Optimal stopping and American options

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Chapter 1 Optimal stopping in discrete time

In this chapter, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a discrete time filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$. For basic results on discrete time martingales, we refer the reader to [115], chapters I to IV (also, note that chapter VI deals with optimal stopping in discrete time). The applications to American option pricing in discrete time models are presented in [98], chapter 2.

1.1 Essential supremum. Uniform integrability

1.1.1 Essential supremum

It is well known that if $(X_n)_{n \in \mathbb{N}}$ is a sequence of real valued random variables, $\sup_{n \in \mathbb{N}} X_n$ is a random variable (with values in $\mathbb{R} \cup \{+\infty\}$). When uncountable families of random variables have to be considered, as occurs in the theory of optimal stopping, the notion of *essential* upper bound is needed.

Theorem 1.1.1 Let $(X_i)_{i \in I}$ be a family of real valued random variables (with a possibly uncountable index set I). There exists a random variable \overline{X} with values in $\overline{\mathbb{R}}$, which is unique up to null events, such that

- 1. For all $i \in I$, $X_i \leq \overline{X}$ a.s..
- 2. If X is a random variable with values in \mathbb{R} satisfying $X_i \leq X$ a.s., for all $i \in I$, then $\overline{X} \leq X$ a.s..

Moreover, there is a countable subset J of I such that $\overline{X} = \sup_{i \in J} X_i$ a.s..

The random variable \bar{X} is called the *essential* upper bound (or essential supremum) of the family $(X_i)_{i \in I}$ and denoted by ess $\sup_{i \in I} X_i$.

Proof: By using a one-to-one increasing mapping from $\overline{\mathbb{R}}$ onto [0,1], we can assume that the X_i 's take on values in [0,1]. Now, given a countable subset J of I, set

$$\bar{X}_J = \sup_{i \in J} X_i$$

This defines, for each J, a random variable with values in [0, 1]. Denote by \mathcal{P}_0 the set of all countable subsets of I and let

$$\alpha = \sup_{J \in \mathcal{P}_0} \mathbb{E} X_J.$$

Consider a sequence $(J_n)_{n\in\mathbb{N}}$ of elements in \mathcal{P}_0 such that $\lim_{n\to\infty} \mathbb{E}\bar{X}_{J_n} = \alpha$. The union $J^* = \bigcup_{n\in\mathbb{N}} J_n$ is a countable subset of I and we have $\alpha = \mathbb{E}\bar{X}_{J^*}$. We will now prove that the random variable $\bar{X} = \bar{X}_{J^*}$ satisfies the required conditions.

First, fix $i \in I$. The set $J^* \cup \{i\}$ is a countable subset of I and $\bar{X}_{J^* \cup \{i\}} = \bar{X} \vee X_i$. Therefore, $\mathbb{E}(\bar{X} \vee X_i) \leq \mathbb{E}\bar{X}$, and $\bar{X} \vee X_i = \bar{X}$ a.s., which means that $X_i \leq \bar{X}$ a.s..

Next, consider a random variable X such that $X \ge X_i$ a.s., for all $i \in I$. Since J^* is countable, we have $X \ge \sup_{i \in J^*} X_i = \overline{X}$ a.s.. \diamond

Definition 1.1.2 A family $(X_i)_{i \in I}$ of real valued random variables is said to have the lattice property *if, for all indices i, j \in I, there exists an index k \in I such that X_k ≥ X_i ∨ X_j a.s..*

Proposition 1.1.3 If $(X_i)_{i \in I}$ has the lattice property, there exists a sequence $(i_n)_{n \in \mathbb{N}}$ of indices such that the sequence $(X_{i_n})_{n \in \mathbb{N}}$ is nondecreasing (up to null events) and ess $\sup_{i \in I} X_i = \sup_{n \in \mathbb{N}} X_{i_n} = \lim_{n \to \infty} X_{i_n}$ a.s..

Proof: Let $(j_n)_{n\in\mathbb{N}}$ be a sequence of indices such that ess $\sup_{i\in I} X_i = \sup_{n\in\mathbb{N}} X_{j_n}$ a.s. Note that the existence of such a sequence follows from Theorem 1.1.1. Since the family $(X_i)_{i\in I}$ has the lattice property, one can construct a sequence $(i_n)_{n\in\mathbb{N}}$ of indices such that, for every integer $n \geq 1$, $X_{i_n} \geq \max(X_{j_0}, \ldots, X_{j_n}, X_{i_{n-1}})$ a.s.. This sequence satisfies the desired properties.

Proposition 1.1.4 Let $(X_i)_{i\in I}$ be a family of nonnegative random variables with the lattice property. We have $\mathbb{E}(\operatorname{ess sup}_{i\in I}X_i) = \sup_{i\in I}\mathbb{E}X_i$ and, more generally, for any sub- σ -field \mathcal{B} , $\mathbb{E}(\operatorname{ess sup}_{i\in I}X_i \mid \mathcal{B}) = \operatorname{ess sup}_{i\in I}\mathbb{E}(X_i \mid \mathcal{B})$ a.s.. These equalities remain valid if the nonnegativity assumption is replaced by $\mathbb{E}\operatorname{ess sup}_{i\in I}|X_i| < \infty$.

Proof: Use the previous proposition together with a monotone convergence argument for the case $X_i \ge 0$, and a dominated convergence argument for the case \mathbb{E} ess $\sup_{i \in I} |X_i| < \infty$. Note that it would be sufficient to have uniform integrability of the family (X_i) (see the next section).

1.1.2 Uniform integrability

Definition 1.1.5 A family $(X_i)_{i \in I}$ of real integrable random variables is called uniformly integrable if $\lim_{a \to +\infty} \sup_{i \in I} \mathbb{E}\left(|X_i| \mathbf{1}_{\{|X_i| \ge a\}}\right) = 0.$

It is easy to verify (exercise!) that a finite family of integrable random variables is uniformly integrable. it can also be proved that if $(X_n)_{n\in\mathbb{N}}$ is bounded in L^p for some p > 1, the sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable. This follows from Hölder's inequality, from which we derive $\mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i|\geq a\}}\right) \leq$

$$||X_i||_p (\mathbb{P}(|X_i| \ge a)^{1-(1/p)} \le ||X_i||_p \left(\frac{||X_i||_1}{a}\right)^{1-(1/p)} \le \frac{||X_i||_p^{2-(1/p)}}{a^{1-(1/p)}}.$$

Proposition 1.1.6 A family $(X_i)_{i \in I}$ of integrable random variables is uniformly integrable if and only if $\sup_{i \in I} \mathbb{E}|X_i| < \infty$ and, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\mathbb{P}(A) \le \eta \quad \Rightarrow \quad \sup_{i \in I} \mathbb{E}\left(|X_i| \mathbf{1}_A\right) \le \varepsilon.$$
(1.1)

Proof: If $(X_i)_{i\in I}$ is uniformly integrable, we can find a > 0 such that $\sup_{i\in I} \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i|\geq a\}}\right) \leq 1$. We then have $\mathbb{E}\left(|X_i|\right) \leq \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i|\geq a\}}\right) + \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i|< a\}}\right) \leq 1 + a$. Hence, $\sup_{i\in I} \mathbb{E}|X_i| < \infty$. On the other hand, for any event A, $\mathbb{E}\left(|X_i|\mathbf{1}_A\right) = \mathbb{E}\left(|X_i|\mathbf{1}_{A\cap\{|X_i|\geq a\}}\right) + \mathbb{E}\left(|X_i|\mathbf{1}_{A\cap\{|X_i|< a\}}\right)$. Given $\varepsilon > 0$, choose a such that $\sup_{i\in I} \mathbb{E}\left(|X_i|\mathbf{1}_{\{|X_i|\geq a\}}\right) \leq \varepsilon/2$. We have $\mathbb{E}\left(|X_i|\mathbf{1}_A\right) \leq (\varepsilon/2) + a\mathbb{P}(A)$ and it suffices to take $\eta = \varepsilon/(2a)$.

Conversely, if $(X_i)_{i\in I}$ is bounded in L^1 and satisfies (1.1), we have $\mathbb{P}(|X_i| \ge a) \le M/a$, with $M = \sup_{i\in I} \mathbb{E}|X_i|$. Hence, given $\varepsilon > 0$ and the associated η given by (1.1), we have, for $a > M/\eta$, $\sup_{i\in I} \mathbb{E}(|X_i|\mathbf{1}_{\{|X_i|\ge a\}}) \le \varepsilon$.

Proposition 1.1.7 If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ is an integrable random variable, the family of all conditional expectations $\mathbb{E}(X|\mathcal{B})$, where \mathcal{B} is any sub- σ -field of \mathcal{F} , is uniformly integrable.

Proof: We have, with the notation $X_{\mathcal{B}} = \mathbb{E}(X|\mathcal{B})$, $\mathbb{E}(|X_{\mathcal{B}}|\mathbf{1}_{\{|X_{\mathcal{B}}|\geq a\}}) \leq \mathbb{E}(\mathbb{E}(|X| | \mathcal{B})\mathbf{1}_{\{|X_{\mathcal{B}}|\geq a\}}) = \mathbb{E}(|X|\mathbf{1}_{\{|X_{\mathcal{B}}|\geq a\}})$, since $X_{\mathcal{B}}$ is \mathcal{B} -mesurable. The single integrable random variable X being uniformly integrable, we know that, for any $\varepsilon > 0$, there exists $\eta > 0$, such that

$$\mathbb{P}(A) < \eta \quad \Rightarrow \quad \mathbb{E}\left(|X|\mathbf{1}_A\right) < \varepsilon$$

Now, $\mathbb{P}(|X_{\mathcal{B}}| \ge a) \le \mathbb{E}|X_{\mathcal{B}}|/a \le \mathbb{E}|X|/a$. Therefore, if $a > \mathbb{E}|X|/\eta$, $\mathbb{E}(|X_{\mathcal{B}}|\mathbf{1}_{\{|X_{\mathcal{B}}| \ge a\}}) \le \varepsilon$.

Proposition 1.1.8 Let $(X_n)_{n \in \mathbb{N}}$ be a uniformly integrable sequence which converges in probability to a random variable X. Then, X is integrable and we have convergence in L^1 .

Proof: The integrability of X comes from the L^1 -boundedness of $(X_n)_{n\in\mathbb{N}}$ together with Fatou's lemma. For L^1 convergence, we note that $\mathbb{E}|X_n-X| \leq \mathbb{E}|X_n-X| \mathbf{1}_{\{|X_n-X|\geq\varepsilon\}} + \varepsilon \leq \mathbb{E}|X_n| \mathbf{1}_{\{|X_n-X|\geq\varepsilon\}} + \mathbb{E}|X| \mathbf{1}_{\{|X_n-X|\geq\varepsilon\}} + \varepsilon$. Using the fact that $\lim_{n\to\infty} \mathbb{P}(|X_n-X|\geq\varepsilon) = 0$ and Proposition 1.1.6, the conclusion follows easily.

1.2 The Snell envelope

Let $Z = (Z_n)_{n \in \mathbb{N}}$ be an (\mathcal{F}_n) -adapted sequence of random variables with $\mathbb{E}\sup_{n \in \mathbb{N}} |Z_n| < \infty$. The optimal stopping problem for Z consists in maximising $\mathbb{E}(Z_{\nu})$ over all finite stopping times ν . The sequence $(Z_n)_{n \in \mathbb{N}}$ is called the *reward* sequence, in reference to gambling. In the setting of American options, Z_n is the profit attached to exercizing the option at time n.

Let \mathcal{T} be the set of all stopping times with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$. We will use the following notation:

$$\begin{aligned} \mathcal{T}_{n,N} &= \{ \nu \in \mathcal{T} \mid \mathbb{P}(\nu \in [n,N]) = 1 \}, \quad 0 \le n \le N, \\ \mathcal{T}_{n,\infty} &= \{ \nu \in \mathcal{T} \mid \mathbb{P}(\nu \in [n,+\infty)) = 1 \}, \quad n \in \mathbb{N}. \end{aligned}$$

The Snell envelope of $(Z_n)_{n \in \mathbb{N}}$ is the sequence $(U_n)_{n \in \mathbb{N}}$ defined by

$$U_n = \operatorname{ess\,sup}_{\nu \in \mathcal{T}_n \, \infty} \mathbb{E} \left(Z_\nu \mid \mathcal{F}_n \right)$$

Note that $(U_n)_{n\in\mathbb{N}}$ is defined up to null events. Let $Z_{\infty}^* = \sup_{n\in\mathbb{N}} |Z_n|$. The random variable Z_{∞}^* is, by assumption, integrable and we have $|Z_{\nu}| \leq Z_{\infty}^*$ a.s., for every $\nu \in \mathcal{T}_{0,\infty}$. Therefore,

$$\forall n \in \mathbb{N}, \quad |U_n| \leq \mathbb{E} \left(Z_{\infty}^* \mid \mathcal{F}_n \right),$$

which proves that the sequence $(U_n)_{n\in\mathbb{N}}$ is bounded in L^1 , and, in fact, uniformly integrable (cf. Proposition 1.1.7). More precisely, we have, for $\nu \in \mathcal{T}_{0,\infty}$, $|U_{\nu}| \leq \mathbb{E}(Z_{\infty}^* | \mathcal{F}_{\nu})$ and the family $(U_{\nu})_{\nu \in \mathcal{T}_{0,\infty}}$ is uniformly integrable.

Theorem 1.2.1 The Snell envelope U of the sequence Z satisfies the following properties:

- 1. For every integer $n \in \mathbb{N}$, $\mathbb{E}U_n = \sup_{\nu \in \mathcal{T}_{n,\infty}} \mathbb{E}(Z_{\nu})$.
- 2. For every integer $n \in \mathbb{N}$, $U_n = \max(Z_n, \mathbb{E}(U_{n+1}|\mathcal{F}_n))$, a.s..
- 3. U is the smallest supermartingale majorant of Z.

The proof of Theorem 1.2.1 relies on the following lemma.

Lemma 1.2.2 Fix $n \in \mathbb{N}$. The family $(\mathbb{E}(Z_{\nu}|\mathcal{F}_n), \nu \in \mathcal{T}_{n,\infty})$ has the lattice property.

Proof: Let $\nu_1, \nu_2 \in \mathcal{T}_{n,\infty}$ and $X_i = \mathbb{E}(Z_{\nu_i}|\mathcal{F}_n)$ (i = 1, 2). Define a stopping time ν by setting:

$$\nu = \nu_1 \mathbf{1}_{\{X_1 \ge X_2\}} + \nu_2 \mathbf{1}_{\{X_1 < X_2\}}.$$

We have $\nu \in \mathcal{T}_{n,\infty}$ and $\mathbb{E}(Z_{\nu}|\mathcal{F}_n) \geq \mathbb{E}(Z_{\nu_i}|\mathcal{F}_n)$, for i = 1, 2.

Proof of Theorem 1.2.1: The first assertion follows immediately from Lemma 1.2.2 and Proposition 1.1.4.

