masses and sizes [Dinamika gravitiruyushchikh tel s peremennymi massami i razmerami]. LAP LAMBERT Academic Publishing. –2012. –P.224. Germany.ISBN:978-3-659-29945-2

The solution of the differential equation (8) is a similar solution to the classical two bodies problem with constant masses.

$$x_i = \gamma_i \rho_i \left[\cos u_i \cdot \cos \Omega_i - \sin u_i \cdot \sin \Omega_i \cdot \cos i_i\right], \quad \rho_i = \frac{p_i}{1 + e_i \cos \theta_i}$$
$$y_i = \gamma_i \rho_i \left[\cos u_i \cdot \sin \Omega_i + \sin u_i \cdot \cos \Omega_i \cdot \cos i_i\right], \quad p_i = a_i\left(1 - e_i^2\right)$$
$$z_i = \gamma_i \rho_i \sin u_i \cdot \sin i_i, \quad r_i = \gamma_i \rho_i, \quad u_i = \theta_i + \omega_i, \tag{9}$$

$$\dot{x}_i = \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) x_i + \gamma_i \rho_i \dot{u}_i \left[-\sin u_i \cos \Omega_i - \cos u_i \sin \Omega_i \cos i_i\right],$$
$$\dot{y}_i = \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) y_i + \gamma_i \rho_i \dot{u}_i \left[-\sin u_i \sin \Omega_i + \cos u_i \cos \Omega_i \cos i_i\right],$$
$$\dot{z}_i = \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) z_i + \gamma_i \rho_i \dot{u}_i \left[\cos u_i \sin i_i\right],$$
$$\dot{\rho}_i = \frac{1}{\gamma_i^2} \frac{\sqrt{\mu_{i0}}}{\sqrt{p_i}} e_i \sin \theta_i, \quad \dot{u}_i = \frac{1}{\gamma_i^2} \frac{\sqrt{\mu_{i0}}\sqrt{p_i}}{\rho_i^2}, \tag{10}$$

Here u_i - an analogue of the latitude argument, θ_i - an analogue of a true anomaly, p_i - an analogue of the focal parameter, $\mu_{i0} = f\left[m_0(t_0) + m_i(t_0)\right]$ - the gravity parameter. Values $a_i, e_i, i_i, \omega_i, \Omega_i, M_i$ are analogues of Keplerian elements.

Solutions (9)-(10) were used as the initial unperturbed motion.

For our purposes, analogues of the second system of canonical Poincare elements $(\Lambda_i, \lambda_i, \xi_i, \eta_i, p_i, q_i)$ are preferred which are introduced according to the following formulas [4]

$$\Lambda_i = \sqrt{\mu_{i0}}\sqrt{a_i}, \quad \lambda_i = l_i + \pi_i, \tag{11}$$

$$\xi_i = \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\left(1 - \sqrt{1 - e_i^2}\right)} Cos\pi_i, \quad \eta_i = -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\left(1 - \sqrt{1 - e_i^2}\right)} Sin\pi_i, \tag{12}$$

$$p_i = \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1 - e_i^2}\left(1 - Cosi_i\right)} Cos\Omega_i, \quad q_i = -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1 - e_i^2}\left(1 - Cosi_i\right)} Sin\Omega_i \tag{13}$$

the following designations are given here

$$l_i = M_i = n_i\left[\phi_i(t) - \phi_i(\tau_i)\right], \quad \pi_i = \Omega_i + \omega_i, \quad n_i = \sqrt{\mu_{i0}} / a_i^{3/2} = const \tag{14}$$

[4] Minglibayev M.Zh. Dynamics of gravitating bodies with variable masses and sizes [Dinamika gravitiruyushchikh tel s peremennymi massami i razmerami]. LAP LAMBERT Academic Publishing. –2012. –P.224. Germany.ISBN:978-3-659-29945-2

where $\phi_i(t)$ - the antiderivative function from expression $\left(\frac{m_0(t)+m_i(t)}{m_0(t_0)+m_i(t_0)}\right)^2$, τ_i - an analogue of the moment of passage through the pericenter, n_i - an analogue of the average movement.