For the second assertion, first consider a stopping time $\nu \in \mathcal{T}_{n,\infty}$. We have (with probability one)

$$\begin{split} \mathbb{E}(Z_{\nu}|\mathcal{F}_{n}) &= Z_{n}\mathbf{1}_{\{\nu=n\}} + \mathbb{E}(Z_{\nu}\mathbf{1}_{\{\nu>n\}}|\mathcal{F}_{n}) \\ &= Z_{n}\mathbf{1}_{\{\nu=n\}} + \mathbf{1}_{\{\nu>n\}}\mathbb{E}(Z_{\nu\vee(n+1)}|\mathcal{F}_{n}) \\ &\leq Z_{n}\mathbf{1}_{\{\nu=n\}} + \mathbf{1}_{\{\nu>n\}}\mathbb{E}(U_{n+1}|\mathcal{F}_{n}) \\ &\leq \max\left(Z_{n}, \mathbb{E}(U_{n+1}|\mathcal{F}_{n})\right). \end{split}$$

It follows that $U_n \leq \max(Z_n, \mathbb{E}(U_{n+1}|\mathcal{F}_n))$ a.s.. The inequality $U_n \geq Z_n$ a.s. comes from the deterministic time *n* being in $\mathcal{T}_{n,\infty}$. On the other hand, the family $(\mathbb{E}(Z_{\nu}|\mathcal{F}_{n+1}), \nu \in \mathcal{T}_{n+1,\infty})$ satisfying the lattice property and being dominated by $\mathbb{E}(Z_{\infty}^* | \mathcal{F}_{n+1})$, we have, using Proposition 1.1.4 again,

$$\mathbb{E}(U_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\left(\operatorname{ess sup}_{\nu \in \mathcal{T}_{n+1,\infty}} \mathbb{E}(Z_{\nu} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\right) = \operatorname{ess sup}_{\nu \in \mathcal{T}_{n+1,\infty}} \mathbb{E}\left(\mathbb{E}(Z_{\nu} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\right),$$

Now, if $\nu \in \mathcal{T}_{n+1,\infty}$,

$$\mathbb{E}(\mathbb{E}(Z_{\nu}|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(Z_{\nu}|\mathcal{F}_n) \le U_n \text{ a.s.}$$

where the last inequality comes from the inclusion $\mathcal{T}_{n+1,\infty} \subset \mathcal{T}_{n,\infty}$. Hence $U_n \ge \max(Z_n, \mathbb{E}(U_{n+1}|\mathcal{F}_n))$ a.s..

From this property, we deduce the inequality $U_n \geq \mathbb{E}(U_{n+1}|\mathcal{F}_n)$, a.s., which means that U is a supermartingale. Now, if a supermartingale $(V_n)_{n\in\mathbb{N}}$ satisfies $V_n \geq Z_n$ a.s. for every $n \in \mathbb{N}$, we have

$$\forall \nu \in \mathcal{T}_{n,\infty}, \quad V_{\nu} \ge Z_{\nu} \text{ a.s.}.$$

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Hence $\mathbb{E}(V_{\nu} \mid \mathcal{F}_n) \geq \mathbb{E}(Z_{\nu} \mid \mathcal{F}_n)$ a.s.. Using the optional sampling theorem, we have, for every $N \geq n$,

$$V_n \ge \mathbb{E}(V_{\nu \wedge N} \mid \mathcal{F}_n) \ge \mathbb{E}(Z_{\nu \wedge N} \mid \mathcal{F}_n),$$

and, by Fatou's lemma (applied to the sequence $(Z_{\nu \wedge N})_{N \in \mathbb{N}}$, which is bounded below by the integrable random variable $-Z_{\infty}^*$), $V_n \geq \mathbb{E}(Z_{\nu} \mid \mathcal{F}_n)$, hence $V_n \geq U_n$ a.s., which proves that U is the smallest supermartingale dominating Z.

Remark 1.2.3 Since U is a supermartingale, so is $(U_{\nu \wedge n})_{n \in \mathbb{N}}$, for every stopping time ν . It follows that $\mathbb{E}U_{\nu \wedge n} \leq \mathbb{E}U_0$ and that, for $\nu \in \mathcal{T}_{0,\infty}$, $\mathbb{E}U_{\nu} \leq \mathbb{E}U_0$, by letting n go to infinity and using the uniform integrability of $(U_{\nu \wedge n})_{n \in \mathbb{N}}$.

Remark 1.2.4 One can define the Snell envelope with finite horizon N, $(U_n^{(N)})_{0 \le n \le N}$, by setting

$$U_n^{(N)} = \operatorname{ess\,sup}_{\nu \in \mathcal{T}_{n,N}} \mathbb{E}(Z_\nu \mid \mathcal{F}_n), \quad 0 \le n \le N.$$

This can be viewed as a special case of the preceding discussion, by replacing Z_n by $Z_{n \wedge N}$ and \mathcal{F}_n by $\mathcal{F}_{n \wedge N}$. In particular, from property 2 of Theorem 1.2.1, the following algorithm, called the dynamic programming algorithm, can be derived:

$$\begin{cases} U_N^{(N)} = Z_N \\ U_n^{(N)} = \max\left\{Z_n, \mathbb{E}\left(U_{n+1}^{(N)} \mid \mathcal{F}_n\right)\right\} & \text{for } 0 \le n \le N-1 \end{cases}$$

The effectiveness of the dynamic programming principle is more transparent in a Markovian setting (see Section 1.5). It is easy to prove that, under the assumptions of this chapter, the Snell envelope with horizon N tends to the Snell envelope with infinite horizon as $N \to \infty$ (for fixed n, $\lim_{N\to\infty} U_n^{(N)} = U_n$ a.s.).

Since the supermartingale $(U_n)_{n \in \mathbb{N}}$ is bounded in L^1 , the sequence $(U_n)_{n \in \mathbb{N}}$ is almost surely convergent (cf. [115], chapitre IV). The following proposition specifies the limit.

Proposition 1.2.5 We have, with probability one, $\lim_{n\to\infty} U_n = \limsup_{n\to\infty} Z_n$.

Proof: From the inequality $U_n \ge Z_n$ a.s., we deduce $\limsup_{n\to\infty} Z_n \le \lim_{n\to\infty} U_n$.

On the other hand, for every $m \in \mathbb{N}$ and for $n \ge m$, we have

$$\forall \nu \in \mathcal{T}_{n,\infty}, \quad \mathbb{E}(Z_{\nu} \mid \mathcal{F}_n) \leq \mathbb{E}\left(\sup_{p \geq m} Z_p \mid \mathcal{F}_n\right),$$

so that $U_n \leq \mathbb{E}\left(\sup_{p\geq m} Z_p \mid \mathcal{F}_n\right)$ a.s.. The random variable $\sup_{p\geq m} Z_p$ is integrable, and measurable with respect to the σ -field \mathcal{F}_{∞} generated by the union of the \mathcal{F}_n 's. Therefore, $\lim_{n\to\infty} \mathbb{E}\left(\sup_{p\geq m} Z_p \mid \mathcal{F}_n\right) = \sup_{p\geq m} Z_p$ almost surely (cf. [115], Proposition II-2-11). Hence, $\lim_{n\to\infty} U_n \leq \sup_{p\geq m} Z_p$ a.s. and, by passing to the limit as m goes to infinity, $\lim_{n\to\infty} U_n \leq \limsup_{n\to\infty} Z_n$.

1.3 Optimal stopping times

The following theorem characterizes *optimal* stopping times, i.e. stopping times ν which maximize $\mathbb{E}Z_{\nu}$.

Theorem 1.3.1 A stopping time $\nu^* \in \mathcal{T}_{0,\infty}$ satisfies $\mathbb{E}Z_{\nu^*} = \sup_{\nu \in \mathcal{T}_{0,\infty}} \mathbb{E}(Z_{\nu})$ if and only if both of the following conditions are satisfied:

- 1. $U_{\nu^*} = Z_{\nu^*} \ a.s.,$
- 2. The stopped sequence $(U_{\nu^* \wedge n})_{n \in \mathbb{N}}$ is a martingale.

Proof: Note that, due to Theorem 1.2.1, ν^* is optimal if and only if $\mathbb{E} Z_{\nu^*} = \mathbb{E} U_0$. On the other hand, we know that, for every $\nu \in \mathcal{T}_{0,\infty}$, $\mathbb{E} Z_{\nu} \leq \mathbb{E} U_{\nu}$, because U dominates Z, and $\mathbb{E} U_{\nu} \leq \mathbb{E} U_0$ from Remark 1.2.3. It follows that the optimality of ν^* is equivalent to the double equality $\mathbb{E} Z_{\nu^*} = \mathbb{E} U_{\nu^*} = \mathbb{E} U_0$. Since $U \geq Z$, the first equality is equivalent to $U_{\nu^*} = Z_{\nu^*}$ a.s., and we are left with proving that $\mathbb{E} U_{\nu^*} = \mathbb{E} U_0$ if and only if $(U_{\nu^* \wedge n})_{n \in \mathbb{N}}$ is a martingale. Since $(U_{\nu^* \wedge n})_{n \in \mathbb{N}}$ is a uniformly integrable supermartingale, we have $\mathbb{E} U_{\nu^*} = \mathbb{E} U_0$ if and only if, for every $n \in \mathbb{N}$, $\mathbb{E} U_{\nu^* \wedge n} = \mathbb{E} U_0$. But the property $\mathbb{E} U_{\nu^* \wedge (n+1)} = \mathbb{E} U_{\nu^* \wedge n}$, together with the inequality $U_{\nu^* \wedge n} \geq \mathbb{E} (U_{\nu^* \wedge (n+1)} \mid \mathcal{F}_n)$, is equivalent to $U_{\nu^* \wedge n} = \mathbb{E} (U_{\nu^* \wedge (n+1)} \mid \mathcal{F}_n)$ a.s.

The following corollary provides a necessary and sufficient condition for the existence of an optimal stopping time and characterizes the smallest optimal stopping time.

Corollary 1.3.2 There exists a stopping time $\nu^* \in \mathcal{T}_{0,\infty}$ such that $\mathbb{E}Z_{\nu^*} = \sup_{\nu \in \mathcal{T}_{0,\infty}} \mathbb{E}Z_{\nu}$ if and only if $\mathbb{P}(\nu_0 < \infty) = 1$, where

$$\nu_0 = \inf\{n \in \mathbb{N} \mid U_n = Z_n\}$$

The stopping time ν_0 is then the smallest optimal stopping time.

Proof: It follows from Theorem 1.3.1 that, if there is an optimal stopping time, we have $\mathbb{P}(\nu_0 < \infty) = 1$, and any optimal stopping time ν^* satisfies $\mathbb{P}(\nu_0 \le \nu^*) = 1$.

In order to prove that, if $\mathbb{P}(\nu_0 < \infty) = 1$, ν_0 is optimal we need to verify that the sequence $(U_{\nu_0 \wedge n})_{n \in \mathbb{N}}$ is a martingale. We have

$$\mathbb{E} \left(U_{\nu_0 \wedge (n+1)} \mid \mathcal{F}_n \right) = \sum_{j=0}^n U_j \mathbf{1}_{\{\nu_0 = j\}} + \mathbb{E} \left(U_{n+1} \mathbf{1}_{\{\nu_0 \ge n+1\}} \mid \mathcal{F}_n \right)$$

$$= \sum_{j=0}^n U_j \mathbf{1}_{\{\nu_0 = j\}} + \mathbf{1}_{\{\nu_0 \ge n+1\}} \mathbb{E} \left(U_{n+1} \mid \mathcal{F}_n \right),$$

where we have used the fact that $\{\nu_0 \ge n+1\} \in \mathcal{F}_n$. On the event $\{\nu_0 \ge n+1\}$, we have $Z_n < U_n$ and, since $U_n = \max(Z_n, \mathbb{E}(U_{n+1} | \mathcal{F}_n)), U_n = \mathbb{E}(U_{n+1} | \mathcal{F}_n)$, whence

$$\mathbb{E}\left(U_{\nu_0 \wedge (n+1)} \mid \mathcal{F}_n\right) = \sum_{j=0}^n U_j \mathbf{1}_{\{\nu_0 = j\}} + \mathbf{1}_{\{\nu_0 \ge n+1\}} U_n = U_{\nu_0 \wedge n}.$$

Remark 1.3.3 This proof shows that the sequence $(U_{\nu_0 \wedge n})_{n \in \mathbb{N}}$ is a martingale, even if $\mathbb{P}(\nu_0 = \infty)$ is positive.

Remark 1.3.4 In the case of a finite horizon, the stopping time ν_0 is obviously finite (because U and Z coincide at terminal time), so that ν_0 is the smallest optimal stopping time. In the infinite horizon case, there may not exist any optimal stopping time: this occurs, in particular, if $(Z_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence.

The following proposition introduces *almost optimal* stopping times.

Proposition 1.3.5 Given $\varepsilon > 0$, the stopping time

$$\nu_{\varepsilon} = \inf\{n \in \mathbb{N} \mid U_n \le Z_n + \varepsilon\}$$

is finite almost surely and we have

$$\mathbb{E} Z_{\nu_{\varepsilon}} \geq \sup_{\nu \in \mathcal{T}_{0,\infty}} \mathbb{E} Z_{\nu} - \varepsilon.$$

In other words ν_{ε} is ε -optimal.

Proof: The fact that $\mathbb{P}(\nu_{\varepsilon} < \infty) = 1$ follows from $\lim_{n \to \infty} U_n = \limsup_{n \to \infty} Z_n$ a.s.. On the other hand, since $\nu_{\varepsilon} \leq \nu_0$ and $(U_{\nu_0 \wedge n})_{n \in \mathbb{N}}$ is a martingale, so is $(U_{\nu_{\varepsilon} \wedge n})_{n \in \mathbb{N}}$. This martingale is uniformly integrable (because $|U_n| \leq \mathbb{E}(Z_{\infty}^* | \mathcal{F}_n)$ a.s.). Therefore, $\mathbb{E}U_{\nu_{\varepsilon}} = \mathbb{E}U_0$ and, since $U_{\nu_{\varepsilon}} \leq Z_{\nu_{\varepsilon}} + \varepsilon$, $\mathbb{E}Z_{\nu_{\varepsilon}} \geq \mathbb{E}U_0 - \varepsilon$.

1.4 The Doob decomposition and the largest optimal stopping time

Recall that a *predictable* sequence is an adapted sequence $(X_n)_{n \in \mathbb{N}}$ such that, for every positive integer n, X_n is \mathcal{F}_{n-1} -measurable.

Theorem 1.4.1 Let $(U_n)_{n \in \mathbb{N}}$ be an integrable supermartingale. There exist a martingale $(M_n)_{n \in \mathbb{N}}$ and a nondecreasing, predictable sequence $(A_n)_{n \in \mathbb{N}}$, with $A_0 = 0$, such that

$$\forall n \in \mathbb{N}, \quad U_n = M_n - A_n$$

The above decomposition is unique up to null events. It is called the Doob decomposition of the supermartingale $(U_n)_{n \in \mathbb{N}}$.

Proof: We obviously have $M_0 = U_0$ and $A_0 = 0$. The martingale property for M yields

$$\mathbb{E}\left(U_{n+1} + A_{n+1} \mid \mathcal{F}_n\right) = U_n + A_n \quad \text{a.s.},$$

and, since A is predictable, $A_{n+1} - A_n = U_n - \mathbb{E}(U_{n+1} | \mathcal{F}_n)$, which determines A_n in a unique way. From this characterization of A and the supermartingale property of U, we deduce that A is nondecreasing. We then have $M_n = U_n + A_n$ and, by construction, $(M_n)_{n \in \mathbb{N}}$ is a martingale.