The differential equations of n bodies motion in the osculating analogues of the second system of Poincare (11)-(13) variables have the form [5]

$$\dot{\lambda}_i = -\frac{\partial R_i^*}{\partial \Lambda_i} = \frac{\mu_{i0}^2}{\gamma_i^2 \Lambda_i^3} - \frac{\partial W_i}{\partial \Lambda_i}, \qquad \dot{\Lambda}_i = \frac{\partial R_i^*}{\partial \lambda_i} = \frac{\partial W_i}{\partial \lambda_i},$$
$$\dot{\eta}_i = -\frac{\partial R_i^*}{\partial \xi_i} = -\frac{\partial W_i}{\partial \xi_i}, \qquad \dot{\xi}_i = \frac{\partial R_i^*}{\partial \eta_i} = \frac{\partial W_i}{\partial \eta_i},$$
$$\dot{q}_i = -\frac{\partial R_i^*}{\partial p_i} = -\frac{\partial W_i}{\partial p_i}, \qquad \dot{p}_i = \frac{\partial R_i^*}{\partial q_i} = \frac{\partial W_i}{\partial q_i}. \qquad (15)$$

here R_i^*- hamiltonian

$$R_i^* = \frac{\mu_{i0}^2}{2\Lambda_i^2} \cdot \frac{1}{\gamma_i^2(t)} + W_i(t, \Lambda_i, \xi_i, p_i, \lambda_i, \eta_i, q_i) \qquad (16)$$

In the expression of the perturbing function W_{gi} it is advisable to highlight the main and indirect part

$$W_{gi} = f\sum_{j=1}^{n}{}' m_j\left(\frac{1}{\Delta_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_j^3}\right) = f\sum_{j=1}^{n}{}' m_j\left(\frac{1}{\Delta_{ij}}\right) - f\sum_{j=1}^{n}{}' m_j\left(\frac{r_i \cdot r_j \cdot Cos\psi_{ij}}{r_j^3}\right) = f\sum_{j=1}^{n}{}' m_j\left(\frac{1}{\Delta_{ij}}\right) - f\sum_{j=1}^{n}{}' m_j\left(r_i \cdot \left(\frac{1}{r_j^2}\right) \cdot Cos\psi_{ij}\right). \qquad (17)$$

The main part of the perturbing function W_{gi}

$$W_{gi,\text{main}} = f\sum_{j=1}^{n}{}' m_k\left(\frac{1}{\Delta_{ij}}\right)$$

The indirect part of the perturbing function W_{gi}

$$W_{gi,\text{add}} = -f\sum_{j=1}^{n}{}' m_j\left(r_i \cdot \left(\frac{1}{r_j^2}\right) Cos(\psi_{ij})\right) \qquad (18)$$

To obtain an analytical expression of the main part of the perturbing function (18), in canonical osculating elements, it is necessary to have decomposition of following expression

$$\left(\frac{1}{\Delta_{ij}}\right), \qquad i, j = 1, 2, .., n, \qquad i \neq j \qquad (19)$$

We describe the decomposition scheme of the main part of the perturbing function for the inner planets. From the equality $\vec{r}_{is} = \vec{r}_s - \vec{r}_i$ follows

$$r_{is} = \Delta_{is}, \quad r_{is}^2 = r_s^2 - 2\vec{r}_s \cdot \vec{r}_i + r_i^2 = r_s^2 - 2r_s r_i \cos\psi_{is} + r_i^2, \quad (s < i) \qquad (20)$$

here ψ_{is} - the angle between $\vec{r}_i$, $\vec{r}_s$

[5] Prokopenya A. N., Minglibayev M. Zh., Kosherbaeva A. B. Derivation of Evolutionary Equations in the Many-Body Problem with Isotropically Varying Masses Using Computer Algebra // Programming and Computer Software. –2022. –V.48(2). –P.107–115. DOI:10.1134/S0361768822020098