Now, let $(Z_n)_{n \in \mathbb{N}}$ be an adapted sequence with $\mathbb{E} \sup_{n \in \mathbb{N}} |Z_n| < \infty$ and denote by $(U_n)_{n \in \mathbb{N}}$ its Snell envelope. The Doob decomposition $U_n = M_n - A_n$ of the Snell envelope leads to additional information about optimal stopping times. **Theorem 1.4.2** A stopping time ν_{max} can be defined by setting

$$\nu_{\max} = \inf\{n \in \mathbb{N} \mid A_{n+1} > 0\}$$

Any optimal stopping time ν^* satisfies $\nu^* \leq \nu_{\max}$ a.s. and, if $\mathbb{P}(\nu_{\max} < \infty) = 1$, ν_{\max} is an optimal stopping time.

Proof: The stopping time property for ν_{max} follows from the predictability of A.

Let ν^* be an optimal stopping time. We know that the stopped sequence $(U_{\nu^* \wedge n})_{n \in \mathbb{N}}$ is a martingale. Therefore, $\mathbb{E}U_{\nu^* \wedge n} = \mathbb{E}U_0 = \mathbb{E}M_0 - \mathbb{E}A_{\nu^* \wedge n}$, for every natural integer n. Since $M_0 = U_0$, we deduce that $\mathbb{E}A_{\nu^* \wedge n} = 0$ for every n and, by monotone convegence, $\mathbb{E}A_{\nu^*} = 0$, hence $\mathbb{P}(A_{\nu^*} = 0) = 1$, so that $\nu^* \leq \nu_{\max}$ a.s..

We will now prove that, if $\mathbb{P}(\nu_{\max} < \infty) = 1$, ν_{\max} is optimal. It follows from the definition of ν_{\max} that $U_{\nu_{\max} \wedge n} = M_{\nu_{\max} \wedge n}$. Therefore, the stopped sequence $(U_{\nu_{\max} \wedge n})_{n \in \mathbb{N}}$ is a martingale. It remains to show that $Z_{\nu_{\max}} = U_{\nu_{\max}}$. We have

$$U_{\nu_{\max}} = \sum_{j=0}^{\infty} U_j \mathbf{1}_{\{\nu_{\max}=j\}}.$$

On $\{\nu_{\max} = j\}$, we have $U_j = M_j$ and $\mathbb{E}(U_{j+1}|\mathcal{F}_j) = M_j - A_{j+1} < U_j$, so that $U_j = \max(Z_j, \mathbb{E}(U_{j+1}|\mathcal{F}_j)) = Z_j$. Hence $Z_{\nu_{\max}} = U_{\nu_{\max}}$ a.s..

1.5 Optimal stopping in a Markovian setting

1.5.1 Finite horizon

Recall that a transition probability (or transition kernel) on a measurable space (E, \mathcal{E}) , is a family $(P(x, \cdot))_{x \in E}$ of probability measures on (E, \mathcal{E}) such that, for every $A \in \mathcal{E}$, the mapping $x \mapsto P(x, A)$ is measurable.

If $P = (P(x, \cdot))_{x \in E}$ is a transition kernel on (E, \mathcal{E}) and if f is a nonnegative Borel-measurable function on E, the function Pf, defined by $Pf(x) = \int_E P(x, dy)f(y)$ is measurable and nonnegative. For more details on these notions, we refer to [114], chapter 3.

Definition 1.5.1 Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$ be a filtered probability space and $(P_n)_{n \in \mathbb{N}}$ a sequence of transition kernels on a measurable space (E, \mathcal{E}) . A sequence $(X_n)_{n \in \mathbb{N}}$ of random variables with values in (E, \mathcal{E}) is an \mathbb{F} -Markov chain with transition kernels (P_n) if $(X_n)_{n \in \mathbb{N}}$ is \mathbb{F} -adapted and, for every nonnegative measurable function f on (E, \mathcal{E}) , we have $\mathbb{E}(f(X_{n+1}) | \mathcal{F}_n) = P_n f(X_n)$, for every $n \in \mathbb{N}$.

The probability measure $P_n(x, \cdot)$ can be viewed as the conditional distribution of X_{n+1} given $\{X_n = x\}$. If the kernel P_n does not depend on n ($P_n = P$ for $n \in \mathbb{N}$), the Markov chain is said to be homogeneous.

Now, consider $(X_n)_{n \in \mathbb{N}}$, an \mathbb{F} -Markov chain with transition kernels (P_n) and a reward sequence Z_n given by

$$Z_n = f(n, X_n), \quad n \in \mathbb{N},$$

where, for every $n \in \mathbb{N}$, f(n, .) is a nonnegative measurable function such that the random variable $f(n, X_n)$ is integrable.

Proposition 1.5.2 Under the above assumptions, the Snell envelope with horizon N of the sequence $(Z_n)_{n \in \mathbb{N}}$ is given by

$$U_n^{(N)} = V(n, X_n) \quad a.s.$$

where the functions $V(n, \cdot)$ (n = 0, ..., N) are determined by the following dynamic programming algorithm:

$$\begin{cases} V(N,x) = f(N,x) \\ V(n,x) = \max \{ f(n,x), P_n[V(n+1,\cdot)](x) \}, & \text{for } 0 \le n \le N-1. \end{cases}$$

Proof: Since f is nonnegative, the functions $V(n, \cdot)$ (n = 0, ..., N) are well defined (with values in $[0, +\infty]$) by the dynamic programming algorithm. We have, using the Markov property,

$$V(n, X_n) = \max \{ f(n, X_n), P_n[V(n+1, \cdot)](X_n) \}$$

= max \{ f(n, X_n), \mathbb{E} (V(n+1, X_{n+1}) | \mathcal{F}_n) \} a.s.,

which, together with the terminal condition $V(N, X_N) = Z_N$, proves that the sequence $(V(n, X_n))_{0 \le n \le N}$ satisfies the characteristic properties of the Snell envelope with horizon N (see Remark 1.2.4).

Remark 1.5.3 In this Markovian setting, with finite horizon N, the smallest optimal stopping time is given by

$$\nu_0 = \inf \left\{ n \in [0, N] \mid V(n, X_n) = f(n, X_n) \right\}.$$

The set of all (n, x) such that V(n, x) = f(n, x) is called the stopping region, the set of all (n, x) such that V(n, x) > f(n, x) is called the continuation region.

Remark 1.5.4 The value function V(n, x) can be interpreted as follows. Fix $n \in \{0, ..., N\}$ and denote by $X^{n,x} = (X_m^{n,x})_{n \le m \le N}$ a Markov chain with respect to the filtration $(\mathcal{F}_m)_{n \le m \le N}$, with transition kernels P_m , which satisfies $X_n^{n,x} = x$ almost surely. Then, we have $V(n, x) = \sup_{\nu \in \mathcal{T}_{n,N}} \mathbb{E}f(\nu, X_{\nu}^{n,x})$.

1.5.2 Infinite horizon

We now consider a homogeneous **F**-Markov chain $(X_n)_{n \in \mathbb{N}}$, with transition kernel P and a reward sequence of the form

$$Z_n = \rho^n f(X_n),$$

where f is nonnegative and measurable and $\rho > 0$. We also assume that $\sup_{n \in \mathbb{N}} Z_n$ is integrable, so that the results of the beginning of this chapter apply.

Proposition 1.5.5 The sequence of functions $(u_n)_{n \in \mathbb{N}}$, defined by

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \max(f(x), \rho P u_n(x)), \quad n \in \mathbb{N}$$

is nondecreasing and the Snell envelope of the sequence $(Z_n)_{n\in\mathbb{N}}$ is given by $U_n = \rho^n u(X_n)$ a.s., where $u(x) = \lim_{n\to\infty} u_n(x)$. Moreover, the function u solves the equation $u = \max(f, \rho P u)$.

Note that, in particular, we have $u(X_0) = \text{ess sup}_{\nu \in \mathcal{T}_{0,\infty}} \mathbb{E}(\rho^{\nu} f(X_{\nu}) \mid \mathcal{F}_0)$ a.s..

Proof: A simple induction argument yields $u_{n+1} \ge u_n$. Next, by monotone convergence, we have $u = \max(f, \rho P u)$. Using dynamic programming, it can be checked that the Snell envelope with horizon N of Z is given by $U_n^{(N)} = \rho^n u_{N-n}(X_n)$ a.s.. By passing to the limit as N goes to infinity, we get $U_n = \rho^n u(X_n)$ a.s.. \diamond

Chapter 2

Optimal stopping in continuous time

The main references for this chapter are: [60] (see also appendix D of [84]), and, for Markov processes, [124, 117].

2.1 Preliminary results

In this chapter the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a continuous time filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. We assume that the so called *usual conditions* are satisfied, which means that the filtration is right continuous and complete (see [81], chapter 1 or [121], chapter I).

We denote by \mathcal{T} the set of all stopping times with respect to the filtration \mathbb{F} and introduce the following subsets of \mathcal{T} :

$$\begin{aligned} \mathcal{T}_{t,T} &= \{ \tau \in \mathcal{T} \mid \mathbb{P}(\tau \in [t,T]) = 1 \}, \quad 0 \le t \le T < \infty, \\ \mathcal{T}_{t,\infty} &= \{ \tau \in \mathcal{T} \mid \mathbb{P}(\tau \in [t,+\infty)) = 1 \}, \quad t \ge 0. \end{aligned}$$

We now recall some basic results on martingales and supermartingales in continuous time (proofs can be found in [121], chapitre II).

Theorem 2.1.1 Let $(M_t)_{t\geq 0}$ be a right-continuous martingale. If σ , τ are bounded stopping times with $\sigma \leq \tau$, the random variables M_{σ} and M_{τ} are integrable and

$$\mathbb{E}\left(M_{\tau} \mid \mathcal{F}_{\sigma}\right) = M_{\sigma} \quad \text{a.s..}$$

Theorem 2.1.2 Let $(X_t)_{t\geq 0}$ be a nonnegative right-continuous supermartingale. The limit $X_{\infty} = \lim_{t\to\infty} X_t$ exists with probability 1, and, if σ , τ are stopping times with $\sigma \leq \tau$, we have

$$\mathbb{E}\left(X_{\tau} \mid \mathcal{F}_{\sigma}\right) \leq X_{\sigma} \quad \text{a.s..}$$

Note that in the above statement the stopping times σ and τ may be infinite.

Theorem 2.1.3 Let $(X_t)_{t\geq 0}$ be an integrable \mathbb{F} -supermartingale. If $t \mapsto \mathbb{E}X_t$ is right-continuous, the process $(X_t)_{t\geq 0}$ has a càdlàg¹ modification which is an \mathbb{F} -supermartingale.

We will use the following terminology.

Definition 2.1.4 An adapted right-continuous process $(X_t)_{t\geq 0}$ is said to be

 $^{^{1}}$ A càdlàg process is a process with sample paths that are right-continuous and have left-limits everywhere.

- regular if, for every $\tau \in \mathcal{T}_{0,\infty}$, X_{τ} is integrable and, for every nondecreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\tau = \lim_{n\to\infty} \tau_n$, we have $\lim_{n\to\infty} \mathbb{E}(X_{\tau_n}) = \mathbb{E}(X_{\tau})$;
- of class D if the family $(X_{\tau})_{\tau \in \mathcal{T}_{0,\infty}}$ is uniformly integrable.

Note that a regular process may have discontinuous paths. Example: let $(N_t)_{t\geq 0}$ be an \mathbb{F} -Poisson process with intensity λ , we have $\mathbb{E}N_{\tau} = \lambda \mathbb{E}\tau$ for any stopping time τ and the process $(N_{t\wedge 1})_{t\geq 0}$ is regular.

2.2 The Snell envelope in continuous time

Throughout this chapter, we consider an adapted right-continuous process $Z = (Z_t)_{t \ge 0}$, satisfying

$$\forall t \ge 0, \quad Z_t \ge 0 \quad \text{and} \quad \mathbb{E}\left(\sup_{t\ge 0} Z_t\right) < \infty.$$

The process Z is obviously of class D. The introduction of the Snell envelope of Z relies on the following theorem.

Theorem 2.2.1 For $t \ge 0$, set

$$U_t = \operatorname{ess \, sup}_{\tau \in \mathcal{T}_t} \mathbb{E}(Z_\tau \mid \mathcal{F}_t).$$

- 1. The process $(U_t)_{t>0}$ is a supermartingale.
- 2. For every $t \ge 0$, $\mathbb{E}(U_t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}(Z_{\tau})$.
- 3. U admits a right-continuous modification.

The right-continuous modification of U is called the Snell envelope of Z and will still be denoted by U. For the proof of Theorem 2.2.1, we will need the following lemma, which can be proved in the same way as Lemma 1.2.2.

Lemma 2.2.2 Fix $t \ge 0$. The family $(\mathbb{E}(Z_{\tau}|\mathcal{F}_t), \tau \in \mathcal{T}_{t,\infty})$ has the lattice property.

Proof of Theorem 2.2.1: 1) Suppose s and t are nonnegative numbers, with $s \le t$. It follows from Lemma 2.2.2 and Proposition 1.1.4 that

$$\begin{split} \mathbb{E}(U_t | \mathcal{F}_s) &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}\left(\mathbb{E}(Z_\tau | \mathcal{F}_t) \mid \mathcal{F}_s\right) \\ &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t \infty} \mathbb{E}(Z_\tau | \mathcal{F}_s) \leq U_s, \quad \text{a.s.} \end{split}$$

where we have used the inclusion $\mathcal{T}_{t,\infty} \subset \mathcal{T}_{s,\infty}$.

2) The second property follows again from the lattice property and Proposition 1.1.4.

3) In order to prove the existence of a right-continuous modification, it suffices, according to Theorem 2.1.3, to prove that $t \mapsto \mathbb{E}(U_t)$ is right-continuous. If $(t_n)_{n \in \mathbb{N}}$ is a nonincreasing sequence with $\lim_{n\to\infty} t_n = t$, we have, for every n, $\mathbb{E}(U_{t_n}) = \sup_{\tau \in \mathcal{T}_{t_{n,\infty}}} \mathbb{E}(Z_{\tau}) \leq \mathbb{E}(U_t)$. On the other hand, if $\tau \in \mathcal{T}_{t,\infty}$, the stopping time $\tau_n = \tau \vee t_n$ is in $\mathcal{T}_{t_{n,\infty}}$ and $\lim_{n\to\infty} Z_{\tau_n} = Z_{\tau}$, by the right-continuity of Z. Hence, $\mathbb{E}(Z_{\tau}) \leq \liminf_{n\to\infty} \mathbb{E}(Z_{\tau_n}) \leq \liminf_{n\to\infty} \mathbb{E}(U_{t_n})$. The inequality being valid for every $\tau \in \mathcal{T}_{t,\infty}$, we have $\mathbb{E}(U_t) \leq \liminf_{n\to\infty} \mathbb{E}(U_{t_n})$, so that $\lim_{n\to\infty} \mathbb{E}(U_{t_n}) = \mathbb{E}(U_t)$. **Remark 2.2.3** Note that U is of class D. Indeed, we have $0 \le U \le M$, where M is the uniformly integrable martingale defined by $M_t = \mathbb{E} \left(\sup_{s>0} Z_s | \mathcal{F}_t \right)$.

Corollary 2.2.4 The Snell envelope U is the smallest right-continuous supermartingale dominating Z.

Proof: If V is a right-continuous supermartingale with $V \ge Z$, we have, using Theorem 2.1.2,

$$\forall \tau \in \mathcal{T}_{t,\infty}, \quad \mathbb{E}(Z_{\tau}|\mathcal{F}_t) \leq \mathbb{E}(V_{\tau}|\mathcal{F}_t) \leq V_t \quad \text{a.s.}.$$

Hence $U_t \leq V_t$ a.s..