Let's rewrite expression (20) as $r_{is}^2 = r_i^2 - 2r_i r_s \cos\psi_{is} + r_s^2 = \left[\sigma_{is}^2\right] + \left[-2r_i r_s \tilde{\Psi}_{is}\right]$, $\sigma_{is}^2 = r_i^2 + r_s^2 - 2r_i r_s \cos(v_s - v_i)$, $\tilde{\Psi}_{is} = \cos\psi_{is} - \cos(v_s - v_i)$ (21)

here $v_i = \theta_i + \pi_i$, $v_s = \theta_s + \pi_s$ - the true longitudes of the planets P_i, P_s, respectively. As a result, we have

$$r_{is}^2 = \sigma_{is}^2\left[1 - \frac{2r_i r_s}{\sigma_{is}^2}\tilde{\Psi}_{is}\right], \quad \frac{1}{r_{is}} = \frac{1}{\sigma_{is}} \cdot \frac{1}{\sqrt{1 - \frac{2r_i r_s}{\sigma_{is}^2}\tilde{\Psi}_{is}}}, \quad \frac{1}{r_{is}} = \frac{1}{\sigma_{is}} + \left(r_i r_s \tilde{\Psi}_{is}\right)\frac{1}{\sigma_{is}^3} + \frac{3}{2}\left(r_i r_s \tilde{\Psi}_{is}\right)^2 \frac{1}{\sigma_{is}^5} + \frac{5}{2}\left(r_i r_s \tilde{\Psi}_{is}\right)^3 \frac{1}{\sigma_{is}^7} + \ldots \tag{22}$$

The decomposition of the perturbing function for the outer planets is obtained in a completely similar way

$$\frac{1}{r_{ik}} = \frac{1}{\sigma_{ik}} + \left(r_i r_k \tilde{\Psi}_{ik}\right)\frac{1}{\sigma_{ik}^3} + \frac{3}{2}\left(r_i r_k \tilde{\Psi}_{ik}\right)^2 \frac{1}{\sigma_{ik}^5} + \frac{5}{2}\left(r_i r_k \tilde{\Psi}_{ik}\right)^3 \frac{1}{\sigma_{ik}^7} + \ldots, \qquad (i < k) \tag{23}$$

Substituting the resulting decomposition (19) into the system (15), we have the equations of perturbed motion in explicit form.

In the case of the absence of resonance, averaging over average longitudes λ_i, we obtain the equations of secular perturbations

$$\begin{aligned}
\dot{\lambda}_i &= -\frac{\partial R_i^*}{\partial \Lambda_i} = \frac{\mu_{i0}^2}{\gamma_i^2 \Lambda_i^3} - \frac{\partial W_i^{(\text{sec})}}{\partial \Lambda_i}, & \dot{\Lambda}_i &= 0, \\
\dot{\eta}_i &= -\frac{\partial R_i^*}{\partial \xi_i} = -\frac{\partial W_i^{(\text{sec})}}{\partial \xi_i}, & \dot{\xi}_i &= \frac{\partial R_i^*}{\partial \eta_i} = \frac{\partial W_i^{(\text{sec})}}{\partial \eta_i}, \\
\dot{q}_i &= -\frac{\partial R_i^*}{\partial p_i} = -\frac{\partial W_i^{(\text{sec})}}{\partial p_i}, & \dot{p}_i &= \frac{\partial R_i^*}{\partial q_i} = \frac{\partial W_i^{(\text{sec})}}{\partial q_i}.
\end{aligned} \tag{24}$$

here

$$W_i^{(\text{sec})} = W_{gi}^{(\text{sec})} + W_{ri}^{(\text{sec})}$$

$$W_{gi}^{(\text{sec})} = W_{gi,main}^{(\text{sec})} = W_{is,main}^{(\text{sec})} + W_{ik,main}^{(\text{sec})} \tag{25}$$

$$W_{ri}^{(\text{sec})} = -\frac{\ddot{\gamma}_i \Lambda_i^4}{2\gamma_i \mu_{i0}^2}\left(1 + \frac{3}{2\Lambda_i}\left(\xi_i^2 + \eta_i^2\right)\right)$$