As in the discrete case, and with a similar proof, we can identify the limit of the Snell envelope at infinity.

Proposition 2.2.5 The Snell enveloppe U of the process Z satisfies $\lim_{t\to\infty} U_t = \limsup_{t\to\infty} Z_t$ a.s..

2.3 Optimal stopping times and the Doob-Meyer decomposition

2.3.1 A characterisation of optimal stopping times

We will say that a stopping time $\tau^* \in \mathcal{T}_{0,\infty}$ is *optimal* if $\mathbb{E}(Z_{\tau^*}) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}(Z_{\tau})$. We have, as in the discrete case, a characterization of optimal stopping times.

Theorem 2.3.1 A stopping time $\tau^* \in \mathcal{T}_{0,\infty}$ is optimal if and only if the following two conditions are satisfied:

- 1. $U_{\tau^*} = Z_{\tau^*}$, almost surely.
- 2. The stopped process U^{τ^*} , defined by $U_t^{\tau^*} = U_{\tau^* \wedge t}$, $0 \leq t \leq T$, is a martingale.

Proof: The stopping time τ^* is optimal if and only if $\mathbb{E}Z_{\tau^*} = \mathbb{E}U_0$. On the other hand, using $U \geq Z$ and Theorem 2.1.2 applied to the supermartingale U, we have $\mathbb{E}Z_{\tau^*} \leq \mathbb{E}U_{\tau^*} \leq \mathbb{E}U_0$. Therefore, τ^* is optimal if and only if $\mathbb{E}(Z_{\tau^*}) = \mathbb{E}(U_{\tau^*}) = \mathbb{E}(U_0)$. The equality $\mathbb{E}(Z_{\tau^*}) = \mathbb{E}(U_{\tau^*})$ is equivalent to $Z_{\tau^*} = U_{\tau^*}$ a.s., since $U \geq Z$. The equality $\mathbb{E}(U_{\tau^*}) = \mathbb{E}(U_0)$ is equivalent to the martingale property for the stopped process U^{τ^*} . Indeed, if U^{τ^*} is a martingale, we have $\mathbb{E}(U_{\tau^* \wedge t}) = \mathbb{E}U_0$ for every $t \geq 0$ and, since U is of class D, $\mathbb{E}(U_{\tau^*}) = \mathbb{E}U_0$ by passing to the limit as t goes to infinity. Conversely, if $\mathbb{E}(U_{\tau^*}) = \mathbb{E}U_0$, we have (by applying again Theorem 2.1.2 to U) $\mathbb{E}(U_{\tau^* \wedge t}) = \mathbb{E}U_0$, for every $t \geq 0$, and we conclude, as in the discrete case, that U^{τ^*} is a martingale.

2.3.2 The Doob-Meyer decomposition and ε -optimal stopping times

The following theorem is the analogue, for continuous time, of Theorem 1.4.1. We refer to [53], [119] or [81] for a proof.

Theorem 2.3.2 Let $U = (U_t)_{t\geq 0}$ be a right-continuous supermartingale of class D. There exists a martingale $(M_t)_{t\geq 0}$ and a nondecreasing right-continuous predictable process $A = (A_t)_{t\geq 0}$, with $A_0 = 0$, which are unique up to indistinguishability, uniformly integrable, such that

$$U_t = M_t - A_t, \quad t \ge 0.$$

Moreover, if U is a regular process, the process A has continuous paths with probability one.

 \diamond

The above decomposition is called the Doob-Meyer decomposition of the supermartingale U.

Theorem 2.3.3 Let U be the Snell envelope of the process Z. For $t \ge 0$ and $\varepsilon > 0$, define

$$D_t^{\varepsilon} = \inf\{s \ge t \mid Z_s \ge U_s - \varepsilon\}.$$

We have $D_t^{\varepsilon} \in \mathcal{T}_{t,\infty}$, $\mathbb{E}U_{D_t^{\varepsilon}} = \mathbb{E}U_t$ and $\mathbb{E}(Z_{D_t^{\varepsilon}}) \geq \mathbb{E}U_t - \varepsilon$.

In particular, we have $\mathbb{E}(Z_{D_0^{\varepsilon}}) \geq \mathbb{E}U_0 - \varepsilon = \sup_{t \in \mathcal{T}_{0,\infty}} \mathbb{E}Z_{\tau} - \varepsilon$. The stopping time D_0^{ε} is said to be ε -optimal. In fact, D_t^{ε} is ε -optimal among stopping times in $\mathcal{T}_{t,\infty}$.

Note that D_t^{ε} is the hitting time of the closed set $[0, \varepsilon]$ for the right-continuous adapted process U - Z. The fact that it is a stopping time is a consequence of the usual conditions for the filtration (see [121], chapter I, section 4). This stopping time is finite almost surely because $\lim_{t\to\infty} U_t = \limsup_{t\to\infty} Z_t$ a.s..

In order to complete the proof of the theorem, we will rely on the following lemma.

Lemma 2.3.4 Let A be the nondecreasing process in the Doob-Meyer decomposition of the Snell envelope U. We have, for $t \ge 0$ and $\varepsilon > 0$, $A_{D_t^{\varepsilon}} = A_t$ almost surely and the processes $(A_{D_t^{\varepsilon}})$ and A are undistinguishable.

Proof: We know that $\mathbb{E}U_t = \sup_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}Z_{\tau}$. Introduce a sequence $(\tau_j)_{j \in \mathbb{N}}$ with $\tau_j \in \mathcal{T}_{t,\infty}$ and $\lim_{j \to \infty} \mathbb{E}Z_{\tau_j} = \mathbb{E}U_t$. We have

$$\mathbb{E}Z_{\tau_j} \le \mathbb{E}U_{\tau_j} = \mathbb{E}M_t - A_{\tau_j} = \mathbb{E}U_t - \mathbb{E}\left(A_{\tau_j} - A_t\right)$$

Therefore $\lim_{j\to\infty} \mathbb{E} \left(U_{\tau_j} - Z_{\tau_j} \right) = \lim_{j\to\infty} \mathbb{E} \left(A_{\tau_j} - A_t \right) = 0$. By passing to a sub-sequence, we can assume, without loss of generality, that $\lim_{j\to\infty} \left(U_{\tau_j} - Z_{\tau_j} \right) = \lim_{j\to\infty} \left(A_{\tau_j} - A_t \right) = 0$ almost surely. The equality $\lim_{j\to\infty} \left(U_{\tau_j} - Z_{\tau_j} \right) = 0$ implies $D_t^{\varepsilon} \leq \tau_j$ for j large enough, so that $A_{D_t^{\varepsilon}} \leq A_{\tau_j}$ and, since $\lim_{j\to\infty} \left(A_{\tau_j} - A_t \right) = 0$ a.s., we have $A_{D_t^{\varepsilon}} \leq A_t$ a.s. and the equality follows from the fact that A is nondecreasing. In order to prove that the processes are undistinguishable, consider an event $\hat{\Omega}$ with $\mathbb{P}(\hat{\Omega}) = 1$, on which the paths of U, Z, A are right-continuous, and such that, for every $\omega \in \hat{\Omega}$ and every rational number $t \geq 0$, $A_{D_t^{\varepsilon}(\omega)}(\omega) = A_t(\omega)$. For an arbitrary t, take a sequence $(t_n)_{n\in\mathbb{N}}$ of rational numbers satisfying $t_n \geq t$ and $\lim_{n\to\infty} t_n = t$. We have $A_t(\omega) \leq A_{D_t^{\varepsilon}(\omega)}(\omega) = A_{t_n}(\omega)$, and, due to the right-continuity, $A_{D_t^{\varepsilon}(\omega)}(\omega) = A_t(\omega)$.

Proof of Theorem 2.3.3: It follows from the definition of D_t^{ε} and from the right-continuity of U - Z that $Z_{D_t^{\varepsilon}} \geq U_{D_t^{\varepsilon}} - \varepsilon$. From Lemma 2.3.4, we derive $\mathbb{E}U_{D_t^{\varepsilon}} = \mathbb{E}U_t$. Hence, $\mathbb{E}Z_{D_t^{\varepsilon}} \geq \mathbb{E}U_t - \varepsilon$.

2.3.3 The regular case

Theorem 2.3.5 If the reward process Z is regular, so is its Snell envelope. In this case, the existence of an optimal stopping time is equivalent to $\mathbb{P}(\tau_0 < \infty) = 1$, where

$$\tau_0 = \inf\{t \ge 0 \mid U_t = Z_t\},\$$

and τ_0 then is the smallest optimal stopping time.

Proof: Fix $\tau \in \mathcal{T}_{0,\infty}$ and let $(\tau_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of stopping times with $\lim_{n \to \infty} \tau_n =$

 τ . From the supermartingale property of U, we deduce that $(\mathbb{E}(U_{\tau_n}))_{n \in \mathbb{N}}$ is nonincreasing and we have $\mathbb{E}(U_{\tau_n}) \geq \mathbb{E}(U_{\tau})$ for all n.

On the other hand, for a fixed $\varepsilon > 0$, $D_{\tau_n}^{\varepsilon} = \inf\{s \ge \tau_n \mid Z_s \ge U_s - \varepsilon\}$ is a stopping time and, according to Lemma 2.3.4, $A_{D_{\tau_n}^{\varepsilon}} = A_{\tau_n}$ almost surely, which yields $\mathbb{E}U_{D_{\tau_n}^{\varepsilon}} = EU_{\tau_n}$. Hence $\mathbb{E}Z_{D_{\tau_n}^{\varepsilon}} \ge \mathbb{E}U_{D_{\tau_n}^{\varepsilon}} - \varepsilon = \mathbb{E}U_{\tau_n} - \varepsilon$. The sequence $(D_{\tau_n}^{\varepsilon})_{n \in \mathbb{N}}$ is nondecreasing and dominated by D_{τ}^{ε} . The stopping time $\bar{\tau} = \lim_{n \to \infty} D_{\tau_n}^{\varepsilon}$ satisfies $\tau \le \bar{\tau} \le D_{\tau}^{\varepsilon}$ and, from the regularity of Z, we derive $\mathbb{E}(Z_{\bar{\tau}}) = \lim_{n \to \infty} \mathbb{E}(Z_{D_{\tau_n}^{(\varepsilon)}})$. Hence,

$$\mathbb{E}(U_{\tau}) \ge \mathbb{E}(U_{\bar{\tau}}) \ge \mathbb{E}(Z_{\bar{\tau}}) = \lim_{n \to \infty} \mathbb{E}(Z_{D_{\tau_n}^{\varepsilon}}) \ge \lim_{n \to \infty} \mathbb{E}U_{\tau_n} - \varepsilon.$$

The regularity of U follows.

We know from Theorem 2.3.1 that the existence of an optimal stopping time implies $\mathbb{P}(\tau_0 < \infty) = 1$. Conversely, assume $\mathbb{P}(\tau_0 < \infty) = 1$ and consider a nonincreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. The sequence $(\tau_n^*)_{n\in\mathbb{N}}$, defined by $\tau_n^* = D_0^{\varepsilon_n}$ is nondecreasing and dominated by τ_0 , and we have

$$\mathbb{E}(U_0) = \mathbb{E}(U_{\tau_n^*}) \le \mathbb{E}(Z_{\tau_n^*}) + \varepsilon_n.$$

By taking limits as n goes to infinity and using the regularity of Z, we see that the stopping time $\tau^* = \lim_{n\to\infty} \tau_n^*$ is optimal. This implies (due to Theorem 2.3.1) that $Z_{\tau^*} = U_{\tau^*}$ a.s., hence $\tau^* \ge \tau_0$ a.s., so that $\tau^* = \tau_0$ a.s., which proves that τ_0 is optimal. According to Theorem 2.3.1, it is minimal among optimal stopping times.

In the regular case, the largest optimal stopping time can also be identified. Note that if Z is regular, its Snell envelope U is a regular supermartingale and the Doob-Meyer decomposition of U reads $U_t = M_t - A_t$, with A a nondecreasing *continuous* adapted process.

Theorem 2.3.6 Assume the reward process Z is regular and define the stopping time

$$\tau_{\max} = \inf\{t \ge 0 \mid A_t > 0\}$$

If τ^* is optimal, we have $\mathbb{P}(\tau^* \leq \tau_{\max}) = 1$ and the stopping time $\tau_{\max} \mathbf{1}_{\{\tau_{\max} < \infty\}} + \tau^* \mathbf{1}_{\{\tau_{\max} = \infty\}}$ is optimal as well. If $\mathbb{P}(\tau_{\max} < \infty) = 1$, τ_{\max} is the largest optimal stopping time.

Lemma 2.3.7 For $t \in [0, +\infty)$ define a (possibly infinite) stopping time $D_t^0 = \inf\{s \ge t \mid U_s = Z_s\}$. The process $(A_{D_t^0})_{t\ge 0}$ (defined with the convention $A_{\infty} = \lim_{t\to\infty} A_t$) is undistinguishable from A.

Proof: We know from Lemma 2.3.4 that, for $\varepsilon > 0$, we have (up to a negligible set) $A_t = A_{D_t^{\varepsilon}}$, for every $t \ge 0$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. The sequence $(D_t^{\varepsilon_n})_{n\in\mathbb{N}}$ is a nondecreasing sequence of stopping times with $D_t^{\varepsilon_n} \le D_t^0$. Set $\tau_t^* = \lim_{n\to\infty} D_t^{\varepsilon_n}$. This limit is a stopping time, with values in $[0, +\infty]$ and we have $\tau_t^* \le D_t^0$. Moreover, since A is continuous, $A_{\tau_t^*} = A_t$. In order to prove the lemma, it suffices to derive the inequality $D_t^0 \le \tau_t^*$ a.s. (this will yield $A_t = A_{D_t^0}$ a.s., for fixed t, and undistinguishability will follow as in the proof of Lemma 2.3.4). For T > t, we have

$$\mathbb{E}Z_{D_t^{\varepsilon_n} \wedge T} = \mathbb{E}\left(Z_{D_t^{\varepsilon_n}} \mathbf{1}_{\{D_t^{\varepsilon_n} \leq T\}} + Z_T \mathbf{1}_{\{D_t^{\varepsilon_n} > T\}}\right)$$

$$\geq \mathbb{E}\left((U_{D_t^{\varepsilon_n}} - \varepsilon_n) \mathbf{1}_{\{D_t^{\varepsilon_n} \leq T\}} + Z_T \mathbf{1}_{\{D_t^{\varepsilon_n} > T\}}\right)$$

$$\geq \mathbb{E}\left(U_{D_t^{\varepsilon_n} \wedge T} \mathbf{1}_{\{D_t^{\varepsilon_n} \leq T\}} + Z_T \mathbf{1}_{\{D_t^{\varepsilon_n} > T\}}\right) - \varepsilon_n$$

$$= \mathbb{E}\left(U_{D_t^{\varepsilon_n} \wedge T} + (Z_T - U_T) \mathbf{1}_{\{D_t^{\varepsilon_n} > T\}}\right) - \varepsilon_n.$$

From the regularity of the processes Z and U we deduce $\lim_{n\to\infty} \mathbb{E}Z_{D_t^{\varepsilon_n}\wedge T} = \mathbb{E}Z_{\tau_t^*\wedge T}$ and $\lim_{n\to\infty} \mathbb{E}U_{D_t^{\varepsilon_n}\wedge T} = \mathbb{E}U_{\tau_t^*\wedge T}$. On the other hand, by monotone convergence, $\lim_{n\to\infty} \mathbb{E}(U_T - Z_T)\mathbf{1}_{\{D_t^{\varepsilon_n}>T\}} = \mathbb{E}(U_T - Z_T)\mathbf{1}_{\{\tau_t^*>T\}}$. Hence

$$\mathbb{E}Z_{\tau_t^* \wedge T} \geq \mathbb{E}\left(U_{\tau_t^* \wedge T} + (Z_T - U_T)\mathbf{1}_{\{\tau_t^* > T\}}\right)$$
$$= \mathbb{E}\left(U_{\tau_t^*}\mathbf{1}_{\{\tau_t^* \le T\}} + Z_T\mathbf{1}_{\{\tau_t^* > T\}}\right).$$

Therefore, $\mathbb{E}Z_{\tau_t^*} \mathbf{1}_{\{\tau_t^* \leq T\}} \geq \mathbb{E}U_{\tau_t^*} \mathbf{1}_{\{\tau_t^* \leq T\}}$. It follows that $Z_{\tau_t^*} = U_{\tau_t^*}$ a.s. on $\{\tau_t^* < \infty\}$, hence $D_t^0 \leq \tau_t^*$ a.s. on $\{\tau_t^* < \infty\}$. The inequality clearly holds on $\{\tau_t^* = \infty\}$ as well.