Secular perturbation function for inner planets $(s < i)$

$$W_{is,main}^{(\text{sec})} = f\sum_{s=1}^{i-1} m_s \left(\frac{A_0^{is}}{2} + \Pi_{ii}^{is}\frac{\eta_i^2 + \xi_i^2}{2\Lambda_i} + \Pi_{is}^{is}\frac{\eta_i\eta_s + \xi_i\xi_s}{\sqrt{\Lambda_i\Lambda_s}} + \Pi_{ss}^{is}\frac{\eta_s^2 + \xi_s^2}{2\Lambda_s} - B_1^{is}\left(\frac{p_i^2 + q_i^2}{8\Lambda_i} - \frac{p_i p_s + q_i q_s}{4\sqrt{\Lambda_i\Lambda_s}} + \frac{p_s^2 + q_s^2}{8\Lambda_s}\right)\right) \tag{26}$$

Secular perturbation function for the outer planets $(i < k)$

$$W_{ik,main}^{(\text{sec})} = f\sum_{k=i+1}^{n} m_k \left(\frac{A_0^{ik}}{2} + \Pi_{ii}^{ik}\frac{\eta_i^2 + \xi_i^2}{2\Lambda_i} + \Pi_{ik}^{ik}\frac{\eta_i\eta_k + \xi_i\xi_k}{\sqrt{\Lambda_i\Lambda_k}} + \Pi_{kk}^{ik}\frac{\eta_k^2 + \xi_k^2}{2\Lambda_k} - B_1^{ik}\left(\frac{p_i^2 + q_i^2}{8\Lambda_i} - \frac{p_i p_k + q_i q_k}{4\sqrt{\Lambda_i\Lambda_k}} + \frac{p_k^2 + q_k^2}{8\Lambda_k}\right)\right) \tag{27}$$

The evolution equations of the n planetary problem with variable masses have the following form

$$\dot{\xi}_i = f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\eta_i,$$

$$\dot{\eta}_i = -f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\xi_i,$$

$$\dot{p}_i = -f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right) \quad (28)$$

$$\dot{q}_i = f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) + f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}}\right)$$

$$\dot{\lambda}_i = \frac{\mu_{i0}^2}{\gamma_i^2\Lambda_i^3} - \frac{\partial W_i^{(\mathrm{sec})}}{\partial\Lambda_i}, \quad \dot{\Lambda}_i = 0, \qquad \alpha_{is} = \frac{\gamma_s a_s}{\gamma_i a_i} = \alpha_{is}(t) < 1, \quad \alpha_{ik} = \frac{\gamma_i a_i}{\gamma_k a_k} = \alpha_{ik}(t) < 1, \quad \begin{matrix}(s<i)\\(i<k)\end{matrix}$$

here

$$\Pi_{ii}^{is} = -\frac{3\alpha_{is}}{4}B_0^{is} - \frac{1}{2}B_1^{is} + \frac{15+6\alpha_{is}^2}{8}C_0^{is} - \frac{3\alpha_{is}}{2}C_1^{is} - \frac{9}{8}C_2^{is}$$

$$\Pi_{is}^{is} = \frac{1}{8}\left(9B_0^{is} + B_2^{is}\right) - \frac{9\left(1+\alpha_{is}^2\right)}{8\alpha_{is}}C_0^{is} + \frac{21}{16}C_1^{is} + \frac{3\left(1+\alpha_{is}^2\right)}{8\alpha_{is}}C_2^{is} + \frac{3}{16}C_3^{is}$$

$$\Pi_{ii}^{ik} = -\frac{3\alpha_{ik}}{4}B_0^{ik} - \frac{1}{2}B_1^{ik} + \frac{15+6\alpha_{ik}^2}{8}C_0^{ik} - \frac{3\alpha_{ik}}{2}C_1^{ik} - \frac{9}{8}C_2^{ik} \quad (29)$$

$$\Pi_{ik}^{ik} = \frac{1}{8}\left(9B_0^{ik} + B_2^{ik}\right) - \frac{9\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}}C_0^{ik} + \frac{21}{16}C_1^{ik} + \frac{3\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}}C_2^{ik} + \frac{3}{16}C_3^{ik}$$

The Laplace coefficients $B_0^{ij}, B_1^{ij}, B_2^{ij}, C_0^{ij}, C_1^{ij}, C_2^{ij}, C_3^{ij}$ $(i \neq j)$ are given in the work [6]

[6] Murray, K.D. and Dermot, S.F., *Solar System Dynamics*, Cambridge: Cambridge Univ. Press, 1999.