Proof of Theorem 2.3.6: We know that if τ^* is optimal, we have $\mathbb{E}U_{\tau^*} = \mathbb{E}Z_{\tau^*} = \mathbb{E}U_0$, so that $\mathbb{E}A_{\tau^*} = 0$ and $\tau^* \leq \tau_{\max}$ a.s.. To complete the proof, it suffices to prove that, on $\{\tau_{\max} < \infty\}$, $Z_{\tau_{\max}} = U_{\tau_{\max}}$ a.s.. We deduce from Lemma 2.3.7 that, on $\{\tau_{\max} < \infty\}$, $\inf\{s \geq \tau_{\max} \mid U_s = Z_s\} = \tau_{\max}$ a.s. and, by the right-continuity, $U_{\tau_{\max}} = Z_{\tau_{\max}}$ a.s..

Remark 2.3.8 If Z is a continuous semimartingale, there are interesting connections between A and the local time at 0 of the semimartingale U - Z (see, in particular, [78]).

2.3.4 Finite horizon

When dealing with an optimal stopping problem with finite horizon T, one needs to introduce the Snell envelope with horizon T, which is defined by

$$U_t^{(T)} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left(Z_\tau \mid \mathcal{F}_t \right).$$

We then have $U_T^{(T)} = Z_T$ a.s. and, if the process $(Z_t)_{0 \le t \le T}$ is nonnegative, right-continuous, regular and satisfies $\mathbb{E} \sup_{0 \le t \le T} Z_t < \infty$, the stopping time $\tau_0 = \inf\{t \ge 0 \mid U_t = Z_t\}$ is in $\mathcal{T}_{0,T}$, since $Z_T = U_T^{(T)}$, and is minimal among all optimal stopping times. The following theorem is a characterization of the Snell envelope in terms of its Doob-Meyer decomposition. It can be used to establish the relation between optimal stopping problems and variational inequalities (see Section 4.3 below).

Theorem 2.3.9 Assume the reward process $(Z_t)_{0 \le t \le T}$ is regular. Let $\hat{U} = (\hat{U}_t)_{0 \le t \le T}$ be a regular right-continuous supermartingale of class D, with Doob-Meyer decomposition $\hat{U} = \hat{M} - \hat{A}$. \hat{U} is the Snell envelope (with horizon T) of Z if and only if the following conditions are satisfied:

- 1. $\hat{U} \geq Z$,
- 2. $\hat{U}_T = Z_T, a.s.,$
- 3. for every $t \in [0, T]$, $\hat{A}_t = \hat{A}_{\hat{\tau}_t}$ where $\hat{\tau}_t = \inf\{s \ge t \mid \hat{U}_s = Z_s\}$.

Proof: The first two conditions are obviously necessary and the third one is also, as a consequence of Lemma 2.3.7. Conversely, if \hat{U} satisfies these three conditions, we have $\hat{U} \ge U$ from Corollary 2.2.4. Moreover, we have

$$\hat{A}_t = \hat{A}_{\hat{\tau}_t} \quad \Rightarrow \quad \mathbb{E}(\hat{U}_t) = \mathbb{E}(\hat{U}_{\hat{\tau}_t}) = \mathbb{E}(Z_{\hat{\tau}_t}).$$

Hence $\mathbb{E}(\hat{U}_t) \leq \mathbb{E}(U_t)$ and $\hat{U} = U$.

 \diamond

2.4 The dual approach to optimal stopping problems

We complete this chapter by introducing the dual approach to optimal stopping problems, which was independently developed by Rogers [122] and Haugh and Kogan [74] (see also [52]). This approach can be used for the computation of American option prices by Monte-Carlo methods (see Chapter 4). For simplicity, we restrict the presentation to the case of finite horizon T.

Theorem 2.4.1 Let $(Z_t)_{0 \le t \le T}$ be a nonnegative, right-continuous, regular process satisfying $\mathbb{E} \sup_{0 \le t \le T} Z_t < \infty$. Then we have

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(Z_{\tau}) = \inf_{M \in \mathcal{M}_0} \mathbb{E}\left(\sup_{0 \le s \le T} (Z_s - M_s)\right),\tag{2.1}$$

where \mathcal{M}_0 is the set of all right continuous martingales $M = (M_t)_{0 \le t \le T}$ with $M_0 = 0$.

Proof: Suppose that $M \in \mathcal{M}_0$. Then, for any stopping time $\tau \in \mathcal{T}_{0,T}$, we have

$$\mathbb{E}(Z_{\tau}) = \mathbb{E}(Z_{\tau} - M_{\tau}) \le \mathbb{E}\left(\sup_{0 \le s \le T} (Z_s - M_s)\right),$$

therefore

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(Z_{\tau}) \le \inf_{M \in \mathcal{M}_0} \mathbb{E}\left(\sup_{0 \le s \le T} (Z_s - M_s)\right).$$
(2.2)

On the other hand, the Snell envelope (with horizon T) of Z admits a Doob-Meyer decomposition

$$U_t^{(T)} = M_t - A_t$$

where M is the martingale part and A the increasing process. The process $\tilde{M} = M - M_0$ is in \mathcal{M}_0 and we have

$$U_t^{(T)} = U_0^{(T)} + \tilde{M}_t - A_t,$$

so that $U_0^{(T)} = A_t - \tilde{M}_t + U_t^{(T)} \ge Z_t - \tilde{M}_t$. Hence

$$\sup_{\tau \in \mathcal{I}_{0,T}} \mathbb{E}(Z_{\tau}) = \mathbb{E}(U_0^{(T)}) \ge \sup_{0 \le t \le T} \mathbb{E}\left(\sup_{0 \le s \le T} (Z_s - \tilde{M}_s)\right),$$

which proves that we have equality in (2.2) and that the infimum is achieved by taking $M = \tilde{M}$.

Chapter 3

Pricing and hedging American options in complete markets

In this chapter, we present the theory of American options in a complete market, as it developed from [25, 82, 83]. For a slightly more general set up, see [84].

3.1 The model

We consider a financial market in which there are d risky assets and one riskless asset. We denote by S_t^0 the unit price of the riskless asset at time t and by S_t^i (i = 1, ..., d) the prices of the risky assets at time t. The model is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration \mathbb{F} is the completion of the natural filtration of a standard d-dimensional Brownian motion $B = (B_t^1, \ldots, B_t^d)_{t\geq 0}$ (which means that the coordinates are independent standard real Brownian motions). It is well known that this filtration satisfies the usual conditions.

The time interval in which the model is studied is a bounded interval [0, T] and we assume that the price of the riskless asset is given by

$$S_t^0 = e^{\int_0^t r_s ds}.$$

where $(r_t)_{0 \le t \le T}$ is a measurable adapted process which is uniformly bounded¹. Here, r_t represents the instantaneous interest rate at time t. For the evolution of the prices of the risky assets, we assume the following equations:

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^d \sigma_{ij}(t) dB_t^j, \quad i = 1, \dots, d,$$
(3.1)

where the processes $(\mu_t^i)_{0 \le t \le T}$ and $(\sigma_{ij}(t))_{0 \le t \le T}$ $(1 \le i, j \le d)$ are adapted, measurable and uniformly bounded. We assume that the *i*-th risky asset continuously distributes dividends at rate δ_t^i , which means that the holder of one unit of this asset at time *t* receives the wealth $\delta_t^i S_t^i dt$ during the infinitesimal time interval [t, t + dt]. The processes $(\delta_t^i)_{0 \le t \le T}$ (i = 1, ..., d) are assumed to be adapted, measurable, and uniformly bounded.

¹A real process $(X_t)_{0 \le t \le T}$ is said to be uniformly bounded if $\sup_{0 \le t \le T} \sup_{\omega \in \Omega} |X_t(\omega)| < \infty$. For the theory of stochastic integration of measurable adapted processes with respect to Brownian motion, we refer to [81]. Replacing measurable adapted by progressively measurable throughout this chapter would not entail any real loss of generality.

We will also assume that, for every $t \in [0, T]$, the matrix $\sigma_t = (\sigma_{ij}(t))_{1 \le i,j \le d}$ is invertible and that the process $(\sigma_t^{-1})_{0 \le t \le T}$ (with values in the space of $d \times d$ matrices) is uniformly bounded. The process $\theta = (\theta_t)_{0 \le t \le T}$ (with values in \mathbb{R}^d) defined by

$$\theta_t = \sigma_t^{-1} \bar{\mu}_t,$$

where $\bar{\mu}_t$ is the vector with coordinates $\bar{\mu}_t^i = \mu_t^i + \delta_t^i - r_t$, for $i = 1, \ldots, d$ is then measurable, adapted and uniformly bounded and, if we set

$$W_t = B_t + \int_0^t \theta_s ds,$$

equation (3.1) can be rewritten as

$$\frac{dS_t^i}{S_t^i} = (r_t - \delta_t^i)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_t^j, \quad i = 1, \dots, d.$$
(3.2)

For $0 \leq t \leq T$, let

$$L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right),$$

with the notations $\theta_s dB_s = \sum_{i=1}^d \theta_s^i dB_s^i$ and $|\theta_s|^2 = \sum_{i=1}^d (\theta_s^i)^2$. The process $(L_t)_{0 \le t \le T}$ is a martingale. We denote by \mathbb{P}^* the probability with density L_T with respect to \mathbb{P}

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = L_T$$

According to Girsanov's theorem (cf. [81], Section 3.5), under probability \mathbb{P}^* , the process $(W_t)_{0 \le t \le T}$ is an \mathbb{F} -standard Brownian motion with values in \mathbb{R}^d . For $i = 1, \ldots, d$, let

$$\hat{S}_t^i = \exp\left(\int_0^t (\delta_s^i - r_s) ds\right) S_t^i, \quad 0 \le t \le T.$$

Under probability \mathbb{P}^* , the processes $(\hat{S}_t^i)_{0 \leq t \leq T}$ are martingales. Note that we have $d\hat{S}_t^i/\hat{S}_t^i = \sum_{j=1}^d \sigma_{ij}(t) dW_t^j$ and

$$\hat{S}_{t}^{i} = S_{0}^{i} \exp\left(\sum_{j=1}^{d} \int_{0}^{t} \sigma_{ij}(s) dW_{s}^{j} - \frac{1}{2} \int_{0}^{t} \sum_{j=1}^{d} \sigma_{ij}^{2}(s) ds\right).$$
(3.3)

We also introduce the discount factor, defined by

$$\beta_t = \frac{1}{S_t^0} = e^{-\int_0^t r_s ds}.$$

The discounted prices of risky assets at time t are given by

$$\tilde{S}_t^i = \beta_t S_t^i = \exp\left(-\int_0^t r_s ds\right) S_t^i, \quad i = 1, \dots, d.$$

Note that

$$\frac{d\tilde{S}_t^i}{\tilde{S}_t^i} = \sum_{j=1}^d \sigma_{ij}(t) dW_t^j - \delta_t^i dt.$$
(3.4)

3.2 Admissible strategies

A strategy is defined as a measurable adapted process $(H_t^0, H_t^1, \ldots, H_t^d)_{0 \le t \le T}$, with values in \mathbb{R}^{d+1} , where the coordinate H_t^i stands for the number of units of asset number *i* that are held at time *t*. The value at time *t* of the portfolio associated with this strategy is given by

$$V_t = \sum_{j=0}^d H_t^j S_t^j.$$

In order to state the so called *self-financing condition*, which means that there is no external source of wealth, or that the evolution of the wealth is completely determined by the dividends, the asset price variations and consumption, we need the following integrability condition.

$$\int_{0}^{T} |H_{t}^{0}| dt + \int_{0}^{T} \sum_{i=1}^{d} |H_{t}^{i}|^{2} dt < \infty \quad \text{a.s.}$$
(3.5)

The self-financing condition can now be written as follows:

$$V_t = V_0 + \int_0^t H_s^0 dS_s^0 + \sum_{j=1}^d \int_0^t H_s^j (dS_s^j + \delta_s^j S_s^j ds) - C_t, \quad 0 \le t \le T,$$
(3.6)

where $(C_t)_{0 \le t \le T}$ is a nondecreasing adapted continuous process with $C_0 = 0$, which represents the cumulative consumption up to time t. The equality (3.6) must be interpreted as the undistinguishability of two processes (and therefore implies that V is continuous).

Definition 3.2.1 A strategy defined by a measurable adapted process $(H_t^0, H_t^1, \ldots, H_t^d)_{0 \le t \le T}$ is said to be admissible if conditions (3.5) and (3.6) hold and

$$\forall t \in [0, T], \quad V_t \ge 0 \quad a.s.. \tag{3.7}$$

The following proposition shows how the self-financing condition can be expressed in terms of discounted quantities.

Proposition 3.2.2 Let (H^0, H^1, \ldots, H^d) be a measurable adapted process with values in \mathbb{R}^{d+1} , satisfying (3.5). The self-financing condition (3.6) holds if and only if we have, with probability one,

$$\tilde{V}_t = V_0 + \sum_{j=1}^d \int_0^t H_s^j (d\tilde{S}_s^j + \delta_s^j \tilde{S}_s^j ds) - \int_0^t \beta_s dC_s, \quad 0 \le t \le T,$$
(3.8)

where $\tilde{V}_t = V_t/S_t^0 = \beta_t V_t$ is the discounted value of the portfolio at time t.

Proof: If condition (3.6) is satisfied, we have, by differentiating the product $\beta_t V_t$,

$$\begin{split} d\beta_{t}V_{t} &= \beta_{t}dV_{t} + V_{t}d\beta_{t} \\ &= \beta_{t}\left(H_{t}^{0}dS_{t}^{0} + \sum_{j=1}^{d}H_{t}^{j}(dS_{t}^{j} + \delta_{t}^{j}S_{t}^{j}dt) - dC_{t}\right) + V_{t}d\beta_{t} \\ &= \beta_{t}\left(H_{t}^{0}dS_{t}^{0} + \sum_{j=1}^{d}H_{t}^{j}(dS_{t}^{j} + \delta_{t}^{j}S_{t}^{j}dt) - dC_{t}\right) + (H_{t}^{0}S_{t}^{0} + \sum_{j=1}^{d}H_{t}^{j}S_{t}^{j})d\beta_{t} \\ &= H_{t}^{0}\left(\beta_{t}dS_{t}^{0} + S_{t}^{0}d\beta_{t}\right) + \sum_{j=1}^{d}H_{t}^{j}\left(\beta_{t}(dS_{t}^{j} + \delta_{t}^{j}S_{t}^{j}dt) + S_{t}^{j}d\beta_{t}\right) - \beta_{t}dC_{t} \\ &= \sum_{j=1}^{d}H_{t}^{j}\left(\beta_{t}dS_{t}^{j} + S_{t}^{j}d\beta_{t} + \delta_{t}^{j}\tilde{S}_{t}^{j}dt\right) - \beta_{t}dC_{t} \\ &= \sum_{j=1}^{d}H_{t}^{j}\left(d\tilde{S}_{t}^{j} + \delta_{t}^{j}\tilde{S}_{t}^{j}dt\right) - \beta_{t}dC_{t}, \end{split}$$

which, by integrating, leads to (3.8). The converse implication is obtained similarly, by differentiating the product $S_t^0 \tilde{V}_t$.

Proposition 3.2.3 Under probability \mathbb{P}^* , the discounted value of an admissible strategy is a supermartingale.

Proof: We deduce from Proposition 3.2.2 and (3.4) that, if $(\tilde{V}_t)_{0 \le t \le T}$ is the discounted value of an admissible strategy (H^0, H^1, \ldots, H^d) , we have

$$\tilde{V}_t = V_0 + \sum_{i=1}^d \int_0^t H_s^i \tilde{S}_s^i \left(\sum_{j=1}^d \sigma_{ij}(s) dW_s^j \right) - \int_0^t \beta_s dC_s.$$

The process M, defined by

$$M_t = V_0 + \sum_{i=1}^d \int_0^t H_s^i \tilde{S}_s^i \left(\sum_{j=1}^d \sigma_{ij}(s) dW_s^j \right),$$

is a nonnegative \mathbb{P}^* -local martingale (because $\tilde{V}_t \ge 0$ and C is nondecreasing, with $C_0 = 0$), hence a supermartingale. The same is true for \tilde{V} , since the process $(\int_0^t \beta_s dC_s)_{0 \le t \le T}$ is nondecreasing. \diamond

3.3 American options and the Snell envelope

An American option with maturity T is characterized by a non-negative adapted continuous process $Z = (Z_t)_{0 \le t \le T}$. The number Z_t stands for the profit attached to exercizing the option at time t. In

the case of a call (resp. put) with strike price K on the first risky asset, we have $Z_t = (S_t^1 - K)_+$ (resp. $(K - S_t^1)_+$). We will also impose the following integrability condition:

$$\mathbb{E}^* \sup_{0 \le t \le T} Z_t < \infty. \tag{3.9}$$

Note that this assumption is equivalent to $\mathbb{E}^* \sup_{0 \le t \le T} \tilde{Z}_t < \infty$, with $\tilde{Z}_t = \beta_t Z_t$, because the process r is uniformly bounded.

Definition 3.3.1 A hedging strategy for the American option defined by the payoff process Z is an admissible strategy with value $V = (V_t)_{0 \le t \le T}$ such that, with probability one,

$$\forall t \in [0, T], \quad V_t \ge Z_t.$$

Proposition 3.3.2 Consider an American option defined by a nonnegative, continuous, adapted process $Z = (Z_t)_{0 \le t \le T}$, which satisfies (3.9). Denote by \tilde{U} the Snell envelope, under \mathbb{P}^* , of the process \tilde{Z} , with $\tilde{Z}_t = \beta_t Z_t$ and let U be the process defined by $U_t = S_t^0 \tilde{U}_t$. We have

$$U_t = \operatorname{ess \, sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^* \left(e^{-\int_t^\tau r_s ds} Z_\tau \mid \mathcal{F}_t \right), \quad 0 \le t \le T,$$
(3.10)

and, if V is the value process of any hedging strategy for the American option, we have, almost surely, $V_t \ge U_t$, for every $t \in [0, T]$.

Proof: We know, from Proposition 3.2.3, that \tilde{V} is a supermartingale. Since we have a hedging strategy, $V \geq Z$ and $\tilde{V} \geq \tilde{Z}$, so that the Proposition follows from the Snell envelope \tilde{U} being the smallest supermartingale majorant of \tilde{Z} .

The following theorem asserts that, in the model under study, there exists an admissible strategy with value V equal to U. This strategy has minimal value among all hedging strategies, and can be used to define the *fair price* of the option.

Theorem 3.3.3 Under the assumptions of the previous Proposition, there exists an admissible strategy with value V satisfying V = U, where U is given by (3.10).

Proof: We start from the Doob-Meyer decomposition of \tilde{U} , the Snell envelope (under \mathbb{P}^*) of \tilde{Z} . We have

$$\tilde{U}_t = M_t - \tilde{C}_t,$$

where M is a martingale under \mathbb{P}^* and \tilde{C} is a nondecreasing, continuous, adapted processs with $C_0 = 0$. The continuity of \tilde{C} follows from the regularity of the continuous process \tilde{Z} . Since M is a \mathbb{P}^* -martingale, $(M_t L_t)_{0 \le t \le T}$ is a \mathbb{P} -martingale. Since \mathbb{F} is the natural filtration of Brownian motion B, we may apply the martingale representation theorem (cf. [81], Section 3.4 D), which states the existence of a measurable adapted process $\alpha = (\alpha_t^1, \ldots, \alpha_t^d)_{0 \le t \le T}$ with $\int_0^T |\alpha_t|^2 dt < \infty$, a.s. (where $|\alpha_t|$ is the Euclidean norm of the vector $\alpha_t = (\alpha_t^1, \ldots, \alpha_t^d)$), and

$$M_t L_t = M_0 + \int_0^t \alpha_s . dB_s, \quad 0 \le t \le T.$$

from this equality, we derive (write $M_t = M_t L_t / L_t$ and use the equality $d(1/L_t) = \theta_s \cdot dW_s / L_s$)

$$M_t = M_0 + \int_0^t \hat{\alpha}_s . dW_s,$$

with $\hat{\alpha}_s = \frac{\alpha_s}{L_s} + M_s \theta_s$. Now define $H_t = (H_t^1, \dots, H_t^d)$ by

$$\sum_{i=1}^{d} H_t^i \tilde{S}_t^i \sigma_{ij}(t) = \hat{\alpha}_t^j$$

This determines H_t in a unique way, due to the invertibility of the matrix σ_t . It can easily be verified that the process H is measurable and adapted and satisfies $\int_0^T |H_t|^2 dt < \infty$ a.s.. By construction, we have

$$M_t = M_0 + \sum_{i=1}^d \int_0^t H_s^i \left(d\tilde{S}_s^i + \delta_s^i \tilde{S}_s^i ds \right).$$

We now define the process H^0 by

$$H_t^0 = \tilde{U}_t - \sum_{i=1}^d H_t^i \tilde{S}_t^i.$$

The process (H^0, H^1, \ldots, H^d) defines an admissible strategy with value equal to U.

 \diamond

 \diamond

Remark 3.3.4 The condition $\tilde{C}_{\tau_0^*} = \tilde{C}_0 = 0$, where $\tau_0^* = \inf\{t \ge 0 \mid U_t = Z_t\}$, means that there is no consumption if the option is exercised at time τ_0^* .

The following proposition shows that the price of an American call on an asset which does not distribute dividends is equal to the price of the European call, as long as the interest rate is nonnegative.

Proposition 3.3.5 Suppose $r_t \ge 0$ and $\delta_t^1 = 0$ for every $t \in [0, T]$. Then, for every K > 0, we have, for $t \in [0, T]$,

ess
$$\sup_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}^* \left(e^{-\int_t^\tau r_s ds} \left(S_\tau^1 - K \right)_+ \mid \mathcal{F}_t \right) = \mathbb{E}^* \left(e^{-\int_t^T r_s ds} \left(S_T^1 - K \right)_+ \mid \mathcal{F}_t \right), \quad a.s.$$

Proof: Let $\tau \in \mathcal{T}_{t,T}$. We have

$$\mathbb{E}^*\left(e^{-\int_t^\tau r_s ds} \left(S_\tau^1 - K\right)_+ \mid \mathcal{F}_t\right) = S_t^0 \mathbb{E}^*\left(\left(\tilde{S}_\tau^1 - K e^{-\int_0^\tau r_s ds}\right)_+ \mid \mathcal{F}_t\right)$$

Since δ^1 is nought, the process $(\tilde{S}^1_t)_{0 \le t \le T}$ is a \mathbb{P}^* -martingale and, by Jensen's inequality, (or direct reasoning)

$$\mathbb{E}^* \left(\left(\tilde{S}_T^1 - K e^{-\int_0^T r_s ds} \right)_+ \mid \mathcal{F}_\tau \right) \geq \left(\mathbb{E}^* \left(\tilde{S}_T^1 - K e^{-\int_0^T r_s ds} \mid \mathcal{F}_\tau \right) \right)_+ \\ = \left(\tilde{S}_\tau^1 - \mathbb{E}^* \left(K e^{-\int_0^T r_s ds} \mid \mathcal{F}_\tau \right) \right)_+ \\ \geq \left(\tilde{S}_\tau^1 - K e^{-\int_0^\tau r_s ds} \right)_+,$$

where the last inequality follows from the estimate $Ke^{-\int_0^\tau r_s ds} \ge Ke^{-\int_0^T r_s ds}$, which is a consequence of $r \ge 0$. Multiplying by S_t^0 and taking the conditional expectation with respect to \mathcal{F}_t yield

$$\mathbb{E}^*\left(e^{-\int_t^T r_s ds} \left(S_T^1 - K\right)_+ \mid \mathcal{F}_t\right) \ge \mathbb{E}^*\left(e^{-\int_t^\tau r_s ds} \left(S_\tau^1 - K\right)_+ \mid \mathcal{F}_t\right),$$

which proves the proposition.

Remark 3.3.6 A similar argument shows that, if the process r is zero, the American put is equivalent to its European version.

Chapter 4

Price functions. Numerical methods

4.1 Optimal stopping and stochastic differential equations

Consider a stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t).dW_t, \tag{4.1}$$

where $W = (W_t^1, \ldots, W_t^l)_{0 \le t \le T}$ is a standard *l*-dimensional Brownian motion with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, *b* is a continuous function from $[0, T] \times \mathbb{R}^d$ into \mathbb{R}^d , σ a continuous function from $[0, T] \times \mathbb{R}^d$ into the space $\mathcal{M}_{d,l}$ of real matrices with *d* rows and *l* columns. We assume that the functions *b* and σ satisfy a Lipschitz condition with respect to *x*, which is uniform over time, in other words, there exists a constant C > 0 such that

$$\forall (t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le C|x-y|,$$

where $|\cdot|$ denotes the Euclidian norm on \mathbb{R}^d or on the space $\mathcal{M}_{d,l}$ (viewed as \mathbb{R}^{dl}). We then have existence and uniqueness of a strong solution for equation (4.1) (cf. [81]). For $(t,x) \in [0,T] \times \mathbb{R}^d$, let $(X_s^{t,x})_{t \leq s \leq T}$ be the unique solution of (4.1) on the time interval [t,T], such that $X_t^{t,x} = x$. Let $r: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a continuous nonnegative function and $f: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ a continuous

Let $r: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a continuous nonnegative function and $f: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ a continuous nonnegative function, with $f(t,x) \leq C(1+|x|^k)$ for every $(t,x) \in [0,T] \times \mathbb{R}^d$, where C and k are positive constants. We are interested in the optimal stopping problem with reward process $Z_t = e^{-\int_0^t r(s,X_s)ds} f(t,X_t)$, where X is a solution of (4.1) (with X_0 deterministic). The processus Z is continuous (and regular) and satisfies $\mathbb{E}\sup_{0\leq t\leq T} Z_t < \infty$. The following theorem shows that the problem reduces to the computation of a function F of t and x, called the *value function* of the optimal stopping problem. This type of result can be proved in a very general Markov setting (see [61, 124]).

Theorem 4.1.1 The function F, defined on $[0,T] \times \mathbb{R}^d$ by

$$F(t,x) = \sup_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}\left(\beta_{\tau}^{t,x} f(\tau, X_{\tau}^{t,x})\right), \qquad (4.2)$$

with $\beta_s^{t,x} = \exp(-\int_t^s r(\theta, X_{\theta}^{t,x})d\theta)$ is continuous and if X is a solution of (4.1) (with X_0 deterministic), the process $(\beta_t F(t, X_t))_{0 \le t \le T}$, where $\beta_t = \exp(-\int_0^t r(s, X_s)ds)$, is the Snell envelope of $Z = (\beta_t f(t, X_t))_{0 \le t \le T}$.

Moreover, if the functions r, b and σ do not depend on time, we have

$$F(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E} \left(\beta_{\tau}^{0,x} f(t+\tau, X_{\tau}^{0,x}) \right).$$

4.2 Call and put prices in the Black-Scholes model with dividends

4.2.1 Price functions

In the Black-Scholes model, there is just one risky asset, with price S_t at time t and the various coefficients (namely the interest rate r, the volatility σ , the dividend rate δ) are constants. Under the risk neutral probability measure, which, from now on, we denote by \mathbb{P} , we have

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t, \tag{4.3}$$

where $(W_t)_{0 \le t \le T}$ is standard Brownian motion. We derive the following proposition from the results of the previous chapter and from Theorem 4.1.1.

Proposition 4.2.1 Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous function with sublinear growth. The value at time t of an American option with payoff process $Z_t = \psi(S_t)$ is given by $V(t, S_t)$, where

$$V(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}e^{-r\tau}\psi\left(xe^{\left(r-\delta-\frac{\sigma^2}{2}\right)\tau+\sigma W_{\tau}}\right)$$

The next proposition establishes a symmetry relation between call and put prices. It was observed in the context of foreign exchange options by O. Grabbe [73] and later by various people (see [43], [108]). To clarify our statement, we highlight the dependence of prices with respect to parameters K, r, δ . Namely, we set

$$C(t,x;K,r,\delta) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}e^{-r\tau} \left(x e^{(r-\delta - \frac{\sigma^2}{2})\tau + \sigma W_{\tau}} - K \right)_{+}$$

and

$$P(t,x;K,r,\delta) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}e^{-r\tau} \left(K - xe^{(r-\delta - \frac{\sigma^2}{2})\tau + \sigma W_\tau} \right)_+$$

Proposition 4.2.2 We have

$$C(t, x; K, r, \delta) = P(t, K; x, \delta, r) = xP(t, K/x; 1, \delta, r)$$

Proof: For $\tau \in \mathcal{T}_{0,T-t}$, we have, with the notation \hat{W}_t for $W_t - \sigma t$ and $\hat{\mathbb{P}}$ for the probability measure with density given by $d\hat{\mathbb{P}}/d\mathbb{P} = e^{\sigma W_T - (\sigma^2/2)T}$,

$$\mathbb{E}e^{-r\tau} \left(xe^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_{\tau}} - K \right)_{+} = \mathbb{E}e^{-\delta\tau}e^{\sigma W_{\tau}-(\sigma^2/2)\tau} \left(x - Ke^{(\delta-r+\frac{\sigma^2}{2})\tau-\sigma W_{\tau}} \right)_{+} \\ = \mathbb{E}e^{-\delta\tau}e^{\sigma W_{T}-(\sigma^2/2)T} \left(x - Ke^{(\delta-r-\frac{\sigma^2}{2})\tau-\sigma \hat{W}_{\tau}} \right)_{+},$$

where the last equality comes from the fact that $(e^{\sigma W_t - (\sigma^2/2)t})_{t \ge 0}$ is a martingale. Therefore,

$$\mathbb{E}e^{-r\tau}\left(xe^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_{\tau}}-K\right)_{+}=\hat{\mathbb{E}}e^{-\delta\tau}\left(x-Ke^{(\delta-r-\frac{\sigma^2}{2})\tau-\sigma \hat{W}_{\tau}}\right)_{+}.$$

Now, under probability $\hat{\mathbb{P}}$, the process $(\hat{W}_t)_{0 \le t \le T}$ is a standard Brownian motion, as well as, by symmetry, the process $(-\hat{W}_t)_{0 \le t \le T}$. Hence, $C(t, x; K, r, \delta) = P(t, K; x, \delta, r)$. The other equality is trivial.