Approximate formulas for the relationship of various systems of osculating elements, as initial assumptions, have the form

$$\xi_i \approx \sqrt{\Lambda_i} e_i \cos\pi_i, \quad \eta_i \approx -\sqrt{\Lambda_i} e_i \sin\pi_i, \quad \Lambda_i e_i^2 \approx \xi_i^2 + \eta_i^2, \quad tg\pi_i = -\eta_i / \xi_i,$$
$$p_i \approx \sqrt{\Lambda_i} \sin i_i \cos\Omega_i, \quad q_i \approx -\sqrt{\Lambda_i} \sin i_i \sin\Omega_i, \quad \Lambda_i \sin^2 i_i \approx p_i^2 + q_i^2, \quad tg\Omega_i = -q_i / p_i. \qquad (30)$$

Systems of equations of eccentric elements

$$\dot{\xi}_i = f\sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s \right) + f\sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k \right) - \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\eta_i,$$
$$\dot{\eta}_i = -f\sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s \right) - f\sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k \right) + \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\xi_i \qquad (31)$$

Systems of equations of oblique elements

$$\dot{p}_i = -f\sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}} \right) - f\sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}} \right)$$
$$\dot{q}_i = f\sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}} \right) + f\sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}} \right) \qquad (32)$$

To transition to dimensionless variables, we use the following physical units of measurement: t - year , a_i - astronomical unit, m_i - the mass of the Sun.
In the evolutionary equations (31) – (32) we turn to dimensionless variables $t^* = \tau = \omega_1 t$, $a_i^* = a_i / a_1$, $m_i^* = m_i / m_{00}$, here

$$\omega_1 = \frac{\sqrt{f m_{00}}}{a_1^{3/2}} = const, \quad T_1^t = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{f m_{00}}} a_1^{3/2} = const, \quad m_i = m_{00} m_i^*, \quad a_i = a_1 a_i^*, \quad m_{00} = m_0(t_0) = const, \quad a_1 = a_1(t_0) = const, \quad \frac{d}{d\tau} = (\ )', \qquad (33)$$

Then we can write

$$\xi_i = \xi_i^* \left(f m_{00} a_1 \right)^{1/4}, \quad \eta_i = \eta_i^* \left(f m_{00} a_1 \right)^{1/4}, \quad p_i = p_i^* \left(f m_{00} a_1 \right)^{1/4}, \quad q_i = q_i^* \left(f m_{00} a_1 \right)^{1/4}, \quad \Lambda_i = \sqrt{f m_{00}} \sqrt{a_1} \Lambda_i^* \qquad (34)$$

here

$$\Lambda_i^* = \sqrt{\mu_{i0}^*} \sqrt{a_i^*}, \quad \mu_{i0}^* = 1 + \frac{m_{i0}}{m_{00}} = const, \quad \frac{3\ddot{\gamma}_i \Lambda_i^3}{2\gamma_i \mu_{i0}^2} = \omega_1 \frac{3\gamma_i''}{2\gamma_i} \frac{\Lambda_i^{*3}}{\mu_{i0}^{*2}}, \quad \frac{d^2}{d\tau^2} = (\ )'' \qquad (35)$$

Thus, dimensionless eccentric and oblique elements have the following form

$$\begin{aligned} \xi_i^* &= \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\left(1-\sqrt{1-e_i^2}\right)} Cos\pi_i, \quad \eta_i^* = -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\left(1-\sqrt{1-e_i^2}\right)} Sin\pi_i, \\ p_i^* &= \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}\,(1-Cosi_i)} Cos\Omega_i, \quad q_i^* = -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}\,(1-Cosi_i)} Sin\Omega_i. \end{aligned} \qquad (36)$$

As a result, by dividing the left and right sides of equation (31) – (32) by a common multiplier $\omega_1 \left(f m_{00} a_1 \right)^{1/4} = const$ and omitting the sign (*) we obtain dimensionless differential equations for eccentric elements

$$\begin{aligned} \xi_i' &= \sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i \Lambda_s}}\eta_s \right) + \sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{ii}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i \Lambda_k}}\eta_k \right) - \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\eta_i, \\ \eta_i' &= -\sum_{s=1}^{i-1} m_s \left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i \Lambda_s}}\xi_s \right) - \sum_{k=i+1}^{n} m_k \left(\frac{\Pi_{ii}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i \Lambda_k}}\xi_k \right) + \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\xi_i \end{aligned} \qquad (37)$$

Dimensionless differential equations for oblique elements have the form

$$\begin{aligned} p_i' &= -\sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i \Lambda_s}} \right) - \sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i \Lambda_k}} \right) \\ q_i' &= \sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i \Lambda_s}} \right) + \sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i \Lambda_k}} \right) \end{aligned} \qquad (38)$$

The evolutionary equations of the four-planet exosystem *V1298 Tau* [7] in explicit form

The system of equations of eccentric elements consists of eight equations

$$\begin{aligned}
\xi_1 &= (\beta_2^{1,2}+\beta_2^{1,3}+\beta_2^{1,4}+\beta_3^{1})\cdot\eta_1+\beta_1^{1,2}\cdot\eta_2+\beta_1^{1,3}\cdot\eta_3+\beta_1^{1,4}\cdot\eta_4,\\
\xi_2 &= \beta_1^{2,1}\cdot\eta_1+(\beta_2^{2,1}+\beta_2^{2,3}+\beta_2^{2,4}+\beta_3^{2})\cdot\eta_2+\beta_1^{2,3}\cdot\eta_3+\beta_1^{2,4}\cdot\eta_4,\\
\xi_3 &= \beta_1^{3,1}\cdot\eta_1+\beta_1^{3,2}\cdot\eta_2+(\beta_2^{3,1}+\beta_2^{3,2}+\beta_2^{3,4}+\beta_3^{3})\cdot\eta_3+\beta_1^{3,4}\cdot\eta_4,\\
\xi_4 &= \beta_1^{4,1}\cdot\eta_1+\beta_1^{4,2}\cdot\eta_2+\beta_1^{4,3}\cdot\eta_3+(\beta_2^{4,1}+\beta_2^{4,2}+\beta_2^{4,3}+\beta_3^{4})\cdot\eta_4,
\end{aligned}
\qquad
\begin{aligned}
\eta_1 &= -(\beta_2^{1,2}+\beta_2^{1,3}+\beta_2^{1,4}+\beta_3^{1})\cdot\xi_1-\beta_1^{1,2}\cdot\xi_2-\beta_1^{1,3}\cdot\xi_3-\beta_1^{1,4}\cdot\xi_4,\\
\eta_2 &= -\beta_1^{2,1}\cdot\xi_1-(\beta_2^{2,1}+\beta_2^{2,3}+\beta_2^{2,4}+\beta_3^{2})\cdot\xi_2-\beta_1^{2,3}\cdot\xi_3-\beta_1^{2,4}\cdot\xi_4,\\
\eta_3 &= -\beta_1^{3,1}\cdot\xi_1-\beta_1^{3,2}\cdot\xi_2-(\beta_2^{3,1}+\beta_2^{3,2}+\beta_2^{3,4}+\beta_3^{3})\cdot\xi_3-\beta_1^{3,4}\cdot\xi_4,\\
\eta_4 &= -\beta_1^{4,1}\cdot\xi_1-\beta_1^{4,2}\cdot\xi_2-\beta_1^{4,3}\cdot\xi_3-(\beta_2^{4,1}+\beta_2^{4,2}+\beta_2^{4,3}+\beta_3^{4})\cdot\xi_4
\end{aligned}
\qquad (39)$$