4.2.2 Analytic properties of the put price

We now restrict our attention to the American put. Proposition 4.2.2 shows that this entails no loss of generality. We will use the notation P(t, x) for $P(t, x; K, r, \delta)$. We have

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}e^{-r\tau} \psi\left(xe^{(r-\delta - \frac{\sigma^2}{2})\tau + \sigma W_{\tau}}\right),\tag{4.4}$$

with

$$\psi(x) = (K - x)_+.$$

We also assume r > 0 since, if r = 0, the American put is equivalent to the European put (see Remark 3.3.6).

The following properties follow easily from (4.4):

- 1. For every $x \in [0, +\infty)$, $t \mapsto P(t, x)$ is a nonincreasing function.
- 2. For every $t \in [0,T]$, $x \mapsto P(t,x)$ is a nonincreasing convex function.
- 3. For every $(t, x) \in [0, T] \times [0, +\infty), P(t, x) \ge \psi(x) = P(T, x).$

The second assertion is a consequence of the convexity and monotonicity properties of ψ .

Proposition 4.2.3 1. For every $(t, x) \in [0, T] \times [0, +\infty)$, we have

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{0,1}} \mathbb{E}e^{-r\tau(T-t)}\psi\left(xe^{(r-\delta-\frac{\sigma^2}{2})\tau(T-t)+\sigma\sqrt{T-t}W_{\tau}}\right).$$
(4.5)

 \diamond

- 2. For every $t \in [0,T]$, and for $x, y \ge 0$, $|P(t,x) P(t,y)| \le |x-y|$.
- 3. There exits a positive constant C such that, for $x \in [0, +\infty)$, and for $t, s \in [0, T]$,

$$|P(t,x) - P(s,x)| \le C \left| \sqrt{T-t} - \sqrt{T-s} \right|.$$

Proof: The first equality is a consequence of the scaling property of Brownian motion. Indeed, let $\bar{\mathcal{F}}_s = \mathcal{F}_{(T-t)s}$. We have $\tau \in \mathcal{T}_{0,T-t}$ if and only if $\tau/(T-t) \in \bar{\mathcal{T}}_{0,1}$, where $\bar{\mathcal{T}}_{0,1}$ is the set of all stopping times with respect to the filtration $(\bar{\mathcal{F}}_s)_{0 \le s \le 1}$, with values in [0, 1]. Therefore,

$$P(t,x) = \sup_{\tau \in \overline{\mathcal{I}}_{0,1}} \mathbb{E}e^{-r\tau(T-t)}\psi\left(xe^{(r-\delta - \frac{\sigma^2}{2})\tau(T-t) + \sigma W_{\tau(T-t)}}\right)$$

Now, observe that $(\bar{\mathcal{F}}_s)_{0 \le s \le 1}$ is the (completion of) the natural filtration of the process $(W_{s(T-t)})_{0 \le s \le 1}$ and that $(W_{s(T-t)})_{0 \le s \le 1}$ has the same law as $(\sqrt{T-t}W_s)_{0 \le s \le 1}$.

For the second statement, note that, for $\tau \in \mathcal{T}_{0,T-t}$,

$$\left|\psi\left(xe^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_{\tau}}\right)-\psi\left(ye^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_{\tau}}\right)\right|\leq |x-y|e^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_{\tau}},$$

where we have used the Lipschitz property of ψ . The desired inequality can now be derived from $\delta \geq 0$ and $\mathbb{E}e^{\sigma W_{\tau} - \frac{\sigma^2}{2}\tau} = 1$.

The third part of the proposition can be deduced in a similar way from (4.5).

Remark 4.2.4 It follows from the Lipschitz properties of P, as given by Proposition 4.2.3, that the first order partial derivatives of P (in the sense of distributions) are locally bounded on the open set $(0,T) \times (0,+\infty)$. More precisely, we have $||\partial P/\partial x||_{L^{\infty}([0,T]\times[0,+\infty))} \leq 1$ and, for $t \in [0,T[,$ $||(\partial P/\partial t)(t,\cdot)||_{L^{\infty}([0,+\infty))} \leq C/\sqrt{T-t}$. We refer to [123] for the basics of distribution theory.

4.3 The variational inequality

4.3.1 Heuristics

With the assumptions and notation of Section 4.1, we aim at a PDE-characterization of the value function F, given by (4.2). We introduce the so-called *Dynkin operator* \mathcal{D} , associated with equation (4.1):

$$\mathcal{D} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i},$$
(4.6)

where the matrix $a(t,x) = (a_{ij}(t,x))_{1 \le i,j \le d}$ is the product of the matrix $\sigma(t,x)$ with its transpose $(a(t,x) = \sigma(t,x)\sigma^*(t,x))$.

If X is a solution of (4.1) on the interval [0,T] and if $\beta_t = \exp\left(-\int_0^t r(s,X_s)ds\right)$, it follows from Ito's formula that, for a function F of class $C^{1,2}$ on $[0,T] \times \mathbb{R}^d$, we have

$$\beta_t F(t, X_t) = F(0, X_0) + \int_0^t \beta_s (\mathcal{D}F - rF)(s, X_s) ds + \int_0^t \beta_s \nabla F(s, X_s) . \sigma(s, X_s) . dW_s,$$

with the notation $\nabla F(s, X_s) \cdot \sigma(s, X_s) \cdot dW_s = \sum_{i=1}^d \frac{\partial F}{\partial x_i}(s, X_s) \sum_{j=1}^d \sigma_{ij}(s, X_s) dW_s^j$. If ∇F is bounded, the

stochastic integral is a martingale. In order to ensure that the process $(\beta_t F(t, X_t))_{0 \le t \le T}$ be the Snell envelope of the discounted payoff process $(\beta_t f(t, X_t))_{0 \le t \le T}$, we are lead to imposing $\mathcal{D}F - rF \le 0$ on one hand (so that we have a supermartingale), and, on the other hand, with the three conditions of Theoreme 2.3.9 in mind, we need $F \ge f$, $F(T, \cdot) = f(T, \cdot)$ and $\mathcal{D}F - rF = 0$ on the set $\{F > f\}$. The last condition ensures that the nondecreasing process in the Doob-Meyer decomposition increases only on the set of times t which satisfy $F(t, X_t) = f(t, X_t)$. All these conditions can be synthesized in the following form:

$$\begin{cases} \max \left(\mathcal{D}F - rF, f - F \right) = 0\\ F(T, \cdot) = f(T, \cdot). \end{cases}$$

This kind of partial differential equation is called a variational inequality. In fact, the precise connection between the variational inequality and the value function of the optimal stopping problem is delicate, as the function F is in general not of class $C^{1,2}$. For that reason, a notion of weak solution needs to be introduced. The concept of viscosity solution has the merit of requiring little regularity on the payoff function f and on the solution F. If the function f belongs to a suitable Sobolev space, variational methods lead to solutions in a stronger sense (see. [23, 24, 65] for the theory of variational inequalities). This approach was applied to American options in [79] for diffusion models and in [130, 131, 132] for jump-diffusion models. For recent results about American options on several assets, see [33, 126]. Regularity results based on viscosity solutions can be found in [19, 20]. In the case of one-dimensional diffusions, the variational inequality can be derived for very general payoff functions (see [101, 102]). There are also results for exponential Lévy models in [100].

4.3.2 Application to the American put price in the Black-Scholes model

We will now derive some consequences of the variational inequality satisfied by the American put price in the Black-Scholes model. If we set $X_t = \log(S_t)$ in (4.3), we see that X satisfies

$$dX_t = \mu dt + \sigma dW_t, \tag{4.7}$$

with $\mu = r - \delta - \frac{\sigma^2}{2}$. Denote by X^x the solution of (4.7) with $X_0^x = x$, so that $X_t^x = x + \mu t + \sigma W_t$. The Dynkin operator of diffusion X is an operator with constant coefficients, given by

$$\mathcal{D} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}$$

The price function of the American put can be written $P(t, x) = F(t, \log x)$, where the function F is defined by

$$F(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}e^{-r\tau} f(X^x_{\tau}),$$

with

$$f(x) = (K - e^x)_+.$$

The verification of the variational inequality is rather straightforward in this setting, due to the fact that the process X is Brownian motion with drift. Indeed, it can be proved (see for instance [96], proof of Proposition 10.3.7) that, if φ is a bounded continuous function on $[0,T] \times \mathbb{R}$ and $(W_t)_{0 \leq t \leq T}$ standard Brownian motion, the process $(\varphi(t, W_t))_{0 \leq t \leq T}$ is a supermartingale, if and only if the function φ satisfies $(\partial \varphi/\partial t) + (1/2)(\partial^2 \varphi/\partial x^2) \leq 0$ in the sense of distributions in the open set $(0,T) \times \mathbb{R}$. By applying this result to $(t,x) \mapsto e^{-rt}F(t,\mu t + \sigma x)$, we derive the inequality $\mathcal{D}F - rF \leq 0$ in the sense of distributions. On the other hand, we have estimates for $\partial F/\partial t$ and $\partial F/\partial x$ which come from the Lipschitz property of f (see Proposition 4.2.3 and Remark 4.2.4). Moreover, the convexity of the function $x \mapsto P(t, x)$ implies that the second derivative $\partial^2 P/\partial x^2$ (in the sense of distributions) is a nonnegative measure. It follows that $(\partial^2 F/\partial x^2) - (\partial F/\partial x)$ is a nonnegative measure. The functions $\partial F/\partial t$ and $\partial F/\partial x$ being locally bounded, we obtain, when putting together the inequalities $\mathcal{D}F - rF \leq 0$ and $(\partial^2 F/\partial x^2) - (\partial F/\partial x) \geq 0$, that the second derivative $\partial^2 F/\partial x^2$ is a locally bounded function. This provides enough regularity to apply a generalized Ito formula (see [89], chapter 2) and prove that the value function solves the variational inequality. This can be sumarized in the following theorem (see [96] for a detailed proof).

Theorem 4.3.1 1. The partial derivatives $\partial F/\partial x$, $\partial F/\partial t$ and $\partial^2 F/\partial x^2$ are locally bounded. More precisely, $\partial F/\partial x$ is uniformly bounded on $[0,T] \times \mathbb{R}$ and there exists a positive constant C_1 such that

$$\forall t \in [0,T), \qquad \left| \left| \frac{\partial F}{\partial t}(t,\cdot) \right| \right|_{L^{\infty}(\mathbb{R})} + \left| \left| \frac{\partial^2 F}{\partial x^2}(t,\cdot) \right| \right|_{L^{\infty}(\mathbb{R})} \leq \frac{C_1}{\sqrt{T-t}}$$

2. The function F satisfies the following variational inequality:

$$\max\left(\mathcal{D}F(t,x) - rF(t,x), f(x) - F(t,x)\right) = 0, \quad dtdx \ a.e. \ in \ (0,T) \times \mathbb{R}$$

with terminal condition $F(T, \cdot) = f$.

Corollary 4.3.2 The function $(\partial F/\partial x)$ is continuous on $[0,T) \times \mathbb{R}$.

Proof: Based on Theorem 4.3.1, it suffices to prove that if U(t, x) is a continuous function on $\mathbb{R} \times \mathbb{R}$ with partial derivatives $\partial U/\partial x$, $\partial U/\partial t$ and $\partial^2 U/\partial x^2$ that are uniformly bounded functions on $\mathbb{R} \times \mathbb{R}$, then $\partial U/\partial x$ is a continuous function (a standard localization procedure will allow to extend the result to the case of locally bounded derivatives). This is a classical result in analysis (see [91], chapter 2, Lemma 3.1). Indeed, one can prove that if U is of class C^2 on $\mathbb{R} \times \mathbb{R}$, with bounded derivatives $\partial U/\partial x$, $\partial U/\partial t$ and $\partial^2 U/\partial x^2$, the function $\partial U/\partial x$ is Hölder continuous with exponent 1/2 with respect to time, and Lipschitz continuous with respect to the space variable, with a control on the constants by the bounds on the derivatives.

For this, fix $s, t, x_0 \in \mathbb{R}$. We have, for every real number x,

$$\int_{x_0}^x \frac{\partial U}{\partial x}(t,y)dy - \int_{x_0}^x \frac{\partial U}{\partial x}(s,y)dy = U(t,x) - U(t,x_0) - U(s,x) + U(s,x_0)$$
$$= U(t,x) - U(s,x) + U(s,x_0) - U(t,x_0).$$

Hence,

$$\left| \int_{x_0}^x \frac{\partial U}{\partial x}(t,y) dy - \int_{x_0}^x \frac{\partial U}{\partial x}(s,y) dy \right| \le 2 \left| \left| \frac{\partial U}{\partial t} \right| \right|_{L^{\infty}} |t-s|.$$

On the other hand, for $\tau = t, s$, we have

$$\left|\frac{\partial U}{\partial x}(\tau, x_0) - \frac{1}{x - x_0} \int_{x_0}^x \frac{\partial U}{\partial x}(\tau, y) dy\right| \le \left|\left|\frac{\partial^2 U}{\partial x^2}\right|\right|_{L^{\infty}} |x - x_0|.$$

Therefore,

$$\left|\frac{\partial U}{\partial x}(t,x_0) - \frac{\partial U}{\partial x}(s,x_0)\right| \le 2 \left|\left|\frac{\partial U}{\partial t}\right|\right|_{L^{\infty}} \frac{|t-s|}{|x-x_0|} + 2 \left|\left|\frac{\partial^2 U}{\partial x^2}\right|\right|_{L^{\infty}} |x-x_0|.$$

By choosing x so that $|x - x_0| = \sqrt{t - s}$, we get

$$\left|\frac{\partial U}{\partial x}(t,x_0) - \frac{\partial U}{\partial x}(s,x_0)\right| \le 2\sqrt{|t-s|} \left(\left|\left|\frac{\partial U}{\partial t}\right|\right|_{L^{\infty}} + \left|\left|\frac{\partial^2 U}{\partial x^2}\right|\right|_{L^{\infty}} \right).$$

Moreover, we have, for $t, x, y \in \mathbb{R}$,

$$\left|\frac{\partial U}{\partial x}(t,x) - \frac{\partial U}{\partial x}(t,y)\right| \le \left|\left|\frac{\partial^2 U}{\partial x^2}\right|\right|_{L^{\infty}} |x-y|.$$

Remark 4.3.3 The continuity of $\partial F/\partial x$ is a refined version of the so-called "smooth-fit property".
This property can be approached by more probabilistic methods (see [62, 78, 128, 63] and [117]). This
property has been studied in the context of exponential Lévy models in [5] for the perpetual case,
and, for finite maturities, in [132] (jump-diffusion case, see also [21]).

The continuity of $(\partial F/\partial t)$ on $[0, T) \times \mathbb{R}$ can be derived from quadratic estimates for the second order derivatives of F (see [64] and [87], chapter 8). We also refer to [28] for recent results on the continuity of the time derivative.

Remark 4.3.4 The limit of the American put price as T tends to infinity can be computed explicitly. This is the price of a *perpetual* put. To be more specific, if

$$P_{\infty}(x) = \sup_{\tau \in \mathcal{T}_{0,+\infty}} \mathbb{E}e^{-r\tau} \psi\left(xe^{(r-\delta - \frac{\sigma^2}{2})\tau + \sigma W_{\tau}}\right),\tag{4.8}$$

with $\psi(x) = (K - x)_+$ and $\mathcal{T}_{0,+\infty}$ the set of all finite stopping times, it can be proved that $P_{\infty}(x) = K - x$, if $x \leq x^*$ and $P_{\infty}(x) = (K - x^*)(x/x^*)^{-\gamma}$, for $x > x^*$, with $x^* = K\gamma/(1 + \gamma)$ and

$$\gamma = \frac{1}{\sigma^2} \left[\left(r - \delta - \frac{\sigma^2}{2} \right) + \sqrt{\left(r - \delta - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right].$$

These formulae go back to [109] (see also [98] chapter 4, section 4, for a more probabilistic approach). They can be extended in various ways (see, for instance, [68]). For recent results on perpetual options in models based on Lévy processes, see [110].

 \diamond

4.3.3 The exercise boundary and the early exercise premium

For $t \in [0, T)$, let

$$s^*(t) = \inf\{x \in [0, +\infty) \mid P(t, x) > \psi(x) = (K - x)_+\}.$$

The number $s^*(t)$ is called *critical price* at time t. We clearly have $0 \leq s^*(t) < K$ for every $t \in [0, T[$. The inequality $s^*(t) < K$ follows from P(t, x) > 0. Indeed, we have $P(t, x) \geq \mathbb{E}e^{-r(T-t)}\psi\left(xe^{(r-\delta-\frac{\sigma^2}{2})(T-t)+\sigma W_{T-t}}\right)$. On the other hand, $s^*(t) \geq x^*$, where x^* is defined in Remark 4.3.4. This follows from the inequality $P(t, x) \leq P_{\infty}(x)$.

We deduce from the convexity of $x \mapsto P(t, x)$ that

$$\forall x \leq s^*(t), \quad P(t,x) = K - x \quad \text{and} \quad \forall x > s^*(t), \quad P(t,x) > (K - x)_+.$$

By translating the variational inequality satisfied by $F(t, x) = P(t, e^x)$ into an inequality satisfied by P, we get, dtdx almost everywhere on $(0, T) \times (0, +\infty)$,

$$\frac{\partial P}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 P}{\partial x^2}(t,x) + (r-\delta)x\frac{\partial P}{\partial x}(t,x) - rP(t,x) = (\delta x - rK)\mathbf{1}_{\{x \le s^*(t)\}}.$$

The function s^* is called the free boundary or exercise boundary. On the set $\{P > \psi\}$, called the *continuation region*, the function P solves

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 P}{\partial x^2} + (r-\delta)x\frac{\partial P}{\partial x} - rP = 0.$$

The set $\{P = \psi\}$ is called the *stopping region*. Using the generalized Ito formula (see [89], chapter 2, we get

$$e^{-rt}P(t,S_t) = P(0,S_0) + \int_0^t e^{-ru}\sigma S_u \frac{\partial P}{\partial x}(u,S_u)dW_u + \int_0^t e^{-ru}(\delta S_u - rK)\mathbf{1}_{\{S_u \le s^*(u)\}}du.$$

Thus, we have obtained an explicit form of the Doob-Meyer decomposition of the supermartingale $(e^{-rt}P(t, S_t))_{0 \le t \le T}$. Incidentally, observe that the amount of risky asset in the minimal hedging strategy is given by $H_t = (\partial P/\partial x)(t, S_t)$.

By letting $t \to T$ and taking expectations, we obtain

$$P_e(0, S_0) = P(0, S_0) + \int_0^T e^{-ru} \mathbb{E}\left((\delta S_u - rK) \mathbf{1}_{\{S_u \le s^*(u)\}}\right) du,$$

where $P_e(t,x) = \mathbb{E}e^{-r(T-t)}\psi\left(xe^{(r-\delta-\frac{\sigma^2}{2})(T-t)+\sigma W_{T-t}}\right)$. Note that P_e is the function price of the *European* put. We are thus lead to the following relation between P(t,x) and $P_e(t,x)$.

$$P(t,x) = P_e(t,x) + \int_0^{T-t} \left(rKe^{-ru} N(d_1(x,t,u)) - \delta x e^{-\delta u} N(d_2(x,t,u)) \right) du,$$
(4.9)

where N is the standard normal cumulative distribution function,

$$d_1(x, t, u) = \frac{\log(s^*(t+u)/x) - (r - \delta - \frac{\sigma^2}{2})u}{\sigma\sqrt{u}},$$
$$d_2(x, t, u) = \frac{\log(s^*(t+u)/x) - (r - \delta + \frac{\sigma^2}{2})u}{\sigma\sqrt{u}}.$$

Remark 4.3.5 The relation (4.9) gives an explicit expression for the quantity $P - P_e$, called the *early* exercise premium (as it is the additional price you have to pay for the possibility of exercising before maturity). It can be used to derive an integral equation satisfied by the function s^* . It seems to have been discovered and used by several persons at essentially the same time (see [86], [43], [39], [76]). The proof of [76] is probabilistic and does not rely on the variational inequality.

Remark 4.3.6 The function s^* is clearly nondecreasing on [0, T]. It can be proved that $\lim_{t\to T} s^*(t) = K \wedge (rK/\delta)$ (cf. [86]), that s^* is differentiable on [0, T) (cf. [64], [87], [29], [44]). The behavior of $s^*(t)$ as t approaches T has been studied in [15] (see also [93]) in the case $\delta = 0$ and in [99] in the case $\delta \neq 0$. More precise results have recently been derived in [44]. The convexity of s^* was proved in [59, 45] in the no-dividend case, but does not seem to be true in general. See also [118, 129, 22] for results in the case of jump diffusions, [100] for general Lévy processes and [47] for local volatility models. For higher dimensional problems, the exercise region has been studied by a number of authors (see [33], [126], [46], [55]). We also refer to the recent monograph [117] for a systematic study of free boundary problems in connection with optimal stopping.

4.4 Numerical methods

During the last ten years, a large number of numerical methods for American options appeared in the literature. A first set of methods can be related to *analytic approaches*, based on the analytic characterization of the value function (variational inequality, free boundary problem). Other methods are closer to the probabilistic formulation and may involve Monte-Carlo methods. Note that many numerical methods have been implemented in the **Premia** software (which can be downloaded from the web-site http://www-rocq.inria.fr/mathfi/Premia/index.html).

In this section, we will first survey numerical methods from a historical perspective, then we will present with more details the linear regression method proposed by Longstaff and Schwartz [103]. For comparisons of various methods, we refer to [70, 32, 34, 4] and, for more recent comparisons to [67, 37].

4.4.1 Analytic methods

Analytic methods are based on the solution of the variational inequality (or the free boundary problem) satisfied by the price function. The finite difference method of Brennan and Schwartz [31] was the firs to appear in the financial literature (see [79] for a rigorous justification and [131, 132] for a complete proof of the result of strong convergence and an extension to models with jumps). More complex options were treated in [54] and [17, 18]. Among recent papers using numerical analysis, one can mention [127] (ADI methods), [1] (finite elements), [106, 107, 75] (wavelet methods), [26] (finite volumes).

The quadratic approximation of MacMillan [105] (see also [16, 48]) is a quasi-explicit approximation of the American put price in the Black-Scholes model, which has some extensions to models with jumps [131]. The paper [40] can be seen as a refinement of this method, which leads to a more accurate approximation. In a somewhat different spirit, approximations of American prices by European options can be given (see [80]). Asymptotic expansions based on the integral equation can be found in [44].

4.4.2 Probabilistic methods based on approximations of the stopping times or of the underlying

A natural idea for approximating the price of an American option consists in restricting the set of stopping times to discrete stopping times (with values in some subdivision of the interval [0, T]).

This amounts to replacing the American option by a "Bermudean" one (i.e. an option which can be exercized at a given finite number of dates). This method appears (for the case of the put in the Black-Scholes model) in [116], [69] but the exact computation of the Bermudean put can be achieved only if the number of exercise dates is small. On the other hand a precise estimate of the approximation error can be given (cf. [42]) and, in infinite horizon, an expansion of the error (cf [57]). This approach has been extended to jump-diffusion models in [111]. Note that there are other ways of restricting the set of stopping times (in particular, one can consider hitting times, see [27] or [32]). A "randomization" of stopping times has also been proposed by [41], and analysed in [30]. There are also iterative constructions of the optimal stopping time of Bermudean options (cf. [88]).

When the number of exercise dates of the Bermudean option is large, or when the underlying is multidimensional, additional approximations are needed. The simplest method consists in approximating Brownian motion by a random walk. The so called *binomial* method (see [50]) and more generally *tree* methods fit in this framework. Convergence results for this type of approximation in a general setting go back to [90] and [2] and have been applied to financial models in [6, 7]. Another approach to convergence has been developed in [92] (see also [112] for results on the functional convergence of envelopes). It seems to be difficult to establish the precise rate of convergence for these approximations (for some results in this direction, see [94], [95], [85]).

4.4.3 Quantization methods

Quantization methods the computation Bermudean have been develfor of options oped by Bally, Pagès and Printems (see [9, 10, 11, 12, 13]). The approach is based on dynamic programming, quantization techniques are used for the approximation conditional expectations. Information on quantization methods can be found on of http://www.quantification.finance-mathematique.com/

4.4.4 Monte-Carlo methods

Monte-Carlo methods for American (or, rather, Bermudean) option pricing were initiated by Broadie and Glasserman [35, 36], see also the recent monograph [72]. Techniques involving Malliavin calculus have also been developed (cf. [66, 120] and, for a comparison with other methods, [37, 67]).

As mention in Section 2.4, the duality approach can be used to compute American option prices by Monte-Carlo techniques (cf. [122], see also [74]). In fact, using the representation (2.1), one can compute an upper bound by choosing a martingale vanishing at 0 and running simulations of the payoff process and the martingale. The problem is to find a good martingale (the martingale which achieves the minimum in (2.1) is not known explicitly).

Another recent approach is based on regression methods (cf. [38, 103, 125]). We will show how this approach develops in the Longstaff-Schwartz algorithm (see [103]).

Let $(Z_j)_{j=0,...,L}$ be a sequence of square integrable real random variables. We assume that this sequence is adapted to some filtration $(\mathcal{F}_j)_{j=0,...,L}$. Denote by $\mathcal{T}_{j,L}$ the set of all stopping times with values in $\{j, \ldots, L\}$. We have

$$\sup_{\tau \in \mathcal{T}_{j,L}} \mathbb{E} Z_{\tau} = U_0$$

with

$$\begin{cases} U_L = Z_L \\ U_j = \max\left(Z_j, \mathbb{E}\left(U_{j+1} \mid \mathcal{F}_j\right)\right), & 0 \le j \le L - 1. \end{cases}$$

We also have $U_j = \mathbb{E} (Z_{\tau_j} | \mathcal{F}_j)$, with

$$\tau_j = \min\{k \ge j \mid U_k = Z_k\},$$

In particular, $\mathbb{E}U_0 = \sup_{\tau \in \mathcal{T}_{0,L}} \mathbb{E}Z_{\tau} = \mathbb{E}Z_{\tau_0}.$

The dynamic programming principle can be written as follows:

$$\begin{cases} \tau_L = L \\ \tau_j = j \mathbf{1}_{\{Z_j \ge \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\}} + \tau_{j+1} \mathbf{1}_{\{Z_j < \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\}}, & 0 \le j \le L-1 \end{cases}$$

and, if $Z_j = f(j, X_j)$, where $(X_j)_{j=0,...,L}$ is an (\mathcal{F}_j) -Markov chain, we can replace $\mathbb{E}(\cdot|\mathcal{F}_j)$ by $\mathbb{E}(\cdot|X_j)$. The first approximation consists in replacing $\mathbb{E}(\cdot|X_j)$ by the linear regression on $e_1(X_j), \ldots, e_m(X_j)$, where e_1, e_2, \ldots form a basis of functions. Namely, we set (with $e^m = (e_1, \ldots, e_m)$)

$$\begin{cases} \tau_L^{[m]} = L \\ \tau_j^{[m]} = j \mathbf{1}_{\{Z_j \ge \alpha_j^m \cdot e^m(X_j)\}} + \tau_{j+1}^{[m]} \mathbf{1}_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}}, \quad 1 \le j \le L-1, \end{cases}$$

where

$$\alpha_j^m = \arg\min_{a \in \mathbb{R}^m} \mathbb{E}\left(Z_{\tau_{j+1}^{[m]}} - a \cdot e^m(X_j)\right)^2.$$

This provides an approximation of U_0 by

$$U_0^m = \max\left(Z_0, \mathbb{E}Z_{\tau_1^{[m]}}\right).$$

The second approximation consists in using a Monte-Carlo method for the computation of $\mathbb{E}Z_{\tau_1^{[m]}}$. We assume that we can simulate N independent samples $(X_j^{(1)}), \ldots, (X_j^{(n)}), \ldots, (X_j^{(N)})$ of the Markov chain (X_j) and we denote by $Z_j^{(n)}$ $(Z_j^{(n)} = f(j, X_j^{(n)}))$ the associated payoff for $j = 0, \ldots, L$, and $n = 1, \ldots, N$. For the path numbered n, the times $\tau_j^{[m]}$ can be approached by:

$$\begin{cases} \tau_L^{n,m,N} = L \\ \tau_j^{n,m,N} = j \mathbf{1}_{\left\{Z_j^{(n)} \ge \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}} + \tau_{j+1}^{n,m,N} \mathbf{1}_{\left\{Z_j^{(n)} < \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}} \\ (1 \le j \le L-1) \end{cases}$$

where $\alpha_j^{(m,N)}$ is the least squares estimator:

$$\alpha_{j}^{(m,N)} = \arg\min_{a \in \mathbb{R}^{m}} \sum_{n=1}^{N} \left(Z_{\tau_{j+1}^{n,m,N}}^{(n)} - a \cdot e^{m}(X_{j}^{(n)}) \right)^{2},$$

We then have the following approximation for U_0^m :

$$U_0^{m,N} = \max\left(Z_0, \frac{1}{N}\sum_{n=1}^N Z_{\tau_1^{n,m,N}}^{(n)}\right).$$

Under some moment and regularity hypotheses it can be proved that the vector

$$\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} (Z_{\tau_{j}^{n,m,N}}^{(n)} - \mathbb{E}Z_{\tau_{j}^{[m]}}), \sqrt{N}(\alpha_{j}^{(m,N)} - \alpha_{j}^{m})\right)_{j=1,\dots,L-1}$$

converges in distribution towards a Gaussian vector as N goes to infinity. For the proofs of the convergence results, we refer to [49] (see also [58]).

Remark 4.4.1 In fact, the real Longstaff-Schwartz algorithm involves only at the money samples, namely those for which $Z_j^{(n)} > 0$, which improves the efficiency of the algorithm (cf. [103]), but does not alter the analysis of convergence (see [49], Remark 2.1). The practical implementation of the method requires a good choice of the basis functions, of the truncation number m, and of the number of trials (see [71] for more comments).

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