The system of equations of oblique elements consists of eight equations

$$\begin{aligned}
p_1 &= -(\chi_2^{1,2}+\chi_2^{1,3}+\chi_2^{1,4})\cdot q_1+\chi_1^{1,2}\cdot q_2+\chi_1^{1,3}\cdot q_3+\chi_1^{1,4}\cdot q_4,\\
p_2 &= \chi_1^{2,1}\cdot q_1-(\chi_2^{2,1}+\chi_2^{2,3}+\chi_2^{2,4})\cdot q_2+\chi_1^{2,3}\cdot q_3+\chi_1^{2,4}\cdot q_4,\\
p_3 &= \chi_1^{3,1}\cdot q_1+\chi_1^{3,2}\cdot q_2-(\chi_2^{3,1}+\chi_2^{3,2}+\chi_2^{3,4})\cdot q_3+\chi_1^{3,4}\cdot q_4,\\
p_4 &= \chi_1^{4,1}\cdot q_1+\chi_1^{4,2}\cdot q_2+\chi_1^{4,3}\cdot q_3-(\chi_2^{4,1}+\chi_2^{4,2}+\chi_2^{4,3})\cdot q_4,
\end{aligned}
\qquad
\begin{aligned}
q_1 &= (\chi_2^{1,2}+\chi_2^{1,3}+\chi_2^{1,4})\cdot p_1-\chi_1^{1,2}\cdot p_2-\chi_1^{1,3}\cdot p_3-\chi_1^{1,4}\cdot p_4,\\
q_2 &= -\chi_1^{2,1}\cdot p_1+(\chi_2^{2,1}+\chi_2^{2,3}+\chi_2^{2,4})\cdot p_2-\chi_1^{2,3}\cdot p_3-\chi_1^{2,4}\cdot p_4,\\
q_3 &= -\chi_1^{3,1}\cdot p_1-\chi_1^{3,2}\cdot p_2+(\chi_2^{3,1}+\chi_2^{3,2}+\chi_2^{3,4})\cdot p_3-\chi_1^{3,4}\cdot p_4,\\
q_4 &= -\chi_1^{4,1}\cdot p_1-\chi_1^{4,2}\cdot p_2-\chi_1^{4,3}\cdot p_3+(\chi_2^{4,1}+\chi_2^{4,2}+\chi_2^{4,3})\cdot p_4
\end{aligned}
\qquad (40)$$

here

$$\beta_1^{i,s}=\frac{m_s\Pi_{i,s}^{i,s}}{\sqrt{\Lambda_s\Lambda_i}},\quad \beta_1^{i,k}=\frac{m_k\Pi_{i,k}^{i,k}}{\sqrt{\Lambda_i\Lambda_k}},\quad \beta_3^{i}=-\frac{3\Lambda_i^3}{2\mu_{i0}^2}\frac{\gamma_i''(t)}{\gamma_i},\quad \chi_1^{i,s}=\frac{1}{4}\frac{m_sB_1^{i,s}}{\sqrt{\Lambda_s\Lambda_i}},\quad \chi_1^{i,k}=\frac{1}{4}\frac{m_kB_1^{i,k}}{\sqrt{\Lambda_i\Lambda_k}},$$

$$\beta_2^{i,s}=\frac{m_s\Pi_{i,i}^{i,s}}{\Lambda_i},\quad \beta_2^{i,k}=\frac{m_k\Pi_{i,i}^{i,k}}{\Lambda_i},\quad \chi_2^{i,s}=\frac{1}{4}\frac{m_sB_1^{i,s}}{\Lambda_i},\quad \chi_2^{i,k}=\frac{1}{4}\frac{m_kB_1^{i,k}}{\Lambda_i}.$$

Further, the evolutionary equations (39)-(40) are investigated numerically.

[7] David Trevor J., Petigura Erik A., Luger Rodrigo, Foreman-Mackey Daniel, Livingston John H., Mamajek Eric E., Hillenbrand Lynne A. Four Newborn Planets Transiting the Young Solar Analog V1298 Tau // The Astrophysical Journal Letters, - 2019, Vol. 885(1) :L12, 10pp. DOI: 10.3847/2041-8213/ab4c99

Conclusion

The main result:

- ✓ We obtained the canonical equations of perturbed motion of a multiplanetary problem with isotropically varying masses at different rates in analogues of the second system of Poincare elements.
- ✓ We obtained the equations of secular perturbed of a multiplanetary problem with isotropically varying masses at different rates in analogues of the second system of Poincare elements.
- ✓ The secular equations obtained by us can be used for any n planetary problem with variable masses at different rates.
- ✓ We have explicitly written the evolutionary equations for the V1298 Tau exosystem.
- ✓ The obtained equations of secular perturbations can be investigated by various numerical methods.

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Please write your questions, I will be happy to answer everyone *kosherbaevaayken@gmail.com*

Thank you for all your attention !